ISRAEL JOURNAL OF MATHEMATICS **241** (2021), 465–500
DOI: 10.1007/s11856-021-2105-5

CONCENTRATION PHENOMENA FOR NONLINEAR MAGNETIC SCHRÖDINGER EQUATIONS WITH CRITICAL GROWTH

BY

CHAO JI

Department of Mathematics, East China University of Science and Technology
Shanghai, 200237, China
e-mail: jichao@ecust.edu.cn

AND

VICENŢIU D. RĂDULESCU

Faculty of Applied Mathematics, AGH University of Science and Technology
30-059 Kraków, Poland
and
Department of Mathematics, University of Craiova, 200585 Craiova, Romania
e-mail: radulescu@inf.ucv.ro

ABSTRACT

In this paper, we are concerned with the following nonlinear magnetic Schrödinger equation with critical growth:

$$\begin{cases} (\frac{\varepsilon}{i}\nabla - A(x))^2 u + V(x)u = f(|u|^2)u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$ is a parameter, $N \geq 3$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, $V : \mathbb{R}^N \to \mathbb{R}$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ are continuous potentials, $f : \mathbb{R} \to \mathbb{R}$ is a subcritical nonlinear term. Under a local assumption on the potential V, by the variational methods, the penalization technique and the Ljusternic–Schnirelmann theory, we prove the multiplicity and concentration of nontrivial solutions of the above problem for ε small. For the problem, the function f is only continuous, which allows to consider larger classes of nonlinearities in the reaction.

Received July 20, 2020 and in revised form December 25, 2020

1. Introduction and main results

In this paper, we study multiplicity and concentration results for the following nonlinear magnetic Schrödinger equation with critical growth:

$$(1.1)\qquad \begin{cases} (\frac{\varepsilon}{i}\nabla - A(x))^2 u + V(x)u = f(|u|^2)u + |u|^{2^*-2}u & \text{in } \mathbb{R}^N,\\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$ is a parameter, $N \geq 3$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is Hölder continuous with exponent $\alpha \in (0, 1]$, and $-\Delta_A u$ is the magnetic Laplace with the following form:

$$-\Delta_A u := \Big(\frac{1}{i}\nabla - A(x)\Big)^2 u = -\Delta u - \frac{2}{i}A(x)\cdot \nabla u + |A(x)|^2 u - \frac{1}{i}u\,\mathrm{div}(A(x)).$$

Problem (1.1) arises when one looks for standing wave solutions

$$\psi(x,t) := e^{-iEt/\hbar}u(x)$$

(with $E \in \mathbb{R}$) of the nonlinear evolution system

$$i\hbar\frac{\partial\psi}{\partial t} = \Big(\frac{\hbar}{i}\nabla - A(x)\Big)^2\psi + U(x)\psi - f(|\psi|^2)\psi, \quad x \in \mathbb{R}^N.$$

From a physical point of view, the existence of such solutions and the study of their shape in the semiclassical limit, namely, as $\hbar \to 0^+$ (or, equivalently, as $\varepsilon \to 0^+$ in (1.1)), is of the greatest importance, since the transition from quantum mechanics to classical mechanics can be formally performed by sending to zero the Planck constant $\hbar$.

For problem (1.1), there is a vast literature concerning the existence and multiplicity of bound states for the case without magnetic field, namely if $A \equiv 0$. The first result in this direction was given by Floer and Weinstein [29], who considered the case $N = 1$ and $f = i_{\mathbb{R}}$. Later on, several authors generalized this result to larger values of N, using different methods. For instance, del Pino and Felmer [27] studied the existence and concentration of solutions to the following problem

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = f(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a possibly unbounded domain in $\mathbb{R}^N$ ($N \geq 3$), the potential V is locally Hölder continuous, bounded from below away from zero, there exists a bounded open set $\Lambda \subset \Omega$ such that

$$\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x), \tag{1.2}$$

and the nonlinearity f satisfies some subcritical growth conditions. In [2], Alves and Figueiredo considered the following quasilinear elliptic equation

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = f(u), & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where V is a positive continuous function and satisfies the local assumption (1.2), $f \in C^1$ is a function having subcritical and superlinear growth. By using the Nehari manifold method and the Ljusternik–Schnirelmann category theory, the authors obtained the multiplicity of positive solutions. In order to apply the Nehari manifold method, the authors assumed that $f \in C^1$, which ensures that the Nehari manifold is a C^1-manifold. If f is only continuous, then the Nehari manifold is only a topological manifold, thus the arguments developed in [2] collapse. We notice that Szulkin and Weth in [41] considered the multiple solutions for the nonlinear stationary Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$, here f is superlinear, subcritical and continuous. In order to use the method of the Nehari manifold, they developed a new approach. In [30], He and Zou considered the following fractional Schrödinger equation:

$$\epsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + u^{2_s^*-1}, \quad x \in \mathbb{R}^N,$$

where V satisfies the local assumption (1.2), and f is subcritical. By using the Nehari manifold method and the Ljusternik–Schnirelmann category theory, the authors obtained the multiplicity of positive solutions. We notice that f is only continuous in [30], the Nehari manifold is only a topological manifold, thus the critical points theory in the C^1 manifold can not be applied. To overcome the difficulty, He and Zou [30] applied the method that Szulkin and Weth developed in [41]. For further results about existence, multiplicity and qualitative properties of semiclassical states with various types of concentration behaviors, which have been established under various assumptions on the potential V and on the nonlinearity f, see [4, 5, 6, 8, 9, 10, 16, 17, 18, 26, 34, 35, 39, 40, 43] and the references therein (see [7] for the fractional case). We also refer to [12, 15, 19, 21, 45] for related contributions to the study of concentration phenomena associated to various Schrödinger-type equations.

On the other hand, the magnetic nonlinear Schrödinger system (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [3, 13, 14, 20, 23, 22, 24, 25, 28, 32, 31, 33, 36, 38] and references therein). To the best of our knowledge, the first result involving the magnetic field was obtained by Esteban and Lions [28]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we want to mention [3] where the authors used the method of the Nehari manifold, the penalization method, and the Ljusternik–Schnirelmann category theory for a subcritical nonlinearity $f \in C^1$. We point out that if f is only continuous, then the arguments developed in [3] fail. In [31], Ji and Rădulescu used the method of the Nehari manifold, the penalization technique and Ljusternik–Schnirelmann category theory to study the multiplicity and concentration results for a magnetic Schrödinger equation in which the subcritical nonlinearity f is only continuous.

It is quite natural to consider the multiplicity and concentrating phenomena of nontrivial solutions for problem (1.1) with critical growth. Inspired by [30, 31], the main purpose of this paper is to investigate multiplicity and concentration of nontrivial solutions for problem (1.1) by combining a local assumption on V and adapting the penalization technique and Ljusternik–Schnirelmann category theory.

Throughout the paper, we make the following assumptions on the potential V:

- $(V1)$ There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$.
- $(V2)$ There exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying:

- $(f1)$ $f(t) = 0$ if $t \leq 0$;
- $(f2)$ there exists $\sigma, q \in (2, 2^*)$ and $\mu > 0$ such that

$$f(t) \geq \mu t^{\sigma - 2/2} \ \forall t > 0, \quad \lim_{t \to +\infty} \frac{f(t^2)t}{t^{q-1}} = 0;$$

($f3$) there is a positive constant $\theta > 2$ such that

$$0 < \frac{\theta}{2}F(t) \leq tf(t), \quad \forall\, t > 0, \text{ where } F(t) = \int_0^t f(s)ds;$$

($f4$) $f(t)$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following:

Theorem 1.1: *Assume that V satisfies ($V1$), ($V2$) and f satisfies ($f1$)–($f4$). Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ε_n} one of these solutions of (1.1) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^N$ the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

The proof of Theorem 1.1 is inspired from [30, 31]. Notice that, due to the presence of the magnetic field $A(x)$, problem (1.1) cannot be changed into a pure real-valued problem, hence we must deal directly with a complex-valued problem, which causes several new difficulties in employing the methods in dealing with our problem. On the other hand, since the problem we deal with has critical growth, we need more refined estimates to overcome the lack of compactness. The plan of the paper is as follows: in Section 2 we introduce the functional setting and give some preliminaries. In Section 3, we study the modified problem. We prove the Palais–Smale condition for the modified functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the associated autonomous problem. It allows us to show the modified problem has the multiple solutions. Finally, in Section 5, we give the proof of Thereom 1.1.

Notation.

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- $B_R(y)$ denotes the open ball centered at $y \in \mathbb{R}^N$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^N$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^N, \mathbb{R})$, $L^q(\mathbb{R}^N, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^N$. $\langle \cdot, \cdot \rangle_0$ denotes the inner product of the space $H^1(\mathbb{R}^N, \mathbb{R})$.

2. The variational framework and the limit problem

For $u : \mathbb{R}^N \to \mathbb{C}$, let us denote

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u$$

and

$$H^1_A(\mathbb{R}^3, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N, \mathbb{R})\}.$$

The space $H^1_A(\mathbb{R}^N, \mathbb{C})$ is Hilbert space endowed with the scalar product

$$\langle u, v\rangle_H := \mathrm{Re}\int_{\mathbb{R}^N} (\nabla_A u\overline{\nabla_A v} + u\overline{v})dx, \quad \text{for any } u, v \in H^1_A(\mathbb{R}^N, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover, we denote by $\|u\|_A$ the norms induced by inner product $\langle u, v\rangle_A$.

On $H^1_A(\mathbb{R}^N, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see, e.g., [37, Theorem 7.21]):

$$|\nabla_A u(x)| \geq |\nabla|u(x)||. \tag{2.1}$$

Moreover, making a simple change of variables, we can see that (1.1) is equivalent to

$$\Big(\frac{1}{i}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = f(|u|^2)u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Let H_ε be the Hilbert spaces obtained as the closure of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_\epsilon := \mathrm{Re}\int_{\mathbb{R}^N} \Big(\nabla_{A_\varepsilon} u\overline{\nabla_{A_\varepsilon} v} + V_\varepsilon(x)u\overline{v}\Big)dx$$

and let us denote by $\|\cdot\|_\varepsilon$ the norm induced by inner product $\langle\cdot,\cdot\rangle_\epsilon$.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ and $\|u\| \leq C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$ is continuous for $2 \leq r \leq 2^*$ and the embedding $H_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ is compact for $1 \leq r < 2^*$.

For compact supported functions in $H^1(\mathbb{R}^N, \mathbb{R})$, we have the following result, which will be very important for some estimates below.

Lemma 2.1: *If $u \in H^1(\mathbb{R}^N, \mathbb{R})$ and u has compact support, then*

$$\omega := e^{iA(0)\cdot x}u \in H_\varepsilon.$$

Proof. Assume that $\mathrm{supp}(u) \subset B_R(0)$. Since V is continuous, it is clear that

$$\int_{\mathbb{R}^N} V_\varepsilon(x)|\omega|^2 dx = \int_{B_R(0)} V_\varepsilon(x)|\omega|^2 dx \leq C\|u\|_2^2 < +\infty.$$

Moreover, since V and A are continuous, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon}\omega|^2 dx &= \int_{\mathbb{R}^N} |\nabla\omega|^2 dx + \int_{\mathbb{R}^N} |A_\varepsilon(x)|^2|\omega|^2 dx + 2\mathrm{Re}\int_{\mathbb{R}^N} iA_\varepsilon(x)\overline{\omega}\nabla\omega dx \\ &\leq 2\int_{\mathbb{R}^N} |\nabla\omega|^2 dx + 2\int_{\mathbb{R}^N} |A_\varepsilon(x)|^2|\omega|^2 dx \\ &\leq C\left[\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx\right] < +\infty\end{aligned}$$

and we conclude. ∎

3. The modified problem

To study (1.1), or equivalently, (2.2) by variational methods, we shall modify suitably the nonlinearity f so that, for $\varepsilon > 0$ small enough, the solutions of such a modified problem are also solutions of the original one. More precisely, we choose $K > 2$, there exists a unique number $a_0 > 0$ verifying

$$f(a_0) + a_0^{(2^*-2)/2} = V_0/K$$

by $(f4)$, where V_0 is given in $(V1)$, and we consider the function

$$\tilde{f}(t) := \begin{cases} f(t) + (t^+)^{(2^*-2)/2}, & t \leq a_0, \\ V_0/K, & t > a_0, \end{cases}$$

and introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(x,t) := \chi_\Lambda(x)(f(t) + (t^+)^{(2^*-2)/2}) + (1 - \chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where χ_Λ is the characteristic function on Λ and

$$G(x,t) := \int_0^t g(x,s)ds.$$

In view of $(f1)$–$(f4)$, we have that g is a Carathéodory function satisfying the following properties:

- $(g1)$ $g(x,t) = 0$ for each $t \leq 0$;
- $(g2)$ $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$;
- $(g3)$ $g(x,t) \leq f(t) + t^{(2^*-2)/2}$ for all $t \geq 0$ and any $x \in \mathbb{R}^N$;

($g4$) $0 < \theta G(x,t) \le 2g(x,t)t$ for each $x \in \Lambda$ and $t > 0$;

($g5$) $0 < G(x,t) \le g(x,t)t \le V_0 t/K$, for each $x \in \Lambda^c$, $t > 0$;

($g6$) for each $x \in \Lambda$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0,+\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0, a_0)$.

Then we consider the modified problem

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = g(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Note that if u is a nontrivial solution of problem (3.2) with

$$|u(x)|^2 \le a_0 \quad \text{for all } x \in \Lambda_\varepsilon^c,\ \Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\},$$

then u is a nontrivial solution of problem (2.2).

The functional associated to problem (3.2) is

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} G(\varepsilon x, |u|^2)dx \quad \text{for all } u \in H_\varepsilon,$$

defined in H_ε. By standard arguments we obtain that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its critical points are the weak solutions of the modified problem (3.2).

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of J_ε, that is,

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \backslash \{0\} : J'_\varepsilon(u)[u] = 0\},$$

and define the number c_ε by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Let H_ε^+ be the open subset H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\text{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S_\varepsilon^+ = S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε. Note that S_ε^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in H_ε^+. Therefore,

$$H_\varepsilon = T_u S_\varepsilon^+ \bigoplus \mathbb{R}u$$

for each $u \in T_u S_\varepsilon^+$, where

$$T_u S_\varepsilon^+ = \{v \in H_\varepsilon : \langle u, v\rangle_\epsilon = 0\}.$$

Arguing as in Lemma 3.1 in [31], we can show that the functional J_ε satisfies the mountain pass geometry (see [11, 44]).

LEMMA 3.1: *For any fixed $\varepsilon > 0$, the functional J_ε satisfies the following properties:*

(i) *there exist $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$;*
(ii) *there exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > r$ such that $J_\varepsilon(e) < 0$.*

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S_ε^+.

LEMMA 3.2: *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ are satisfied, then the following properties hold:*

(A1) *For any $u \in H_\varepsilon^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\varepsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞).*
(A2) *There is a $\tau > 0$ independent on u such that $t_u \geq \tau$ for all $u \in S_\varepsilon^+$. Moreover, for each compact $\mathcal{W} \subset S_\varepsilon^+$ there is $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$.*
(A3) *The map $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ given by $\widehat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \widehat{m}_\varepsilon|_{S_\varepsilon^+}$ is a homeomorphism between S_ε^+ and $\mathcal{N}_\varepsilon$. Moreover,*

$$m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}.$$

(A4) *If there is a sequence $\{u_n\} \subset S_\varepsilon^+$ such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Proof. (A1) Arguing as in Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t \geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ verifying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Now, we show that t_u is unique. Arguing by contradiction, suppose that there exist $t_1 > t_2 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i \int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx = \int_{\mathbb{R}^N} g(\varepsilon x, t_i^2|u|^2) t_i |u|^2 dx.$$

Hence,

$$\frac{\int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u|^2 + V_\varepsilon(x)|u|^2)dx}{t_i^2} = \int_{\mathbb{R}^N} \frac{g(\varepsilon x, t_i^2|u|^2)|u|^2}{t_i^2} dx,$$

which implies that

$$\begin{aligned}
&\Big(\frac{1}{t_1^2}-\frac{1}{t_2^2}\Big)\Big(\int_{\mathbb{R}^N}(|\nabla_{A_\varepsilon}u|^2+V_\varepsilon(x)|u|^2)dx\Big)\\
&\qquad=\int_{\mathbb{R}^N}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
&\qquad\geq\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
&\qquad\quad+\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}\Big(\frac{g(\varepsilon x,t_1^2|u|^2)}{t_1^2|u|^2}-\frac{g(\varepsilon x,t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4dx\\
&\qquad=\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2}-\frac{f(t_2^2|u|^2)+t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4dx\\
&\qquad\quad+\frac{1}{K}\Big(\frac{1}{t_1^2}-\frac{1}{t_2^2}\Big)\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}V_0|u|^2dx.
\end{aligned}$$

Since $t_1>t_2>0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla_{A_\varepsilon}u|^2+V_\varepsilon(x)|u|^2)dx\\
&\qquad\leq\frac{t_1^2t_2^2}{t_2^2-t_1^2}\int_{\Lambda_\varepsilon^c\cap\{t_2^2|u|^2\leq a_0\leq t_1^2|u|^2\}}\Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2}-\frac{f(t_2^2|u|^2)+t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4dx\\
&\qquad\quad+\frac{1}{K}\int_{\Lambda_\varepsilon^c\cap\{a_0\leq t_2^2|u|^2\}}V_0|u|^2dx\\
&\qquad\leq\frac{1}{K}\int_{\Lambda_\varepsilon^c}V_0|u|^2dx\leq\frac{1}{K}\|u\|_\varepsilon^2,
\end{aligned}$$

which is a contradiction. Therefore, $\max_{t\geq 0}g_u(t)$ is achieved at a unique $t=t_u$ so that $g_u'(t)=0$ and $t_uu\in\mathcal{N}_\varepsilon$.

(A2) For $\forall\, u\in S_\varepsilon^+$, by (A1), there exists a unique $t_u>0$ such that

$$t_u=\int_{\mathbb{R}^N}g(\varepsilon x,t_u^2|u|^2)t_u|u|^2dx.$$

From $(g2)$, $(g3)$, the Sobolev embeddings and $2<q<2^*$, we get

$$t_u\leq\zeta t_u\int_{\mathbb{R}^N}|u|^2dx+C_\zeta t_u^{2^*-1}\int_{\mathbb{R}^N}|u|^{2^*}dx\leq C_1\zeta t_u+C_2t_u^{2^*-1},$$

which implies that $t_u\geq\tau$ for some $\tau>0$. If $\mathcal{W}\subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\}\subset\mathcal{W}$ with $t_n:=t_{u_n}\to\infty$. Since $\mathcal{W}$ is compact, there exists a $u\in\mathcal{W}$ such that $u_n\to u$ in H_ε. Moreover, using the proof of Lemma 3.1(ii), we have that $J_\varepsilon(t_nu_n)\to-\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\varepsilon$. Then from the definition of g and $(g4)$, $(g5)$ and $\theta > 2$, it follows that

$$\begin{aligned} J_\varepsilon(v_n) =& J_\varepsilon(v_n) - \frac{1}{\theta} J'_\varepsilon(v_n)[v_n] \\ \geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|v_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |v_n|^2)|v_n|^2 - \frac{1}{2} G(\varepsilon x, |v_n|^2)\Big) dx \\ \geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|v_n\|_\varepsilon^2 - \frac{1}{K}\int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2 dx\Big) \\ \geq& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big)\|v_n\|_\varepsilon^2. \end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\varepsilon = t_n$, we obtain

$$0 < \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(1 - \frac{1}{K}\Big) \leq \frac{J_\varepsilon(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves (A2).

(A3) First of all, we note that $\widehat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (A2), for each $u \in H_\varepsilon^+$, there is a unique $\widehat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in H_\varepsilon^+$. Otherwise, we have $|\mathrm{supp}(u) \cap \Lambda_\varepsilon| = 0$ and by $(g5)$, it follows that

$$\begin{aligned} \|u\|_\varepsilon^2 &= \int_{\mathbb{R}^N} g(\varepsilon x, |u|^2)|u|^2 dx \\ &= \int_{\Lambda_\varepsilon^c} g(\varepsilon x, |u|^2)|u|^2 dx \\ &\leq \frac{1}{K}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2 dx \\ &\leq \frac{1}{K}\|u\|_\varepsilon^2 \end{aligned}$$

which is impossible since $K > 2$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$ is well defined and continuous. From

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\varepsilon} = u, \ \forall\, u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection.

Now we prove that $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous. Let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ such that $u_n \to u$ in H_ε. By (A2), there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\varepsilon$, we obtain

$$t_n^2\|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(\varepsilon x, t_n^2|u_n|^2)t_n^2|u_n|^2 dx, \ \forall\, n \in N,$$

and passing to the limit as $n \to \infty$ in the last inequality, we have

$$t_0^2\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(\varepsilon x, t_0^2|u|^2)t_0^2|u|^2 dx,$$

which implies that $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. This proves that $\widehat{m}_\varepsilon(u_n) \to \widehat{m}_\varepsilon(u)$ in H_ε^+. Thus, $\widehat{m}_\varepsilon$ and m_ε are continuous functions and (A3) is proved.

(A4) Let $\{u_n\} \subset S_\varepsilon^+$ be a subsequence such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$, then for each $v \in \partial S_\varepsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ε. Therefore, by $(V1)$, $(V2)$ and the Sobolev embedding, for any $t \in [2, 2^*]$, there exists a constant $C_t > 0$ such that

$$\begin{aligned}
\|u_n\|_{L^t(\Lambda_\varepsilon)} &\leq \inf_{v \in \partial S_\varepsilon^+} \|u_n - v\|_{L^t(\Lambda_\varepsilon)} \\
&\leq C_t \bigg(\inf_{v \in \partial S_\varepsilon^+} \int_{\Lambda_\varepsilon} (|\nabla_{A_\varepsilon} u_n - v|^2 + V_\varepsilon(x)|u_n - v|^2) dx \bigg)^{\frac{1}{2}} \\
&\leq C_t \operatorname{dist}(u_n, \partial S_\varepsilon^+)
\end{aligned}$$

for all $n \in N$. By $(g2)$, $(g2)$ and $(g5)$, for each $t > 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} G(\varepsilon x, t^2|u_n|^2) dx \leq& \int_{\Lambda_\varepsilon} \Big(F(t^2|u_n|^2) + \frac{t^{2^*}|u_n|^{2^*}}{2^*} \Big) dx \\
&+ \frac{t^2}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2 dx \\
\leq& C_1 t^2 \int_{\Lambda_\varepsilon} |u_n|^2 dx + C_2 t^q \int_{\Lambda_\varepsilon} |u_n|^q dx \\
&+ \frac{t^{2^*}}{2^*} \int_{\Lambda_\varepsilon} |u_n|^{2^*} dx + \frac{t^2}{K}\|u_n\|_\varepsilon^2 \\
\leq& C_3 t^2 \operatorname{dist}(u_n, \partial S_\varepsilon^+)^2 + C_4 t^q \operatorname{dist}(u_n, \partial S_\varepsilon^+)^q \\
&+ C_5 t^{2^*} \operatorname{dist}(u_n, \partial S_\varepsilon^+)^{2^*} + \frac{t^2}{K}.
\end{aligned}$$

Therefore,

$$\limsup_n \int_{\mathbb{R}^N} G(\varepsilon x, t^2|u_n|^2) dx \leq \frac{t^2}{K}, \quad \forall\, t > 0.$$

On the other hand, from the definition of m_ε and the last inequality, for all $t > 0$, we obtain

$$\begin{aligned}\liminf_n J_\varepsilon(m_\varepsilon(u_n)) &\geq \liminf_n J_\varepsilon(tu_n)\\ &\geq \liminf_n \frac{t^2}{2}\|u_n\|_\varepsilon^2 - \frac{t^2}{K}\\ &= \frac{K-2}{2K}t^2.\end{aligned}$$

This implies that

$$\liminf_n \frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 \geq \liminf_n J_\varepsilon(m_\varepsilon(u_n)) \geq \frac{K-2}{2K}t^2, \quad \forall\, t > 0.$$

From the arbitrariness of $t > 0$, it follows that

$$\|m_\varepsilon(u_n)\|_\varepsilon \to \infty \quad \text{and} \quad J_\varepsilon(m_\varepsilon(u_n)) \to \infty \quad \text{as } n \to \infty.$$

We conclude the proof of Lemma 3.2. ∎

Now we define the function

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R}$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and denote $\Psi_\varepsilon := (\widehat{\Psi}_\varepsilon)|_{S_\varepsilon^+}$.

From Lemma 3.2 , arguing as in [42, Corollary 10] we may obtain the following result.

Lemma 3.3: *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ are satisfied. Then:*

(B1) $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\varepsilon'(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\epsilon}{\|u\|_\epsilon} J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall\, u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon.$$

(B2) $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*

$$\Psi_\varepsilon'(u)v = \|m_\varepsilon(u)\|_\epsilon J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall\, v \in T_u S_\varepsilon^+.$$

(B3) *If $\{u_n\}$ is a $(PS)_c$ sequence of Ψ_ε, then $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence of J_ε. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_c$ sequence of J_ε, then $\{m_\varepsilon^{-1}(u_n)\}$ is a $(PS)_c$ sequence of Ψ_ε.*

(B4) *u is a critical point of Ψ_ε if and only if $m_\varepsilon(u)$ is a critical point of J_ε. Moreover, the corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon. = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

As in [42], we have the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u\in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu). \tag{3.3}$$

The following result is important to prove the $(PS)_c$ condition for the functional J_ε.

Lemma 3.4: *Let $c > 0$ and $\{u_n\}$ is a $(PS)_c$ sequence for J_ϵ. Then $\{u_n\}$ is bounded in H_ϵ.*

Proof. Assume that $\{u_n\} \subset H_\epsilon$ is a $(PS)_c$ sequence for J_ϵ, that is, $J_\epsilon(u_n) \to c$ and $J'_\epsilon(u_n) \to 0$. By $(g4)$ and $(g5)$, we have

$$\begin{aligned}
c + o_n(1)+&o_n(1)\|u_n\|_\varepsilon \\
&=J_\varepsilon(u_n) - \frac{1}{\theta}J'_\varepsilon(u_n)[u_n] \\
&=\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^N}\Big(\frac{1}{\theta}g(\varepsilon x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(\varepsilon x,|u_n|^2)\Big)dx \\
&\geq\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c}\Big(\frac{1}{\theta}g(\varepsilon x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(\varepsilon x,|u_n|^2)\Big)dx \\
&\geq\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2}\int_{\Lambda_\varepsilon^c} G(\varepsilon x,|u_n|^2)dx \\
&\geq\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 - \frac{1}{2K}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2dx \\
&\geq\Big(\frac{1}{2}-\frac{1}{\theta}-\frac{1}{2K}\Big)\|u_n\|_\varepsilon^2.
\end{aligned}$$

Since $K>2$, from the last inequality we obtain that $\{u_n\}$ is bounded in H_ε. ∎

The following property provides a range of levels in which the energy functional J_ϵ verifies the Palais–Smale condition.

Lemma 3.5: *The functional J_ϵ satisfies the $(PS)_c$ condition for any $c \in (0, \frac{1}{N}S^{N/2})$, where S is the best constant for the Sobolev inequality*

$$S\Big(\int_{\mathbb{R}^N}|v|^{2^*}dx\Big)^{2/2^*} \leq \int_{\mathbb{R}^N}(|\nabla v|^2+|v|^2)dx, \quad \textit{for } v\in H^1(\mathbb{R}^N,\mathbb{R}).$$

Proof. Let $(u_n) \subset H_\epsilon$ be a $(PS)_c$ for J_ϵ. By Lemma 3.4, the sequence (u_n) is bounded in H_ϵ. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ϵ and $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^N,\mathbb{C})$ for all $1 \leq r < 2^*$ as $n \to +\infty$.

Step 1. For some fixed $\epsilon > 0$, let $R > 0$ be such that $\Lambda_\epsilon \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$ and if R is large enough,

$$\limsup_n \int_{B_R^c(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)dx \le \zeta. \tag{3.4}$$

Let $\phi_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0\ x \in B_{R/2}(0), \quad \phi_R = 1\ x \in B_R^c(0), \quad 0 \le \phi_R \le 1, \quad \text{and} \quad |\nabla \phi_R| \le C/R$$

where $C > 0$ is a constant independent of R. Since the sequence $(\phi_R u_n)$ is bounded in H_ϵ, we have

$$J'_\epsilon(u_n)[\phi_R u_n] = o_n(1),$$

hence

$$\begin{aligned}\mathrm{Re} \int_{\mathbb{R}^N} \nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon}(\phi_R u_n)} dx + \int_{\mathbb{R}^N} V_\epsilon(x)|u_n|^2 \phi_R dx \\ = \int_{\mathbb{R}^N} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R dx + o_n(1).\end{aligned}$$

Since

$$\overline{\nabla_{A_\epsilon}(\phi_R u_n)} = i\overline{u_n}\nabla \phi_R + \phi_R \overline{\nabla_{A_\epsilon} u_n},$$

using $(g5)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)\phi_R dx \\ &\quad = \int_{\mathbb{R}^N} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R dx - \mathrm{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\epsilon} u_n \nabla \phi_R dx + o_n(1) \\ &\quad \le \frac{1}{K} \int_{\mathbb{R}^N} V_\epsilon(x)|u_n|^2 \phi_R dx + C\left| \mathrm{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\epsilon} u_n \nabla \phi_R dx \right| + o_n(1).\end{aligned}$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of (u_n) in H_ε, we obtain

$$\begin{aligned}\left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx &\le \frac{C}{R}\|u_n\|_2 \|\nabla_{A_\varepsilon} u_n\|_2 + o_n(1) \\ &\le \frac{C_1}{R} + o_n(1)\end{aligned}$$

and so (3.4) holds.

Step 2. Now, we prove that for any $R > 0$, the following limit holds

$$\limsup_n \int_{B_R(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)dx = \int_{B_R(0)} (|\nabla_{A_\epsilon} u|^2 + V_\epsilon(x)|u|^2)dx. \tag{3.5}$$

Let $\phi_\rho \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a cut-off function such that

$$\phi_\rho = 1 \quad x \in B_\rho(0), \quad \phi_\rho = 0 \quad x \in B^c_{2\rho}(0), \quad 0 \le \phi_\rho \le 1, \quad \text{and} \quad |\nabla \phi_\rho| \le C/\rho$$

where $C > 0$ is a constant independent of ρ. Let

$$P_n(x) = |\nabla_{A_\varepsilon} u_n - \nabla_{A_\varepsilon} u|^2 + V_\epsilon(x)|u_n - u|^2.$$

For the fixed $R > 0$, choosing $\rho > R > 0$, we have

$$\begin{aligned}\int_{B_R} P_n(x)dx &\le \int_{\mathbb{R}^N} P_n(x)\phi_\rho(x)dx \\ &= \int_{\mathbb{R}^N} |\nabla_{A_\varepsilon} u_n - \nabla_{A_\varepsilon} u|^2 \phi_\rho(x)dx + \int_{\mathbb{R}^N} V_\epsilon(x)|u_n - u|^2\phi_\rho(x)dx \\ &= J^1_{n,\rho} - J^2_{n,\rho} + J^3_{n,\rho} + J^4_{n,\rho},\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}J^1_{n,\rho} &= \int_{\mathbb{R}^N} |\nabla_{A_\varepsilon} u_n|^2\phi_\rho(x)dx + \int_{\mathbb{R}^N} V_\epsilon(x)|u_n|^2\phi_\rho(x)dx \\ &\quad - \int_{\mathbb{R}^N} g(\epsilon x, |u_n|^2)|u_n|^2\phi_\rho dx, \\ J^2_{n,\rho} &= \mathrm{Re}\int_{\mathbb{R}^N} \nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon} u}\phi_\rho(x)dx + \mathrm{Re}\int_{\mathbb{R}^N} V_\epsilon(x)u_n\overline{u}\phi_\rho(x)dx \\ &\quad - \mathrm{Re}\int_{\mathbb{R}^N} g(\epsilon x, |u_n|^2)u_n\overline{u}\phi_\rho(x)dx, \\ J^3_{n,\rho} &= -\mathrm{Re}\int_{\mathbb{R}^N} (\nabla_{A_\epsilon} u_n - \nabla_{A_\epsilon} u)\overline{\nabla_{A_\epsilon} u}\phi_\rho(x)dx \\ &\quad - \mathrm{Re}\int_{\mathbb{R}^N} V_\epsilon(x)(u_n - u)\overline{u}\phi_\rho(x)dx,\end{aligned}$$

and

$$J^4_{n,\rho} = \mathrm{Re}\int_{\mathbb{R}^N} g(\epsilon x, |u_n|^2)u_n\overline{(u_n - u)}\phi_\rho(x)dx.$$

It is easy to see that

$$J^1_{n,\rho} = J'_\epsilon(u_n)[\phi_\rho u_n] - \mathrm{Re}\int_{\mathbb{R}^N} i\overline{u_n}\nabla_{A_\epsilon} u_n \nabla\phi_\rho dx$$

and

$$J^2_{n,\rho} = J'_\epsilon(u_n)[\phi_\rho u] - \mathrm{Re}\int_{\mathbb{R}^N} i\overline{u}\nabla_{A_\epsilon} u_n \nabla\phi_\rho dx.$$

Then

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^1_{n,\rho}| = 0, \quad \lim_{\rho\to\infty}\limsup_{n\to\infty}|J^2_{n,\rho}| = 0.$$

On the other hand, since the sequence (u_n) is bounded in H_ϵ, it is easy to see that

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^3_{n,\rho}| = 0.$$

Now we prove that

$$\lim_{\rho\to\infty}\limsup_{n\to\infty}|J^4_{n,\rho}| = 0. \tag{3.7}$$

We first show that

$$\lim_n \int_{\Lambda_\varepsilon} |u_n|^{2^*} dx = \int_{\Lambda_\varepsilon} |u|^{2^*} dx. \tag{3.8}$$

Using the boundedness of (u_n) in H_ϵ again, and the diamagnetic inequality (2.1), we may assume that

$$|\nabla|u_n||^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^{2^*} \rightharpoonup \nu \tag{3.9}$$

in the sense of measures. Moreover, by the diamagnetic inequality (2.1) and (3.4), (u_n) is a tight sequence in $H^1(\mathbb{R}^3,\mathbb{R})$, thus, using the concentration-compactness principle in [44], we can find an at most countable index I, sequences $(x_i) \subset \mathbb{R}^N, (\mu_i), (\nu_i) \subset (0,\infty)$ such that

$$\begin{aligned} \mu &\geq |\nabla|u||^2 dx + \sum_{i\in I}\mu_i\delta_{x_i}, \\ \nu &= |u|^{2^*} + \sum_{i\in I}\nu_i\delta_{x_i} \quad \text{and} \quad S\nu_i^{2/2^*} \leq \mu_i \end{aligned} \tag{3.10}$$

for any $i \in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $(x_i)_{i\in I} \cap \Lambda_\varepsilon = \emptyset$. Assume, by contradiction, that $x_i \in \Lambda_\varepsilon$ for some $i \in I$. For any $\rho > 0$, we define

$$\psi_\rho(x) = \psi\Big(\frac{x - x_i}{\rho}\Big)$$

where $\psi \in C_0^\infty(\mathbb{R}^N,[0,1])$ is such that $\psi = 1$ in B_1, $\psi = 0$ in $\mathbb{R}^N \setminus B_2$ and

$$\|\nabla\psi\|_{L^\infty(\mathbb{R}^N,\mathbb{R})} \leq 2.$$

We suppose that $\rho > 0$ is such that $\operatorname{supp}(\psi_\rho) \subset \Lambda_\varepsilon$. Since $(\psi_\rho u_n)$ is bounded in H_ε, we can see that

$$J'_\varepsilon(u_n)[\psi_\rho u_n] = o_n(1),$$

that is,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon} u_n|^2 \psi_\rho dx + \operatorname{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \psi_\rho dx + \int_{\mathbb{R}^N} V_\epsilon(x)|u_n|^2 \psi_\rho dx \\ = \int_{\mathbb{R}^N} g(\varepsilon x, |u_n|^2)|u_n|^2 \psi_\rho dx + o_n(1) \\ = \int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2 \psi_\rho dx + \int_{\mathbb{R}^N} |u_n|^{2^*} \psi_\rho dx + o_n(1).\end{aligned}$$

Using the diamagnetic inequality (2.1) again, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla |u_n||^2 \psi_\rho dx + \operatorname{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \psi_\rho dx \\ \le \int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2 \psi_\rho dx + \int_{\mathbb{R}^N} |u_n|^{2^*} \psi_\rho dx + o_n(1).\end{aligned} \tag{3.11}$$

Due to the fact that f has the subcritical growth and ψ_ρ has the compact support, we have that

$$\lim_{\rho\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} f(|u_n|^2)|u_n|^2 \psi_\rho dx = \lim_{\rho\to 0} \int_{\mathbb{R}^N} f(|u|^2)|u|^2 \psi_\rho dx = 0. \tag{3.12}$$

Now, we show that

$$\lim_{\rho\to 0} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \psi_\rho dx \right| = 0. \tag{3.13}$$

Because of the boundedness of (u_n) in H_ε, using the Hölder inequality, the strong convergence of $(|u_n|)$ in $L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$, $|u| \in L^{2^*}(\mathbb{R}^N, \mathbb{R})$, $|\nabla \psi_\rho| \le C\rho^{-1}$ and $|B_{2\rho}(x_i)| \sim \rho^N$, we have that

$$\begin{aligned}0 \le &\lim_{\rho\to 0} \limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \psi_\rho dx \right| \le \lim_{\rho\to 0} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\overline{u_n} \nabla \psi_\rho||\nabla_{A_\varepsilon} u_n| dx \\ \le &\lim_{\rho\to 0} \lim_{n\to\infty} \left(\int_{B_{2\rho}(x_i)} |\overline{u_n} \nabla \psi_\rho|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon} u_n|^2 dx \right)^{1/2} \\ \le &C \lim_{\rho\to 0} \left(\int_{B_{2\rho}(x_i)} |u|^2 dx \right)^{1/2} = 0\end{aligned}$$

which shows that (3.13) holds.

Then, taking into account (3.9), (3.11), (3.12) and (3.13), we can conclude that $\nu_i \geq \mu_i$. Together with the inequality $S\nu_i^{2/2^*} \leq \mu_i$ in (3.10), we have

$$\nu_i \geq S^{N/2}. \tag{3.14}$$

Now, from $(f3)$, $(g4)$ and $(g5)$, we have

$$\begin{aligned} c =& J_\varepsilon(u_n) - \frac{1}{2}J'_\varepsilon(u_n)[u_n] + o_n(1) \\ =& \int_{\mathbb{R}^N} \Big(\frac{1}{2}g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2}G(\varepsilon x, |u_n|^2)\Big)dx + o_n(1) \\ \geq& \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{2}g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2}G(\varepsilon x, |u_n|^2)\Big)dx + \frac{1}{N}\int_{\Lambda_\varepsilon} |u_n|^{2^*}dx + o_n(1) \\ \geq& \frac{1}{N}\int_{\Lambda_\varepsilon} |u_n|^{2^*}dx + o_n(1) \\ \geq& \frac{1}{N}\int_{\Lambda_\varepsilon} \psi_\rho |u_n|^{2^*}dx + o_n(1). \end{aligned}$$

From the above arguments and relations (3.10) and (3.14), we obtain

$$\begin{aligned} c \geq& \frac{1}{N}\sum_{\{i\in I: x_i \in \Lambda_\varepsilon\}} \psi_\rho(x_i)\nu_i \\ \geq& \frac{1}{N}\nu_i \geq \frac{1}{N}S^{N/2} = c_0 \end{aligned}$$

which gives a contradiction. This means that (3.8) holds.

We now observe that

$$\begin{aligned} |J^4_{n,\rho}| \leq& \int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap B_{2\rho}(0)} |g(\epsilon x, |u_n|^2)u_n\overline{(u_n-u)}|dx \\ &+ \int_{\Lambda_\varepsilon\cap B_{2\rho}(0)} |g(\epsilon x, |u_n|^2)u_n\overline{(u_n-u)}|dx. \end{aligned}$$

By the Sobolev compact embedding $H_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ for $1 \leq r < 2^*$, and $(g5)$, we have

$$\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap B_{2\rho}(0)} |g(\epsilon x, |u_n|^2)u_n\overline{(u_n-u)}|dx \to 0, \quad \text{as } n\to\infty.$$

Moreover, using the Sobolev compact embedding $H_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$ for $1 \leq r < 2^*$, again, and (3.8), we have

$$\int_{\Lambda_\varepsilon\cap B_{2\rho}(0)} |g(\epsilon x, |u_n|^2)u_n\overline{(u_n-u)}|dx \to 0, \quad \text{as } n\to\infty.$$

Thus, (3.7) holds. Moreover, by (3.6), it follows that

$$0 \leq \limsup_n \int_{B_R} P_n(x)dx \leq \limsup_n (|J^1_{n,\rho}| + |J^2_{n,\rho}| + |J^3_{n,\rho}| + |J^4_{n,\rho}|) = 0,$$

hence

$$\limsup_n \int_{B_R} P_n(x)dx = 0.$$

Thus, relation (3.5) holds.

Step 3. From (3.4) and (3.5), we have

$$\begin{aligned}\|u\|^2_\varepsilon &\leq \liminf_n \|u_n\|^2_\varepsilon \leq \limsup_n \|u_n\|^2_\varepsilon \\ &\leq \limsup_n \left\{ \int_{B_R(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)dx \right. \\ &\qquad \left. + \int_{B^c_R(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)dx \right\} \\ &\leq \int_{B_R(0)} (|\nabla_{A_\epsilon} u|^2 + V_\epsilon(x)|u|^2)dx + \zeta.\end{aligned}$$

Passing to the limit as $\zeta \to 0$ we have $R \to \infty$, which implies that

$$\|u\|^2_\varepsilon \leq \liminf_n \|u_n\|^2_\varepsilon \leq \limsup_n \|u_n\|^2_\varepsilon \leq \|u\|^2_\varepsilon.$$

Then $u_n \to u$ in H_ϵ and we conclude. ∎

Since f is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

Corollary 3.1: *The functional Ψ_ε satisfies the $(PS)_c$ condition on S^+_ε at any level $c \in (0, \frac{1}{N}S^{N/2})$.*

Proof. Let $\{u_n\} \subset S^+_\varepsilon$ be a $(PS)_c$ sequence for Ψ_ε with $c \in (0, \frac{1}{N}S^{N/2})$. Then $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi'_\varepsilon(u_n)\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n}S^+_\varepsilon)^*$. By Lemma 3.3(B3), we know that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for J_ε in H_ε. From Lemma 3.5, we know that there exists a $u \in S^+_\varepsilon$ such that, up to a subsequence, $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in H_ε. By Lemma 3.2(A3), we obtain

$$u_n \to u \text{ in } S^+_\varepsilon,$$

and the proof is complete. ∎

4. Multiple solutions for the modified problem

4.1. The autonomous problem. For our scope, we need also to study the following limit problem

$$(4.1) \qquad -\Delta u + V_0 u = f(|u|^2)u + u^{2^*-1}, \quad u : \mathbb{R}^N \to \mathbb{R}, \text{ and } u > 0,$$

whose associated C^1-functional, defined in $H^1(\mathbb{R}^N, \mathbb{R})$, is

$$I_{V_0}(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0 u^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} F(u^2)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} (u^+)^{2^*} dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : I'_{V_0}(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_{V_0}(u).$$

By $(f1)$ and $(f4)$, for each $u \in H^1(\mathbb{R}^N, \mathbb{R})\setminus\{0\}$, there is a unique $t(u) > 0$ such that

$$I_{V_0}(t(u)u) = \max_{t \geq 0} I_{V_0}(tu) \quad \text{and} \quad t(u)u \in \mathcal{N}_{V_0}.$$

Then, using the assumptions on f, arguing as in [44, Lemma 4.1 and Theorem 4.2] we have that

$$0 < c_{V_0} = \inf_{u \in H^1(\mathbb{R}^N, \mathbb{R})\setminus\{0\}} \max_{t \geq 0} I_{V_0}(tu).$$

Let

$$H_0 := H^1(\mathbb{R}^N, \mathbb{R})$$

and define by H_0^+ the open set of H_0 given by

$$H_0^+ = \{u \in H_0 : |\text{supp}(u^+)| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where S_0 be the unit sphere of H_0.

As in Section 3, S_0^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_0 and contained in H_0^+. Therefore,

$$H_0 = T_u S_0^+ \bigoplus \mathbb{R}u$$

for each $u \in T_u S_0^+$, where

$$T_u S_0^+ = \{v \in H_0 : \langle u, v \rangle_0 = 0\}.$$

Arguing as in Lemma 3.2, we have the following property.

Lemma 4.1: *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied. Then the following properties hold:*

(a1) *For any $u \in H_0^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_{V_0}(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞).*

(a2) *There is a $\tau > 0$ independent on u such that $t_u > \tau$ for all $u \in S_0^+$. Moreover, for each compact $\mathcal{W} \subset S_0^+$ there is $C_{\mathcal{W}}$ such that $t_u \le C_{\mathcal{W}}$, for all $u \in \mathcal{W}$.*

(a3) *The map $\widehat{m} : H_0^+ \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m_0 = \widehat{m}_0|_{S_0^+}$ is a homeomorphism between S_0^+ and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$.*

(a4) *If there is a sequence $\{u_n\} \subset S_0^+$ such that $\operatorname{dist}(u_n, \partial S_0^+) \to 0$, then $\|m(u_n)\|_0 \to \infty$ and $I_{V_0}(m(u_n)) \to \infty$.*

We shall consider the functional defined by

$$\widehat{\Psi}_0(u) = I_{V_0}(\widehat{m}(u)) \quad \text{and} \quad \Psi_0 := \widehat{\Psi}_0|_{S_0^+}.$$

Arguing as in [42, Proposition 9 and Corollary 10], we have

Lemma 4.2: *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied. Then:*

(b1) *$\widehat{\Psi}_0 \in C^1(H_0^+, \mathbb{R})$ and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_{V_0}'(\widehat{m}(u))[v], \quad \forall\, u \in H_0^+ \text{ and } \forall v \in H_0.$$

(b2) *$\Psi_0 \in C^1(S_0^+, \mathbb{R})$ and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_{V_0}'(\widehat{m}(u))[v], \quad \forall v \in T_u S_0^+.$$

(b3) *If $\{u_n\}$ is a $(PS)_c$ sequence of Ψ_0, then $\{m(u_n)\}$ is a $(PS)_c$ sequence of I_{V_0}. If $\{u_n\} \subset \mathcal{N}_0$ is a bounded $(PS)_c$ sequence of I_{V_0}, then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence of Ψ_0.*

(b4) *u is a critical point of Ψ_0 if and only if $m(u)$ is a critical point of I_{V_0}. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_{V_0}.$$

Similar to the previous argument, we have the following variational characterization of the infimum of I_{V_0} over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u\in\mathcal{N}_0} I_{V_0}(u) = \inf_{u\in H_0^+}\sup_{t>0} I_{V_0}(tu) = \inf_{u\in S_0^+}\sup_{t>0} I_{V_0}(tu). \tag{4.2}$$

Arguing as in [1], we have

Lemma 4.3: *Assume that $(f1)$–$(f4)$ hold; problem (4.1) has a positive ground state solution ω which is radially symmetric and a classical solution, that is $\omega \in C^2(\mathbb{R}^N,\mathbb{R}) \cap L^\infty(\mathbb{R}^N,\mathbb{R})$. Moreover, $0 < c_{V_0} < \frac{1}{N}S^{N/2}$.*

Moreover, we have the following important property.

Lemma 4.4: *Let $(u_n) \subset \mathcal{N}_0$ be such that $I_{V_0}(u_n) \to c_{V_0}$. Then (u_n) has a convergent subsequence in H_0.*

Proof. Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.1(a3), Lemma 4.2(b4) and the definition of c_{V_0}, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \quad \forall n \in N,$$

and

$$\Psi_0(v_n) = I_{V_0}(u_n) \to c_{V_0} = \inf_{u\in S_0^+} \Psi_0(u).$$

Although S_0^+ is not a complete C^1 manifold, we still can apply Ekeland's variational principle to the functional $\mathcal{E}_0 : H \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathcal{E}_0(u) := \widehat{\Psi}_0(u) \quad \text{if } u \in S_0^+$$

and

$$\mathcal{E}_0(u) := \infty \quad \text{if } u \in \partial S_0^+,$$

where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric

$$d(u,v) := \|u-v\|_0.$$

In fact, by Lemma 4.1(a4), $\mathcal{E}_0 \in C(H, \mathbb{R}\cup\{\infty\})$, and from Lemma 4.2(b4), $\mathcal{E}_0$ is bounded below. Therefore, there exists a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0^+ and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Arguing as in Lemma 3.5, we conclude. ∎

Now, we show that relationship between c_ϵ and c_{V_0}.

Lemma 4.5: *The numbers c_ϵ and c_{V_0} satisfy the following inequality:*

$$\lim_{\epsilon\to 0} c_\epsilon = c_{V_0} < \frac{1}{N} S^{N/2}.$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function such that $\eta = 1$ in $B_{\rho/2}$ and $\mathrm{supp}(\eta) = B_\rho \subset \Lambda$ for some $\rho > 0$. Let us define

$$\omega_\epsilon(x) := \eta_\epsilon(x)\omega(x)e^{iA(0)\cdot x},$$

where

$$\eta_\epsilon(x) = \eta(\epsilon x) \quad \text{for } \epsilon > 0,$$

ω is a positive and radial ground state solution of problem (4.1), and we observe that $|\omega_\epsilon| = \eta_\epsilon \omega$ and $\omega_\epsilon \in H_\varepsilon$ in view of Lemma 2.1. Arguing as in [25, Lemma 4.1] or [31, Lemma 4.6], we have that

$$\lim_{\epsilon\to 0} \|\omega_\epsilon\|_\epsilon^2 = \|\omega\|_{V_0}^2 \tag{4.3}$$

and

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} |\nabla_{A_\varepsilon}\omega_\epsilon|^2 dx = \int_{\mathbb{R}^N} |\nabla\omega|^2 dx. \tag{4.4}$$

It is also easy to verify that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} |\nabla\omega_\epsilon|^{2^*} dx = \int_{\mathbb{R}^N} |\nabla\omega|^{2^*} dx. \tag{4.5}$$

Now let $t_\epsilon > 0$ be the unique number such that

$$J_\epsilon(t_\epsilon\omega_\epsilon) = \max_{t\geq 0} J_\epsilon(t\omega_\epsilon).$$

We observe that t_ϵ satisfies

$$\begin{aligned} t_\epsilon^2\Big(\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon}\omega_\epsilon|^2 dx &+ \int_{\mathbb{R}^N} V_\varepsilon(x)|\omega_\epsilon|^2 dx\Big) \\ &= \int_{\mathbb{R}^N} g(\epsilon x, t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2 dx \\ &= \int_{\mathbb{R}^N} f(t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^N} t_\epsilon^{2^*}|\omega_\epsilon|^{2^*} dx, \end{aligned}$$

where we use $\mathrm{supp}(\eta) \subset \Lambda$ and the definition of $g(x,t)$. Moreover, using that $\eta = 1$ in $B_{\rho/2}$, u is a positive continuous function and $(f4)$, we have

$$\begin{aligned}
\frac{1}{t_\epsilon^2}&\left(\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon}\omega_\epsilon|^2 dx + \int_{\mathbb{R}^N} V_\varepsilon(x)|\omega_\epsilon|^2 dx\right)\\
&=\frac{1}{t_\epsilon^2}\int_{\mathbb{R}^N} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^N} t_\epsilon^{2^*-4}|\omega_\epsilon|^{2^*} dx\\
&\geq\frac{1}{t_\epsilon^2}\int_{\mathbb{R}^N} f(t_\epsilon^2\eta^2(|\epsilon x|)\omega^2(x))\eta^2(|\epsilon x|)\omega^2(x) dz\\
&\geq\frac{1}{t_\epsilon^2}\int_{B_{\rho/(2\epsilon)}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z) dz\\
&\geq\frac{1}{t_\epsilon^2}\int_{B_{\rho/2}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z) dz\\
&\geq\frac{f(t_\epsilon^2\gamma^2)}{t_\epsilon^2}\int_{B_{\rho/2}(0)} \omega^2(z) dz
\end{aligned}$$

for all $0 < \epsilon < 1$ and where $\gamma = \min\{\omega(z) : |z| \leq \rho/2\}$.

If $t_\epsilon \to +\infty$ as $\epsilon \to 0$, by $(f4)$, we contradict (4.4). Therefore, up to a subsequence, we may assume that $t_\epsilon \to t_0 \geq 0$ as $\epsilon \to 0$.

If $t_\epsilon \to 0$, using the fact that f is increasing, the Lebesgue dominated convergence theorem and (4.5), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla_{A_\varepsilon}\omega_\epsilon|^2 + &\int_{\mathbb{R}^N} V_\varepsilon(x)|\omega_\epsilon|^2 dx\\
&= \int_{\mathbb{R}^N} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2 dx + \int_{\mathbb{R}^N} t_\epsilon^{2^*-2}|\omega_\epsilon|^{2^*} dx \to 0, \quad \text{as } \epsilon \to 0
\end{aligned}$$

which contradicts (4.3). Thus, we have $t_0 > 0$ and

$$t_0^2\int_{\mathbb{R}^N} (|\nabla\omega|^2 + V_0\omega^2) dx = \int_{\mathbb{R}^N} f(t_0^2\omega^2)t_0^2\omega^2 dx + \int_{\mathbb{R}^N} t_0^{2^*}|\omega|^{2^*} dx,$$

so that $t_0\omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^N} F(|t_\epsilon\omega_\epsilon|^2) dx = \int_{\mathbb{R}^N} F(\omega^2) dx.$$

Hence

$$\lim_{\epsilon\to 0} J_\epsilon(t_\epsilon\omega_\epsilon) = I_{V_0}(u) = c_{V_0}.$$

Since

$$c_\epsilon \le \max_{t\ge 0} J_\epsilon(t\omega_\epsilon) = J_\epsilon(t_\epsilon\omega_\epsilon),$$

we can conclude that $\limsup_{\epsilon\to 0} c_\epsilon \le c_{V_0}$. Moreover, by (3.3), (4.2) and $I_{V_0}(|u|) \le J_\epsilon(u)$ for any $u \in H_\varepsilon$, we have $c_{V_0} \le c_\epsilon$. Then $c_{V_0} \le \liminf_{\epsilon\to 0} c_\epsilon$. Combining with the previous arguments, we conclude that

$$\lim_{\epsilon\to 0} c_\epsilon = c_{V_0} < \frac{1}{N} S^{N/2}. \qquad ∎$$

Remark 4.1: From Lemma 4.1 and Lemma 3.5, we see that for $\epsilon > 0$ small, the problem (3.2) has a ground state solution u_ϵ such that

$$J_\varepsilon(u_\epsilon) = c_\epsilon \quad \text{and} \quad J'_\varepsilon(u_\epsilon) = 0.$$

4.2. Technical results. In this subsection, we prove a multiplicity result for the modified problem (3.2) using the Ljusternik–Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^N, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that

$$\eta(t) = 1 \quad \text{if } 0 \le t \le \delta/2$$

and

$$\eta(t) = 0 \quad \text{if } t \ge \delta.$$

For any $y \in M$, let us introduce the function

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\Big(\frac{\varepsilon x - y}{\varepsilon}\Big) \exp\Big(i\tau_y\Big(\frac{\varepsilon x - y}{\varepsilon}\Big)\Big),$$

where

$$\tau_y(x) := \sum_i^N A_i(y)x_i.$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t\ge 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}).$$

Note that $t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.

Let us define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon\Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Moreover, arguing as in Lemma 4.1, the energy of the above functions has the following behavior as $\varepsilon \to 0^+$.

Lemma 4.6: *The limit*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ defined by setting

$$\Upsilon(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|u(x)|^2 dx.$$

We have the following asymptotic uniform estimate.

Lemma 4.7: *The limit*

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that there exist $\kappa > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \kappa. \tag{4.6}$$

Using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} (\Upsilon(\varepsilon_n z + y_n) - y_n)\eta^2(|\varepsilon_n z|)\omega^2(z)dz}{\int_{\mathbb{R}^N} \eta^2(|\varepsilon_n z|)\omega^2(z)dz}.$$

Taking into account $(y_n) \subset M \subset M_\delta \subset B_\rho$ and the Lebesgue dominated convergence theorem, we can obtain that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.6). ∎

Now, we prove the following useful compactness result.

Proposition 4.1: *Let* $\varepsilon_n \to 0^+$ *and* $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ *be such that* $J_{\varepsilon_n}(u_n) \to c_{V_0}$. *Then there exists* $(\tilde{y}_n) \subset \mathbb{R}^N$ *such that the sequence* $(|v_n|) \subset H^1(\mathbb{R}^N, \mathbb{R})$, *where* $v_n(x) := u_n(x + \tilde{y}_n)$, *has a convergent subsequence in* $H^1(\mathbb{R}^N, \mathbb{R})$. *Moreover, up to a subsequence,* $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ *as* $n \to +\infty$.

Proof. Since $J'_{\varepsilon_n}(u_n)[u_n] = 0$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\varepsilon_n} \leq C$ for all $n \in \mathbb{N}$.

Arguing as in Lemma 3.1(ii), we have that $c_{V_0} > 0$. Moreover, arguing as in the proof of Lemma 3.2, we have that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants R, $\beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \geq \beta. \tag{4.7}$$

Now, let us consider the sequence $\{|v_n|\} \subset H^1(\mathbb{R}^N, \mathbb{R})$, where

$$v_n(x) := u_n(x + \tilde{y}_n).$$

By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^N, \mathbb{R})$, and using (4.7), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^N, \mathbb{R})$ for some $v \neq 0$.

Let $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \varepsilon_n \tilde{y}_n$.

Using the diamagnetic inequality (2.1) again, we have

$$c_{V_0} \leq I_{V_0}(\tilde{v}_n) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.

Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^N, \mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, then (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, that is $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$. Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, and, by Lemma 4.4, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^N, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$.

Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - \varepsilon_n|x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using (V_1) and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla|v_n||^2 + V_0|v_n|^2)dx \\
&\leq \int_{\mathbb{R}^N} |\nabla_{A_\varepsilon} v_n|^2 + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|v_n|^2 dx \\
&\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\
&\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} |v_n|^{2^*} dx.
\end{aligned} \tag{4.8}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\tilde{f}(t) \leq V_0/K$, we can see that (4.8) yields

$$\min\Big\{1, V_0\Big(1 - \frac{1}{K}\Big)\Big\} \int_{\mathbb{R}^N} (|\nabla|v_n||^2 + |v_n|^2)dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, which contradicts $v \not\equiv 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^N$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned}
c_{V_0} &= I_{V_0}(\tilde{v}) < \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla \tilde{v}|^2 + V(y_0)|\tilde{v}|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}|^2)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |\tilde{v}|^{2^*} dx \\
&\leq \liminf_n \Big(\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla \tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2)dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}_n|^2)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |\tilde{v}|^{2^*} dx \Big) \\
&= \liminf_n \Big(\frac{t_n^2}{2}\int_{\mathbb{R}^N}(|\nabla|u_n||^2 + V(\varepsilon_n z)|u_n|^2)dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^N} F(|t_n u_n|^2)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |t_n u_n|^{2^*} dx \Big) \\
&\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0}
\end{aligned}$$

which is impossible and the proof is complete. ∎

Let now

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Fixing $y \in M$, by Lemma 4.6,

$$|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0 \quad \text{as} \quad \varepsilon \to 0^+,$$

we get that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 4.8: *We have*

$$\lim_{\varepsilon \to 0^+} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u \in \tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_{V_0}(t|u_n|) \le J_{\varepsilon_n}(tu_n)$ for any $t \ge 0$. Therefore, recalling that $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can deduce that

$$c_{V_0} \le \max_{t \ge 0} I_{V_0}(t|u_n|) \le \max_{t \ge 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \le c_{V_0} + h(\varepsilon_n) \tag{4.9}$$

which implies that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$.

Thus, by Proposition 4.1, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough.

Making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} (\Upsilon(\varepsilon_n z + y_n) - y_n)|u_n(z + \tilde{y}_n)|^2 dz}{\int_{\mathbb{R}^N} |u_n(z + \tilde{y}_n)|^2 dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\varepsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^N$, we conclude. ∎

4.3. Multiplicity of solutions for problem (3.2). Finally, we present a relation between the topology of M and the number of solutions of the modified problem (3.2).

Theorem 4.1: *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions.*

Proof. For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S_\varepsilon^+$ by

$$\pi_\epsilon(y) = m_\varepsilon^{-1}(\Phi_\epsilon(y)), \quad \forall y \in M.$$

By Lemma 4.6 and Lemma 3.3(B4), we obtain

$$\lim_{\epsilon \to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon \to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \quad \text{uniformly in } y \in M.$$

Hence, there is a number $\hat{\epsilon} > 0$ such that the set

$$\tilde{S}_\varepsilon^+ := \{u \in S_\varepsilon^+ : \Psi_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\}$$

is nonempty, for all $\epsilon \in (0, \hat{\epsilon})$, since $\pi_\epsilon(M) \subset \tilde{S}_\varepsilon^+$. Here h is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$.

Given $\delta > 0$, by Lemma 4.6, Lemma 3.2(A3), Lemma 4.7, and Lemma 4.8, we can find $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and continuous. From Lemma 4.7, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\varepsilon(\Phi_\varepsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define $H(t, z) = z + (1 - t)\Theta(\epsilon, z)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly,

$$H(0, z) = \beta_\varepsilon(\Phi_\varepsilon(z)) \quad H(1, z) = z$$

for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \pi_\epsilon$ and the embedding $\iota : M \to M_\delta$. Thus, this fact implies that

$$\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \ge \mathrm{cat}_{M_\delta}(M). \tag{4.10}$$

By Corollary 3.1 and the abstract category theorem [42], Ψ_ε has at least $\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on S_ε^+. Therefore, from Lemma 3.3(B4) and (4.10), we have that J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\varepsilon$ which implies that problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions. ∎

5. Proof of Theorem 1.1

In this section we prove our main result. The idea is to show that the solutions u_ε obtained in Theorem 4.1 satisfy

$$|u_\varepsilon(x)|^2 \le a_0 \quad \text{for } x \in \Lambda_\varepsilon^c$$

for ε small. Arguing as in [31, Lemma 5.1], we have the following important result.

Lemma 5.1: *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of problem* (3.2) *for $\varepsilon = \varepsilon_n$. Then $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^N, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ solution of problem (3.2), then

$$\|u_\varepsilon\|^2_{L^\infty(\Lambda_\varepsilon^c)} \le a_0 \tag{5.1}$$

holds. We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ such that for every n there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ which satisfies $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda_{\varepsilon_n}^c)} > a_0. \tag{5.2}$$

Arguing as in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n := \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all n large enough.

Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \le a_0$ in $B_R^c(0)$ and n large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \le a_0$ in $B_R^c(\tilde{y}_n)$ and n large enough. Moreover, if n is so large that $r/\varepsilon_n > R$, then $\Lambda_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n)$, which gives $|u_n|^2 \le a_0$ for any $x \in \Lambda_{\varepsilon_n}^c$. This contradicts (5.2) and proves the claim.

Let now $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$, where $\tilde{\varepsilon}_\delta > 0$ is given by Theorem 4.1. Then we have $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.2). If $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is one of these solutions, then, by (5.1) and the definition of g, we conclude that u_ε is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\varepsilon|$, where $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a solution to problem (1.1), as $\varepsilon \to 0^+$.

Take $\varepsilon_n \to 0^+$ and the sequence (u_n) where each u_n is a solution of (3.2) for $\varepsilon = \varepsilon_n$. From the definition of g, there exists $\gamma \in (0, a_0)$ such that

$$g(\varepsilon x, t^2)t^2 \leq \frac{V_0}{K} t^2, \quad \text{for all } x \in \mathbb{R}^N,\ |t| \leq \gamma.$$

Arguing as above we can take $R > 0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{5.3}$$

Up to a subsequence, we may also assume that for n large enough

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.4}$$

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < \gamma$. Thus, since $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, using $(g5)$ and the diamagnetic inequality (2.1) that

$$\int_{\mathbb{R}^N} (|\nabla |u_n||^2 + V_0|u_n|^2)dx \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, |u_n|^2)|u_n|^2 dx \leq \frac{V_0}{K} \int_{\mathbb{R}^N} |u_n|^2 dx$$

and, with $K > 2$, $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points p_n of $|u_{\varepsilon_n}|$ belong to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of the problem (1.1) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point η_{ε_n} of $|\hat{u}_n|$ is

$$\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n.$$

Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

The proof is now complete. ∎

Acknowledgements. C. Ji is partially supported by Natural Science Foundation of Shanghai (20ZR1413900, 18ZR1409100). The authors would like to thank the referee for several useful comments.

References

[1] C. O. Alves, D. C. de Morais Filho and M. A. S. Souto, *Radially symmetric solutions for a class of critical exponent elliptic problems in* $\mathbb{R}^N$, Electronic Journal of Differential Equations **7** (1996).

[2] C. O. Alves and G. M. Figueiredo, *Multiplicity of positive solutions for a quasilinear problem in* $\mathbb{R}^N$ *via penaization method*, Advanced Nonlinear Studies **5** (2005), 551–572.

[3] C. O. Alves, G. M. Figueiredo and M. F. Furtado, *Multiple solutions for a nonlinear Schrödinger equation with magnetic fields*, Communications in Partial Differential Equations **36** (2011), 1565–1586.

[4] C. O. Alves and C. Ji, *Existence of a positive solution for a logarithmic Schrödinger equation with saddle-like potential*, Manuscripta Mathematica, to appear, `https://link.springer.com/article/10.1007/s00229-020-01197-z`.

[5] C. O. Alves and C. Ji, *Multiple positive solutions for a Schrödinger logarithmic equation*, Discrete and Continuous Dynamical Systems **40** (2020), 2671–2685.

[6] C. O. Alves and C. Ji, *Existence and concentration of positive solutions for a logarithmic Schrödinger equation via penalization method*, Calculus of Variations and Partial Differential Equations **59** (2020), Article no. 21.

[7] C. O. Alves and O. H. Miyagaki, *Existence and concentration of solution for a class of fractional elliptic equation in* $\mathbb{R}^N$ *via penalization method*, Calculus of Variations and Partial Differential Equations **55** (2016), Article no. 47.

[8] C. O. Alves and M. A. S. Souto, *Multiplicity of positive solutions for a class of problems with exponentical critical growth in* $\mathbb{R}^2$, Journal of Differential Equations **244** (2008), 1502–1520.

[9] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Archive for Rational Mechanics and Analysis **140** (1997), 285–300.

[10] A. Ambrosetti, A. Malchiodi and S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Archive for Rational Mechanics and Analysis **159** (2001), 253–271.

[11] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theoryand applications*, Journal of Functional Analysis **14** (1973), 349–381.

[12] V. Ambrosio, *Existence and concentration results for some fractional Schrödinger equations in* $\mathbb{R}^N$ *with magnetic fields*, Communications in Partial Differential Equations **44** (2019), 637–680.

[13] G. Arioli and A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Archive for Rational Mechanics and Analysis **170** (2003), 277–295.

[14] S. Barile, S. Cingolani and S. Secchi, *Single-peaks for a magnetic Schrödinger equation with critical growth*, Advances in Differential Equations **11** (2006), 1135–1166.

[15] T. Bartsch, E. N. Dancer and S. Peng, *On multi-bump semi-classical bound states of nonlinear Schrödinger equations with electromagnetic fields*, Advances in Differential Equations **11** (2006), 781–812.

[16] V. Benci and G. Cerami, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calculus of Variations and Partial Differential Equations **2** (1994), 29–48.

[17] J. Byeon and K. Tanaka, *Semi-classical standing waves for nonlinear Schrödinger equations at structurally stable critical points of the potential*, Journal of the European Mathematical Society **15** (2013), 1859–1899.

[18] J. Byeon and K. Tanaka, *Semiclassical standing waves with clustering peaks for nonlinear Schrödinger equations*, Memoirs of the American Mathematical Society **229** (2014).

[19] J. Chabrowski and A. Szulkin, *On the Schrödinger equation involving a critical Sobolev exponent and magnetic field*, Topological Methods in Nonlinear Analysis **25** (2005), 3–21.

[20] S. Cingolani, *Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field*, Journal of Differential Equations **188** (2003), 52–79.

[21] S. Cingolani and M. Gallo, *On the fractional NLS equation and the effects of the potential well's topology*, Advanced Nonlinear Studies, to appear, `https://doi.org/10.1515/ans-2020-2114`.

[22] S. Cingolani, L. Jeanjean and S. Secchi, *Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions*, ESAIM Control Optimization and Calculus of Variations **15** (2009), 653–675.

[23] S. Cingolani, L. Jeanjean and K. Tanaka, *Multiple complex-valued solutions for nonlinear magnetic Schrödinger equations*, Journal of Fixed Point Theory and Applications **19** (2017), 37–66.

[24] S. Cingolani and S. Secchi, *Semiclassical states for NLS equations with magnetic potentials having polynomial growths*, Journal of Mathematical Physics **46** (2005), Article no. 053503.

[25] P. d'Avenia and C. Ji, *Multiplicity and concentration results for a magnetic Schrödinger equation with exponential critical growth in* $\mathbb{R}^2$, International Mathematics Research Notices, to appear, `https://doi.org/10.1093/imrn/rnaa074`.

[26] P. d'Avenia, A. Pomponio and D. Ruiz, *Semiclassical states for the nonlinear Schrödinger equation on saddle points of the potential via variational methods*, Journal of Functional Analysis **262** (2012), 4600–4633.

[27] M. del Pino and P. L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calculus of Variations and Partial Differential Equations **4** (1996), 121–137.

[28] M. J. Esteban and P. L. Lions, *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field*, in *Partial Differential Equations and the Calculus of Variations. Vol. I*, Progress in Nonlinear Differential Equations and their Applications, Vol. 1, Birkhäuser, Boston, MA, 1989, pp. 401–449.

[29] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, Journal of Functional Analysis **69** (1986), 397–408.

[30] X. M. He and W. M. Zou, *Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities*, Calculus of Variations and Partial Differential Equations **55** (2016), Article no. 91,

[31] C. Ji and V. D. Rădulescu, *Multiplicity and concentration of solutions to the nonlinear magnetic Schrodinger equation*, Calculus of Variations and Partial Differential Equations **59** (2020), Article no. 115.

[32] C. Ji and V. D. Rădulescu, *Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in* $\mathbb{R}^2$, Manuscripta Mathematica, to appear, https://doi.org/10.1007/s00229-020-01195-1.

[33] C. Ji and V. D. Rădulescu, *Multiplicity properties and concentration phenomena for magnetic Kirchhoff equations*, preprint.

[34] C. Ji and A. Szulkin, *A logarithmic Schrödinger equation with asymptotic conditions on the potential*, Journal of Mathematical Analysis and Applications **437** (2016), 241-254.

[35] C. Ji, Z.-Q. Wang and Y. Z. Wu, *A monotone property of the ground state energy to the scalar field equation and applications*, Journal of the London Mathematical Society **100** (2019), 804–824.

[36] S. Liang and S. Shi, *On multi-bump solutions of nonlinear Schrödinger equation with electromagnetic fields and critical nonlinearity in* $\mathbb{R}^N$, Calculus of Variations and Partial Differential Equations **56** (2017), Article no. 25.

[37] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, RI, 2001.

[38] Y. L. Liu, X. Li and C. Ji, *Multiplicity of concentrating solutions for a class of magnetic Schrödinger-Poisson type equation*, Advnced Nonlinear Analysis **10** (2021), 131–151.

[39] Y. G. Oh, *Existence of semi-classical bound state of nonlinear Schrödinger equations with potential on the class of* $(V)_a$, Communications in Partial Differential Equations **13** (1998), 1499–1519.

[40] Y. G. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Communications in Mathematical Physics **131** (1990), 223–253.

[41] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, Journal of Functional Analysis **257** (2009), 3802–3822.

[42] A. Szulkin and T. Weth, *The method of Nehari manifold*, in *Handbook of Nonconvex Analysis and Applications*, International Press, Somerville, MA, 2010, pp. 2314–2351.

[43] X. F. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Communications in Mathematical Physics **153** (1993), 229–244.

[44] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser, Boston, MA, 1996.

[45] Y. Zhang, X. Tang and V. D. Rădulescu, *Small perturbations for nonlinear Schrödinger equations with magnetic potential*, Milan Journal of Mathematics **88** (2020), 479–506.