Gradient Estimates for Multi-Phase Problems in Campanato Spaces

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ABSTRACT. We establish a new Campanato-type estimate for the weak solutions of a class of multi-phase problems. The problem under consideration is characterized by the fact that both ellipticity and growth switch between three different types of polynomial according to the position, which describes a feature of strongly anisotropic materials. The results obtained in this paper are different from the BMO-type estimates for the usual p-Laplacian equation due to DiBenedetto and Manfredi. The content of this paper is in close relationship with the recent pioneering contributions of Marcellini and Mingione in the qualitative analysis of multi-phase problems.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain. This paper deals with the following multi-phase problem in divergence form:

$$\begin{aligned} &\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u + b(x)|\nabla u|^{s-2}\nabla u\right) \\ &\quad = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F + b(x)|F|^{s-2}F\right) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

which is driven by the (p, q, s)-energy functional

$$\mathcal{F}(u) = \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q + b(x)|\nabla u|^s)\,\mathrm{d}x, \quad 1 < p < q \leq s.$$

The double phase problem ($b(x) \equiv 0$) is characterized by the fact that the ellipticity and growth rates of its integrand radically change with the position

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variable x. It provides a model for describing a feature of strongly anisotropic materials and new examples of Lavrentiev phenomenon. This was initially noted by Zhikov (see, e.g., [39, 40]). The modulating coefficient $a(x) \geq 0$ dictates the geometry of the composite made by two different materials. More precisely, considering two different materials with power hardening exponents p and q, respectively, the variable coefficient $a(\cdot)$ dictates the geometry of a composite of the materials. In the region where $a(x)$ is positive, the q-material is present; otherwise, the p-material is the only one making the composite.

From a regularity point of view, even without the presence of the coefficients $a(\cdot)$ and $b(\cdot)$, functional $\mathcal{F}$ presents a very interesting feature, falling in the class of problems with the non-uniformly elliptic conditions. Thus, it cannot be treated via the standard regularity methods. We refer the readers to the pioneering works by Marcellini [25–28].

If the coefficient $a(x) \neq 0$ or $b(x) \neq 0$, it brought new difficulties in the corresponding regularity theory. Indeed, even basic regularity issues for these double phase problems have remained unsolved for several decades. The first contribution was due to Colombo and Mingione in [10, 11]. By assuming that the modulating coefficient $a(\cdot) \in C^{0,\alpha}(\Omega)$, the Hölder regularity of gradients of the minimizers was proved for double phase functionals of the type

$$\mathcal{P}(u,\Omega) := \int_{\Omega} |\nabla u|^p + a(x)|\nabla u|^q \,\mathrm{d}x, \tag{1.2}$$

and the corresponding Euler-Lagrange equation of the functional is

$$\operatorname{div}(p|\nabla u|^{p-2}\nabla u + a(x)q|\nabla u|^{q-2}\nabla u) = 0. \tag{1.3}$$

They showed that if $q < p + \alpha p/N$, then minimizers of (1.2) are in $C^{1,\beta}$ for some $\beta \in (0,1)$, and if $q \leq p + \alpha$, then bounded minimizers of (1.2) are in $C^{1,\beta}$ for some $\beta \in (0,1)$ as well. Recently, Baroni, Colombo, and Mingione [3] further obtained $C^{1,\beta}$-regularity for general functionals of the double phase problems including the end point case where $q = p + \alpha p/N$, whose models are given by

$$\mathcal{P}(u,\Omega) := \int_{\Omega} F(x,u,\nabla u)\,\mathrm{d}x,$$

and they also proved that if $q < p + \alpha/(1-\gamma)$, then C^{γ}-regulary weak solutions to (1.3) are in $C^{1,\beta}$ for some $\beta \in (0,1)$, by using a different approach from previous ones introduced in [10, 11].

Starting from the remarkable work of Colombo and Mingione, despite its relatively short history, double phase problems have already evolved into an elaborate theory with several connections to other branches. In particular, Colombo and Mingione in [12] proved the following Calderón-Zygmund estimates:

$$(|F|^p + a(x)|F|^q) \in L^{\gamma}_{\mathrm{loc}}(\Omega) \Rightarrow (|\nabla u|^p + a(x)|\nabla u|^q) \in L^{\gamma}_{\mathrm{loc}}(\Omega), \quad \gamma > 1$$

for solutions to equation (1.1) with $b(x) \equiv 0$, where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain. Later on, the previous Calderón-Zygmund estimates were improved by De Filippis and Mingione in [18] for the borderline case where $q/p = 1 + \alpha/N$. Furthermore, Byun, Cho, and Oh in [4] obtained the Calderón-Zygmund type estimates for a class of irregular obstacle problems having non-uniformly elliptic operator in divergence form of (p,q)-growth. For more results, we refer to [1, 2, 5, 6, 14, 16, 17, 20, 29–34, 38] and the references therein.

More recently, De Filippis and Oh in [19] investigated the regularity issues for multi-phase problem, which is the three-phase energy

$$W^{1,H(\cdot)}(\Omega) \ni w \mapsto \mathcal{F}(w) = \int_\Omega H(x, \mathrm{D}w)\,\mathrm{d}x, \quad 1 < p < q \leq s \tag{1.4}$$

with

$$H(x,z) := |z|^p + a(x)|z|^q + b(x)|z|^s$$

whenever $x \in \Omega$ and $z \in \mathbb{R}^N$. They showed that the local minimizers of (1.4) belong to $C^{1,\nu}_{\mathrm{loc}}(\Omega)$ for some $\nu \in (0,1)$, under the assumptions that

$$a(x), b(x) \geq 0, \quad a \in C^{0,\alpha}_{\mathrm{loc}}(\Omega), \quad b \in C^{0,\beta}_{\mathrm{loc}}(\Omega), \quad \alpha, \beta \in (0,1] \tag{1.5}$$

and

$$\frac{q}{p} \leq 1 + \frac{\alpha}{N}, \quad \frac{s}{p} \leq 1 + \frac{\beta}{N}. \tag{1.6}$$

The techniques they introduced allowed proving the regularity results for functional with an arbitrary number of phases, for example,

$$w \mapsto \int_\Omega \left[|\mathrm{D}w|^p + \sum_{n=1}^{m} a_i(x)|\mathrm{D}w|^{p_i} \right] \mathrm{d}x$$

with

$$a_i \in C^{0,\alpha_i}(\Omega),\ \alpha_i \in (0,1],\ 1 < \frac{p_i}{p} \leq 1 + \frac{\alpha_i}{N},\ 1 < p < p_1 \leq \cdots \leq p_m.$$

2. Main Result

The aim of this paper is to establish a new Campanato-type estimate of weak solutions for equation (1.1). The classical Campanato space was introduced by Campanato in [7] (see also [8] for more details and applications to elliptic systems). Thanks to the Morrey-Campanato estimates, one gave different proofs of the Schauder estimates for the Laplace equation, which was traditionally built upon the Newton potential theory.

For any $x \in \Omega$, $\rho > 0$, let $\Omega(x,\rho) := \Omega \cap B(x,\rho)$. In the following, we recall some definitions and basic properties of Campanato space.

Definition 2.1. Let $s \geq 1$, $\mu \geq 0$. The Campanato space $\mathcal{L}^{s,\mu}(\Omega)$ is the class of all functions $u \in L^s(\Omega)$ such that

$$[u]_{s,\mu;\Omega} := \sup_{x\in\Omega,\, 0<\rho<\operatorname{diam}\Omega} \left(\rho^{-\mu}\int_{\Omega(x,\rho)} |u(z)-u_{x,\rho}|^s \,\mathrm{d}z\right)^{1/s} < \infty,$$

where

$$u_{x,\rho} = \fint_{\Omega(x,\rho)} u(z)\,\mathrm{d}z = \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} u(z)\,\mathrm{d}z$$

and $|\Omega(x,\rho)|$ is the Lebesgue measure of $\Omega(x,\rho)$.

For any $u \in \mathcal{L}^{s,\mu}(\Omega)$, we define

$$\|u\|_{\mathcal{L}^{s,\mu}(\Omega)} = \|u\|_{L^s(\Omega)} + [u]_{s,\mu;\Omega}.$$

The Campanato space $(\mathcal{L}^{s,\mu}(\Omega), \|\cdot\|_{\mathcal{L}^{s,\mu}(\Omega)})$ is a Banach space that extends the notion of functions of bounded mean oscillation, which is due to John and Nirenberg [23]. The Campanato space describes situations where the oscillation of the function in a ball is proportional to some power of the radius other than the dimension. They are used in the theory of elliptic partial differential equations, since for certain values of μ, elements of the space $\mathcal{L}^{s,\mu}(\Omega)$ are Hölder continuous in Ω. We also refer to the pioneering contributions of Stampacchia [35, 36] and Campanato and Murthy [9], who proved interpolation properties for Campanato spaces. In relationship with the Riesz-Thorin interpolation theorem, they proved that if A is a linear bounded operator from $L^{t_i}(\Omega)$ to $\mathcal{L}^{s_i,\mu_i}(\Omega)$ (for $i = 1, 2$), then A is bounded from $L^t(\Omega)$ to $\mathcal{L}^{s,\mu}(\Omega)$ for some corresponding intermediate values of s, t, and μ.

Obviously, in the standard situation $p = q = s$, the coefficients $a(x)$ and $b(x)$ act in the energy density as a local perturbation of the main elliptic term. In the case $p < q = s$, it indeed becomes the double phase problem. When $p < q \leq s$, $a(x)$, and $b(x)$ are no longer perturbations, a new phenomenon emerges: the rates of Hölder continuity of $a(x)$ and $b(x)$ interact with the ratios of q/p and s/p in a crucial yet precise way. Indeed, it was shown that the bound $1 + \alpha/N$ and $1 + \beta/N$ is essentially optimal (see [15, 21]).

In the rest of the paper we shall use the notation

$$A(x,z) = |z|^{p-2}z + a(x)|z|^{q-2}z + b(x)|z|^{s-2}z$$

and

$$H(x,z) = |z|^p + a(x)|z|^q + b(x)|z|^s$$

whenever $x \in \Omega$ and $z \in \mathbb{R}^N$.

Definition 2.2. A local distributional solution to (1.1) is a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega A(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega A(x,F)\cdot\nabla\varphi\,\mathrm{d}x. \tag{2.1}$$

If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local distributional solution to (1.1), with the natural integrability assumption $H(x, \nabla u), H(x, F) \in L^1_{\mathrm{loc}}(\Omega)$, then (2.1) still holds for every function φ with $H(x, \nabla\varphi) \in L^1_{\mathrm{loc}}(\Omega)$.

The main theorem is stated in the following regularity property.

Theorem 2.3. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local distributional solution to* (1.1) *with*

$$H(x, \nabla u) \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad H(x, F) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$$

for some $\sigma > 0$. Under the assumptions (1.5), (1.6), *and*

$$1 < p < q \le s < \infty,$$

if $A(x, F) \in \mathcal{L}^{p/(p-1),\mu}_{\mathrm{loc}}(\Omega)$, where $0 < \mu < N$, the estimate

$$\nabla u \in \mathcal{L}^{p,\tilde{\mu}}_{\mathrm{loc}}(\Omega)$$

holds. Here, the constant $\tilde{\mu}$ is defined by

$$\tilde{\mu} = \begin{cases} \mu & \text{if } 2 \le p < q \le s, \\ (p-1)\mu & \text{otherwise.} \end{cases} \tag{2.2}$$

Our results are a natural extension of those in the previous works [4, 12, 13, 19]. The technical approach in our proof is based on the different comparison estimates along with the good properties of homogeneous problems and carefully controlling the interaction between the two phase transitions together with the appropriate localization method. Recall that, when $a(x) = b(x) \equiv 0$, equation (1.1) becomes the usual p-Laplace type equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(|F|^{p-2}F). \tag{2.3}$$

DiBenedetto and Manfredi in [13] established the BMO estimates for the weak solution of (2.3), which states that if $p > 2$ and $|F|^{p-2}F \in \mathrm{BMO}(\mathbb{R}^N)$, then $\nabla u \in \mathrm{BMO}(\mathbb{R}^N)$. Meanwhile, the local counterpart

$$|F|^{p-2}F \in \mathrm{BMO}_{\mathrm{loc}}(\Omega) \Rightarrow \nabla u \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$$

is obtained simultaneously for $p > 2$. We would like to point out that the result obtained here is different from the usual p-Laplace equation because of the presence of variable coefficients $a(\cdot)$ and $b(\cdot)$. It gave rise to an interesting new phenomenon that the BMO-type estimate in [13] is no longer valid. In fact, we only have the Campanato-type estimates of weak solutions for equation (1.1). Note that in this paper we do not touch Sobolev-Morrey and Besov-Morrey type spaces as well as other generalizations of such a kind; we refer for more details to the monograph [37] by Yuan, Sickel, and Yang titled "Morrey and Campanato Meet Besov, Lizorkin and Triebel."

3. PROOF OF MAIN RESULT

Hereafter, let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local distributional solution to (1.1) with the natural integrability assumptions

$$H(x,\nabla u) \in L^1_{\mathrm{loc}}(\Omega) \text{ and } H(x,F) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega) \quad \text{for some } \sigma > 0.$$

For any bounded domain $\Omega_0 \Subset \tilde{\Omega} \Subset \Omega$ and $x_0 \in \Omega_0$, without loss of generality, we suppose that $x_0 = 0$, $0 < R \le \frac{1}{2}\operatorname{dist}(\Omega_0, \partial\tilde{\Omega}) := R_0 \le 1$.

The open ball of $\mathbb{R}^N$ centered at x_0 with positive radius ρ is denoted by $B_\rho(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \rho\}$. We shall simply denote $B_\rho \equiv B_\rho(0)$ and set $K := \int_{B_{2R_0}} H(x,\nabla u)\,\mathrm{d}x$

In the following, we will give two comparison results. We first consider the following homogeneous problem:

$$\begin{cases} \operatorname{div} A(x,\nabla w) = 0 & \text{in } B_{2R}, \\ w \in u + W^{1,p}_0(B_{2R}). \end{cases} \tag{3.1}$$

Lemma 3.1. *There exists a unique distributional solution $w \in u + W^{1,p}_0(B_{2R})$ to (3.1) such that $H(x,\nabla w) \in L^1(B_{2R})$. Moreover, we have the following:*

(1) *For $1 < p < q \le s$*

$$\int_{B_{2R}} H(x,\nabla w)\,\mathrm{d}x \le C \int_{B_{2R}} H(x,\nabla u)\,\mathrm{d}x,$$

where the constant C depends on p, q, s.

(2) *For $2 \le p < q \le s$*

$$\int_{B_{2R}} H(x,\nabla(u-w))\,\mathrm{d}x \le CR^{\mu}[A(x,F)]^{p/(p-1)}_{p/(p-1),\mu;\tilde{\Omega}},$$

where C depends on p, q, s, N;

(3) *For $1 < p < q \le s < 2$*

$$\int_{B_{2R}} H(x,\nabla(u-w))\,\mathrm{d}x \le CR^{\mu(p-1)}([A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}} + 1)^{ps/(2p-s)},$$

where C depends on p, q, s, N, K.

(4) *For $1 < p < 2 \le q \le s$ or $1 < p < q < 2 \le s$*

$$\int_{B_{2R}} H(x,\nabla(u-w))\,\mathrm{d}x \le CR^{\mu(p-1)}([A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}} + 1)^{p/(p-1)},$$

where C depends on p, q, s, N, K.

Proof. Under the assumptions (1.5), (1.6), the proof of the existence and uniqueness result of distributional solutions to (3.1) is analogous to the one in Theorem 3.1 in [12]. Thus, we only need to prove the three properties.

(1) Taking $w - u$ as a test function in (3.1), we find that

$$\int_{B_{2R}} A(x, \nabla w) \cdot \nabla(w - u)\,\mathrm{d}x = 0. \tag{3.2}$$

For any $\varepsilon > 0$, by Young's inequality we have

$$\begin{aligned}\int_{B_{2R}} H(x, \nabla w)\,\mathrm{d}x &= \int_{B_{2R}} A(x, \nabla w) \cdot \nabla u\,\mathrm{d}x \\ &\leq \int_{B_{2R}} (|\nabla w|^{p-1} + a(x)|\nabla w|^{q-1} + b(x)|\nabla w|^{s-1})|\nabla u|\,\mathrm{d}x \\ &\leq \varepsilon \int_{B_{2R}} H(x, \nabla w)\,\mathrm{d}x + C\int_{B_{2R}} H(x, \nabla u)\,\mathrm{d}x.\end{aligned}$$

By choosing $\varepsilon = \frac{1}{2}$, it follows that $\int_{B_{2R}} H(x, \nabla w)\,\mathrm{d}x \leq C \int_{B_{2R}} H(x, \nabla u)\,\mathrm{d}x$, where the constant C depends on p, q, s.

(2) Taking $w - u$ as a test function in (1.1), we get

$$\int_{B_{2R}} A(x, \nabla u) \cdot \nabla(w - u)\,\mathrm{d}x = \int_{B_{2R}} A(x, F) \cdot \nabla(w - u)\,\mathrm{d}x. \tag{3.3}$$

Note that $\operatorname{div}((A(x, F))_{B_{2R}}) = 0$, where

$$(A(x, F))_{B_{2R}} = \fint_{B_{2R}} A(x, F)\,\mathrm{d}x.$$

Subtracting (3.2) from (3.3), we obtain

$$\begin{aligned}&\int_{B_{2R}} (A(x, \nabla u) - A(x, \nabla w)) \cdot \nabla(w - u)\,\mathrm{d}x \\ &\qquad = \int_{B_{2R}} (A(x, F) - (A(x, F))_{B_{2R}}) \cdot \nabla(w - u)\,\mathrm{d}x.\end{aligned} \tag{3.4}$$

When $s \geq q > p \geq 2$, for any $\varepsilon > 0$, we have

$$\begin{aligned}&\int_{B_{2R}} H(x, \nabla(u - w))\,\mathrm{d}x \\ &\quad\leq C\int_{B_{2R}} (A(x, \nabla u) - A(x, \nabla w)) \cdot \nabla(u - w)\,\mathrm{d}x \\ &\quad\leq \varepsilon \int_{B_{2R}} |\nabla(u - w)|^p\,\mathrm{d}x + C\int_{B_{2R}} |A(x, F) - (A(x, F))_{B_{2R}}|^{p/(p-1)}\,\mathrm{d}x.\end{aligned}$$

By choosing $\varepsilon = \frac{1}{2}$, it implies that

$$\int_{B_{2R}} H(x, \nabla(u-w))\,\mathrm{d}x \le C\int_{B_{2R}} |A(x,F) - (A(x,F))_{B_{2R}}|^{p/(p-1)}\,\mathrm{d}x \le CR^{\mu}[A(x,F)]^{p/(p-1)}_{p/(p-1),\mu;\tilde{\Omega}},$$

where the constant C depends on p, q, s, N.

(3) From (3.4), we first give the following estimate:

$$\begin{aligned}\int_{B_{2R}} &\langle A(x,\nabla u) - A(x,\nabla w), \nabla u - \nabla w\rangle\,\mathrm{d}x \\ &\le \int_{B_{2R}} |A(x,F) - (A(x,F))_{B_{2R}}|\cdot|\nabla u - \nabla w|\,\mathrm{d}x \\ &\le \left(\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x\right)^{1/p} \\ &\quad\times\left(\int_{B_{2R}} |A(x,F) - (A(x,F))_{B_{2R}}|^{p/(p-1)}\,\mathrm{d}x\right)^{(p-1)/p} \\ &\le CR^{\mu(p-1)/p}[A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}}\left(\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x\right)^{1/p}.\end{aligned} \tag{3.5}$$

Let $1 < t \in \{p, q, s\} < 2$ and $m(x) \in \{1, a(x), b(x)\}$. By Hölder's inequality,

$$\begin{aligned}\int_{B_{2R}} &m(x)|\nabla u - \nabla w|^t\,\mathrm{d}x \\ &\le \left(\int_{B_{2R}} m(x)(|\nabla u|^2 + |\nabla w|^2)^{(t-2)/2}|\nabla u - \nabla w|^2\,\mathrm{d}x\right)^{t/2} \\ &\qquad\times\left(\int_{B_{2R}} m(x)(|\nabla u|^2 + |\nabla w|^2)^{t/2}\,\mathrm{d}x\right)^{(2-t)/2} \\ &\le 2^t\left(\int_{B_{2R}} m(x)(|\nabla u| + |\nabla w|)^{t-2}|\nabla u - \nabla w|^2\,\mathrm{d}x\right)^{t/2} \\ &\qquad\times\left(\int_{B_{2R}} m(x)(|\nabla u|^t + |\nabla w|^t)\,\mathrm{d}x\right)^{(2-t)/2}.\end{aligned} \tag{3.6}$$

When $1 < t < 2$, from the monotonicity of A, we know that

$$\begin{aligned}c(t)&(|\nabla u| + |\nabla w|)^{t-2}|\nabla u - \nabla w|^2 \\ &\le \langle |\nabla u|^{t-2}\nabla u - |\nabla w|^{t-2}\nabla w, \nabla u - \nabla w\rangle.\end{aligned} \tag{3.7}$$

Denote

$$\begin{aligned}B(x,\nabla u,\nabla w) := \int_{B_{2R}} \Big[&(|\nabla u| + |\nabla w|)^{p-2}|\nabla u - \nabla w|^2 + a(x)(|\nabla u| + |\nabla w|)^{q-2} \\ &\times|\nabla u - \nabla w|^2 + b(x)(|\nabla u| + |\nabla w|)^{s-2}|\nabla u - \nabla w|^2\Big]\,\mathrm{d}x.\end{aligned}$$

Merging (3.5), (3.6), (3.7), and the energy bound (1) of this lemma, we estimate

$$
\begin{aligned}
&\int_{B_{2R}} H(x,\nabla u-\nabla w)\,\mathrm{d}x\\
&\quad\le 2^p\Big(\int_{B_{2R}}(|\nabla u|+|\nabla w|)^{p-2}|\nabla u-\nabla w|^2\,\mathrm{d}x\Big)^{p/2}\\
&\qquad\times\Big(\int_{B_{2R}}(|\nabla u|^p+|\nabla w|^p)\,\mathrm{d}x\Big)^{(2-p)/2}\\
&\qquad+2^q\Big(\int_{B_{2R}}a(x)(|\nabla u|+|\nabla w|)^{q-2}|\nabla u-\nabla w|^2\,\mathrm{d}x\Big)^{q/2}\\
&\qquad\times\Big(\int_{B_{2R}}a(x)(|\nabla u|^q+|\nabla w|^q)\,\mathrm{d}x\Big)^{(2-q)/2}\\
&\qquad+2^s\Big(\int_{B_{2R}}b(x)(|\nabla u|+|\nabla w|)^{s-2}|\nabla u-\nabla w|^2\,\mathrm{d}x\Big)^{s/2}\\
&\qquad\times\Big(\int_{B_{2R}}b(x)(|\nabla u|^s+|\nabla w|^s)\,\mathrm{d}x\Big)^{(2-s)/2}\\
&\quad\le C(p,q,s)\max_{t\in\{p,s\}}\{(B(x,\nabla u,\nabla w))^{t/2}\}\\
&\qquad\times\max_{t\in\{p,s\}}\Big\{\Big(\int_{B_{2R}}H(x,\nabla u)+H(x,\nabla w)\,\mathrm{d}x\Big)^{(2-t)/2}\Big\}\\
&\quad\le C(p,q,s)\max_{t\in\{p,s\}}\Big\{\Big(\int_{B_{2R}}H(x,\nabla u)\,\mathrm{d}x\Big)^{(2-t)/2}\Big\}\\
&\qquad\times\max_{t\in\{p,s\}}\Big\{\Big(\int_{B_{2R}}\langle A(x,\nabla u)-A(x,\nabla w),\nabla u-\nabla w\rangle\,\mathrm{d}x\Big)^{t/2}\Big\}\\
&\quad\le C(p,q,s,K)\max_{t\in\{p,s\}}\Big\{\Big(\int_{B_{2R}}\langle A(x,\nabla u)-A(x,\nabla w),\nabla u-\nabla w\rangle\,\mathrm{d}x\Big)^{t/2}\Big\}\\
&\quad\le C(p,q,s,K)\sum_{t\in\{p,s\}}\Big(\int_{B_{2R}}\langle A(x,\nabla u)-A(x,\nabla w),\nabla u-\nabla w\rangle\,\mathrm{d}x\Big)^{t/2}\\
&\quad\le C(p,q,s,N,K)R^{\mu(p-1)/2}[A(x,F)]^{p/2}_{p/(p-1),\mu;\tilde\Omega}\Big(\int_{B_{2R}}|\nabla u-\nabla w|^p\,\mathrm{d}x\Big)^{1/2}\\
&\qquad+C(p,q,s,N,K)R^{\mu s(p-1)/(2p)}[A(x,F)]^{s/2}_{p/(p-1),\mu;\tilde\Omega}\\
&\qquad\times\Big(\int_{B_{2R}}|\nabla u-\nabla w|^p\,\mathrm{d}x\Big)^{s/(2p)}\\
&\quad\le\varepsilon\int_{B_{2R}}|\nabla u-\nabla w|^p\,\mathrm{d}x\\
&\qquad+C(\varepsilon)(C(p,q,s,N,K)R^{\mu(p-1)/2}[A(x,F)]^{p/2}_{p/(p-1),\mu;\tilde\Omega})^2\\
&\qquad+\varepsilon\int_{B_{2R}}|\nabla u-\nabla w|^p\,\mathrm{d}x\\
&\qquad+C(\varepsilon)(C(p,q,s,N,K)R^{\mu s(p-1)/(2p)}[A(x,F)]^{s/2}_{p/(p-1),\mu;\tilde\Omega})^{2p/(2p-s)}.
\end{aligned}
$$

Taking ε small enough, we deduce that

$$\begin{aligned}
&\int_{B_{2R}} H(x, \nabla u - \nabla w)\,\mathrm{d}x \\
&\quad \le C(p,q,s,N,K)[A(x,F)]^{p}_{p/(p-1),\mu;\tilde{\Omega}} R^{\mu(p-1)} \\
&\qquad + C(p,q,s,K,N)[A(x,F)]^{ps/(2p-s)}_{p/(p-1),\mu;\tilde{\Omega}} R^{\mu s(p-1)/(2p-s)} \\
&\quad \le C(p,q,s,N,K)R^{\mu(p-1)}([A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}} + 1)^{ps/(2p-s)}.
\end{aligned}$$

Here, we have used the fact that $s/(2p-s) > 1$ due to $1 < p < q \le s < 2$ and $0 < R \le 1$.

(4) For $1 < p < 2 \le q \le s$, we have

$$\begin{aligned}
&\int_{B_{2R}} H(x, \nabla u - \nabla w)\,\mathrm{d}x \\
&\le C(p)\Big(\int_{B_{2R}} (|\nabla u| + |\nabla w|)^{p-2}|\nabla u - \nabla w|^2\,\mathrm{d}x\Big)^{p/2} \\
&\qquad \times \Big(\int_{B_{2R}} (|\nabla u|^p + |\nabla w|^p)\,\mathrm{d}x\Big)^{(2-p)/2} \\
&\qquad + C(q)\int_{B_{2R}} a(x)\langle |\nabla u|^{q-2}\nabla u - |\nabla w|^{q-2}\nabla w, \nabla u - \nabla w\rangle\,\mathrm{d}x \\
&\qquad + C(s)\int_{B_{2R}} b(x)\langle |\nabla u|^{s-2}\nabla u - |\nabla w|^{s-2}\nabla w, \nabla u - \nabla w\rangle\,\mathrm{d}x \\
&\le C(p,q,s)(K+1)\max_{t\in\{p,2\}}\Big(\int_{B_{2R}} \langle A(x,\nabla u) - A(x,\nabla w), \nabla u - \nabla w\rangle\,\mathrm{d}x\Big)^{t/2} \\
&\le C(p,q,s,K)\sum_{t\in\{p,2\}}\Big(\int_{B_{2R}} \langle A(x,\nabla u) - A(x,\nabla w), \nabla u - \nabla w\rangle\,\mathrm{d}x\Big)^{t/2} \\
&\le C(p,q,s,N,K)R^{\mu(p-1)/2}[A(x,F)]^{p/2}_{p/(p-1),\mu;\tilde{\Omega}}\Big(\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x\Big)^{1/2} \\
&\qquad + C(p,q,s,N,K)R^{\mu(p-1)/p}[A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}} \\
&\qquad \times \Big(\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x\Big)^{1/p} \\
&\le \varepsilon\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x \\
&\qquad + C(\varepsilon)(C(p,q,s,N,K)R^{\mu(p-1)/2}[A(x,F)]^{p/2}_{p/(p-1),\mu;\tilde{\Omega}})^2 \\
&\qquad + \varepsilon\int_{B_{2R}} |\nabla u - \nabla w|^p\,\mathrm{d}x \\
&\qquad + C(\varepsilon)(C(p,q,s,N,K)R^{\mu(p-1)/p}[A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}})^{p/(p-1)}.
\end{aligned}$$

Noting that $1 < p < 2$ and choosing ε small enough yield that

$$\int_{B_{2R}} H(x, \nabla u - \nabla w)\,\mathrm{d}x \\ \leq C(p,q,s,N,K) R^{\mu(p-1)} \left([A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}} + 1\right)^{p/(p-1)}.$$

The case $1 < p < q < 2 \leq s$ can be obtained analogously. Therefore, we finish the proof. ◻

For later convenience, we denote

$$\mathbf{data} := \begin{cases} \mathbf{data}(N,p,q,s,K,[a]_{0,\alpha;\tilde{\Omega}},[b]_{0,\beta;\tilde{\Omega}}) & \text{if } N \geq p(1+\delta_g/2), \\ \mathbf{data}(N,p,q,s,K,[a]_{0,\alpha;\tilde{\Omega}},[b]_{0,\beta;\tilde{\Omega}},\|H(x,F)\|_{L^{1+\sigma}(\tilde{\Omega})}) & \\ & \text{if } N < p(1+\delta_g/2), \end{cases}$$

as the set of basic parameters intervening in the problem. Here, δ_g ($0 < \delta_g < \sigma$), depending on $N, p, q, s, K, [a]_{0,\alpha;\tilde{\Omega}}$ and $[b]_{0,\beta;\tilde{\Omega}}$, is concluded by Lemmas 4 and 5 in [19].

Remark 3.2. It should be mentioned that the higher integrability assumption $H(\cdot,F) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$ is needed only for treating the borderline case in which equality is assumed in (1.6).

Lemma 3.3. *Let $w \in W^{1,1}(B_{2R})$ be a distributional solution to* (3.1) *such that $H(x,\nabla w) \in L^1(B_{2R})$. For $0 < \delta < N$ and $R \leq R_0$, we have*

$$\int_{B_R} H(x,\nabla w)\,\mathrm{d}x \leq CKR^{N-\delta},$$

where the constant C depends on δ, R_0 and $\mathbf{data}$.

Proof. Much as in the arguments of Theorem 2 in [19], we could obtain the following Morrey-type estimate under the assumptions (1.5) and (1.6). For any $0 < \delta < N$ and $0 < \rho \leq R \leq 1$, we have

$$\int_{B_\rho} H(x,\nabla w)\,\mathrm{d}x \leq C_1 \left(\frac{\rho}{R}\right)^{N-\delta} \int_{B_R} H(x,\nabla w)\,\mathrm{d}x, \tag{3.8}$$

where $B_\rho \subset B_R \Subset \tilde{\Omega} \Subset \Omega$ and C_1 depends on δ and $\mathbf{data}$. By Lemma 3.1,

$$\int_{B_\rho} H(x,\nabla w)\,\mathrm{d}x \leq C_2 \left(\frac{\rho}{R}\right)^{N-\delta} \int_{B_{2R}} H(x,\nabla u)\,\mathrm{d}x,$$

then

$$\int_{B_\rho} H(x,\nabla w)\,\mathrm{d}x \leq C_2 \rho^{N-\delta} R_0^{\delta-N} \int_{B_{2R_0}} H(x,\nabla u)\,\mathrm{d}x \\ \leq CK\rho^{N-\delta}$$

by choosing $R = R_0$, where C depends on δ, R_0 and $\mathbf{data}$. ◻

Next, we introduce further comparison problems. Denote

$$a_0(R) = \inf_{x\in B_R} a(x) \quad \text{and} \quad b_0(R) = \inf_{x\in B_R} b(x). \tag{3.9}$$

Let $v \in W^{1,1}(B_R)$ be the distributional solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}\Big(|\nabla v|^{p-2}\nabla v + a_0(R)|\nabla v|^{q-2}\nabla v \\ \qquad + b_0(R)|\nabla v|^{s-2}\nabla v\Big) = 0 & \text{in } B_R, \\ v \in w + W_0^{1,p}(B_R). \end{cases} \tag{3.10}$$

For later uses, we also introduce the functions

$$\begin{aligned} A_0(z) &:= |z|^{p-2}z + a_0(R)|z|^{q-2}z + b_0(R)|z|^{s-2}z, \\ H_0(z) &:= |z|^p + a_0(R)|z|^q + b_0(R)|z|^s. \end{aligned}$$

Lemma 3.4. *Let $v \in W^{1,1}(B_R)$ be a distributional solution to (3.10) such that $H_0(\nabla v) \in L^1(B_R)$, and $w \in W^{1,1}(B_{2R})$ be a distributional solution to (3.1). Then,*

$$\int_{B_R} H_0(\nabla v)\,\mathrm{d}x \le C\int_{B_R} H_0(\nabla w)\,\mathrm{d}x,$$

where the constant C depends on p, q, s.

Proof. Taking $v - w$ as a test function in (3.10), we find

$$\int_{B_R} (|\nabla v|^{p-2}\nabla v + a_0(R)|\nabla v|^{q-2}\nabla v + b_0(R)|\nabla v|^{s-2}\nabla v)\cdot\nabla(v-w)\,\mathrm{d}x = 0.$$

Then,

$$\begin{aligned} &\int_{B_R} H_0(\nabla v)\,\mathrm{d}x \\ &\qquad \le \int_{B_R} (|\nabla v|^{p-1}|\nabla w| + a_0(R)|\nabla v|^{q-1}|\nabla w| + b_0(R)|\nabla v|^{s-1}|\nabla w|)\,\mathrm{d}x. \end{aligned}$$

Young's inequality further implies that

$$\int_{B_R} H_0(\nabla v)\,\mathrm{d}x \le C\int_{B_R} H_0(\nabla w)\,\mathrm{d}x,$$

where C depends on p, q, s. $\square$

Lemma 3.5. *Let* $w \in W^{1,1}(B_{2R})$ *be a distributional solution to* (3.1), *and let* $v \in W^{1,1}(B_R)$ *be a distributional solution to* (3.10). *For any* $0 < R \le R_0$, *we have*

$$\int_{B_R} H_0(\nabla w)\,\mathrm{d}x \le CKR^{N-\delta}$$

and

$$\int_{B_R} H_0(\nabla v)\,\mathrm{d}x \le CKR^{N-\delta},$$

where the constant C *depends on* δ, R_0 *and* $\mathbf{data}$.

Proof. Recalling Lemma 3.1, (3.8) and (3.9), for any $0 < \rho \le R \le R_0$, we get

$$\begin{aligned}\int_{B_\rho} H_0(\nabla w)\,\mathrm{d}x &= \int_{B_\rho} (|\nabla w|^p + a_0(R)|\nabla w|^q + b_0(R)|\nabla w|^s)\,\mathrm{d}x\\ &\le \int_{B_\rho} H(x,\nabla w)\,\mathrm{d}x\\ &\le C\left(\frac{\rho}{R}\right)^{N-\delta}\int_{B_R} H(x,\nabla w)\,\mathrm{d}x\\ &\le C\left(\frac{\rho}{R}\right)^{N-\delta}\int_{B_{2R}} H(x,\nabla u)\,\mathrm{d}x.\end{aligned}$$

Thus, it follows that

$$\int_{B_\rho} H_0(\nabla w)\,\mathrm{d}x \le CK\rho^{N-\delta},$$

by taking $R = R_0$. From Lemma 3.4, we conclude that

$$\int_{B_R} H_0(\nabla v)\,\mathrm{d}x \le CKR^{N-\delta},$$

where C depends on δ, R_0 and $\mathbf{data}$. We complete the proof. $\square$

Finally, we shall give the following key result, which leads to the main theorem.

Theorem 3.6. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *be a local distributional solution to* (1.1). *For any* $0 < \rho \le R_0/2$, $\delta < N - \mu$, $x \in \Omega$, *we have*

$$\rho^{-\tilde{\mu}}\int_{B_\rho(x)} |\nabla u - (\nabla u)_{B_\rho(x)}|^p\,\mathrm{d}y \le C, \tag{3.11}$$

where

$$\tilde{\mu} = \begin{cases}\mu & \text{if } 2 \le p < q \le s,\\ (p-1)\mu & \text{otherwise,}\end{cases}$$

and the constant C *depends on* R_0, δ, $[A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}}$, $\mathbf{data}$.

Proof. From Lemma 5.1 of the work of Lieberman in [24], there is $\gamma \in (0,1)$ such that for any $0 < \rho < R/2$,

$$\begin{aligned}
&\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^p + a_0(R)|\nabla v - (\nabla v)_{B_\rho}|^q + b_0(R)|\nabla v - (\nabla v)_{B_\rho}|^s \,\mathrm{d}x \\
&\qquad \le C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^p + a_0(R)|\nabla v - (\nabla v)_{B_R}|^q \\
&\qquad\qquad + b_0(R)|\nabla v - (\nabla v)_{B_R}|^s \,\mathrm{d}x.
\end{aligned}$$

Then,

$$\begin{aligned}
&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x \\
&\quad \le C\fint_{B_\rho} |\nabla u - (\nabla v)_{B_\rho}|^p \,\mathrm{d}x \\
&\quad \le C\fint_{B_\rho} |\nabla u - \nabla w|^p \,\mathrm{d}x + C\fint_{B_\rho} |\nabla w - \nabla v|^p \,\mathrm{d}x \\
&\qquad + C\fint_{B_\rho} |\nabla v - (\nabla v)_{B_\rho}|^p \,\mathrm{d}x \\
&\quad \le C\left(\frac{R}{\rho}\right)^N \fint_{B_R} |\nabla u - \nabla w|^p \,\mathrm{d}x + C\left(\frac{R}{\rho}\right)^N \fint_{B_R} |\nabla w - \nabla v|^p \,\mathrm{d}x \\
&\qquad + C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^p + a_0(R)|\nabla v - (\nabla v)_{B_R}|^q \\
&\qquad + b_0(R)|\nabla v - (\nabla v)_{B_R}|^s \,\mathrm{d}x.
\end{aligned}$$

Note that

$$\begin{aligned}
&\fint_{B_R} |\nabla v - (\nabla v)_{B_R}|^p \,\mathrm{d}x \\
&\quad \le C\fint_{B_R} |\nabla v - (\nabla u)_{B_R}|^p \,\mathrm{d}x \\
&\quad \le C\fint_{B_R} |\nabla v - \nabla w|^p \,\mathrm{d}x + C\fint_{B_R} |\nabla w - (\nabla w)_{B_R}|^p \,\mathrm{d}x \\
&\qquad + C\fint_{B_R} |(\nabla w)_{B_R} - (\nabla u)_{B_R}|^p \,\mathrm{d}x \\
&\quad \le C\fint_{B_R} |\nabla v - \nabla w|^p \,\mathrm{d}x + C\fint_{B_R} |\nabla u - \nabla w|^p \,\mathrm{d}x \\
&\qquad + C\fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \,\mathrm{d}x.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x \\
&\quad \le C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |\nabla u - \nabla w|^p \,\mathrm{d}x \\
&\qquad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |\nabla w - \nabla v|^p \,\mathrm{d}x \\
&\qquad + C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} a_0(R)|\nabla v - (\nabla v)_{B_R}|^q + b_0(R)|\nabla v - (\nabla v)_{B_R}|^s \,\mathrm{d}x \\
&\qquad + C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \,\mathrm{d}x.
\end{aligned}
$$

For the third term at the righthand side in the above inequality, we know from Hölder's inequality that

$$
\begin{aligned}
&\fint_{B_R} a_0(R)|\nabla v - (\nabla v)_{B_R}|^q + b_0(R)|\nabla v - (\nabla v)_{B_R}|^s \,\mathrm{d}x \\
&\quad \le C \fint_{B_R} a_0(R)(|\nabla v|^q + |(\nabla v)_{B_R}|^q) + b_0(R)(|\nabla v|^s + |(\nabla v)_{B_R}|^s) \,\mathrm{d}x \\
&\quad = C \fint_{B_R} a_0(R)|\nabla v|^q + b_0(R)|\nabla v|^s \,\mathrm{d}x \\
&\qquad + C a_0(R)\left|\fint_{B_R} \nabla v \,\mathrm{d}x\right|^q + C b_0(R)\left|\fint_{B_R} \nabla v \,\mathrm{d}x\right|^s \\
&\quad \le C \fint_{B_R} a_0(R)|\nabla v|^q + b_0(R)|\nabla v|^s \,\mathrm{d}x.
\end{aligned}
$$

Then, combining these previous inequalities arrives at

$$
\begin{aligned}
&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x \\
&\le C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \,\mathrm{d}x \\
&\quad + C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} a_0(R)|\nabla v|^q + b_0(R)|\nabla v|^s \,\mathrm{d}x \\
&\quad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |\nabla u - \nabla w|^p \,\mathrm{d}x \\
&\quad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |\nabla w - \nabla v|^p \,\mathrm{d}x \\
&\le C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \,\mathrm{d}x \\
&\quad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |\nabla u - \nabla w|^p \,\mathrm{d}x \\
&\quad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] \left(\fint_{B_R} H_0(\nabla v) \,\mathrm{d}x + \fint_{B_R} H_0(\nabla w) \,\mathrm{d}x\right).
\end{aligned}
\tag{3.12}
$$

For any $R \le R_0/2$, merging inequality (3.12) with Lemmas 3.1 and 3.5, we could derive the following results.

Case 1. When $2 \le p < q \le s < \infty$, we get

$$
\begin{aligned}
&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x \\
&\quad \le C\left(\frac{\rho}{R}\right)^{\gamma} \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \,\mathrm{d}x \\
&\qquad + C\left[\left(\frac{R}{\rho}\right)^{N} + \left(\frac{\rho}{R}\right)^{\gamma}\right] R^{\mu-N}[A(x,F)]^{p/(p-1)}_{p/(p-1),\mu;\tilde{\Omega}} \\
&\qquad + C\left[\left(\frac{R}{\rho}\right)^{N} + \left(\frac{\rho}{R}\right)^{\gamma}\right] R^{-\delta}.
\end{aligned}
$$

Denote

$$
\varphi(\rho) = \int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x.
$$

Taking $\delta < N - \mu$, we have

$$
\begin{aligned}
\varphi(\rho) &\le C\left[R^N + \left(\frac{\rho}{R}\right)^{\gamma} \rho^N\right] R^{\mu-N}[A(x,F)]^{p/(p-1)}_{p/(p-1),\mu;\tilde{\Omega}} \\
&\qquad + C\left(\frac{\rho}{R}\right)^{\gamma+N} \varphi(R) + C\left[R^N + \left(\frac{\rho}{R}\right)^{\gamma} \rho^N\right] R^{-\delta} \\
&\le C\left(\frac{\rho}{R}\right)^{\gamma+N} \varphi(R) + CR^{N-\delta} + CR^{\mu} \\
&\le C\left(\frac{\rho}{R}\right)^{\gamma+N} \varphi(R) + CR^{\mu},
\end{aligned}
$$

for any $0 < \rho \le R \le R_0/2$ and $0 < \mu < N$. By using the iteration lemma (see Lemma 7.3 in [22]), we get

$$
\begin{aligned}
\varphi(\rho) &\le C\left[\left(\frac{\rho}{R}\right)^{\mu} \varphi(R) + \rho^{\mu}\right] \\
&\le C\left[\rho^{\mu} R_0^{-\mu} \varphi\left(\frac{R_0}{2}\right) + \rho^{\mu}\right].
\end{aligned}
$$

Then,

$$
\rho^{-\mu} \int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \,\mathrm{d}x \le C.
$$

Case 2. If $1 < p < q \le s < 2$, then we have

$$\begin{aligned}&\fint_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \, \mathrm{d}x \\ &\quad \le C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] R^{\mu(p-1)-N}(1 + [A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}})^{ps/(2p-s)} \\ &\qquad + C\left[\left(\frac{R}{\rho}\right)^N + \left(\frac{\rho}{R}\right)^\gamma\right] R^{-\delta} + C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |\nabla u - (\nabla u)_{B_R}|^p \, \mathrm{d}x.\end{aligned}$$

By choosing $\delta < N - \mu(p-1)$, after some calculations we arrive at

$$\begin{aligned}\varphi(\rho) &\le C\left(\frac{\rho}{R}\right)^{\gamma+N} \varphi(R) + CR^{N-\delta} + CR^{\mu(p-1)} \\ &\le C\left(\frac{\rho}{R}\right)^{\gamma+N} \varphi(R) + CR^{\mu(p-1)},\end{aligned}$$

for any $0 < \rho \le R \le R_0/2$. Note that $\mu(p-1) < \mu < N$. Again using the iteration lemma (see Lemma 7.3 in [22]), we get

$$\begin{aligned}\varphi(\rho) &\le C\left[\left(\frac{\rho}{R}\right)^{\mu(p-1)} \varphi(R) + \rho^{\mu(p-1)}\right] \\ &\le C\left[\rho^{\mu(p-1)} R_0^{-\mu(p-1)} \varphi\left(\frac{R_0}{2}\right) + \rho^{\mu(p-1)}\right].\end{aligned}$$

Therefore, we find

$$\rho^{-\mu(p-1)} \int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \, \mathrm{d}x \le C.$$

Case 3. For the case $1 < p < 2 \le q \le s$ or $1 < p < q < 2 \le s$, the proof is the same as in *Case 2*, so we will not repeat it here. We also derive that

$$\rho^{-\mu(p-1)} \int_{B_\rho} |\nabla u - (\nabla u)_{B_\rho}|^p \, \mathrm{d}x \le C.$$

Consequently, when $\delta < N - \mu$, we could verify that for any $x \in \Omega$,

$$\begin{aligned}&\rho^{-\tilde{\mu}} \int_{B_\rho(x)} |\nabla u - (\nabla u)_{B_\rho(x)}|^p \, \mathrm{d}y \\ &\qquad \le C \int_{B_{R_0/2}(x)} |\nabla u - (\nabla u)_{B_{R_0/2}(x)}|^p \, \mathrm{d}y \le C,\end{aligned}$$

where

$$\tilde{\mu} = \begin{cases} \mu & \text{if } 2 \le p < q \le s, \\ (p-1)\mu & \text{otherwise,} \end{cases}$$

and the constant C depends on R_0, δ, $[A(x,F)]_{p/(p-1),\mu;\tilde{\Omega}}$, **data**. □

Once we have Theorem 3.6, the main result will follow immediately.

Proof of Theorem 2.3. From the estimate (3.11), we can obtain that

$$\nabla u \in \mathcal{L}^{p,\tilde{\mu}}_{\mathrm{loc}}(\Omega),$$

where $\tilde{\mu}$ is defined as in (2.2). This completes the proof. □

Remark 3.7. (i) In this paper, we always assume that $a(x), b(x) \ge 0$. Particularly, when $a(x) = b(x) \equiv 0$, equation (1.1) becomes the usual p-Laplace equation. For this case, if we choose $N < \mu \le N + p/(p-1)$, then we have the assumption

$$A(x,F) \in \mathcal{L}^{p/(p-1),\mu}_{\mathrm{loc}}(\Omega) \simeq C^{0,\delta_1}_{\mathrm{loc}}(\Omega) \quad \text{with } \delta_1 = \frac{(\mu-N)(p-1)}{p}.$$

Hence, from Theorem 2.3 we know that $\nabla u \in C^{0,\delta_2}_{\mathrm{loc}}(\Omega)$, where $\delta_2 = (\mu - N)/p$. If $\mu = N$, it follows from $A(x,F) \in \mathcal{L}^{p/(p-1),\mu}_{\mathrm{loc}}(\Omega)$ that $A(x,F) \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$, then $\nabla u \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$, which recovers the classical BMO estimates in [13].

(ii) We would like to mention that the methods we employ to derive the aforementioned theorem can be generalized to establish the Campanato-type estimates to the following multi-phase equation:

$$\begin{aligned} &\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u + \sum_{n=1}^{m} a_i(x)|\nabla u|^{p_i}\Big) \\ &\qquad = \operatorname{div}\Big(|F|^{p-2}F + \sum_{n=1}^{m} a_i(x)|F|^{p_i}\Big) \quad \text{in } \Omega, \end{aligned}$$

where

$$a_i \in C^{0,\alpha_i}(\Omega),\ \alpha_i \in (0,1],\ 1 < \frac{p_i}{p} \le 1 + \frac{\alpha_i}{N},\ 2 < p < p_1 \le \cdots \le p_m.$$

The main problem is to control the interaction between several potentially degenerate items of the energy, that is, $a_i(x)|\nabla u|^{p_i}$ $(i = 1, 2, \ldots, m)$.

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