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Explosive solutions of elliptic equations with absorption and non-linear gradient term

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Abstract. Let f be a non-decreasing C^1-function such that $f > 0$ on $(0, \infty)$, $f(0) = 0$, $\int_1^\infty 1/\sqrt{F(t)}\mathrm{d}t < \infty$ and $F(t)/f^{2/a}(t) \to 0$ as $t \to \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$ and $a \in (0, 2]$. We prove the existence of positive large solutions to the equation $\Delta u + q(x)|\nabla u|^a = p(x)f(u)$ in a smooth bounded domain $\Omega \subset \mathbf{R}^N$, provided that p, q are non-negative continuous functions so that any zero of p is surrounded by a surface strictly included in Ω on which p is positive. Under additional hypotheses on p we deduce the existence of solutions if Ω is unbounded.

Keywords. Explosive solution; semilinear elliptic problem; entire solution; maximum principle.

1. Introduction and the main results

The aim of this paper is to study the following semilinear elliptic problem

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x)f(u), & \text{in } \Omega \\ u \geq 0,\ u \not\equiv 0, & \text{in } \Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume throughout this paper that $a \leq 2$ is a positive real number, p, q are non-negative functions such that $p \not\equiv 0$, $p, q \in C^{0,\alpha}(\overline{\Omega})$ if Ω is bounded, and $p, q \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, otherwise. The non-linearity f is assumed to fulfill

(f1) $\quad f \in C^1[0, \infty),\ f' \geq 0,\ f(0) = 0$ and $f > 0$ on $(0, \infty)$.

(f2) $\quad \int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

(f3) $\quad \dfrac{F(t)}{f^{2/a}(t)} \to 0$ as $t \to \infty$.

The condition (f2) is called Keller–Osserman condition (see [5,11]). We also point out that the increasing non-linearity f is called an absorption term.

Remarks

(1) The above conditions hold provided that $f(t) = t^k$, $k > 1$ and $0 < a < 2k/(k+1)$ (< 2), or $f(t) = \mathrm{e}^t - 1$, or $f(t) = \mathrm{e}^t - t$ and $a < 2$.

(2) By (f1) and (f3) it follows that $f/F^{a/2} \geq \beta > 0$ for t large enough, that is, $(F^{1-a/2})' \geq \beta > 0$ for t large enough which yields $0 < a \leq 2$.
(3) Conditions (f2) and (f3) imply $\int_1^\infty \mathrm{d}t/f^{1/a}(t) < \infty$.

We are mainly interested in finding properties of *large* (*explosive*) *solutions* of (1), that is, solutions u satisfying $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbf{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbf{R}^N$). In the latter case the solution is called an *entire large* (*explosive*) solution.

Cîrstea and Rădulescu [2] proved the existence of large solutions to (1) in the case $q \equiv 0$. The aim of this paper is to study the influence of the non-linear gradient term $|\nabla u|^a$. It turns out that the presence of this term can have significant influence on the existence of a solution, as well as on its asymptotic behavior. Problems of this type appear in stochastic control theory and have been first studied by Lasry and Lions [8]. The corresponding parabolic equation was considered in Quittner [12]. In terms of the dynamic programming approach, an explosive solution of (1) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see [8]).

Bandle and Giarrusso [1] studied the existence of a large solution of problem (1) in the case $p \equiv 1$, $q \equiv 1$ and Ω bounded, while Lair and Wood [7] studied the sublinear case if $p \equiv 1$. Giarrusso [4] also studied the asymptotic behavior of the explosive solution under the same assumptions as in [1].

As observed in [1], the simplest case is $a = 2$, which can be reduced to a problem without gradient term. Indeed, if u is a solution of (1) for $q \equiv 1$, then the function $v = \mathrm{e}^u$ satisfies

$$\begin{cases} \Delta v = p(x) v f(\ln v) & \text{in } \Omega\,, \\ v(x) \to +\infty & \text{if } \operatorname{dist}(x, \partial\Omega) \to 0. \end{cases}$$

We shall therefore mainly consider the case where $0 < a < 2$.

Our first result concerns the existence of a large solution to problem (1) when Ω is bounded.

Theorem 1. *Suppose Ω is bounded and p satisfies*
(p1) *For every $x_0 \in \Omega$ with $p(x_0) = 0$, there exists a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega_0} \subset \Omega$ and $p > 0$ on $\partial\Omega_0$.*
Then problem (1) *has a positive large solution.*

Note that, by the maximum principle, a solution of (1) provides an upper bound for any solution of

$$\Delta u = p(x) g(u, \nabla u) \qquad \text{in } \Omega,$$

where

$$g(u, \xi) \geq f(u) - |\xi|^a, \qquad \forall u \in \mathbf{R},\ \forall \xi \in \mathbf{R}^N.$$

The next purpose of the paper is to prove the existence of an entire large solution for (1). Our result in this case is

Theorem 2. *Assume that $\Omega = \mathbf{R}^N$ and that problem* (1) *has at least a solution. Suppose that p satisfies the condition*
(p1)$'$ *There exists a sequence of smooth bounded domains $(\Omega_n)_{n\geq 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\mathbf{R}^N = \cup_{n=1}^\infty \Omega_n$, and* (p1) *holds in Ω_n, for any $n \geq 1$.*

Then there exists a classical solution U of (1) *which is a maximal solution if p is positive.*

If p verifies the additional condition

$$\text{(p2)} \qquad \int_0^\infty r\Phi(r)\,\mathrm{d}r < \infty\,, \quad \textit{where } \Phi(r) = \max\{p(x) :\ |x| = r\},$$

then U is an entire large solution of (1).

An example of function p satisfying both the conditions (p1)$'$ and (p2), with p vanishing in every neighborhood of infinity is given in [1].

Theorem 3. *Suppose that $\Omega \neq \mathbf{R}^N$ is unbounded and that problem* (1) *has at least a solution. Assume that p satisfies condition* (p1)$'$ *in Ω. Then there exists a classical solution U of problem* (1) *which is maximal solution if p is positive.*

If $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$ and p satisfies the additional condition (p2), *with $\Phi(r) = 0$ for $r \in [0, R]$, then the solution U of* (1) *is a large solution that blows-up at infinity.*

Our paper is organized as follows. In §2 we give an auxiliary result concerning problem (1) for Ω bounded. In §3 we prove Theorem 1 while in §4 we prove Theorems 2 and 3. In the last part of the paper we prove the following necessary condition for the existence of entire large solutions to eq. (1) if p satisfies (p2), and for which f is not assumed to satisfy (f2), and p is not required to be so regular as before. More precisely, we prove

Theorem 4. *Assume that $p \in C(\mathbf{R}^N)$ is a non-negative and non-trivial function which satisfies* (p2). *Let f be a function satisfying assumption* (f1). *Then condition*

$$\int_1^\infty \frac{\mathrm{d}t}{f(t)} < \infty \tag{2}$$

is necessary for the existence of entire large solutions to (1).

The above results also apply to problems on Riemannian manifolds if Δ is replaced by the Laplace–Beltrami operator

$$\Delta_B = \frac{1}{\sqrt{c}}\,\frac{\partial}{\partial x_i}\left(\sqrt{c}\,a_{ij}(x)\frac{\partial}{\partial x_i}\right), \qquad c := \det(a_{ij})\,,$$

with respect to the metric $\mathrm{d}s^2 = c_{ij}\,\mathrm{d}x_i\mathrm{d}x_j$, where (c_{ij}) is the inverse of (a_{ij}). In this case our results apply to concrete problems arising in Riemannian geometry (see, e.g., Li [9] and Loewner–Nirenberg [10]). For instance, if Ω is replaced by the standard N-sphere (S^N, g_0), Δ is the Laplace–Beltrami operator Δ_{g_0} and $f(u) = (N-2)/[4(N-1)]\,u^{(N+2)/(N-2)}$, we find the prescribing scalar curvature equation on S^N.

The proofs are essentially based on the maximum principle for non-linear elliptic equations and we also use the sub- and super-solutions method.

2. An auxiliary result

Lemma 1. *Let Ω be a bounded domain. Assume that $p, q \in C^{0,\alpha}(\overline{\Omega})$ are non-negative functions,* $0 < a < 2$ *is a real number, f satisfies* (f1) *and $g : \partial\Omega \to (0,\infty)$ is continuous. Then the boundary value problem*

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x)f(u) & in\ \Omega \\ u = g & on\ \partial\Omega \\ u \geq 0,\ u \not\equiv 0 & in\ \Omega \end{cases} \tag{3}$$

has a classical solution. Furthermore, if p is positive and f is strictly increasing, then the solution is unique.

Proof. First we notice that the function $u^+(x) = n$ is a super-solution of problem (3), if n is large enough. In order to find a positive sub-solution, we apply Theorem 5 in [2] (see also [3]). Hence the problem

$$\begin{cases} \Delta u = p(x)f(u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega \end{cases}$$

has a unique classical solution v, which is positive. Thus $u_- = v$ is a positive sub-solution of problem (3). Therefore this problem has at least a positive solution u. Furthermore, taking into account the regularity of p, q and f, a standard bootstrap argument based on Schauder and Hölder regularity shows that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Let us now assume that u_1 and u_2 are arbitrary solutions of (3). In order to prove the uniqueness, it is enough to show that $u_1 \geq u_2$ in Ω. We claim that

$$u_2(x) \leq u_1(x) \qquad \text{for any } x \in \Omega. \tag{4}$$

Suppose the contrary. Due to the fact that (4) is obviously fulfilled on $\partial\Omega$, we deduce that

$$\max_{x\in\overline{\Omega}} \{u_2(x) - u_1(x)\}$$

is achieved in Ω. At that point, say x_0, we have $\nabla(u_1 - u_2)(x_0) = 0$ and

$$\begin{aligned} 0 &\geq \Delta\,(u_2(x_0) - u_1(x_0)) \\ &= p(x_0)\,(f(u_2(x_0)) - f(u_1(x_0))) \\ &\quad -q(x_0)(|\nabla u_1(x_0)|^a - |\nabla u_2(x_0)|^a) \\ &= p(x_0)\,(f(u_2(x_0)) - f(u_1(x_0))) > 0\,. \end{aligned}$$

This contradiction concludes our proof. ∎

3. Existence results for bounded domains

Proof of Theorem 1. By Lemma 1, the boundary value problem

$$\begin{cases} \Delta v_n + q(x)|\nabla v_n|^a = \left(p(x) + \dfrac{1}{n}\right) f(v_n) & \text{in } \Omega \\ v_n = n & \text{on } \partial\Omega \\ v_n \geq 0,\ v_n \not\equiv 0 & \text{in } \Omega \end{cases}$$

has a unique positive solution, for any $n \geq 1$.

Let us notice first that the sequence (v_n) is non-decreasing. Indeed, by Lemma 1, the boundary value problem

$$\begin{cases} \Delta\zeta + q(x)|\nabla\zeta|^a = (\|p\|_\infty + 1)f(\zeta) & \text{in } \Omega \\ \zeta = 1 & \text{on } \partial\Omega \\ \zeta > 0 & \text{in } \Omega \end{cases}$$

has a unique solution. Using the same arguments as in the proof of Lemma 1 we deduce that

$$0 < \zeta \le v_1 \le \cdots \le v_n \le \cdots, \qquad \text{in } \Omega\,. \tag{5}$$

We now claim that

(a) for all $x_0 \in \Omega$ there exist an open set $\mathcal{O} \subset\subset \Omega$ which contains x_0 and $M_0 = M_0(x_0) > 0$ such that $v_n \le M_0$ in $\mathcal{O}$ for all $n \ge 1$.
(b) $\lim_{x\to\partial\Omega} v(x) = \infty$, where $v(x) = \lim_{n\to\infty} v_n(x)$.

We also observe that the statement (a) shows that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Standard elliptic regularity arguments show that v is a solution of problem (1). Then, by virtue of (5) and the statement (b), it follows that v is a large solution of problem (1).

To prove (a) we distinguish two cases:

Case $p(x_0) > 0$. By the continuity of p, there exists a ball $B = B(x_0, r) \subset\subset \Omega$ such that

$$m_0 := \min\{p(x);\ x \in \overline{B}\} > 0.$$

Let w be a positive solution of the problem

$$\begin{cases} \Delta w + q(x)|\nabla w|^a = m_0 f(w) & \text{in } B \\ w(x) \to \infty & \text{as } x \to \partial B. \end{cases}$$

The existence of w follows by considering the problem

$$\begin{cases} \Delta w_n + q(x)|\nabla w_n|^a = m_0 f(w_n) & \text{in } B \\ w_n = n & \text{on } \partial B. \end{cases}$$

The maximum principle implies $w_n \le w_{n+1} \le \theta$, where

$$\begin{cases} \Delta\theta + \|q\|_{L^\infty}|\nabla\theta|^a = m_0 f(\theta) & \text{in } B \\ \theta(x) \to \infty & \text{as } x \to \partial B. \end{cases}$$

We point out that the existence of θ follows as in [1] with the changing of variable $\theta(x) = u(\xi x)$, where $\xi = \|q\|_{L^\infty}^{1/(2-a)}$.

Using the same arguments as in the proof of Lemma 1, it follows that $v_n \le w$ in B. Furthermore, w is bounded in $\overline{B(x_0, r/2)}$. Setting $M_0 = \sup_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0, r/2)$, we obtain (a).

Case $p(x_0) = 0$. Our hypothesis (p1) and the boundedness of Ω imply the existence of a domain $\mathcal{O} \subset\subset \Omega$ which contains x_0 such that $p > 0$ on $\partial\mathcal{O}$. The above case shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in Ω and a constant $M_x > 0$ such that $v_n \le M_x$ on $B(x, r_x/2)$, for any $n \ge 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \dots, k_0$. Setting $M_0 = \max\{M_{x_1}, \dots, M_{x_{k_0}}\}$, we have $v_n \le M_0$ on $\partial\mathcal{O}$, for any $n \ge 1$. Applying the maximum principle (as in the proof of the uniqueness in Lemma 1) we obtain $v_n \le M_0$ in $\mathcal{O}$ and (a) follows.

Let z be the unique solution of the linear problem

$$\begin{cases} -\Delta z = p(x) & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \\ z \ge 0,\ z \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{6}$$

Moreover, by the maximum principle, $z > 0$ in Ω.

We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le z(x) \quad \text{for any } x \in \Omega. \tag{7}$$

By ([2], Lemma 1), the left-hand side of (7) is well-defined in Ω. We choose $R > 0$ so that $\overline{\Omega} \subset B(0, R)$ and fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, let $n_1 = n_1(\varepsilon)$ be such that

$$n_1 > \frac{1}{\varepsilon(N-3)(1+R^2)^{-3/2} + 3\varepsilon(1+R^2)^{-5/2}}, \tag{8}$$

and

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall\, x \in \partial\Omega,\ \forall\, n \ge n_1. \tag{9}$$

In order to prove (7), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \forall\, x \in \Omega,\ \forall n \ge n_1. \tag{10}$$

Indeed, taking $n \to \infty$ in (10) we deduce (7), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (10) fails. Then

$$\max_{x\in\overline{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right\} > 0.$$

Using (9) we see that the point where the maximum is achieved must lie in Ω. At this point, say x_0, for all $n \ge n_1$ we have

$$
\begin{aligned}
0 &\geq \Delta\left(\int_{v_n(x)}^{\infty}\frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2}\right)_{|x=x_0}\\
&= \left(-\frac{1}{f(v_n)}\Delta v_n - \left(\frac{1}{f}\right)'(v_n)\cdot|\nabla v_n|^2 - \Delta z(x)\right)_{|x=x_0}\\
&\quad - \varepsilon(\Delta(1+|x|^2)^{-1/2})_{|x=x_0}\\
&= \left(-p(x) - \frac{1}{n} + q(x)\,\frac{|\nabla v_n|^a}{f(v_n)} - \left(\frac{1}{f}\right)'(v_n)\cdot|\nabla v_n|^2 + p(x)\right)\\
&\quad - \varepsilon(\Delta(1+|x|^2)^{-1/2})_{|x=x_0}\\
&= \left(q(x)\,\frac{|\nabla v_n|^a}{f(v_n)} - \left(\frac{1}{f}\right)'(v_n)\cdot|\nabla v_n|^2\right)_{|x=x_0}\\
&\quad + \varepsilon(N-3)(1+|x_0|^2)^{-3/2} + 3\,\varepsilon(1+|x_0|^2)^{-5/2} - \frac{1}{n}\\
&\geq \left(q(x)\,\frac{|\nabla v_n|^a}{f(v_n)} - \left(\frac{1}{f}\right)'(v_n)\cdot|\nabla v_n|^2\right)_{|x=x_0}\\
&\quad + \varepsilon(N-3)(1+R^2)^{-3/2} + 3\,\varepsilon(1+R^2)^{-5/2} - \frac{1}{n} > 0
\end{aligned}
$$

(for the last inequality from above we have used (8)). This contradiction shows that inequality (9) holds and the proof of Theorem 1 is complete. ∎

4. Existence results for unbounded domains

Proof of Theorem 2. By Theorem 1, the boundary value problem

$$
\begin{cases}
\Delta u_n + q(x)|\nabla u_n|^a = p(x)f(u_n) & \text{in } \Omega_n\\
u_n(x) \to \infty & \text{as } x \to \partial\Omega_n\\
u_n > 0 & \text{in } \Omega_n
\end{cases}
\tag{11}
$$

has solution. Since $\overline{\Omega_n} \subset \Omega_{n+1}$, for each $n \geq 1$, in the same manner as in the uniqueness proof of Lemma 1 we find that $u_n \geq u_{n+1}$ in Ω_n. Since $\mathbf{R}^N = \cup_{n=1}^{\infty}\Omega_n$ and $\overline{\Omega_n} \subset \Omega_{n+1}$ it follows that for every $x_0 \in \mathbf{R}^N$ there exists $n_0 = n_0(x_0)$ such that $x_0 \in \Omega_n$ for all $n \geq n_0$. In view of the monotonicity of the sequence $(u_n(x_0))_{n\geq n_0}$ we can define $U(x_0) = \lim_{n\to\infty} u_n(x_0)$. Applying a standard bootstrap argument (see ([6], Theorem 1)) we find that $U \in C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ and $\Delta U + q(x)|\nabla U|^a = p(x)f(U)$ in $\mathbf{R}^N$.

We now prove that U is the maximal solution of problem (1) under the assumption that p is positive. Indeed, let v be an arbitrary solution of (1). By the maximum principle, we find that $u_n \geq v$ in Ω_n for all $n \geq 1$. Thus the definition of U implies that $U \geq v$ in $\mathbf{R}^N$.

We suppose, in addition, that p satisfies (p2) and we shall prove that U blows-up at infinity. From [2], the problem

$$\begin{cases} \Delta u = p(x) f(u) & \text{in } \Omega \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega\,, \end{cases}$$

admits a classical maximal solution V which, under the above assumption blows-up at infinity. It is sufficient now to show that

$$V(x) \leq u_n(x) + \varepsilon(1+|x|^2)^{-1/2} \qquad \text{for any } x \in \Omega_n \tag{12}$$

where ε is fixed. Suppose it is contrary. Then

$$\max_{x\in\overline{\Omega}_n} \big(\, V(x) - u_n(x) - \varepsilon(1+|x|^2)^{-1/2}\big) > 0\,.$$

Since $u_n(x) \to \infty$ as $x \to \partial\Omega_n$, we find that the point where the maximum is achieved must lie in Ω_n. At that point, say x_0, we have

$$\begin{aligned} 0 &\geq \Delta(V(x) - u_n(x) - \varepsilon(1+|x|^2)^{-1/2})_{|x=x_0} \\ &= p(x_0)\,(f\,(V(x_0)) - f(u_n(x_0))) + q(x)\,|\nabla u_n|^a(x_0) \\ &\quad + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\,\varepsilon(1+|x|^2)^{-5/2} > 0. \end{aligned}$$

This contradiction shows that the inequality (12) holds. Hence $V \leq u_n$ in Ω_n. By definition of U it follows that $V \leq U$ in $\mathbf{R}^N$ and so $U(x) \to \infty$ as $|x| \to \infty$. This completes the proof. ■

Proof of Theorem 3. Let $(\Omega_n)_{n\geq 1}$ be the sequence of bounded smooth domains given by condition (p1)$'$. For $n \geq 1$ fixed, let u_n be a positive solution of problem (11) and recall that $u_n \geq u_{n+1}$ in Ω_n. Set $U(x) = \lim_{n\to\infty} u_n(x)$, for every $x \in \Omega$. With the same arguments as in the proof of Theorem 2 we find that U is a classical solution to (1) and that U is the maximal solution provided that p is positive.

For the second part, in which $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$, we suppose that (p2) is fulfilled, with $\Phi(r) = 0$ for $r \in [0, R]$.

By ([2], Theorem 3), the problem

$$\begin{cases} \Delta v = p(x) f(v) & \text{in } \Omega \\ v \geq 0, \ v \not\equiv 0 & \text{in } \Omega\,, \end{cases}$$

admits a maximal solution V which, under the same assumptions as in Theorem 3, blows-up at infinity. In the same manner as in the proof of Theorem 2 we show that $V \leq U$, hence U blows up at infinity. ■

5. Proof of Theorem 4

Let u be an entire large solution of problem (1). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S = \frac{1}{\omega_N} \int_{|\xi|=1} \left(\int_{a_0}^{u(r\xi)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S,$$

where ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$ and a_0 is chosen such that $a_0 \in (0, u_0)$, where $u_0 = \inf_{\mathbf{R}^N} u > 0$. By the divergence theorem, we have

$$\begin{aligned}
\bar{u}'(r) &= \frac{1}{\omega_N}\int_{|\xi|=1} \frac{1}{f(u(r\xi))}\nabla u(r\xi)\cdot\xi\,\mathrm{d}S \\
&= \frac{1}{\omega_N r^N}\int_{|y|=r} \frac{1}{f(u(y))}\nabla u(y)\cdot y\,\mathrm{d}S \\
&= \frac{1}{\omega_N r^N}\int_{|y|=r} \nabla\left(\int_{a_0}^{u(y)} \frac{\mathrm{d}t}{f(t)}\right)\cdot y\,\mathrm{d}S \\
&= \frac{1}{\omega_N r^{N-1}}\int_{|y|=r} \frac{\partial}{\partial\nu}\left(\int_{a_0}^{u(y)} \frac{\mathrm{d}t}{f(t)}\right)\mathrm{d}S \\
&= \frac{1}{\omega_N r^{N-1}}\int_{B(0,r)} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)}\right)\mathrm{d}x.
\end{aligned}$$

Since u is a positive classical solution it follows that

$$|\bar{u}'(r)| \le Cr \to 0, \qquad \text{as } r \to 0\,.$$

On the other hand

$$\begin{aligned}
\omega_N(R^{N-1}\bar{u}'(R) - r^{N-1}\bar{u}'(r)) &= \int_D \Delta\left(\int_{a_0}^{u(x)} \frac{1}{f(t)}\,\mathrm{d}t\right)\mathrm{d}x \\
&= \int_r^R \left(\int_{|x|=z} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)}\right)\mathrm{d}S\right)\mathrm{d}z,
\end{aligned}$$

where $D = \{x \in \mathbf{R}^N : r < |x| < R\}$. Dividing by $R - r$ and taking $R \to r$ we find

$$\begin{aligned}
\omega_N(r^{N-1}\bar{u}'(r))' &= \int_{|x|=r} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)}\right)\mathrm{d}S \\
&= \int_{|x|=r} \operatorname{div}\left(\frac{1}{f(u(x))}\nabla u(x)\right)\mathrm{d}S \\
&= \int_{|x|=r} \left[\left(\frac{1}{f}\right)'(u(x))\cdot|\nabla u(x)|^2 + \frac{1}{f(u(x))}\Delta u(x)\right]\mathrm{d}S \\
&\le \int_{|x|=r} \frac{p(x)f(u(x))}{f(u(x))}\,\mathrm{d}S \le \omega_N r^{N-1}\Phi(r).
\end{aligned}$$

The above inequality yields by integration

$$\bar{u}(r) \le \bar{u}(0) + \int_0^r \sigma^{1-N}\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\sigma \qquad \forall r \ge 0. \tag{13}$$

On the other hand, according to (p2), for all $r > 0$ we have

$$\begin{aligned}
&\int_0^r \sigma^{1-N}\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\sigma \\
&\quad = \frac{1}{2-N}\int_0^r \frac{\mathrm{d}}{\mathrm{d}\sigma}(\sigma^{2-N})\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\sigma \\
&\quad = \frac{1}{2-N}r^{2-N}\int_0^r \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau - \frac{1}{2-N}\int_0^r \sigma\Phi(\sigma)\,\mathrm{d}\sigma \\
&\quad \le \frac{1}{N-2}\int_0^\infty r\Phi(r)\,\mathrm{d}r < \infty.
\end{aligned}$$

So, by (13),

$$\bar{u}(r) \le \bar{u}(0) + K \qquad \forall r \ge 0.$$

The last inequality implies that $\bar{u}$ is bounded and assuming that (2) is not fulfilled it follows that u cannot be a large solution. ■

We point out that the hypothesis (p2) on p is essential in the statement of Theorem 4. Indeed, let us consider $f(t) = t$, $p \equiv 1$, $\alpha \in (0,1)$, $q(x) = 2^{\alpha-2}\cdot|x|^\alpha$, $a = 2-\alpha \in (1,2)$. The corresponding problem is

$$\begin{cases} \Delta u + 2^{\alpha-2}|x|^\alpha|\nabla u|^a = u & \text{in } \mathbf{R}^N \\ u \ge 0,\ u \not\equiv 0 & \text{in } \mathbf{R}^N \end{cases}$$

which has the entire large solution $u(x) = |x|^2 + 2N$. It is clear that (2) is not fulfilled.

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