Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 347 (2023) 56–103

Journal of Differential Equations

www.elsevier.com/locate/jde

Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction

Wen Zhang [a,b], Jian Zhang [a,b,*], Vicenţiu D. Rădulescu [c,b,d]

[a] College of Science, Hunan University of Technology and Business, 410205 Changsha, Hunan, China

[b] Department of Mathematics, University of Craiova, 200585 Craiova, Romania

[c] Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland

[d] Simion Stoilow Institute of Mathematics of the Romanian Academy, 21 Calea Griviţei Street, 010702 Bucharest, Romania

Received 1 August 2022; revised 12 November 2022; accepted 17 November 2022

Abstract

This paper focuses on the study of multiplicity and concentration phenomena of positive solutions for the singularly perturbed double phase problem with nonlocal Choquard reaction

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \epsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $1 < p < q < N$, $0 < \mu < N$, ϵ is a small positive parameter and V is the absorption potential. Combining variational and topological arguments from Nehari manifold analysis and Ljusternik-Schnirelmann category theory, we prove the existence of positive ground state solutions that concentrate around global minimum points of the potential V. In the second part of this paper, we establish the relationship between the number of positive solutions and the topology of the set where V attains its global minimum. The main results included in this paper complement several recent contributions to the study of concentration phenomena.

* Corresponding author at: College of Science, Hunan University of Technology and Business, 410205 Changsha, Hunan, China.

E-mail addresses: zwmath2011@163.com (W. Zhang), zhangjian433130@163.com (J. Zhang), radulescu@inf.ucv.ro (V.D. Rădulescu).

https://doi.org/10.1016/j.jde.2022.11.033

0022-0396/

MSC: 35A15; 35B09; 35J92; 58E05

Keywords: Double phase problem; Concentration; Multiplicity; Nonlocal reaction; De Giorgi iteration

1. Introduction and main results

In this paper we will consider the following singularly perturbed double phase problem with nonlocal Choquard reaction

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \epsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $1 < p < q < N$, $0 < \mu < N$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, with $r \in \{p, q\}$, is the r-Laplace operator, ϵ is small positive parameter, $*$ represents the convolution between two functions, V is absorption potential and the nonlinear function G is the primitive function of g.

The present paper is inspired by recent fundamental progress in the mathematical analysis of many nonlinear patterns with unbalanced growth and nonlocal reaction. The main purpose of this paper is to investigate the multiplicity and concentration phenomenon of positive solutions for problem (1.1). And the main novelty of the paper is the combination of both, a double phase operator and a nonlocal Choquard reaction term which we describe below. To the best of our knowledge, this is the first paper dealing with the combination of both notions.

Since the contents of the paper are closely concerned with unbalanced double phase problems and nonlocal Choquard problems, we briefly introduce in what follows the related background and applications and recall some pioneering contributions in these fields. For the study of the problems with unbalanced growth, as we know the first work is due to Ball [10] who was interested in models arising in nonlinear elasticity and their qualitative properties (cavitations, discontinuous equilibrium solutions, etc.). Here we point out that the source term of problem (1.1) is driven by a differential operator with unbalanced growth due to the presence of the (p, q)-Laplace operator. This type of problem comes from a general reaction-diffusion system:

$$u_t = \operatorname{div}[A(u)\nabla u] + c(x, u) \text{ and } A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes density or concentration of multi-component substances, $\operatorname{div}[A(u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $A(u)$ and $c(x, u)$ is the reaction and relates to source and loss processes.

More precisely, problem (1.1) combines two interesting phenomena. The first one is the fact that the operator involved in (1.1) is the so-called double phase operator whose behavior switches between two different elliptic situations, which generates an interesting double phase associated energy. Originally, Zhikov [44] was the first who studied the double phase functionals in order to describe models of strongly anisotropic materials. Moreover, the double phase problems are also motivated by numerous models arising in mathematical physics. For example, we can refer to the Born-Infeld equation [14] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, using the Taylor formula, we have

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots \text{ for } |x|<1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we can obtain the double phase problem with $p = 2$ and $q = 4$. Especially, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{3/4}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots.$$

This implies that the fourth-order relativistic operator can be approximated by the following autonomous double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

For more details in the physical backgrounds and other applications, we refer the readers to see Bahrouni-Rădulescu-Repovš [9] (for phenomena associated with transonic flows) and to Benci-D'Avenia-Fortunato-Pisani [13] (for models arising in quantum physics).

The second interesting phenomena in our work is the appearance of a Choquard reaction term on the right-hand side which generates the nonlocal characteristic. This type of nonlocal problem arises in many interesting physical situations in quantum theory and plays an important role in describing the finite-range many-body interactions. For instance, the nonlocal Choquard problem was first introduced in the pioneering work of Pekar [32] to reveal the quantum physics of a polaron at rest. As pointed out by Lieb [20], Choquard used this model to describe an electron trapped in its own hole and to study steady states of the one component plasma approximation in the Hartree-Fock theory. We also refer to a survey [25] for more physical interpretations.

It is well known that, as $\epsilon \to 0$ in (1.1), the existence and asymptotic behavior of the solutions of the singularly perturbed problem (1.1) is known as the semi-classical problem. It is used to describe the transition between of quantum mechanics and classical mechanics, and gives rise to significant physical insights. In this framework, the semi-classical states have very rich dynamic behaviors such as concentration, convergence and decay etc. In order to better understand the dynamic behaviors of semi-classical states, we observe that if the function v is a solution of (1.1) for $x_0 \in \mathbb{R}^N$, then the new function $u = v(x_0 + \epsilon x)$ satisfies

$$-\Delta_p u - \Delta_q u + V(x_0 + \epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), \ \text{ in } \mathbb{R}^N.$$

This suggests some convergence, as $\epsilon \to 0$, of the family of solutions to a solution u_0 of the limit problem

$$-\Delta_p u - \Delta_q u + V(x_0)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), \ \text{ in } \mathbb{R}^N.$$

It is expected that, in the semiclassical limit $\epsilon \to 0$, the dynamic behaviors should be governed by the potential V. In particular, there should be a correspondence between semi-classical solutions of the equation and critical points of the potential.

On the other hand, this paper is also motivated by recent works related to the semi-classical limit of nonlocal Choquard equation

$$-\epsilon^2 \Delta u + V(x)u = (K(x) * |u|^p)|u|^{p-2}u, \ \text{ in } \mathbb{R}^N, \tag{1.2}$$

where V is a potential and K is a response function which possesses information on the mutual interaction between the bosons. Particularly, if $p = q = 2$, problem (1.1) comes back to problem (1.2) with Coulomb kernel. In recent years, there have been some works concerning with the study of the semi-classical analysis of problem (1.2) under various assumptions on the potential. See for example [2,26,35,37,39–41] and the references therein. Among them, the authors are mainly interested in the problem how the property of the potential influence the existence, multiplicity and concentration of semi-classical solutions. To be more precise, when $K(x) = |x|^{-1}$ and $p = 2$, using the Lyapunov-Schmidt reduction method, Wei-Winter [35] constructed families of solutions concentrating to the nondegenerate critical points of potential V. Moroz and Van Schaftingen [26] used variational methods and developed a novel nonlocal penalization technique to study the localized concentration phenomena: the solutions concentrating around local minima of potential V. Alves-Gao-Squassina-Yang [2] studied the existence and concentration behavior of solutions for the critical problem with both linear and nonlinear potentials. A similar result for the critical growth case also can be found in [40]. Yang-Ding [37] established the existence and multiplicity of semi-classical states for Choquard type equations with critical frequency.

When $p = q \neq 2$, (1.1) boils down to the following quasi-linear Choquard equation involving p-Laplacian operator

$$\begin{cases} -\epsilon^p \Delta_p u + V(x)|u|^{p-2}u = \epsilon^{\mu-N}\left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Concerning the investigation of semi-classical limit of problem (1.3), we would like to mention the recent papers by Alves-Yang [4–6]. More specifically, using mountain pass argument combined with the Ljusternik-Schnirelmann category theory, they proved the existence and multiplicity of positive solutions which concentrate at global minimum points of V under the global condition on V (see [5]). Subsequently, under the local condition on V, they used the penalization method developed by [16] to obtain a family of semiclassical states concentrating around local minimum points of V [6]. In [4], they considered the problem with competing potentials and characterized a new concentration behavior: the semi-classical solutions concentrating around global minimum points of linear potential and global maximum points of nonlinear potential.

We point out that the double phase problem with local nonlinear reaction has been considered recently by several authors. Some interesting and meaningful results were established by using various topological and variational arguments. We refer the readers to [11,17,27–30] for the existence and multiplicity results, [3,7,8,38,42,43] for related concentration and multiplicity properties of semi-classical states, and [18] for some regularity and decay results, and so on. We also mention the recent paper by Mingione-Rădulescu [24] in which a comprehensive overview of the recent developments concerning elliptic variational problems with nonstandard growth conditions and related to different kinds of nonuniformly elliptic operators.

To the best of our knowledge, there are no results concerning with the semi-classical analysis for the nonlocal double phase problem driven by a differential operator with unbalanced growth and a nonlocal Choquard reaction up until now. More precisely, motivated by this fact and the above mentioned works, in the present paper we are interested in the qualitative and asymptotic analysis of semi-classical solutions to problem (1.1) and we are mainly concerned with existence and multiplicity properties of solutions, as well as with concentration phenomenon as $\epsilon \to 0$. The features of the present paper are the following:

(1) the problem contains the combined effects of a double phase operator with unbalanced growth and of a Choquard reaction with nonlocal property;

(2) the concentration phenomenon creates a bridge between the global maximum point of the solution versus the global minimum of the absorption potential;

(3) since the lack of compactness caused by the unboundedness of domain, the Palais-Smale sequences do not have the compactness property;

(4) the proofs combine refined analysis techniques, including variational and topological tools.

Before stating our main result, we need introduce the assumptions on the potential V and the nonlinearity g. We first assume that the potential V satisfies the following condition introduced by Rabinowitz [31]:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ and satisfies

$$0 < \inf_{x\in\mathbb{R}^N} V(x) = V_0 < V_\infty = \liminf_{|x|\to\infty} V(x) < \infty.$$

Meanwhile, we assume that the nonlinearity g satisfies the following conditions:

(g_1) $g \in C^1(\mathbb{R}, \mathbb{R})$ and $g(s) = 0$ for all $s < 0$;
(g_2) $g(s) = o(|s|^{p-1})$ as $s \to 0$;
(g_3) there exist $c_0 > 0$ and $\tau \in (\frac{(2N-\mu)q}{2N}, \frac{(N-\mu)q}{N-q})$ such that

$$|g(s)| \le c_0(1 + |s|^{\tau-1}) \text{ for all } s \in \mathbb{R};$$

(g_4) there exists $\theta > q$ such that

$$0 < \theta G(s) = \theta \int_0^s g(t)dt \le 2g(s)s \text{ for all } s > 0;$$

(g_5) there exists $\kappa \in [\frac{\theta}{2}, \frac{(N-\mu)q}{N-q} + \frac{\theta}{2} - q)$ such that

$$g'(s)s^2 - (q + \kappa - \frac{\theta}{2} - 1)g(s)s > 0 \text{ for all } s > 0.$$

The first result is the existence of positive ground state solution, we have the following theorem.

Theorem 1.1. *Let $0 < \mu < p$ and assume that conditions (V) and (g_1)-(g_5) are satisfied, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, problem (1.1) has at least a positive ground state solution.*

Let us point out that the existence of semi-classical solutions is actually related to the existence of minimum point of the potential V. As a consequence, it seems rather natural to ask whether it is possible to relate the multiplicity of solutions for (1.1) to the set of minimum points of V. To answer this question, the other purpose of the present paper is to study the multiplicity result depended on the topology of the set where potential V attains its global minimum.

In order to give the precise statement of multiplicity result, we need introduce some definitions and notations. We first recall the definition of Ljusternik-Schnirelmann category. If Y is a given closed subset of a topological space X, the Ljusternik-Schnirelmann category $\mathrm{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. Let us define the sets

$$\Pi := \{x \in \mathbb{R}^N : V(x) = V_0\} \text{ and } \Pi_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Pi) \leq \delta\} \text{ for } \delta > 0.$$

Without loss of generality, below we may assume $0 \in \Pi$ throughout the paper. On the multiplicity and concentration behavior of positive solutions we have the following theorem.

Theorem 1.2. *Let $0 < \mu < p$ and assume that conditions (V) and (g_1)-(g_5) are satisfied. Then for any $\delta > 0$ there is $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$, problem (1.1) has at least $cat_{\Pi_\delta}(\Pi)$ positive solutions. Furthermore, if u_ϵ denotes one of these solutions and $x_\epsilon \in \mathbb{R}^N$ is its global maximum, then*

$$\lim_{\epsilon \to 0} V(x_\epsilon) = V_0.$$

Let us now outline the strategies and methods to prove Theorem 1.1 and Theorem 1.2. Our arguments are based on topological and variational approaches and refined analysis techniques in order to complete the proofs of main results. More precisely, in order to find the existence result of positive ground state solutions, we intend to make use of the Neahri manifold method to deal with our problem. Besides, to prove the multiplicity result of positive solutions, we take advantage of the Ljusternik-Schnirelmann category theory and the techniques due to Benci-Cerami [12] based on precise comparisons between the category of some sublevel sets of the energy functional and the category of the set Π. However, there exists a common difficulty that we need to overcome, that is the lack of compactness. Therefore, we have to verify that the energy functional possesses certain compactness condition at some minimax level. This goal will be achieved by doing a finer analysis and using the energy comparison method to establish some comparison relationships of the ground state energy value between the original problem and certain auxiliary problems.

On the other hand, the combined effects of the double phase operator with unbalanced growth and of the Choquard reaction with nonlocal nature also bring some difficulties to our analysis, it is difficult to prove the L^∞-estimate of solutions. Since the double phase operator we handled is a general class of quasilinear operator, some standard Moser iteration procedures do not work well, and then we adopt an appropriate De Giorgi iteration argument and some refined analysis techniques to show the L^∞-estimate and decay property of solutions. Moreover, we can see that these properties we obtained are contributed to determine the concentration location of solutions.

The structure of this paper is the following. In Section 2, we introduce the variational setting of problem (1.1) and present some preliminary results. In Section 3, we deal with the autonomous problem associated to problem (1.1). In Section 4, we analyze carefully the Palais-Smale compactness condition and prove the existence of positive ground state solution. In Section 5, we are devoted to the multiplicity and concentration of solutions and we complete the proofs of Theorem 1.1 and Theorem 1.2.

2. Preliminary results

In this paper, for convenience we will use the following notations.

- $|\cdot|_s$ denotes the usual norm of the space $L^s(\mathbb{R}^N)$, $1 \le s \le \infty$;
- c, C, c_i, C_i denote some different positive constants;
- u^+ and u^- denotes the positive and negative parts of function u, respectively, i.e.,

$$u^+ = \max\{u, 0\} \text{ and } u^- = \min\{u, 0\};$$

- $r^* = \frac{Nr}{N-r}$ denotes the embedding critical exponent of the Sobolev space $W^{1,r}(\mathbb{R}^N)$.

In what follows, we introduce some relevant results about the Sobolev spaces. For $p \in (1, \infty)$ and $N > p$, we define $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$|\nabla u|_p^p = \int\limits_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x.$$

Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard norm

$$\|u\|^p = \int\limits_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) \mathrm{d}x.$$

We give the following embedding property for the spaces $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$.

Lemma 2.1. *Let $N > p$, then there exists a constant $S_* > 0$ such that, for any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$,*

$$|u|_{p^*}^p \le S_*^{-1} |\nabla u|_p^p.$$

Furthermore, $W^{1,p}(\mathbb{R}^N)$ is embedded continuously into $L^s(\mathbb{R}^N)$ for any $s \in [p, p^*]$ and compactly into $L^s_{loc}(\mathbb{R}^N)$ for any $s \in [1, p^*)$.

We also have the following Lions concentration compactness lemma due to [21].

Lemma 2.2. *Let $N > p$ and $r \in [p, p^*)$. If $\{u_n\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (p, p^*)$.

Observe that, making the change of variable $x \mapsto \epsilon x$, then problem (1.1) is equivalent to the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

Evidently, if u is a solution of problem (2.1), then $v(x) := u(x/\epsilon)$ is a solution of problem (1.1). Thus, to study the original problem (1.1), we just need to study the equivalent problem (2.1).

For any fixed $\epsilon > 0$, we define the working space

$$E_\epsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)(|u|^p + |u|^q)\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_\epsilon = \|u\|_{V_\epsilon,p} + \|u\|_{V_\epsilon,q},$$

where

$$\|u\|^s_{V_\epsilon,s} = \int_{\mathbb{R}^N} (|\nabla u|^s + V(\epsilon x)|u|^s)\mathrm{d}x \text{ for all } s > 1.$$

From condition (V), we can see that $\|\cdot\|_\epsilon$ and the norm of $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ are equivalent. So, $E_\epsilon = E := W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$.

Moreover, according to Alves and Figueiredo [3], we have the following embedding property.

Lemma 2.3. *E_ϵ embeds continuously into $L^s(\mathbb{R}^N)$ for $s \in [p, q^*]$ and compactly into $L^s_{loc}(\mathbb{R}^N)$ for $s \in [1, q^*)$. Furthermore, there exist positive constant π_s such that*

$$\pi_s |u|_s \leq \|u\|_\epsilon, \text{ for all } s \in [p, q^*]. \tag{2.2}$$

Since we are going to deal with the nonlocal type problem (2.1), we would like to recall the classical Hardy-Littlewood-Sobolev inequality (see [22]) which will be frequently used throughout this paper.

Lemma 2.4. *(Hardy-Littlewood-Sobolev inequality [22]) Let $1 < r, s < +\infty$ and $0 < \mu < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\mu}{N} = 2$. If $\phi \in L^r(\mathbb{R}^N)$ and $\psi \in L^s(\mathbb{R}^N)$, then there exists a sharp constant $C(N, \mu, r, s) > 0$, independent of ϕ and ψ, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\phi(x)\psi(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \le C(N, \mu, r, s)|\phi|_r|\psi|_s.$$

We note that if $G(u) = |u|^\tau$ for some $\tau > 0$, then Lemma 2.4 shows that the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x$$

is well defined if $G(u) \in L^s(\mathbb{R}^N)$ for $s > 1$ with $\frac{2}{s} + \frac{\mu}{N} = 2$. Since we will work with $u \in E$, in order to make the integral be well defined, we have to require that $s\tau \in [p, q^*]$ by Lemma 2.3. Then we can see that τ satisfies the following inequality

$$\frac{(2N-\mu)p}{2N} \le \tau \le \frac{(2N-\mu)q^*}{2N}. \tag{2.3}$$

Here the exponent $\frac{(2N-\mu)p}{2N}$ is called the lower critical exponent and $\frac{(2N-\mu)q^*}{2N}$ is called the upper critical exponent for the double phase problem with nonlocal Choquard reaction.

Generally speaking, if the nonlinearity g satisfies the growth condition of (2.3), we can establish the existence of nontrivial solutions by using variational methods. However, in the present paper, since we not only study the existence of ground state solution, but also intent to investigate some properties of solutions such as positivity, regularity and concentration, we assume a stronger condition for the growth exponent, see conditions (g_2) and (g_3).

From conditions (g_2) and (g_3) we can deduce that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(s)| \le \varepsilon|s|^{p-1} + C_\varepsilon|s|^{\tau-1} \text{ and } |G(s)| \le \varepsilon|s|^p + C_\varepsilon|s|^\tau \text{ for any } s \in \mathbb{R}. \tag{2.4}$$

Moreover, from condition (g_5), we have

$$\sigma := q + \kappa - \frac{\theta}{2} \in [q, \frac{(N-\mu)q}{N-q}), \tag{2.5}$$

and the following monotonicity conclusions

$$s \mapsto \frac{g(s)}{s^{\sigma-1}}, \frac{G(s)}{s^\sigma} \text{ are strictly increasing on } (0, +\infty). \tag{2.6}$$

Evidently, condition (g_5) implies that the first conclusion holds. Next we show that the second conclusion holds. Observe that

$$\left(\frac{G(s)}{s^\sigma}\right)' = \frac{g(s)s^\sigma - \sigma s^{\sigma-1}G(s)}{s^{2\sigma}} = \frac{g(s)s - \sigma G(s)}{s^{\sigma+1}}.$$

Setting $f(s) = g(s)s - \sigma G(s)$, it follows that

$$f(0) = 0 \text{ and } f'(s) = g'(s)s + g(s) - \sigma g(s).$$

We infer from (g_5) that $f'(s) > 0$ for all $s > 0$. So, $f(s)$ is increasing and $f(s) > 0$ for all $s > 0$. Consequently, we know that the second conclusion holds. Meanwhile, from (2.6) we also have

$$s \mapsto \frac{g(s)}{s^{\frac{q}{2}-1}}, \frac{G(s)}{s^{\frac{q}{2}}} \text{ are strictly increasing on } (0, +\infty). \tag{2.7}$$

We define the energy functional associated with problem (2.1)

$$\mathcal{I}_\epsilon(u) = \frac{1}{p}|\nabla u|_p^p + \frac{1}{q}|\nabla u|_q^q + \int_{\mathbb{R}^N} V(\epsilon x)\left[\frac{1}{p}|u|^p + \frac{1}{q}|u|^q\right]\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Using Lemma 2.3, Lemma 2.4 and some standard arguments, we can easily check that $\mathcal{I}_\epsilon$ is well defined on E_ϵ and belongs to C^2 with its derivative given by

$$\begin{aligned}\langle \mathcal{I}_\epsilon'(u), v\rangle = &\int_{\mathbb{R}^N} \left[|\nabla u|^{p-2}\nabla u \cdot \nabla v + |\nabla u|^{q-2}\nabla u \cdot \nabla v\right]\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} V(\epsilon x)\left[|u|^{p-2}u + |u|^{q-2}u\right]v\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))v(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Hence, it is obvious that the solutions of problem (2.1) correspond to critical points of $\mathcal{I}_\epsilon$.

To obtain the positive ground state solutions of problem (2.1), we need to define the Nehari manifold and ground state energy related to $\mathcal{I}_\epsilon$

$$\mathcal{N}_\epsilon := \{u \in E_\epsilon \setminus \{0\} : \langle \mathcal{I}_\epsilon'(u), u\rangle = 0\} \text{ and } c_\epsilon := \inf_{\mathcal{N}_\epsilon} \mathcal{I}_\epsilon.$$

Obviously, $\mathcal{N}_\epsilon$ contains all nontrivial critical points of $\mathcal{I}_\epsilon$, and if c_ϵ is achieved by some $u_\epsilon \in \mathcal{N}_\epsilon$, then u_ϵ is called a ground state solution of problem (2.1).

Next we check some properties for Nehari manifold $\mathcal{N}_\epsilon$, which will be used frequently in the sequel of the paper.

Lemma 2.5. *Assume that (V) and (g_1)-(g_5) hold, there exists $\alpha_0 > 0$, independent of ϵ, such that $\|u\|_\epsilon \geq \alpha_0$ for all $u \in \mathcal{N}_\epsilon$.*

Proof. For any $u \in \mathcal{N}_\epsilon$, from Lemma 2.1, Lemma 2.4, (2.2) and (2.4) we conclude that

$$\begin{aligned}\|u\|_{V_\epsilon,p}^p + \|u\|_{V_\epsilon,q}^q &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \\ &\leq c_1\left(\varepsilon|u|_{\frac{2Np}{2N-\mu}}^p + C_\varepsilon|u|_{\frac{2N\tau}{2N-\mu}}^\tau\right)^2 \\ &\leq c_2\left(\varepsilon\|u\|_\epsilon^{2p} + C_\varepsilon\|u\|_\epsilon^{p+\tau} + C_\varepsilon\|u\|_\epsilon^{2\tau}\right).\end{aligned}$$

If $\|u\|_\epsilon \geq 1$, the conclusion is obvious. If $\|u\|_\epsilon < 1$, then $\|u\|^q_{V_\epsilon,p} \leq \|u\|^p_{V_\epsilon,p} < 1$ since $1 < p < q$. According to the inequality: $a^s + b^s \geq c_s(a+b)^s$ for any $a, b \geq 0$ and $s > 1$, we have

$$c_2\left(\varepsilon\|u\|^{2p}_\epsilon + C_\varepsilon\|u\|^{p+\tau}_\epsilon + C_\varepsilon\|u\|^{2\tau}_\epsilon\right) \geq \|u\|^p_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q} \geq \|u\|^q_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q} \geq c_3\|u\|^q_\epsilon.$$

So, from the above facts, we can see that there exists $\alpha_0 > 0$ such that $\|u\|_\epsilon \geq \alpha_0$. This completes the proof. $\square$

We set

$$\widehat{\mathcal{I}}_\epsilon(u) = \int_{\mathbb{R}^N} \left[|\nabla u|^p + V(\epsilon x)|u|^p + |\nabla u|^q + V(\epsilon x)|u|^q\right] \mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x,$$

then we have $\langle \widehat{\mathcal{I}}_\epsilon(u), u\rangle < 0$, this shows that $\mathcal{N}_\epsilon$ is a smooth complete manifold of codimension 1 in the space E_ϵ. Furthermore, there exists a constant $\alpha > 0$ such that

$$\langle \widehat{\mathcal{I}}_\epsilon(u), u\rangle \leq -\alpha \quad \text{for all } u \in \mathcal{N}_\epsilon. \tag{2.8}$$

Indeed, according to (g_4) and (g_5) we can deduce that

$$\begin{aligned}
&\langle \widehat{\mathcal{I}}_\epsilon(u), u\rangle \\
=&p\int_{\mathbb{R}^N} |\nabla u|^p + V(\epsilon x)|u|^p \mathrm{d}x + q\int_{\mathbb{R}^N} |\nabla u|^q + V(\epsilon x)|u|^q \mathrm{d}x \\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(u(y))u(y)g(u(x))u(x) + G(u(y))g'(u(x))u^2(x) + G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
=&(p-q)\int_{\mathbb{R}^N} |\nabla u|^p + V(\epsilon x)|u|^p \mathrm{d}x + q\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(u(y))u(y)g(u(x))u(x) + G(u(y))g'(u(x))u^2(x) + G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
\leq&(p-q)\int_{\mathbb{R}^N} |\nabla u|^p + V(\epsilon x)|u|^p \mathrm{d}x \\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))\left[(q-1-\frac{\theta}{2})g(u(x))u(x) - g'(u(x))u^2(x)\right]}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
\leq&(p-q)\int_{\mathbb{R}^N} |\nabla u|^p + V(\epsilon x)|u|^p \mathrm{d}x - \kappa\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))u(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

If there exists $\{u_n\} \subset \mathcal{N}_\epsilon$ such that $\langle \widehat{\mathcal{I}}_\epsilon(u_n), u_n\rangle \to 0$, then we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^p + V(\epsilon x)|u_n|^p \mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to 0.$$

So we can get $u_n \to 0$ in E_ϵ, this contradicts with the fact that $\|u\| \geq \alpha_0$ for all $u \in \mathcal{N}_\epsilon$.

The following result shows that the energy functional $\mathcal{I}_\epsilon$ satisfies the geometric structure of the mountain pass theorem.

Lemma 2.6. (mountain pass geometry) *Assume that* (V) *and* (g_1)-(g_5) *hold, then*

(i) *there exist* σ_0, $\varrho > 0$ *such that* $\mathcal{I}_\epsilon(u) \geq \sigma_0$ *with* $\|u\|_\epsilon = \varrho$;
(ii) *there exist* $u_0 \in E_\epsilon$ *and* $R > 0$ *with* $\|u_0\|_\epsilon > R$ *such that* $\mathcal{I}_\epsilon(u_0) < 0$.

Proof. (i) Let $u \in E_\epsilon$, we first take $\varrho \in (0,1)$ with $\|u\|_\epsilon = \varrho$, then we have $\|u\|^q_{V_\epsilon,p} \leq \|u\|^p_{V_\epsilon,p} < 1$. Therefore, using Lemma 2.1, Lemma 2.4, (2.2) and (2.4) we obtain

$$\begin{aligned}
\mathcal{I}_\epsilon(u) &= \frac{1}{p}\|u\|^p_{V_\epsilon,p} + \frac{1}{q}\|u\|^q_{V_\epsilon,q} - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\geq \frac{1}{p}\|u\|^q_{V_\epsilon,p} + \frac{1}{q}\|u\|^q_{V_\epsilon,q} - c_4\left(\varepsilon\|u\|^{2p}_\epsilon + \varepsilon C_\varepsilon\|u\|^{p+\tau}_\epsilon + C_\varepsilon\|u\|^{2\tau}_\epsilon\right) \\
&\geq \frac{c_5}{q}\left(\|u\|_{V_\epsilon,p} + \|u\|_{V_\epsilon,q}\right)^q - c_4\left(\varepsilon\|u\|^{2p}_\epsilon + \varepsilon C_\varepsilon\|u\|^{p+\tau}_\epsilon + C_\varepsilon\|u\|^{2\tau}_\epsilon\right) \\
&= \frac{c_5}{q}\|u\|^q_\epsilon - c_4\left(\varepsilon\|u\|^{2p}_\epsilon + \varepsilon C_\varepsilon\|u\|^{p+\tau}_\epsilon + C_\varepsilon\|u\|^{2\tau}_\epsilon\right).
\end{aligned}$$

Since $q < 2\tau$, then there exists $\sigma_0 > 0$ such that $\mathcal{I}_\epsilon(u) \geq \sigma_0 > 0$ when $\|u\|_\epsilon = \varrho$.

(ii) We fix $u_0 \in E_\epsilon\backslash\{0\}$ with $u_0 > 0$, and we set

$$h(t) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\frac{tu_0(y)}{\|u_0\|_\epsilon})G(\frac{tu_0(x)}{\|u_0\|_\epsilon})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \text{ for } t > 0.$$

We can infer from (g_4) that

$$\begin{aligned}
h'(t) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\frac{tu_0(x)}{\|u_0\|_\epsilon})g(\frac{tu_0(y)}{\|u_0\|_\epsilon})\frac{u_0(x)}{\|u_0\|_\epsilon}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\geq \frac{\theta}{2t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\frac{tu_0(y)}{\|u_0\|_\epsilon})G(\frac{tu_0(x)}{\|u_0\|_\epsilon})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&= \frac{\theta}{t}h(t).
\end{aligned} \tag{2.9}$$

Integrating (2.9) on $[1, s\|u_0\|_\epsilon]$ with $s\|u_0\|_\epsilon > 1$, we have

$$h(s\|u_0\|_\epsilon) \geq h(1)(s\|u_0\|_\epsilon)^\theta,$$

that is

$$\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(su_0(y))G(su_0(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \geq \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(\frac{u_0(y)}{\|u_0\|_\epsilon})G(\frac{u_0(x)}{\|u_0\|_\epsilon})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x(s\|u_0\|_\epsilon)^\theta.$$

Therefore, from the above facts we have

$$\begin{aligned}\mathcal{I}_\epsilon(su_0) =& \frac{s^p}{p}\|u_0\|^p_{V_\epsilon,p} + \frac{s^q}{q}\|u_0\|^q_{V_\epsilon,q} - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(su_0(y))G(su_0(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ \leq& c_6(s^p+s^q) - c_7 s^\theta\end{aligned}$$

for $s > \frac{1}{\|u_0\|_\epsilon}$. Taking $e = su_0$ with s sufficiently large, we can see that the conclusion (ii) holds since $\theta > q$. $\square$

According to Lemma 2.6, we can use a version of mountain pass theorem without the Palais-Smale condition [36] to deduce the existence of a Palais-Smale sequence $\{u_n\}$ at level $\tilde{c}_\epsilon$, namely

$$\mathcal{I}_\epsilon(u_n) \to \tilde{c}_\epsilon \text{ and } \mathcal{I}'_\epsilon(u_n) \to 0,$$

where $\tilde{c}_\epsilon$ is the mountain pass level of $\mathcal{I}_\epsilon$ defined as

$$\tilde{c}_\epsilon = \inf_{\ell\in\Gamma}\max_{t\in[0,1]}\mathcal{I}_\epsilon(\ell(t)),$$

and

$$\Gamma = \{\ell \in C([0,1], E_\epsilon) : \ell(0) = 0, \mathcal{I}_\epsilon(\ell(1)) < 0\}.$$

Lemma 2.7. *Let $u \in E_\epsilon\backslash\{0\}$, then there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\epsilon$. Moreover, $t_u u$ is the unique global maximum of $\mathcal{I}_\epsilon$ on $\mathbb{R}^+u$. In particular, if $u \in \mathcal{N}_\epsilon$, then*

$$\mathcal{I}_\epsilon(u) = \max_{t\geq 0}\mathcal{I}_\epsilon(tu) \geq \mathcal{I}_\epsilon(tu) \text{ for all } t \geq 0.$$

Proof. Let $u \in E_\epsilon\backslash\{0\}$, we define the function $f(t) = \mathcal{I}_\epsilon(tu)$ for $t > 0$. From Lemma 2.6, we can know that $f(0) = 0$, $f(t) > 0$ for t sufficiently small and $f(t) < 0$ for t sufficiently large. Therefore, there is $t = t_u$ such that $\max_{t>0} f(t)$ is achieved at t_u, so $f'(t_u) = 0$ and $t_u u \in \mathcal{N}_\epsilon$.

Next, we claim that t_u is the unique critical point of f. Assume by contradiction that there exist t_1 and t_2 with $0 < t_1 < t_2$ such that $t_1 u, t_2 u \in \mathcal{N}_\epsilon$, then it follows that

$$t_1^{p-q}\|u\|^p_{V_\epsilon,p} + \|u\|^q_{V_\epsilon,q} = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(t_1u(y))g(t_1u(x))u(x)}{t_1^{q-1}|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

and

$$t_2^{p-q}\|u\|_{V_\epsilon,p}^p+\|u\|_{V_\epsilon,q}^q=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(t_2u(y))g(t_2u(x))u(x)}{t_2^{q-1}|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Subtracting term by term in the above equalities, we get

$$(t_1^{p-q}-t_2^{p-q})\|u\|_{V_\epsilon,p}^p=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{G(t_1u(y))g(t_1u(x))u(x)}{t_1^{q-1}|x-y|^\mu}-\frac{G(t_2u(y))g(t_2u(x))u(x)}{t_2^{q-1}|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x.$$

Using (g_1) and (2.7) and recalling that $p<q$, we can deduce that

$$\begin{aligned}0<&(t_1^{p-q}-t_2^{p-q})\|u\|_{V_\epsilon,p}^p\\=&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{\frac{q}{2}}|u(x)|^{\frac{q}{2}}}{|x-y|^\mu}\left[\frac{G(t_1u(y))g(t_1u(x))}{|t_1u(y)|^{\frac{q}{2}}|t_1u(x)|^{\frac{q}{2}-1}}-\frac{G(t_2u(y))g(t_2u(x))}{|t_2u(y)|^{\frac{q}{2}}|t_2u(x)|^{\frac{q}{2}-1}}\right]\mathrm{d}y\mathrm{d}x<0,\end{aligned}$$

which implies a contradiction. The proof is completed. □

Applying Lemma 2.6 and Lemma 2.7, we can see that the ground state energy c_ϵ has a minimax characterization given by

$$c_\epsilon=\tilde{c}_\epsilon=\inf_{u\in E_\epsilon\setminus\{0\}}\max_{t\geq 0}\mathcal{I}_\epsilon(tu).\tag{2.10}$$

The proof can be found in [36], here we omit the details.

Lemma 2.8. *Let $\{u_n\}$ be a Palais-Smale sequence at level $c>0$ for $\mathcal{I}_\epsilon$, then $\{u_n\}$ is bounded in E_ϵ and $\|u_n^-\|_\epsilon=o(1)$.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence at level $c>0$ for $\mathcal{I}_\epsilon$, we conclude from (g_4) that

$$\begin{aligned}c+1+\|u_n\|_\epsilon\geq&\mathcal{I}_\epsilon(u_n)-\frac{1}{\theta}\langle\mathcal{I}_\epsilon'(u_n),u_n\rangle\\&=\left[\frac{1}{p}-\frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p+\left[\frac{1}{q}-\frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,q}^q\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u_n(y))}{|x-y|^\mu}\left[\frac{1}{\theta}g(u_n(x))u_n(x)-\frac{1}{2}G(u_n(x))\right]\mathrm{d}y\mathrm{d}x\\&\geq\left[\frac{1}{p}-\frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p+\left[\frac{1}{q}-\frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,q}^q\\&\geq\left[\frac{1}{q}-\frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p+\|u_n\|_{V_\epsilon,q}^q).\end{aligned}$$

We use a contradiction argument to prove this conclusion, and we assume that $\|u_n\|_\epsilon \to \infty$. In what follows we divide into three cases to finish the proof of the lemma.

Case 1. $\|u_n\|_{V_\epsilon,p} \to \infty$ and $\|u_n\|_{V_\epsilon,q} \to \infty$. Since $p < q$, we can see that $\|u_n\|_{V_\epsilon,q}^q \geq \|u_n\|_{V_\epsilon,q}^p > 1$ for n sufficiently large. Therefore, from the above facts we infer that

$$\begin{aligned} c+1+\|u_n\|_\epsilon &\geq \left[\frac{1}{q}-\frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p+\|u_n\|_{V_\epsilon,q}^q) \\ &\geq \left[\frac{1}{q}-\frac{1}{\theta}\right](\|u_n\|_{V_\epsilon,p}^p+\|u_n\|_{V_\epsilon,q}^p) \\ &\geq c_8(\|u_n\|_{V_\epsilon,p}+\|u_n\|_{V_\epsilon,q})^p = c_8\|u_n\|_\epsilon^p. \end{aligned}$$

Evidently, this is impossible, a contradiction.

Case 2. $\|u_n\|_{V_\epsilon,p} \to \infty$ and $\|u_n\|_{V_\epsilon,q}$ is bounded. According to the following fact

$$c+1+\|u_n\|_{V_\epsilon,p}+\|u_n\|_{V_\epsilon,q} \geq \left[\frac{1}{q}-\frac{1}{\theta}\right]\|u_n\|_{V_\epsilon,p}^p,$$

we have

$$\frac{c_9}{\|u_n\|_{V_\epsilon,p}^p}+\frac{\|u_n\|_{V_\epsilon,p}}{\|u_n\|_{V_\epsilon,p}^p} \geq \frac{1}{q}-\frac{1}{\theta}.$$

Letting $n \to \infty$, we can see that $0 \geq \frac{1}{q}-\frac{1}{\theta} > 0$, which shows a contradiction.

Case 3. $\|u_n\|_{V_\epsilon,p}$ is bounded and $\|u_n\|_{V_\epsilon,q} \to \infty$. We can proceed similarly as in the Case 2.

From the boundedness of $\{u_n\}$ we have $\langle \mathcal{I}_\epsilon'(u_n), u_n^-\rangle = o(1)$. Employing (g_1) and the following inequality

$$|a-b|^{s-2}(a-b)(a^- - b^-) \geq |a^- - b^-|^s \text{ for all } s > 1, \tag{2.11}$$

we get

$$\begin{aligned} \|u_n^-\|_{V_\epsilon,p}^p+\|u_n^-\|_{V_\epsilon,q}^q &\leq \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla u_n^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^{p-2}u_n u_n^- \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \nabla u_n^- \mathrm{d}x + \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^{q-2}u_n u_n^- \mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))u_n^-(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x = o(1), \end{aligned}$$

which implies that $\|u_n^-\|_\epsilon \to 0$ in E_ϵ. Consequently, we may assume that $u_n \geq 0$ for any $n \in \mathbb{N}$. The proof is now complete. □

3. The autonomous problem

For our scope, we shall also investigate the limit problem associated with problem (2.1). To this end, we first discuss in this section the existence of the positive ground state solutions of the autonomous problem.

Let $m > 0$, we consider the following autonomous problem

$$\begin{cases} -\Delta_p u - \Delta_q u + m(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

and define the space

$$E_m = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} m(|u|^p + |u|^q)\mathrm{d}x < \infty \right\}$$

with the norm $\|u\|_m = \|u\|_{m,p} + \|u\|_{m,q}$, where

$$\|u\|_{m,s}^s = \int_{\mathbb{R}^N} (|\nabla u|^s + m|u|^s)\mathrm{d}x \text{ for all } s > 1.$$

The corresponding energy functional of problem (3.1) is defined by

$$\mathcal{J}_m(u) = \frac{1}{p}\|u\|_{m,p}^p + \frac{1}{q}\|u\|_{m,q}^q - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

According to the discussion in Section 2, we can easily see that $\mathcal{J}_m \in C^2(E_m, \mathbb{R})$ and

$$\begin{aligned} \langle \mathcal{J}_m'(u), v\rangle = &\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \cdot \nabla v \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} m[|u|^{p-2}u + |u|^{q-2}u]v\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))v(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \end{aligned}$$

for any $u, v \in E_m$. Accordingly, we use $\mathcal{N}_m$ and c_m to denote the corresponding Nehari manifold and ground state energy of $\mathcal{J}_m$

$$\mathcal{N}_m := \{u \in E_m \setminus \{0\} : \langle \mathcal{J}_m'(u), u\rangle = 0\} \text{ and } c_m = \inf_{\mathcal{N}_m} \mathcal{J}_m.$$

Moreover, analogous to arguments used in Section 2, we can see that $\mathcal{J}_m$, $\mathcal{N}_m$ and c_m have some properties similar to those of $\mathcal{I}_\epsilon$, $\mathcal{N}_\epsilon$ and c_ϵ. By using the mountain pass theorem without the Palais-Smale condition [36], there exists a Palais-Smale sequence $\{u_n\} \subset E_m$ such that

$$\mathcal{J}_m'(u_n) \to 0 \text{ and } \mathcal{J}_m(u_n) \to c_m, \tag{3.2}$$

where

$$c_m = \inf_{u\in E_m\setminus\{0\}} \max_{t\geq 0} \mathcal{J}_m(tu) = \inf_{u\in\mathcal{N}_m} \mathcal{J}_m(u).$$

We now state the main result for the autonomous problem (3.1).

Lemma 3.1. *Let $0<\mu<p$ and assume that conditions (g_1)-(g_5) hold. Then problem (3.1) has at least one positive ground state solution v such that $\mathcal{J}_m(v)=c_m>0$.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence at level $c_m>0$ for $\mathcal{J}_m$, Lemma 2.8 shows that $\{u_n\}$ is bounded in E_m. We claim that

$$\varliminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^q \mathrm{d}x > 0.$$

If the above claim does not hold, by Lemma 2.2 we have $u_n\to 0$ in $L^s(\mathbb{R}^N)$ for any $s\in(q,q^*)$. So, using the Hardy-Littlewood-Sobolev inequality and (2.4) we obtain

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to 0.$$

Consequently, we can infer that

$$\begin{aligned} o(1) = \langle \mathcal{J}_m'(u_n),u_n\rangle &= \|u_n\|_{m,p}^p + \|u_n\|_{m,q}^q - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_n(y))g(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &= \|u_n\|_{m,p}^p + \|u_n\|_{m,q}^q + o(1), \end{aligned}$$

which implies that $\|u_n\|_m\to 0$. This contradicts with the conclusion of Lemma 2.5.

Therefore, there exist $R,\delta>0$ and $\{k_n\}\subset\mathbb{Z}^N$ such that

$$\int\limits_{B_R(k_n)} |u_n|^q \mathrm{d}x \geq \delta.$$

Setting $v_n(x)=u_n(x+k_n)$, we have

$$\int\limits_{B_R(0)} |v_n|^q \mathrm{d}x \geq \delta. \tag{3.3}$$

Since $\mathcal{J}_m$ and $\mathcal{J}_m'$ are both invariants by translation, it follows that

$$\mathcal{J}_m(v_n)\to c_m \text{ and } \mathcal{J}_m'(v_n)\to 0. \tag{3.4}$$

After passing to a subsequence, we assume that $v_n\rightharpoonup v$ in E_m, $v_n\to v$ in $L^s_{loc}(\mathbb{R}^N)$ for $s\in(p,q^*)$, and $v_n(x)\to v(x)$ a.e. in $\mathbb{R}^N$. Moreover, from (3.4) we deduce that $v\neq 0$.

Next we show that v is a critical point of $\mathcal{J}_m$. To do this, after passing to a subsequence, we need to prove $\nabla v_n(x) \to \nabla v(x)$ a.e. on $\mathbb{R}^N$.

Indeed, let us fix $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \le \psi(x) \le 1, \forall x \in \mathbb{R}^N, \psi(x) = 1, \forall x \in B_1(0) \text{ and } \psi(x) = 0, \forall x \in B_2^c(0),$$

and define the function $\psi_R(x) = \psi(x/R)$ for each $R > 0$ and $x \in \mathbb{R}^N$. Moreover, for simplicity we set

$$\begin{aligned}A_n = &\int_{\mathbb{R}^N} \langle |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v\rangle \psi_R \mathrm{d}x\\ &+ \int_{\mathbb{R}^N} \langle |\nabla v_n|^{q-2}\nabla v_n - |\nabla v|^{q-2}\nabla v, \nabla v_n - \nabla v\rangle \psi_R \mathrm{d}x\\ &+ \int_{\mathbb{R}^N} m\left[(|v_n|^{p-2}v_n - |v|^{p-2}v) + (|v_n|^{q-2}v_n - |v|^{q-2}v)\right](v_n - v)\psi_R \mathrm{d}x.\end{aligned}$$

Computing directly, we have

$$\begin{aligned}A_n = &\langle \mathcal{J}_m'(v_n), v_n\psi_R\rangle - \langle \mathcal{J}_m'(v_n), v\psi_R\rangle - \int_{\mathbb{R}^N} m[|v|^{p-2}v + |v|^{q-2}v](v_n - v)\psi_R \mathrm{d}x\\ &- \int_{\mathbb{R}^N} \left[|\nabla v_n|^{p-2}\nabla v_n + |\nabla v_n|^{q-2}\nabla v_n\right]\nabla\psi_R(v_n - v)\mathrm{d}x\\ &- \int_{\mathbb{R}^N} \left[|\nabla v|^{p-2}\nabla v + |\nabla v|^{q-2}\nabla v\right](\nabla v_n - \nabla v)\psi_R \mathrm{d}x\\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[\frac{G(v_n(y))g(v_n(x))v_n(x)\psi_R(x)}{|x-y|^\mu} - \frac{G(v_n(y))g(v_n(x))v(x)\psi_R(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x\end{aligned}$$

From Lemma 2.3 and the weak convergence $v_n \rightharpoonup v$ in E_m, we can infer that

$$\begin{aligned}&\int_{\mathbb{R}^N} m[|v|^{p-2}v + |v|^{q-2}v](v_n - v)\psi_R \mathrm{d}x = o(1),\\ &\int_{\mathbb{R}^N} \left[|\nabla v_n|^{p-2}\nabla v_n + |\nabla v_n|^{q-2}\nabla v_n\right]\nabla\psi_R(v_n - v)\mathrm{d}x = o(1),\\ &\int_{\mathbb{R}^N} \left[|\nabla v|^{p-2}\nabla v + |\nabla v|^{q-2}\nabla v\right](\nabla v_n - \nabla v)\psi_R \mathrm{d}x = o(1).\end{aligned} \tag{3.5}$$

On the other hand, using Lemma 2.3 and Lemma 2.4, we can easily prove that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{G(v_n(y))g(v_n(x))v_n(x)\psi_R(x)}{|x-y|^\mu}-\frac{G(v_n(y))g(v_n(x))v(x)\psi_R(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x=o(1). \quad (3.6)$$

Combining (3.5) and (3.6) and the facts that $\langle \mathcal{J}_m'(v_n), v_n\psi_R\rangle = \langle \mathcal{J}_m'(v_n), v\psi_R\rangle = o(1)$, we deduce that $A_n = o(1)$. Moreover, applying the following inequality

$$\langle |\xi|^{s-2}\xi - |\eta|^{s-2}\eta, \xi-\eta\rangle \geq \begin{cases} c|\xi-\eta|^s, & \text{if } s\geq 2,\\ c(|\xi|+|\eta|)^{s-2}|\xi-\eta|^2, & \text{if } 1<s<2, \end{cases} \quad \forall \xi,\eta\in\mathbb{R}^N \quad (3.7)$$

we conclude for some subsequence of $\{v_n\}$ that $\nabla v_n(x)\to\nabla v(x)$ a.e. $x\in B_R(0)$. Since $R>0$ is arbitrary, we derive that for some subsequence,

$$\nabla v_n(x)\to\nabla v(x) \text{ a.e. } x\in\mathbb{R}^N.$$

This limit allows us to conclude that $\mathcal{J}_m'(v)=0$. Consequently, we have $v\in\mathcal{N}_m$ and $\mathcal{J}_m(v)\geq c_m$. On the other hand, using Fatou's lemma and (g_4), we conclude that

$$\begin{aligned} c_m &= \lim_{n\to\infty}\left[\mathcal{J}_m(v_n)-\frac{1}{\theta}\langle \mathcal{J}_m'(v_n), v_n\rangle\right]\\ &= \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{\theta}\right)\|v_n\|_{m,p}^p+\left(\frac{1}{q}-\frac{1}{\theta}\right)\|v_n\|_{m,q}^q\right.\\ &\quad\left.+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v_n(y))}{|x-y|^\mu}\left(\frac{1}{\theta}g(v_n(x))v_n(x)-\frac{1}{2}G(v_n(x))\right)\mathrm{d}y\mathrm{d}x\right]\\ &\geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\|v\|_{m,p}^p+\left(\frac{1}{q}-\frac{1}{\theta}\right)\|v\|_{m,q}^q\\ &\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v_(y))}{|x-y|^\mu}\left(\frac{1}{\theta}g(v(x))v(x)-\frac{1}{2}G(v(x))\right)\mathrm{d}y\mathrm{d}x\\ &=\mathcal{J}_m(v)-\frac{1}{\theta}\langle \mathcal{J}_m'(v), v\rangle=\mathcal{J}_m(v). \end{aligned}$$

Therefore, $\mathcal{J}_m(v)=c_m$ and v is a ground state solution of problem (3.1).

Moreover, choosing v^- as test function in problem (3.1), and applying (g_1) and (2.11) we infer that

$$\begin{aligned} \|v^-\|_{m,p}^p+\|v^-\|_{m,q}^q &\leq \int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla v^-\mathrm{d}x+m\int_{\mathbb{R}^N}|v|^{p-2}vv^-\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}|\nabla v|^{q-2}\nabla v\nabla v^-\mathrm{d}x+m\int_{\mathbb{R}^N}|v|^{q-2}vv^-\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v(y))g(v(x))v^-(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x=0. \end{aligned}$$

Evidently, this shows that $v^- = 0$, then $v \geq 0$ in $\mathbb{R}^N$.

Now we claim that $v \in L^\infty(\mathbb{R}^N) \cap C^{1,\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0, 1)$. Indeed, setting

$$K(x) = \int_{\mathbb{R}^N} \frac{G(v(y))}{|x-y|^\mu} \mathrm{d}y,$$

we first show that there exists $C > 0$ such that

$$|K(x)| \leq C \quad \text{for all } x \in \mathbb{R}^N. \tag{3.8}$$

We deduce from Lemma 2.3 and (2.4) that

$$\begin{aligned} |K(x)| = &\left| \int_{\mathbb{R}^N} \frac{G(v(y))}{|x-y|^\mu} \mathrm{d}y \right| = \left| \int_{|x-y|\leq 1} \frac{G(v(y))}{|x-y|^\mu} \mathrm{d}y \right| + \left| \int_{|x-y|\geq 1} \frac{G(v(y))}{|x-y|^\mu} \mathrm{d}y \right| \\ \leq & C_1 \int_{|x-y|\leq 1} \frac{|v_n|^p + |v_n|^\tau}{|x-y|^\mu} \mathrm{d}y + C_1 \int_{|x-y|\geq 1} (|v_n|^p + |v_n|^\tau) \mathrm{d}y \\ \leq & C_1 \int_{|x-y|\leq 1} \frac{|v_n|^p + |v_n|^\tau}{|x-y|^\mu} \mathrm{d}y + C_2. \end{aligned} \tag{3.9}$$

Choosing $t \in (\frac{N}{N-\mu}, \frac{N}{N-p})$ such that $N - \frac{t\mu}{t-1} > 0$, and using Hölder inequality we have

$$\begin{aligned} \int_{|x-y|\leq 1} \frac{|v_n|^p}{|x-y|^\mu} \mathrm{d}y \leq & \left[\int_{|x-y|\leq 1} |v_n|^{tp} \mathrm{d}y \right]^{\frac{1}{t}} \left[\int_{|x-y|\leq 1} |x-y|^{-\frac{t\mu}{t-1}} \mathrm{d}y \right]^{\frac{t-1}{t}} \\ \leq & C_3 \left[\int_{|r|\leq 1} |r|^{N-1-\frac{t\mu}{t-1}} \mathrm{d}r \right]^{\frac{t-1}{t}} \leq C_4. \end{aligned} \tag{3.10}$$

Similarly, since $\tau \in (\frac{(2N-\mu)q}{2N}, \frac{(N-\mu)q}{N-q})$, taking $s \in (\frac{N}{N-\mu}, \frac{Nq}{(N-q)\tau})$ such that $N - \frac{s\mu}{s-1} > 0$, we get

$$\begin{aligned} \int_{|x-y|\leq 1} \frac{|v_n|^\tau}{|x-y|^\mu} \mathrm{d}y \leq & \left[\int_{|x-y|\leq 1} |v_n|^{s\tau} \mathrm{d}y \right]^{\frac{1}{s}} \left[\int_{|x-y|\leq 1} |x-y|^{-\frac{s\mu}{s-1}} \mathrm{d}y \right]^{\frac{s-1}{s}} \\ \leq & C_5 \left[\int_{|r|\leq 1} |r|^{N-1-\frac{s\mu}{s-1}} \mathrm{d}r \right]^{\frac{s-1}{s}} \leq C_6. \end{aligned} \tag{3.11}$$

Obviously, from (3.9), (3.10) and (3.11) we can see that (3.8) holds.

According to the above arguments, we know that v is a solution of problem

$$-\Delta_p v - \Delta_q v + m(|v|^{p-2}v + |v|^{q-2}v) = K(x)g(v) \ \text{ in } \mathbb{R}^N,$$

with $K \in L^\infty(\mathbb{R}^N)$. Applying the regularity conclusions in [18], we have $v \in L^\infty(\mathbb{R}^N) \cap C^{1,\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0,1)$. Finally, using the Harnack's inequality in [34], we conclude that $v > 0$ in $\mathbb{R}^N$. □

Next we establish a comparison relation for the ground state energy under different parameters.

Lemma 3.2. *If* $0 < m_1 < m_2$*, then we have* $c_{m_1} < c_{m_2}$.

Proof. Let $u \in \mathcal{N}_{m_2}$ with $\mathcal{J}_{m_2}(u) = c_{m_2}$, then, Lemma 2.7 shows that

$$c_{m_2} = \mathcal{J}_{m_2}(u) = \max_{t \geq 0} \mathcal{J}_{m_2}(tu).$$

By Lemma 2.7 again, there exist $t_0 > 0$ such that $u_0 = t_0 u \in \mathcal{N}_{m_1}$ satisfying

$$\mathcal{J}_{m_1}(u_0) = \max_{t \geq 0} \mathcal{J}_{m_1}(tu_0).$$

According to the above facts we deduce that

$$\begin{aligned} c_{m_2} &= \mathcal{J}_{m_2}(u) \geq \mathcal{J}_{m_2}(u_0) \\ &= \mathcal{J}_{m_1}(u_0) + (m_2 - m_1)\left[\frac{1}{p}|u_0|_p^p + \frac{1}{q}|u_0|_q^q\right] + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_0(y))G(u_0(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &\geq c_{m_1} + (m_2 - m_1)\left[\frac{1}{p}|u_0|_p^p + \frac{1}{q}|u_0|_q^q\right] + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_0(y))G(u_0(x))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Evidently, we have $c_{m_2} > c_{m_1}$. The proof is now completed. □

4. Existence of positive ground state solutions

In this section, we are going to prove the existence of positive ground state solutions to problem (1.1). We begin by analyzing the Palais-Smale compactness condition which play a fundamental role in our analysis.

For simplicity, we use the following symbols

$$\Gamma(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u(y))G(u(y))}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \ \text{ and } \ \langle \Gamma'(u), v\rangle = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u(y))g(u(x))v(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

We have a variant of the Brezis–Lieb lemma for the nonlocal term.

Lemma 4.1. *Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in E_ϵ, and set $v_n = u_n - u$. Then the following conclusions hold:*

$$\Gamma(v_n) = \Gamma(u_n) - \Gamma(u) + o(1),$$
$$\langle \Gamma'(v_n), \psi\rangle = \langle \Gamma'(u_n), \psi\rangle - \langle \Gamma'(u), \psi\rangle + o(1) \text{ uniformly in } \psi \in E_\epsilon.$$

Proof. We only prove the first conclusion because the second one can be obtained using similar arguments. We first claim that

$$G(u_n) - G(v_n) - G(u) \to 0 \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N). \tag{4.1}$$

Using the mean value theorem, Young's inequality and (2.4), we can see that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\begin{aligned} |G(u_n) - G(v_n)|^{\frac{2N}{2N-\mu}} \le & \left| \int_0^1 g(u_n - tu)u\,\mathrm{d}t \right|^{\frac{2N}{2N-\mu}} \\ \le & c_{10}\left[|u|(|u_n| + |u|)^{p-1} + |u|(|u_n| + |u|)^{\tau-1} \right]^{\frac{2N}{2N-\mu}} \\ \le & \epsilon c_{11}\left[|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}} \right] + c_{12}C_\epsilon \left[|u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\tau}{2N-\mu}} \right]. \end{aligned}$$

This, together with the following estimate

$$|G(u)|^{\frac{2N}{2N-\mu}} \le c_{13}\left[|u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\tau}{2N-\mu}} \right]$$

implies that

$$\begin{aligned} & |G(u_n) - G(v_n) - G(u)|^{\frac{2N}{2N-\mu}} \\ \le & \epsilon c_{11}\left[|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}} \right] + (c_2 C_\epsilon + c_{13})\left[|u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\tau}{2N-\mu}} \right] \\ \le & \epsilon c_{11}\left[|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}} - |u|^{\frac{2Np}{2N-\mu}} - |u|^{\frac{2N\tau}{2N-\mu}} \right] + c_{14}\left[|u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\tau}{2N-\mu}} \right]. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{G}_{\epsilon,n} = \max\Bigg\{ & |G(u_n) - G(v_n) - G(u)|^{\frac{2N}{2N-\mu}} \\ & - \epsilon c_{11}\left[|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}} - |u|^{\frac{2Np}{2N-\mu}} - |u|^{\frac{2N\tau}{2N-\mu}} \right], 0 \Bigg\} \end{aligned}$$

and we note that $\mathcal{G}_{\epsilon,n} \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and

$$0 \le \mathcal{G}_{\epsilon,n} \le c_{14}\left[|u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\tau}{2N-\mu}}\right] \in L^1(\mathbb{R}^N).$$

According to the definition of $\mathcal{G}_{\epsilon,n}$, we immediately obtain

$$|G(u_n) - G(v_n) - G(u)|^{\frac{2N}{2N-\mu}} \le \mathcal{G}_{\epsilon,n} + \epsilon c_{11}\left[|u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\tau}{2N-\mu}}\right],$$

which together with the boundedness of $\{u_n\}$, yields that

$$\limsup_{n\to\infty} \int\limits_{\mathbb{R}^N} |G(u_n) - G(v_n) - G(u)|^{\frac{2N}{2N-\mu}} \,\mathrm{d}x \le c_{15}\epsilon.$$

Evidently, this shows that (4.1) holds since the arbitrariness of ϵ.

By a direct computation, we obtain

$$\begin{aligned}
\Gamma(u_n) - \Gamma(v_n) - \Gamma(u) =& \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_n)G(u_n) - G(v_n)G(v_n) - G(u)G(u)}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x \\
=& \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u_n)\left[G(u_n) - G(v_n) - G(u)\right]}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x \\
&+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(v_n)\left[G(u_n) - G(v_n) - G(u)\right]}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x \\
&+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u)\left[G(u_n) - G(v_n) - G(u)\right]}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x \\
&+2 \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u)G(v_n)}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x \\
=& \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.
\end{aligned}$$

We deduce from the boundedness of $\{u_n\}$ and Lemma 2.3 that

$$\int\limits_{\mathbb{R}^N} |G(u_n)|^{\frac{2N}{2N-\mu}} \mathrm{d}y \le C \text{ and } \int\limits_{\mathbb{R}^N} |G(v_n)|^{\frac{2N}{2N-\mu}} \mathrm{d}y \le C.$$

So, according to Lemma 2.4 we have

$$|\Sigma_1| \le \left[\int\limits_{\mathbb{R}^N} |G(u_n)|^{\frac{2N}{2N-\mu}} \mathrm{d}y\right]^{\frac{2N-\mu}{2N}} \left[\int\limits_{\mathbb{R}^N} |G(u_n) - G(v_n) - G(u)|^{\frac{2N}{2N-\mu}} \mathrm{d}x\right]^{\frac{2N-\mu}{2N}} \to 0.$$

Similarly, we also show that $\Sigma_2 \to 0$ and $\Sigma_3 \to 0$. On the other hand, we observe that

$$G(v_n) \rightharpoonup 0 \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^\mu} \mathrm{d}y \in L^{\frac{2N}{\mu}}(\mathbb{R}^N).$$

Consequently, it is easy to see that $\Sigma_4 \to 0$, and we complete the proof of the first conclusion. □

Following from [23, Theorem 3.3], we can obtain the following technical result without proof.

Lemma 4.2. *Let $\varphi_n : \mathbb{R}^N \to \mathbb{R}^m$, $m \geq 1$, with $\varphi_n \in L^s(\mathbb{R}^N) \times \cdots \times L^s(\mathbb{R}^N)$ $(s > 1)$, $\varphi_n \to 0$ a.e. in $\mathbb{R}^m$ and $A(y) = |y|^{s-2}y$, $y \in \mathbb{R}^m$. Then, if $|\varphi_n|_s \leq c$ for all $n \in \mathbb{N}$, there holds*

$$\int_{\mathbb{R}^N} |A(\varphi_n + u) - A(\varphi_n) - A(u)|^{\frac{s}{s-1}} \mathrm{d}x = o(1)$$

for each $u \in L^s(\mathbb{R}^N) \times \cdots \times L^s(\mathbb{R}^N)$.

Employing Lemma 4.1 and Lemma 4.2 and using some standard arguments, we can show the following result holds.

Lemma 4.3. *Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in E_ϵ, and set $v_n = u_n - u$. Then we have*

$$\mathcal{I}_\epsilon(v_n) = \mathcal{I}_\epsilon(u_n) - \mathcal{I}_\epsilon(u) + o(1),$$
$$\langle \mathcal{I}'_\epsilon(v_n), \psi \rangle = \langle \mathcal{I}'_\epsilon(u_n), \psi \rangle - \langle \mathcal{I}'_\epsilon(u), \psi \rangle + o(1)$$

uniformly in $\psi \in E_\epsilon$.

Proof. According to the Brezis–Lieb lemma and Lemma 4.1, it is easy to see that the first conclusion holds. Next we prove that the second conclusion holds. Indeed, for $s \in \{p, q\}$, Lemma 4.2 implies that

$$\int_{\mathbb{R}^N} |A(v_n) - A(u_n) + A(u)|^{\frac{s}{s-1}} \mathrm{d}x = o(1). \tag{4.2}$$

Moreover, following the proof of Theorem 3.3 in [23], we obtain

$$\int_{\mathbb{R}^N} V(\epsilon x)||v_n|^{s-2}v_n - |u_n|^{s-2}u_n + |u|^{s-2}u|^{\frac{s}{s-1}} \mathrm{d}x = o(1). \tag{4.3}$$

Taking advantage of the Hölder inequality, for any $\psi \in E_\epsilon$ with $\|\psi\|_\epsilon \leq 1$, we infer that

$$|\langle \mathcal{J}'_\epsilon(v_n) - \mathcal{J}'_\epsilon(u_n) + \mathcal{J}'_\epsilon(u), \psi \rangle|$$
$$\leq \left[\int_{\mathbb{R}^N} |A(v_n) - A(u_n) + A(u)|^{\frac{p}{p-1}} \mathrm{d}x \right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^N} |\nabla \psi|^p \mathrm{d}x \right]^{\frac{1}{p}}$$

$$
\begin{aligned}
&+\left[\int_{\mathbb{R}^N} |A(v_n)-A(u_n)+A(u)|^{\frac{q}{q-1}}\mathrm{d}x\right]^{\frac{q-1}{q}}\left[\int_{\mathbb{R}^N}|\nabla\psi|^q\mathrm{d}x\right]^{\frac{1}{q}}\\
&+\left[\int_{\mathbb{R}^N} V(\epsilon x)||v_n|^{p-2}v_n-|u_n|^{p-2}u_n+|u|^{p-2}u|^{\frac{p}{p-1}}\mathrm{d}x\right]^{\frac{p-1}{p}}\left[\int_{\mathbb{R}^N}V(\epsilon x)|\psi|^p\mathrm{d}x\right]^{\frac{1}{p}}\\
&+\left[\int_{\mathbb{R}^N} V(\epsilon x)||v_n|^{q-2}v_n-|u_n|^{q-2}u_n+|u|^{q-2}u|^{\frac{q}{q-1}}\mathrm{d}x\right]^{\frac{q-1}{q}}\left[\int_{\mathbb{R}^N}V(\epsilon x)|\psi|^q\mathrm{d}x\right]^{\frac{1}{q}}\\
&+\langle\Gamma'(v_n)-\Gamma'(u_n)+\Gamma'(u),\psi\rangle.
\end{aligned}
$$

Evidently, Lemma 4.1, (4.2) and (4.3) show the second conclusion holds. The proof is completed. □

To study the compactness issue, we need to consider the limit problem of (2.1)

$$
\begin{cases}
-\Delta_p u-\Delta_q u+V_\infty(|u|^{p-2}u+|u|^{q-2}u)=\left(\frac{1}{|x|^\mu}*G(u)\right)g(u), & \text{in } \mathbb{R}^N,\\
u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), u>0, & \text{in } \mathbb{R}^N.
\end{cases}
\tag{4.4}
$$

Also we use $\mathcal{J}_{V_\infty}$, $\mathcal{N}_{V_\infty}$ and c_{V_∞} to denote the energy functional, Nehari manifold and ground state energy of problem (4.4), respectively.

Lemma 4.4. *Let $\{u_n\}$ be a Palais-Smale sequence at level $c>0$ for $\mathcal{I}_\epsilon$ with $u_n \rightharpoonup u$ in E_ϵ. Then we have either $u_n\to u$ in E_ϵ along a subsequence, or $c-\mathcal{I}_\epsilon(u)\geq c_{V_\infty}$.*

Proof. We set $v_n=u_n-u$ and assume that $v_n\nrightarrow 0$ in E_ϵ. From Lemma 2.7, we see that there is a unique $\{t_n\}\subset(0,\infty)$ such that $\{t_nv_n\}\subset\mathcal{N}_{V_\infty}$. We will divide our proof into three steps.

Step 1. The sequence $\{t_n\}$ satisfies

$$\limsup_{n\to\infty} t_n\leq 1.$$

Indeed, arguing by contradiction we assume that there exist $\nu>0$ and a subsequence of $\{t_n\}$, still denoted by itself, such that

$$t_n\geq 1+\nu \text{ for all } n\in\mathbb{N}.$$

From Lemma 4.3 we know that $\langle\mathcal{I}_\epsilon'(v_n),v_n\rangle=o(1)$, and combining $\{t_nv_n\}\subset\mathcal{N}_{V_\infty}$ we deduce that

$$|\nabla v_n|_p^p+|\nabla v_n|_q^q+\int_{\mathbb{R}^N}V(\epsilon x)(|v_n|^p+|v_n|^q)\mathrm{d}x-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(v_n)g(v_n)v_n}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x=o(1)$$

and

$$t_n^{p-q}|\nabla v_n|_p^p + |\nabla v_n|_q^q + V_\infty \int_{\mathbb{R}^N} (t_n^{p-q}|v_n|^p + |v_n|^q)\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(t_n v_n)g(t_n v_n)v_n}{t_n^{q-1}|x-y|^\mu}\mathrm{d}y\mathrm{d}x = 0.$$

Consequently, we immediately obtain from the above formulas

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[\frac{G(t_n v_n)g(t_n v_n)v_n}{t_n^{q-1}|x-y|^\mu} - \frac{G(v_n)g(v_n)v_n}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x \\ =&(t_n^{p-q}-1)|\nabla v_n|_p^p + \int_{\mathbb{R}^N} \left[t_n^{p-q}V_\infty - V(\epsilon x)\right]|v_n|^p\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} [V_\infty - V(\epsilon x)]\,|v_n|^q\mathrm{d}x + o(1). \end{aligned} \tag{4.5}$$

From condition (V), we know that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that

$$V(\epsilon x) \geq V_\infty - \varepsilon > V_\infty / t_n^{q-p} - \varepsilon \text{ for any } |x| \geq R. \tag{4.6}$$

Since $v_n \rightharpoonup 0$ in E_ϵ, Lemma 2.3 yields that $v_n \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for $s \in [1, q^*)$. By (4.5) and (4.6) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left[\frac{G(t_n v_n)g(t_n v_n)v_n}{t_n^{q-1}|x-y|^\mu} - \frac{G(v_n)g(v_n)v_n}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} \left[(t_n^{p-q}V_\infty - V(\epsilon x))|v_n|^p + (V_\infty - V(\epsilon x))|v_n|^q\right]\mathrm{d}x + o(1) \\ &\leq \varepsilon \int_{|x|\geq R} (|v_n|^p + |v_n|^q)\mathrm{d}x + 2V_{\max} \int_{|x|\leq R} (|v_n|^p + |v_n|^q)\mathrm{d}x + o(1) \\ &= c_{16}\varepsilon + o(1). \end{aligned} \tag{4.7}$$

Since $v_n \nrightarrow 0$ in E_ϵ and $\mathcal{I}'_\epsilon(v_n) \to 0$, we can show there exist $\bar{R}$, $\delta > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_{\bar{R}}(y_n)} |v_n|^q\mathrm{d}x \geq \delta. \tag{4.8}$$

Otherwise, Lemma 2.2 implies that $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (q, q^*)$. Since $\langle \mathcal{I}'_\epsilon(v_n), v_n\rangle = o(1)$, then, using Lemma 2.4 and (2.4) we get

$$
\begin{aligned}
o(1) &= \langle \mathcal{I}'_\epsilon(v_n), v_n\rangle \\
&= \|v_n\|^p_{V_\epsilon,p} + \|v_n\|^q_{V_\epsilon,q} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(v_n)g(v_n)v_n}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\geq \|v_n\|^p_{V_\epsilon,p} + \|v_n\|^q_{V_\epsilon,q} - c_{17}\left[\varepsilon|v_n|^p_{\frac{2Np}{2N-\mu}} + C_\varepsilon|v_n|^\tau_{\frac{2N\tau}{2N-\mu}}\right]^2.
\end{aligned}
$$

Obviously, this which implies that $v_n \to 0$ in E_ϵ. So, (4.8) holds.

Setting $\tilde{v}_n = v_n(x+y_n)$, we assume that, after passing to a subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ in E_ϵ and $\tilde{v}_n(x) \to \tilde{v}(x)$ a.e. in $\mathbb{R}^N$. Thus,

$$
\int_{B_{\tilde{R}}(0)} |\tilde{v}_n|^q \mathrm{d}x \geq \delta,
$$

showing that $\tilde{v} \neq 0$. Moreover, using the fact that $v_n \geq 0$ for all $n \in \mathbb{N}$, we have that $\tilde{v}(x) \geq 0$ a.e. in $\mathbb{R}^N$. Hence, there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $\tilde{v}(x) > 0$ for all $x \in \Omega$. Consequently, it follows from (2.7) and (4.7) that

$$
\begin{aligned}
0 &< \int_\Omega\int_\Omega \frac{|v_n(y)|^{\frac{q}{2}}|v_n(x)|^{\frac{q}{2}}}{|x-y|^\mu}\left[\frac{G((1+\nu)v_n(y))g((1+\nu)v_n(x))}{|(1+\nu)v_n(y)|^{\frac{q}{2}}|(1+\nu)v_n(x)|^{\frac{q}{2}-1}} - \frac{G(v_n(y))g(v_n(x))}{|v_n(y)|^{\frac{q}{2}}|v_n(x)|^{\frac{q}{2}-1}}\right]\mathrm{d}y\mathrm{d}x \\
&= \int_\Omega\int_\Omega\left[\frac{G((1+\nu)v_n(y))g((1+\nu)v_n(x))v_n(x)}{(1+\nu)^q|x-y|^\mu} - \frac{G(v_n(y))g(v_n(x))v_n(x)}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x \\
&\leq c_{16}\varepsilon + o(1).
\end{aligned}
$$

Letting $n \to \infty$ in the last inequality and employing Fatou's lemma, we have

$$
0 < \int_\Omega\int_\Omega\left[\frac{G((1+\nu)\tilde{v})g((1+\nu)\tilde{v})\tilde{v}}{(1+\nu)^q|x-y|^\mu} - \frac{G(\tilde{v})g(\tilde{v})\tilde{v}}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x \leq c_{16}\varepsilon,
$$

which is a contradiction, since the arbitrariness of ε.

According to Step 1, we conclude that

$$
\limsup_{n\to\infty} t_n = 1 \text{ or } \limsup_{n\to\infty} t_n = t_0 < 1.
$$

Next we study each one of these possibilities.

Step 2. The sequence $\{t_n\}$ satisfies

$$
\limsup_{n\to\infty} t_n = 1.
$$

For this case, there exists a subsequence, such that $t_n \to 1$. Using $\mathcal{J}_{V_\infty}(t_n v_n) \geq c_{V_\infty}$ and Lemma 4.3 we have

$$\begin{aligned} c - \mathcal{I}_\epsilon(u) + o(1) = \mathcal{I}_\epsilon(v_n) &= \mathcal{I}_\epsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + \mathcal{J}_{V_\infty}(t_n v_n) \\ &\geq \mathcal{I}_\epsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) + c_{V_\infty}. \end{aligned} \tag{4.9}$$

Observe that,

$$\begin{aligned} &\mathcal{I}_\epsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) \\ &= \frac{(1-t_n^p)}{p} \int\limits_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x + \frac{1}{p} \int\limits_{\mathbb{R}^N} \left[V(\epsilon x) - t_n^p V_\infty\right] |v_n|^p \mathrm{d}x \\ &+ \frac{(1-t_n^q)}{q} \int\limits_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x + \frac{1}{q} \int\limits_{\mathbb{R}^N} \left[V(\epsilon x) - t_n^q V_\infty\right] |v_n|^q \mathrm{d}x \\ &+ \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \left[\frac{G(t_n v_n) G(t_n v_n)}{|x-y|^\mu} - \frac{G(v_n) G(v_n)}{|x-y|^\mu}\right] \mathrm{d}y \mathrm{d}x. \end{aligned} \tag{4.10}$$

It follows from (4.6) that

$$V(\epsilon x) - t_n^p V_\infty = [V(\epsilon x) - V_\infty] + (1-t_n^p) V_\infty \geq -\varepsilon + (1-t_n^p) V_\infty \text{ for any } |x| \geq R,$$

then by $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ and $t_n \to 1$ we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} \left[V(\epsilon x) - t_n^p V_\infty\right] |v_n|^p \mathrm{d}x \\ &= \int\limits_{|x|\leq R} \left[V(\epsilon x) - t_n^p V_\infty\right] |v_n|^p \mathrm{d}x + \int\limits_{|x|\geq R} \left[V(\epsilon x) - t_n^p V_\infty\right] |v_n|^p \mathrm{d}x \\ &\geq (V_0 - t_n^p V_\infty) \int\limits_{|x|\leq R} |v_n|^p \mathrm{d}x - \varepsilon \int\limits_{|x|\geq R} |v_n|^p \mathrm{d}x + V_\infty(1-t_n^p) \int\limits_{|x|\geq R} |v_n|^p \mathrm{d}x \\ &\geq o(1) - c_{18}\varepsilon. \end{aligned} \tag{4.11}$$

Similarly, we obtain

$$\int\limits_{\mathbb{R}^N} \left[V(\epsilon x) - t_n^q V_\infty\right] |v_n|^q \mathrm{d}x \geq o(1) - c_{19}\varepsilon. \tag{4.12}$$

Using the mean value theorem and $t_n \to 1$ we can prove

$$\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \left[\frac{G(t_n v_n) G(t_n v_n)}{|x-y|^\mu} - \frac{G(v_n) G(v_n)}{|x-y|^\mu}\right] \mathrm{d}y \mathrm{d}x = o(1). \tag{4.13}$$

Therefore, from (4.9), (4.10) (4.11), (4.12) and (4.13), we conclude that

$$c - \mathcal{I}_\epsilon(u) \ge o(1) - c_{20}\varepsilon + c_{V_\infty},$$

and taking the limit as $\varepsilon \to 0$ we get

$$c - \mathcal{I}_\epsilon(u) \ge c_{V_\infty}.$$

Step 3. The sequence $\{t_n\}$ satisfies

$$\limsup_{n\to\infty} t_n = t_0 < 1.$$

We assume that there exists a subsequence, still denoted by $\{t_n\}$, such that $t_n \to t_0 < 1$. First, according to (g_5) we can easily check that

$$G(s) \text{ and } h(s) = \frac{1}{q}g(s)s - \frac{1}{2}G(s) \text{ are increasing in } (0,+\infty). \tag{4.14}$$

Moreover, according to the above arguments, we can get

$$\int_{\mathbb{R}^N} [V_\infty - V(\epsilon x)]|v_n|^p \mathrm{d}x = o(1). \tag{4.15}$$

Since $\langle \mathcal{I}'_\epsilon(v_n), v_n\rangle = o(1)$, then we have

$$\begin{aligned} c - \mathcal{I}_\epsilon(u) + o(1) &= \mathcal{I}_\epsilon(v_n) - \frac{1}{q}\langle \mathcal{I}'_\epsilon(v_n), v_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|v_n\|^p_{V_\epsilon,p} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(v_n)}{|x-y|^\mu}\left[\frac{1}{q}g(v_n)v_n - \frac{1}{2}G(v_n)\right]\mathrm{d}y\mathrm{d}x. \end{aligned} \tag{4.16}$$

Using $t_n v_n \in \mathcal{N}_{V_\infty}$, (4.14), (4.15) and (4.16) we deduce that

$$\begin{aligned} c_{V_\infty} \le& \mathcal{J}_{V_\infty}(t_n v_n) \\ =& \mathcal{J}_{V_\infty}(t_n v_n) - \frac{1}{q}\langle \mathcal{J}'_{V_\infty}(t_n v_n), t_n v_n\rangle \\ =& \left(\frac{1}{p} - \frac{1}{q}\right)\|t_n v_n\|^p_{V_\infty,p} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(t_n v_n)}{|x-y|^\mu}\left[\frac{1}{q}g(t_n v_n)t_n v_n - \frac{1}{2}G(t_n v_n)\right]\mathrm{d}y\mathrm{d}x \\ \le& \left(\frac{1}{p} - \frac{1}{q}\right)\|v_n\|^p_{V_\infty,p} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(v_n)}{|x-y|^\mu}\left[\frac{1}{q}g(v_n)v_n - \frac{1}{2}G(v_n)\right]\mathrm{d}y\mathrm{d}x \\ =& \left(\frac{1}{p} - \frac{1}{q}\right)\|v_n\|^p_{V_\epsilon,p} + \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} [V_\infty - V(\epsilon x)]|v_n|^p \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(v_n)}{|x-y|^{\mu}}\left[\frac{1}{q}g(v_n)v_n-\frac{1}{2}G(v_n)\right]\mathrm{d}y\mathrm{d}x\\
=&\mathcal{I}_\epsilon(v_n)-\frac{1}{q}\langle\mathcal{I}_\epsilon'(v_n),v_n\rangle+o(1)\\
=&c-\mathcal{I}_\epsilon(u)+o(1).
\end{aligned}
$$

Taking the limit as $n\to\infty$, we get

$$c-\mathcal{I}_\epsilon(u)\geq c_{V_\infty}.$$

We finish the proof of the lemma. □

Combining Lemma 4.3 and Lemma 4.4, we have the following compactness result.

Lemma 4.5. *Let $\{u_n\}$ be a bounded Palais-Smale sequence at level $c<c_{V_\infty}$ for $\mathcal{I}_\epsilon$. Then $\{u_n\}$ has a convergent subsequence in E_ϵ.*

Proof. Let $\{u_n\}$ be a bounded Palais-Smale sequence, up to a subsequence, we may assume that $u_n\rightharpoonup u$ in E_ϵ, $u_n\to u$ in L^s_{loc} for $s\in[1,q^*)$ and $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$. From the proof of Lemma 3.1, we can see that $\mathcal{I}_\epsilon'(u)=0$. Therefore, we deduce from (g_4) that

$$
\begin{aligned}
\mathcal{I}_\epsilon(u)=&\mathcal{I}_\epsilon(u)-\frac{1}{\theta}\langle\mathcal{I}_\epsilon'(u),u\rangle\\
=&\left[\frac{1}{p}-\frac{1}{\theta}\right]\|u\|^p_{V_\epsilon,p}+\left[\frac{1}{q}-\frac{1}{\theta}\right]\|u\|^q_{V_\epsilon,q}\\
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u)}{|x-y|^{\mu}}\left[\frac{1}{\theta}g(u)u-\frac{1}{2}G(u)\right]\mathrm{d}y\mathrm{d}x\geq 0,
\end{aligned}\tag{4.17}
$$

which implies that $c-\mathcal{I}_\epsilon(u)\leq c<c_{V_\infty}$. Finally, from Lemma 4.4 we can conclude that $u_n\to u$ in E_ϵ. This completes the proof. □

Employing Lemma 4.5 we can prove $\mathcal{I}_\epsilon$ satisfies the Palais-Smale condition on $\mathcal{N}_\epsilon$.

Lemma 4.6. *Let $\{u_n\}$ be any Palais-Smale sequence restricted in $\mathcal{N}_\epsilon$ and assume that $c<c_{V_\infty}$. Then $\{u_n\}$ has a convergent subsequence E_ϵ.*

Proof. Let $\{u_n\}\subset\mathcal{N}_\epsilon$ be any Palais-Smale sequence for $\mathcal{I}_\epsilon$ on $\mathcal{N}_\epsilon$ at level c, namely

$$\mathcal{I}_\epsilon(u_n)\to c \text{ and } \mathcal{I}_\epsilon'|_{\mathcal{N}_\epsilon}(u_n)\to 0.$$

Then, there exists $\lambda_n\in\mathbb{R}$ such that

$$\mathcal{I}_\epsilon'(u_n)=\lambda_n\widehat{\mathcal{I}}_\epsilon'(u_n)+o(1).$$

From (2.8), we know that $\langle \widehat{\mathcal{I}}_\epsilon(u_n), u_n\rangle \le -\alpha$. Then, we can deduce that $\lambda_n \to 0$. Consequently, $\{u_n\}$ is indeed a Palais-Smale sequence of $\mathcal{I}_\epsilon$. Since $c < c_{V_\infty}$, from Lemma 4.5, it is easy to see that the conclusion holds. This ends the proof. $\square$

Consider the following problem

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \left(\frac{1}{|x|^\mu} * G(u)\right) g(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.18}$$

where V_0 is given in (V). In view of Lemma 3.1, we know that problem (4.18) possesses a positive ground state solution u_0 satisfying

$$\mathcal{J}_{V_0}(u_0) = c_{V_0} = \inf_{\mathcal{N}_{V_0}} \mathcal{J}_{V_0}.$$

Next we give the comparison relationship of the ground state energy level between problem (2.1) and problem (4.18), which is very important in our arguments.

Lemma 4.7. $\limsup_{\epsilon\to 0} c_\epsilon \le c_{V_0}$.

Proof. Let u be a positive ground state solution of problem (4.18). From Lemma 2.7 we see that

$$c_{V_0} = \mathcal{J}_{V_0}(u) = \max_{t\ge 0} \mathcal{J}_{V_0}(tu). \tag{4.19}$$

By Lemma 2.7 again, there exists $t_\epsilon > 0$ such that $t_\epsilon u \in \mathcal{N}_\epsilon$, and

$$c_\epsilon \le \mathcal{I}_\epsilon(t_\epsilon u) = \max_{t\ge 0} \mathcal{I}_\epsilon(tu). \tag{4.20}$$

It is clear to see that $\{t_\epsilon\}$ is bounded. Therefore, after passing to a subsequence, we assume that $t_\epsilon \to t_0$. Observe that

$$\mathcal{I}_\epsilon(t_\epsilon u) = \mathcal{J}_{V_0}(t_\epsilon u) + \frac{t_\epsilon^p}{p} \int_{\mathbb{R}^N} [V(\epsilon x) - V_0]|u|^p \mathrm{d}x + \frac{t_\epsilon^q}{q} \int_{\mathbb{R}^N} [V(\epsilon x) - V_0]|u|^q \mathrm{d}x. \tag{4.21}$$

According to the boundedness of t_ϵ and the fact $V(\epsilon x) \to V_0$ in a bounded domain, we have

$$\frac{t_\epsilon^p}{p} \int_{\mathbb{R}^N} [V(\epsilon x) - V_0]|u|^p \mathrm{d}x = o_\epsilon(1) \text{ and } \frac{t_\epsilon^q}{q} \int_{\mathbb{R}^N} [V(\epsilon x) - V_0]|u|^q \mathrm{d}x = o_\epsilon(1). \tag{4.22}$$

From (4.21) and (4.22) we infer that

$$\mathcal{I}_\epsilon(t_\epsilon u) = \mathcal{J}_{V_0}(t_0 u) + o_\epsilon(1).$$

Combining (4.19) and (4.20), as $\epsilon \to 0$, we obtain

$$c_\epsilon \le \mathcal{I}_\epsilon(t_\epsilon u) \to \mathcal{J}_{V_0}(t_0 u) \le \max_{t\ge 0} \mathcal{J}_{V_0}(tu) = \mathcal{J}_{V_0}(u) = c_{V_0}.$$

Consequently, we have

$$\limsup_{\epsilon\to 0} c_\epsilon \le c_{V_0}.$$

The proof is now complete. □

Next, we prove the existence result of positive ground state solutions of problem (2.1).

Theorem 4.1. *Assume that* (V) *and* (g_1)*-*(g_5) *are satisfied. Then, there exists* $\epsilon_0 > 0$ *such that problem* (2.1) *has a positive ground state solution* u_ϵ *for all* $\epsilon < \epsilon_0$*.*

Proof. According to Lemma 2.6, we see that $\mathcal{I}_\epsilon$ satisfies the mountain pass geometry, then there exists a Palais-Smale sequence $\{u_n\}$ at level c_ϵ, namely

$$\mathcal{I}_\epsilon(u_n) \to c_\epsilon \text{ and } \mathcal{I}'_\epsilon(u_n) \to 0.$$

Lemma 2.8 shows that $\{u_n\}$ is bounded. Then, up to a subsequence, we assume that $u_n \rightharpoonup u_\epsilon$ in E_ϵ. Moreover, from the proof of Lemma 3.1, we can see that $\mathcal{I}'_\epsilon(u_\epsilon) = 0$. By (V) and Lemma 3.2, we get $c_{V_0} < c_{V_\infty}$. Moreover, according to Lemma 4.7 we can deduce that there exists $\epsilon_0 > 0$ such that $c_\epsilon \le c_{V_0} < c_{V_\infty}$ for $\epsilon < \epsilon_0$. Therefore, Lemma 4.5 shows that $\mathcal{I}_\epsilon$ satisfies the Palais-Smale condition for $\epsilon < \epsilon_0$. Applying the Fatou's lemma we can see that u_ϵ is a ground state solution of problem (2.1). Finally, the positivity of ground state solution follows with same arguments as in the proof of Lemma 3.1, we omit the details here. We complete the proof of the theorem. □

5. Multiplicity and concentration

In this section we are going to investigate the multiplicity and concentration phenomenon of positive ground state solutions.

Let u be a positive ground state solution of problem (4.18) and ζ be a smooth nonincreasing cut-off function in $[0, +\infty)$ such that $\zeta(s) = 1$ if $0 \le s \le \frac{1}{2}$ and $\zeta(s) = 0$ if $s \ge 1$. For any $z \in \Pi$, we define the function

$$\Psi_{\epsilon,z}(x) = \zeta(|\epsilon x - z|)u(\frac{\epsilon x - z}{\epsilon}).$$

It follows from Lemma 2.7 that there exists $t_\epsilon > 0$ such that

$$\max_{t\ge 0} \mathcal{I}_\epsilon(t\Psi_{\epsilon,z}) = \mathcal{I}_\epsilon(t_\epsilon \Psi_{\epsilon,z}).$$

So, we define $\Phi_\epsilon : \Pi \to \mathcal{N}_\epsilon$ by $\Phi_\epsilon(z) = t_\epsilon \Psi_{\epsilon,z}$. According to the construction of $\Psi_{\epsilon,z}$, we can see that $\Phi_\epsilon(z)$ has compact support for any $z \in \Pi$. The following lemma describes an important relationship between Φ_ϵ and the set Π.

Lemma 5.1. *We have the limit*

$$\lim_{\epsilon\to 0}\mathcal{I}_\epsilon(\Phi_\epsilon(z)) = c_{V_0} \text{ uniformly in } z\in\Pi.$$

Proof. We argue by contradiction. Assume that there exist $\varepsilon_0>0$, $\{z_n\}\subset\Pi$ and $\epsilon_n\to 0$ such that

$$|\mathcal{I}_{\epsilon_n}(\Phi_{\epsilon_n}(z)) - c_{V_0}| \geq \varepsilon_0. \tag{5.1}$$

Observe that, using the Lebesgue's dominated convergence theorem, we can easily check that

$$|\nabla\Psi_{\epsilon_n,z_n}|_p^p + \int_{\mathbb{R}^N} V(\epsilon_n x)|\Psi_{\epsilon_n,z_n}|^p \mathrm{d}x \to |\nabla u|_p^p + \int_{\mathbb{R}^N} V_0|u|^p\mathrm{d}x, \tag{5.2}$$

$$|\nabla\Psi_{\epsilon_n,z_n}|_q^q + \int_{\mathbb{R}^N} V(\epsilon_n x)|\Psi_{\epsilon_n,z_n}|^q \mathrm{d}x \to |\nabla u|_q^q + \int_{\mathbb{R}^N} V_0|u|^q\mathrm{d}x, \tag{5.3}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\Psi_{\epsilon_n,z_n})G(\Psi_{\epsilon_n,z_n})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{G(u)G(u)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \tag{5.4}$$

Since $\langle \mathcal{I}'_{\epsilon_n}(t_{\epsilon_n}\Psi_{\epsilon_n,y_n}), t_{\epsilon_n}\Psi_{\epsilon_n,y_n}\rangle = 0$ and making the change of variable

$$\widehat{y} = \frac{\epsilon_n y - z_n}{\epsilon_n} \text{ and } \widehat{x} = \frac{\epsilon_n x - z_n}{\epsilon_n},$$

we have

$$\begin{aligned}
&t_{\epsilon_n}^p|\nabla\Psi_{\epsilon_n,z_n}|_p^p + t_{\epsilon_n}^q|\nabla\Psi_{\epsilon_n,z_n}|_q^q + \int_{\mathbb{R}^N} V(\epsilon_n x)(|t_{\epsilon_n}\Psi_{\epsilon_n,z_n}|^p + |t_{\epsilon_n}\Psi_{\epsilon_n,z_n}|^q)\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(y))g(t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(x))t_{\epsilon_n}\Psi_{\epsilon_n,z_n}(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{y}|)u(\widehat{y}))g(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})}{|\widehat{x}-\widehat{y}|^\mu}\mathrm{d}\widehat{y}\mathrm{d}\widehat{x}.
\end{aligned} \tag{5.5}$$

We show that $t_{\epsilon_n}\to 1$. We first need to prove that $\{t_{\epsilon_n}\}$ is bounded. Indeed, we assume by contradiction that $t_{\epsilon_n}\to\infty$. Using (5.5) and (2.7) we have

$$
\begin{aligned}
& t_{\epsilon_n}^{p-q}|\nabla\Psi_{\epsilon_n,z_n}|_p^p+|\nabla\Psi_{\epsilon_n,z_n}|_q^q+\int\limits_{\mathbb{R}^N} V(\epsilon_n x)(t_{\epsilon_n}^{p-q}|\Psi_{\epsilon_n,z_n}|^p+|\Psi_{\epsilon_n,z_n}|^q)\mathrm{d}x \\
&=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{y}|)u(\widehat{y}))g(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})}{|\widehat{x}-\widehat{y}|^\mu}\mathrm{d}\widehat{y}\mathrm{d}\widehat{x} \\
&=\int\limits_{B_{\epsilon_n^{-1}}(0)}\int\limits_{B_{\epsilon_n^{-1}}(0)}\frac{G(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{y}|)u(\widehat{y}))g(t_{\epsilon_n}\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x}))\zeta(|\epsilon_n\widehat{x}|)u(\widehat{x})}{t_{\epsilon_n}^{q-1}|\widehat{x}-\widehat{y}|^\mu}\mathrm{d}\widehat{y}\mathrm{d}\widehat{x} \\
&\geq\frac{G(t_{\epsilon_n}u(\widehat{x_0}))}{(t_{\epsilon_n}u(\widehat{x_0}))^{\frac{q}{2}}}\frac{g(t_{\epsilon_n}u(\widehat{x_0}))}{(t_{\epsilon_n}u(\widehat{x_0}))^{\frac{q}{2}-1}}\int\limits_{B_{2^{-1}}(0)}\int\limits_{B_{2^{-1}}(0)}u(\widehat{y})^{\frac{q}{2}}u(\widehat{x})^{\frac{q}{2}}\mathrm{d}\widehat{y}\mathrm{d}\widehat{x},
\end{aligned}
\tag{5.6}
$$

where $u(\widehat{x_0})=\min\{u(\widehat{x}):|\widehat{x}|\leq\frac{1}{2}\}>0$ (we recall that $u\in C(\mathbb{R}^N)$ by Lemma 3.1). Since $p<q$, then from (g_4) and (5.6), we can deduce that $\|\Psi_{\epsilon_n,z_n}\|_{V_\epsilon,q}^q\to\infty$. Evidently, this contradicts relation (5.3). Hence, $\{t_{\epsilon_n}\}$ is bounded. Up to a subsequence, we may assume that $t_{\epsilon_n}\to t_0\geq 0$. If $t_0=0$, by (g_2), (5.3) and (5.5) we can derive that $\|\Psi_{\epsilon_n,z_n}\|_{V_\epsilon,p}^p\to 0$, this contradicts relation (5.2). So, we conclude that $t_0>0$.

We claim that $t_0=1$. Letting $n\to\infty$ in (5.5), we obtain

$$
t_0^{p-q}|\nabla u|_p^p+|\nabla u|_q^q+V_0\int\limits_{\mathbb{R}^N}(t_0^{p-q}|u|^p+|u|^q)\mathrm{d}x=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(t_0u)g(t_0u)t_0u}{t_0^q|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \tag{5.7}
$$

Since u is a positive ground state solution of problem (4.18), we have

$$
|\nabla u|_p^p+|\nabla u|_q^q+V_0\int\limits_{\mathbb{R}^N}(|u|^p+|u|^q)\mathrm{d}x=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(u)g(u)u}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x. \tag{5.8}
$$

From (5.7) and (5.8) we have

$$
(t_0^{p-q}-1)\|u\|_{V_0,p}^p=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\left[\frac{G(t_0u)g(t_0u)t_0u}{t_0^q|x-y|^\mu}-\frac{G(u)g(u)u}{|x-y|^\mu}\right]\mathrm{d}y\mathrm{d}x.
$$

Evidently, by (2.7) we get $t_0=1$. Therefore, we infer from (5.2), (5.3) and (5.4) that

$$
\begin{aligned}
\mathcal{I}_{\epsilon_n}(\Phi_{\epsilon_n}(z_n))=&\frac{t_{\epsilon_n}^p}{p}|\nabla\Psi_{\epsilon_n,z_n}|_p^p+\frac{t_{\epsilon_n}^q}{q}|\nabla\Psi_{\epsilon_n,z_n}|_q^q \\
&+\int\limits_{\mathbb{R}^N}V(\epsilon_n x)\left[\frac{t_{\epsilon_n}^p}{p}|\Psi_{\epsilon_n,z_n}|^p+\frac{t_{\epsilon_n}^q}{q}|\Psi_{\epsilon_n,z_n}|^q\right]\mathrm{d}x \\
&-\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{G(t_{\epsilon_n}\Psi_{\epsilon_n,z_n})G(t_{\epsilon_n}\Psi_{\epsilon_n,z_n})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\to \frac{1}{p}\|u\|^p_{V_0,p} + \frac{1}{q}\|u\|^q_{V_0,q} - \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{G(u)G(u)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&=\mathcal{J}_{V_0}(u) = c_{V_0}.
\end{aligned}
$$

According to (5.1) we can see that this is impossible. Thus, we finish the proof of the lemma. □

Next we are in the position to introduce the barycenter map. For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $\Pi_\delta \subset B_\rho(0)$. We define $\eta : \mathbb{R}^N \to \mathbb{R}^N$ as follows

$$
\eta(x) = x \text{ for } |x| \le \rho \text{ and } \eta(x) = \frac{\rho x}{|x|} \text{ for } |x| \ge \rho.
$$

The barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^N$ is defined by

$$
\beta_\epsilon(u) = \frac{\int_{\mathbb{R}^N} \eta(\epsilon x)(|u|^p + |u|^q)\mathrm{d}x}{\int_{\mathbb{R}^N}(|u|^p + |u|^q)\mathrm{d}x}.
$$

Combining the above definitions, we can prove the following result.

Lemma 5.2. *We have the limit*

$$
\lim_{\epsilon\to 0} \beta_\epsilon(\Phi_\epsilon(z)) = z \textit{ uniformly in } z \in \Pi.
$$

Proof. Arguing by contradiction, we assume that there exist $\sigma_0 > 0$, $\{z_n\} \subset \Pi$ and $\epsilon_n \to 0$ such that

$$
|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) - z_n| \ge \sigma_0 > 0. \tag{5.9}
$$

According to the definitions of Φ_{ϵ_n} and β_{ϵ_n}, and making the change of variable $y = (\epsilon_n x - z_n)/\epsilon_n$ we immediately obtain

$$
\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) = z_n + \frac{\int_{\mathbb{R}^N}[\eta(\epsilon_n y + z_n) - z_n](|\zeta(|\epsilon_n y|)u(y)|^p + |\zeta(|\epsilon_n y|)u(y)|^q)\mathrm{d}y}{\int_{\mathbb{R}^N}(|\zeta(|\epsilon_n y|)u(y)|^p + |\zeta(|\epsilon_n y|)u(y)|^q)\mathrm{d}y}.
$$

Since $\{z_n\} \subset \Pi \subset B_\rho(0)$, employing the Lebesgue dominating convergence theorem, we can get

$$
|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(z_n)) - z_n| \to 0,
$$

which contradicts relation (5.9). □

Now, we prove the following useful compactness result.

Lemma 5.3. *Let $\epsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\epsilon_n}$ be a sequence satisfying $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_0}$. Then there exists $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $v_n = u_n(x + \tilde{z}_n)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n \to z \in \Pi$, where $z_n = \epsilon_n \tilde{z}_n$.*

Proof. Since $\{u_n\} \subset \mathcal{N}_{\epsilon_n}$ and $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_0}$, then, a standard argument shows that $\{u_n\}$ is bounded. We claim that there are $R_0, \delta > 0$ and $\tilde{z}_n \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int\limits_{B_{R_0}(\tilde{z}_n)} |u_n|^q \mathrm{d}x \geq \delta. \tag{5.10}$$

Otherwise, Lemma 2.2 implies that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (q, q^*)$. According to (2.4) and Lemma 2.4, we can easily prove that $u_n \to 0$ in E_ϵ, this is impossible because $\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_0} > 0$. Consequently, (5.10) holds.

Let us define $v_n(x) = u_n(x + \tilde{z}_n)$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v \neq 0$. By virtue of Lemma 2.7, there exists $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{N}_{V_0}$. Then we have

$$c_{V_0} \leq \mathcal{J}_{V_0}(\tilde{v}_n) = \mathcal{J}_{V_0}(t_n u_n) \leq \mathcal{I}_{\epsilon_n}(t_n u_n) \leq \mathcal{I}_{\epsilon_n}(u_n) \to c_{V_0},$$

which shows $\mathcal{J}_{V_0}(\tilde{v}_n) \to c_{V_0}$. Therefore, $\{\tilde{v}_n\} \subset \mathcal{N}_{V_0}$ is a minimizing sequence, and using the Ekeland's variational principle, we may also assume it is a bounded Palais-Smale sequence at c_{V_0} for $\mathcal{J}_{V_0}$. Thus, after passing to subsequence, we have $\tilde{v}_n \rightharpoonup \tilde{v}$ with $\tilde{v} \neq 0$. Moreover, $\mathcal{J}'_{V_0}(\tilde{v}) = 0$. According to Lemma 4.3 we obtain

$$\mathcal{J}_{V_0}(\tilde{v}_n - \tilde{v}) \to c_{V_0} - \mathcal{J}_{V_0}(\tilde{v}) \text{ and } \mathcal{J}'_{V_0}(\tilde{v}_n - \tilde{v}) \to 0.$$

Using (g_4) and employing Fatou's lemma, we obtain

$$\begin{aligned}
c_{V_0} &= \lim_{n\to\infty} \left[\mathcal{J}_{V_0}(\tilde{v}_n) - \frac{1}{\theta} \langle \mathcal{J}'_{V_0}(\tilde{v}_n), \tilde{v}_n \rangle \right] \\
&= \lim_{n\to\infty} \left[\left(\frac{1}{p} - \frac{1}{\theta}\right) \|\tilde{v}_n\|^p_{V_0,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|\tilde{v}_n\|^q_{V_0,q} \right. \\
&\quad \left. + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{G(\tilde{v}_n)}{|x-y|^\mu} \left[\frac{1}{\theta} g(\tilde{v}_n)\tilde{v}_n - \frac{1}{2} G(\tilde{v}_n) \right] \mathrm{d}y\mathrm{d}x \right] \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|\tilde{v}\|^p_{V_0,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right) \|\tilde{v}\|^q_{V_0,q} \\
&\quad + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{G(\tilde{v})}{|x-y|^\mu} \left[\frac{1}{\theta} g(\tilde{v})\tilde{v} - \frac{1}{2} G(\tilde{v}) \right] \mathrm{d}y\mathrm{d}x \\
&= \mathcal{J}_{V_0}(\tilde{v}) - \frac{1}{\theta} \langle \mathcal{J}'_{V_0}(\tilde{v}), \tilde{v} \rangle \\
&= \mathcal{J}_{V_0}(\tilde{v}) \geq c_{V_0}.
\end{aligned}$$

Consequently, it follows that

$$\mathcal{J}_{V_0}(\tilde{v}_n - \tilde{v}) \to 0 \text{ and } \mathcal{J}'_{V_0}(\tilde{v}_n - \tilde{v}) \to 0. \tag{5.11}$$

Moreover, using again (g_4) and (5.11) we have

$$
\begin{aligned}
o(1) =& \mathcal{J}_{V_0}(\tilde{v}_n - \tilde{v}) - \frac{1}{\theta}\langle \mathcal{J}'_{V_0}(\tilde{v}_n - \tilde{v}), \tilde{v}_n - \tilde{v}\rangle \\
=& \left(\frac{1}{p} - \frac{1}{\theta}\right)\|\tilde{v}_n - \tilde{v}\|^p_{V_0,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|\tilde{v}_n - \tilde{v}\|^q_{V_0,q} \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\tilde{v}_n - \tilde{v})}{|x-y|^\mu}\left[\frac{1}{\theta} g(\tilde{v}_n - \tilde{v})(\tilde{v}_n - \tilde{v}) - \frac{1}{2} G(\tilde{v}_n - \tilde{v})\right] \mathrm{d}y\mathrm{d}x \\
\geq& \left(\frac{1}{p} - \frac{1}{\theta}\right)\|\tilde{v}_n - \tilde{v}\|^p_{V_0,p} + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|\tilde{v}_n - \tilde{v}\|^q_{V_0,q},
\end{aligned}
$$

which implies that $\tilde{v}_n \to \tilde{v}$ in E_{V_0}. Since $\{t_n\}$ is bounded, we can assume that $t_n \to t_0 > 0$, and so, $v_n \to v$ in E_{V_0}.

Next, we verify that $\{z_n\} = \{\epsilon_n \tilde{z}_n\}$ has a subsequence satisfying $z_n \to z \in \Pi$. We first claim that $\{z_n\}$ is bounded. Indeed, suppose by contradiction that $\{z_n\}$ is not bounded. Then, up to a subsequence, we assume $|z_n| \to \infty$. From $\tilde{v}_n \to \tilde{v}$ in E_{V_0} and $V_0 < V_\infty$, we can conclude that

$$
\begin{aligned}
c_{V_0} =& \frac{1}{p}\|\tilde{v}\|^p_{V_0,p} + \frac{1}{q}\|\tilde{v}\|^q_{V_0,q} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\tilde{v})G(\tilde{v})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
<& \frac{1}{p}\|\tilde{v}\|^p_{V_\infty,p} + \frac{1}{q}\|\tilde{v}\|^q_{V_\infty,q} - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\tilde{v})G(\tilde{v})}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
\leq& \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} |\nabla \tilde{v}_n|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla \tilde{v}_n|^q \mathrm{d}x \right. \\
&\left. + \int_{\mathbb{R}^N} V(\epsilon_n x + z_n)\left[\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right]\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(\tilde{v}_n)G(\tilde{v}_n)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\right] \\
\leq& \liminf_{n\to\infty}\left[\frac{1}{p}\int_{\mathbb{R}^N} |\nabla t_n u_n|^p \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla t_n u_n|^q \mathrm{d}x \right. \\
&\left. + \int_{\mathbb{R}^N} V(\epsilon_n x)\left[\frac{1}{p}|t_n u_n|^p + \frac{1}{q}|t_n u_n|^q\right]\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{G(t_n u_n)G(t_n u_n)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\right] \\
=& \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(t_n u_n) \\
\leq& \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(u_n) \\
=& c_{V_0},
\end{aligned}
$$

which is a contradiction. Thus, $\{z_n\}$ is bounded and, passing to a subsequence, we may assume that $z_n \to z$. If $z \notin \Pi$, then $V_0 < V(z)$, and according to the above steps we get a contradiction. Consequently, we conclude that $z \in \Pi$. □

Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function given by

$$
\vartheta(\epsilon) = \max_{z\in\Pi} |\mathcal{I}_\epsilon(\Phi_\epsilon(z)) - c_{V_0}|.
$$

It follows from Lemma 5.1 that $\vartheta(\epsilon) \to 0$ as $\epsilon \to 0$. We introduce a subset $\tilde{\mathcal{N}_\epsilon}$ of $\mathcal{N}_\epsilon$, and set

$$\tilde{\mathcal{N}_\epsilon} := \{u \in \mathcal{N}_\epsilon : \mathcal{I}_\epsilon(u) \le c_{V_0} + \vartheta(\epsilon)\}.$$

Since $\Phi_\epsilon(z) \in \tilde{\mathcal{N}_\epsilon}$ for all $z \in \Pi$, then we can deduce that $\tilde{\mathcal{N}_\epsilon} \neq \emptyset$. Moreover, we have the following result.

Lemma 5.4. *For any $\delta > 0$, then the following limit holds*

$$\lim_{\epsilon \to 0} \sup_{u \in \tilde{\mathcal{N}_\epsilon}} \inf_{z \in \Pi_\delta} |\beta_\epsilon(u) - z| = 0.$$

Proof. Let $\epsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists $\{u_n\} \subset \tilde{\mathcal{N}_{\epsilon_n}}$, such that

$$\inf_{z \in \Pi_\delta} |\beta_{\epsilon_n}(u_n) - z| = \sup_{u \in \tilde{\mathcal{N}_{\epsilon_n}}} \inf_{z \in \Pi_\delta} |\beta_{\epsilon_n}(u) - z| + o(1).$$

Hence, it is sufficient to prove that there exists $\{z_n\} \subset \Pi_\delta$ such that

$$\lim_{n \to \infty} |\beta_{\epsilon_n}(u_n) - z_n| = 0.$$

In fact, since $\{u_n\} \subset \tilde{\mathcal{N}_{\epsilon_n}}$, then we have

$$c_{V_0} \le c_{\epsilon_n} \le \mathcal{I}_{\epsilon_n}(u_n) \le c_{V_0} + \vartheta(\epsilon_n),$$

which implies that

$$\mathcal{I}_{\epsilon_n}(u_n) \to c_{V_0} \text{ and } \{u_n\} \subset \mathcal{N}_{\epsilon_n}.$$

According to Lemma 5.3, there exists $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{z}_n)$ has a convergent subsequence. Moreover, up to a subsequence, $z_n = \epsilon_n \tilde{z}_n \to z \in \Pi$, and we can conclude that

$$\begin{aligned}
\beta_{\epsilon_n}(u_n) =& \frac{\int_{\mathbb{R}^N} \eta(\epsilon_n x)(|u_n|^p + |u_n|^q)\mathrm{d}x}{\int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q)\mathrm{d}x} \\
=& \frac{\int_{\mathbb{R}^N} \eta(\epsilon_n y + z_n)(|u_n(y + \tilde{z}_n)|^p + |u_n(y + \tilde{z}_n)|^q)\mathrm{d}y}{\int_{\mathbb{R}^N} (|u_n(y + \tilde{z}_n)|^p + |u_n(y + \tilde{z}_n)|^q)\mathrm{d}y} \\
=& z_n + \frac{\int_{\mathbb{R}^N} [\eta(\epsilon_n y + z_n) - z_n](|v_n(y)|^p + |v_n(y)|^q)\mathrm{d}y}{\int_{\mathbb{R}^N} (|v_n(y)|^p + |v_n(y)|^q)\mathrm{d}y} \\
\to& z \in \Pi.
\end{aligned}$$

Therefore, there exists $\{z_n\} \subset \Pi_\delta$ such that

$$\lim_{n \to \infty} |\beta_{\epsilon_n}(u_n) - z_n| = 0.$$

The proof is now complete. $\square$

To investigate the concentration phenomenon of solutions, next we will apply an appropriate De Giorgi iteration argument and some refined analysis techniques to show the L^∞-estimate and decay property of solutions, which plays a fundamental role in the study of the behavior of the maximum points of solutions.

Lemma 5.5. *Let v_n be a solution of the following problem*

$$\begin{cases} -\Delta_p v_n - \Delta_q v_n + V_n(x)(|v_n|^{p-2}v_n + |v_n|^{q-2}v_n) = \left(\frac{1}{|x|^\mu} * G(v_n)\right) g(v_n), & \text{in } \mathbb{R}^N, \\ v_n \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), v_n > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n \tilde{z}_n)$. If $v_n \to v$ in E_ϵ for some $v \neq 0$. Then we have $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Moreover,

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

Proof. Let $\{v_n\}$ be a sequence of positive solutions, and $v_n \to v$ in E_ϵ. Define

$$K_n(x) = \int_{\mathbb{R}^N} \frac{G(v_n(y))}{|x-y|^\mu} \mathrm{d}y.$$

According to the boundedness of $\{v_n\}$ and following the proof of (3.8), we have

$$|K_n(x)| \le C \text{ for some } C > 0 \text{ and any } n \in \mathbb{N}. \tag{5.12}$$

We adapt some ideas from [1,19] to prove the conclusion of the lemma. Let $x_0 \in \mathbb{R}^N$, $R_0 > 1$ and $0 < t < s < 1$, and let smooth function $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying

$$0 \le \psi(x) \le 1, \ \operatorname{supp}\psi \subset B_s(x_0), \psi(x) = 1, \forall x \in B_t(x_0) \text{ and } |\nabla\psi| \le \frac{2}{s-t}.$$

For $l \ge 1$ and $\rho > 0$, we set $\Lambda_{n,l,\rho} = \{x \in B_\rho(x_0) : v_n(x) > l\}$ and

$$J_n = \int_{\Lambda_{n,l,s}} (|\nabla v_n|^p + |\nabla v_n|^q)\psi^q \mathrm{d}x.$$

We observe that for any $\varphi \in E_\epsilon$ the following relation holds

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} V_n(x)(v_n^{p-1} + v_n^{q-1})\varphi \mathrm{d}x = \int_{\mathbb{R}^N} K_n(x) g(v_n)\varphi \mathrm{d}x.$$

Using $\varphi_n = \psi^q (v_n - l)_+$ as test function, we have

$$
\begin{aligned}
& q \int_{\Lambda_{n,l,s}} \psi^{q-1}(v_n-l)_+|\nabla v_n|^{p-2}\nabla v_n\cdot\nabla\psi \mathrm{d}x + \int_{\Lambda_{n,l,s}} \psi^q|\nabla v_n|^p \mathrm{d}x \\
&+q \int_{\Lambda_{n,l,s}} \psi^{q-1}(v_n-l)_+|\nabla v_n|^{q-2}\nabla v_n\cdot\nabla\psi \mathrm{d}x + \int_{\Lambda_{n,l,s}} \psi^q|\nabla v_n|^q \mathrm{d}x \\
&+ \int_{\Lambda_{n,l,s}} V_n(x)(v_n^{p-1}+v_n^{q-1})\psi^q(v_n-l)_+\mathrm{d}x = \int_{\Lambda_{n,l,s}} K_n(x)g(v_n)\psi^q(v_n-l)_+\mathrm{d}x,
\end{aligned}
$$

which, together with (V), yields that

$$
\begin{aligned}
J_n \le & C_6 \int_{\Lambda_{n,l,s}} \psi^{q-1}(v_n-l)_+|\nabla\psi|(|\nabla v_n|^{p-1}+|\nabla v_n|^{q-1})\mathrm{d}x \\
& - \int_{\Lambda_{n,l,s}} V_0\psi^{q-1}(v_n-l)_+(v_n^{p-1}+v_n^{q-1})\mathrm{d}x + \int_{\Lambda_{n,l,s}} K_n(x)g(v_n)\psi^q(v_n-l)_+\mathrm{d}x.
\end{aligned}
$$

Moreover, by (2.4) and (5.12), we can deduce that

$$
J_n \le C_7\left[\int_{\Lambda_{n,l,s}} \psi^{q-1}(v_n-l)_+|\nabla\psi|(|\nabla v_n|^{p-1}+|\nabla v_n|^{q-1})\mathrm{d}x + \int_{\Lambda_{n,l,s}} v_n^{q^*-1}\psi^q(v_n-l)_+\mathrm{d}x\right].
$$

Since $0<s-t<1$, using Young inequality and Hölder inequality, we can check that

$$
\begin{aligned}
& \int_{\Lambda_{n,l,s}} v_n^{q^*-1}\psi^q(v_n-l)_+\mathrm{d}x \le \int_{\Lambda_{n,l,s}} (|v_n-l|+l)^{q^*-1}\psi^q(v_n-l)_+\mathrm{d}x \\
& \le C_8\left[\int_{\Lambda_{n,l,s}} |v_n-l|^{q^*}\mathrm{d}x + l^{q^*-1}\int_{\Lambda_{n,l,s}} |v_n-l|\mathrm{d}x\right] \\
& \le C_8\left[\int_{\Lambda_{n,l,s}} \left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x + l^{q^*}\left[\int_{\Lambda_{n,l,s}} \left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x\right]^{\frac{1}{q^*}} |\Lambda_{n,l,s}|^{\frac{q^*-1}{q^*}}\right] \\
& \le C_9\left[\int_{\Lambda_{n,l,s}} \left|\frac{v_n-l}{s-t}\right|^{q^*}\mathrm{d}x + l^{q^*}|\Lambda_{n,l,s}|\right].
\end{aligned}
$$

Using the properties of Young functions and following the proof of Theorem 3.1 in [33], for some $\varepsilon_1\in(0,1)$ we can get

$$\int\limits_{\Lambda_{n,l,s}} (|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1})\psi^{q-1}|\nabla\psi|(v_n - l)_+ \mathrm{d}x$$

$$\leq \varepsilon_1 \int\limits_{\Lambda_{n,l,s}} (|\nabla v_n|^p + |\nabla v_n|^q)\psi^q \mathrm{d}x + C_{\varepsilon_1}\left[\int\limits_{\Lambda_{n,l,s}} \left|\frac{v_n - l}{s-t}\right|^{q^*} \mathrm{d}x + |\Lambda_{n,l,s}|\right].$$

Consequently, combining the above facts we obtain

$$J_n \leq C_{10}\left[\int\limits_{\Lambda_{n,l,s}} \left|\frac{v_n - l}{s-t}\right|^{q^*} \mathrm{d}x + (l^{q^*}+1)|\Lambda_{n,l,s}|\right].$$

Exploiting the definition of ψ, we conclude that

$$\int\limits_{\Lambda_{n,l,t}} |\nabla v_n|^q \mathrm{d}x \leq C_{10}\left[\int\limits_{\Lambda_{n,l,s}} \left|\frac{v_n - l}{s-t}\right|^{q^*} \mathrm{d}x + (l^{q^*}+1)|\Lambda_{n,l,s}|\right], \tag{5.13}$$

where C_{10} does not depend on l and $l \geq l_0 \geq 1$ for some constant l_0.

We fix $R_1 \in (0,1)$ and define

$$\sigma_j = \frac{R_1}{2}\left(1+\frac{1}{2^j}\right), \bar{\sigma}_j = \frac{1}{2}(\sigma_j + \sigma_{j+1}), l_j = \frac{l_0}{2}\left(1 - \frac{1}{2^{j+1}}\right)$$

and

$$J_{j,n} = \int\limits_{\Lambda_{n,l_j,\sigma_j}} (v_n - l_j)_+^{q^*} \mathrm{d}x \text{ and } \xi_j = \xi\left(\frac{2^{j+1}}{R_1}\left(|x-x_0| - \frac{R_1}{2}\right)\right),$$

where $\xi \in C^1(\mathbb{R})$ satisfies

$$0 \leq \xi \leq 1, \xi(s) = 1 \text{ for } s \leq \frac{1}{2}, \xi(s) = 0 \text{ for } s \geq \frac{3}{4} \text{ and } |\xi'| \leq c_0.$$

Evidently, we know that

$$\sigma_j \to \frac{R_1}{2} \text{ (decreasing)},\ l_j \to \frac{l_0}{2} \text{ (increasing) and } \sigma_{j+1} < \bar{\sigma}_j < \sigma_j < 1.$$

Since $\xi_j = 1$ in $B_{\sigma_{j+1}}(x_0)$ and $\xi_j = 0$ outside $B_{\bar{\sigma}_j}(x_0)$, using Lemma 2.1 we have

$$
\begin{aligned}
J_{j+1,n} &= \int\limits_{\Lambda_{n,l_{j+1},\sigma_{j+1}}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x \le \int\limits_{B_{R_1}(x_0)} \left[(v_n - l_{j+1})_+ \xi_j\right]^{q^*} \mathrm{d}x \\
&\le C_{11} \left[\int\limits_{B_{R_1}(x_0)} |\nabla \left[(v_n - l_{j+1})_+ \xi_j\right]|^q \mathrm{d}x \right]^{\frac{q^*}{q}} \\
&\le C_{12} \left[\int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} |\nabla v_n|^q \mathrm{d}x + 2^{jp} \int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^q \mathrm{d}x \right]^{\frac{q^*}{q}} .
\end{aligned}
\tag{5.14}
$$

From (5.13) we can see that

$$
\begin{aligned}
\int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} |\nabla v_n|^q \mathrm{d}x &\le C_{13} \left[\int\limits_{\Lambda_{n,l_{j+1},\sigma_j}} \left| \frac{v_n - l_{j+1}}{\sigma_j - \bar{\sigma}_j} \right|^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1)|\Lambda_{n,l_{j+1},\sigma_j}| \right] \\
&\le C_{14} \left[2^{jq^*} \int\limits_{\Lambda_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1)|\Lambda_{n,l_{j+1},\sigma_j}| \right].
\end{aligned}
\tag{5.15}
$$

Using Hölder inequality and Young's inequality, we have

$$
\begin{aligned}
\int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^q \mathrm{d}x &\le \left[\int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x \right]^{\frac{q}{q^*}} |\Lambda_{n,l_{j+1},\bar{\sigma}_j}|^{\frac{q^*-q}{q^*}} \\
&\le C_{15} \left[\int\limits_{\Lambda_{n,l_{j+1},\bar{\sigma}_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + |\Lambda_{n,l_{j+1},\bar{\sigma}_j}| \right].
\end{aligned}
\tag{5.16}
$$

From (5.14), (5.15) and (5.16) we can deduce that

$$
\begin{aligned}
J_{j+1,n}^{\frac{q}{q^*}} &\le C_{16} \left[(2^{jq^*} + 2^{jp}) \int\limits_{\Lambda_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + (l_{j+1}^{q^*} + 1 + 2^{jq})|\Lambda_{n,l_{j+1},\sigma_j}| \right] \\
&\le C_{17} \left[(2^{jq^*} + 2^{jp}) \int\limits_{\Lambda_{n,l_{j+1},\sigma_j}} (v_n - l_{j+1})_+^{q^*} \mathrm{d}x + 2^{jq}|\Lambda_{n,l_{j+1},\sigma_j}| \right].
\end{aligned}
\tag{5.17}
$$

We note that

$$J_{j,n} \geq \int\limits_{\Lambda_{n,l_{j+1},\sigma_j}} (v_n - l_j)_+^{q^*} \mathrm{d}x \geq (l_{j+1} - l_j)^{q^*} |\Lambda_{n,l_{j+1},\sigma_j}|,$$

which implies that

$$|\Lambda_{n,l_{j+1},\sigma_j}| \leq \left(\frac{1}{l_{j+1} - l_j}\right)^{q^*} J_{j,n} = \left(\frac{2^{j+3}}{l_0}\right)^{q^*} J_{j,n}. \tag{5.18}$$

Combining (5.17) and (5.18), we immediately obtain

$$J_{j+1,n}^{\frac{q}{q^*}} \leq C_{18}\left[(2^{jq^*} + 2^{jp}) J_{j,n} + 2^{j(q^*+q)} J_{j,n}\right] \leq C_{19} 2^{j(q^*+q)} J_{j,n}.$$

Therefore, we have the following iteration formula

$$J_{j+1,n} \leq C_{20} B^j J_{j,n}^{1+\beta},$$

where C_{20} depends on N, q, R_1, l_0, $B = 2^{(q^*+q)q/q^*} > 1$ and $\beta = q^*/q - 1$.

Since $v_n \to v$ in E_ϵ, we get

$$\limsup_{l_0\to\infty}\left[\limsup_{n\to\infty} J_{0,n}\right] = \limsup_{l_0\to\infty}\left[\limsup_{n\to\infty} \int\limits_{\Lambda_{n,l_0,\sigma_0}} (v_n - \frac{l_0}{4})_+^{q^*} \mathrm{d}x\right] = 0.$$

So, there exists N_0 and $L_0 > 0$ such that

$$J_{0,n} \leq C^{-\frac{1}{\beta}} B^{-\frac{1}{\beta^2}} \text{ for } n \geq N_0 \text{ and } l_0 \geq L_0.$$

Exploiting [19, Lemma 4.7], we see that

$$\lim_{j\to\infty} J_{j,n} = 0 \text{ for } n \geq N_0.$$

On the other hand,

$$\lim_{j\to\infty} J_{j,n} = \lim_{j\to\infty} \int\limits_{\Lambda_{n,l_j,\sigma_j}} (v_n - l_j)_+^{q^*} \mathrm{d}x = \int\limits_{\Lambda_{n,\frac{l_0}{2},\frac{R_1}{2}}} (v_n - \frac{l_0}{2})_+^{q^*} \mathrm{d}x.$$

Then, we obtain

$$\int\limits_{\Lambda_{n,\frac{l_0}{2},\frac{R_1}{2}}} (v_n - \frac{l_0}{2})_+^{q^*} \mathrm{d}x = 0 \text{ for all } n \geq N_0,$$

and consequently,

$$v_n(x) \le \frac{l_0}{2} \text{ for a.e. } x \in B_{\frac{R_1}{2}}(x_0) \text{ and for all } n \ge N_0.$$

From the arbitrariness of $x_0 \in \mathbb{R}^N$, we can see that

$$v_n(x) \le \frac{l_0}{2} \text{ for a.e. } x \in \mathbb{R}^N \text{ and for all } n \ge N_0,$$

that is,

$$|v_n|_\infty \le \frac{l_0}{2} \text{ for all } n \ge N_0.$$

Setting $C = \max\{\frac{l_0}{2}, |v_1|_\infty, \ldots, |v_{N_0-1}|_\infty\}$, we have $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Moreover, using the regularity conclusion found in [18] (see Theorem 1 and Theorem 2), we can see that $v_n \in C^{1,\nu}_{loc}(\mathbb{R}^N)$ for some $\nu \in (0, 1)$.

Finally, we prove that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. In fact, following the above arguments, for each $\varepsilon > 0$, we have that

$$\limsup_{|x_0|\to\infty}\left[\limsup_{n\to\infty} J_{0,n}\right] = \limsup_{|x_0|\to\infty}\left[\limsup_{n\to\infty} \int\limits_{\Lambda_{n,l_0,\sigma_0}} (v_n - \frac{\varepsilon}{4})_+^{q^*} \mathrm{d}x\right] = 0.$$

Thereby, employing [19, Lemma 4.7], there exist $R_* > 0$ and $N_0 \in \mathbb{N}$ such that

$$\lim_{j\to\infty} J_{j,n} = 0 \text{ if } |x_0| > R_* \text{ and } n \ge N_0,$$

this shows that

$$v_n(x) \le \frac{\varepsilon}{4} \text{ for } x \in B_{\frac{R_1}{2}}(x_0) \text{ and } |x_0| > R_*, n \ge N_0.$$

Now, increasing R_* if necessary, it follows that

$$v_n(x) \le \frac{\varepsilon}{4} \text{ for } |x_0| > R_* \text{ and for all } n \in \mathbb{N}.$$

According to the arbitrariness of ε, we can see that

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N},$$

finishing the proof of the lemma. □

Lemma 5.6. *There exists $v_0 > 0$ such that $|v_n|_\infty \ge v_0$ for all $n \in \mathbb{N}$.*

Proof. Arguing as in the proof of (5.10) we can show that

$$\int_{B_R(0)} |v_n(x)|^q \mathrm{d}x \geq \delta > 0$$

for some $\delta > 0$, $R > 0$ and $n \geq N_0$. Assume by contradiction that $|v_n|_\infty \to 0$ as $n \to +\infty$, then

$$0 < \delta \leq \int_{B_R(0)} |v_n(x)|^q \mathrm{d}x \leq |B_R(0)||v_n(x)|_\infty^q \to 0 \text{ as } n \to \infty,$$

which implies a contradiction. This completes the proof. □

Finally, we are in a position to complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Existence: From Theorem 4.1, we can see that there exists $\epsilon_0 > 0$ such that problem (2.1) has a positive ground state solution u_ϵ for all $\epsilon < \epsilon_0$. Evidently, $\widehat{u}_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is a positive ground state solution of problem (1.1).

Proof of Theorem 1.2. Multiplicity: For any $\delta > 0$, combining Lemma 5.1 and Lemma 5.4, we can see that there exists $\epsilon_\delta > 0$ such that the diagram

$$\Pi \xrightarrow{\Phi_\epsilon} \tilde{\mathcal{N}_\epsilon} \xrightarrow{\beta_\epsilon} \Pi_\delta \tag{5.19}$$

is well defined for any $\epsilon \in (0, \epsilon_\delta)$. By Lemma 5.2, there exists a function $\gamma(\epsilon, z)$ with $|\gamma(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in \Pi$ for all $\epsilon \in (0, \epsilon_\delta)$, such that $\beta_\epsilon(\Phi_\epsilon(z)) = z + \gamma(\epsilon, z)$ for all $z \in \Pi$. We define the function $H(t, z) = z + (1 - t)\gamma(\epsilon, z)$. Then, $H : [0, 1] \times \Pi \to \Pi_\delta$ is continuous, $H(0, z) = \beta_\epsilon(\Phi_\epsilon(z))$ and $H(1, z) = z$ for all $z \in \Pi$. Moreover, from (5.19), we know that $\beta_\epsilon \circ \Phi_\epsilon$ is homotopic to the inclusion mapping $id : \Pi \to \Pi_\delta$. Applying the argument of [12] and the conclusion of [15, Lemma 2.2] we can obtain

$$\operatorname{cat}_{\tilde{\mathcal{N}_\epsilon}}(\tilde{\mathcal{N}_\epsilon}) \geq \operatorname{cat}_{\Pi_\delta}(\Pi).$$

On the other hand, let us choose a function $\pi(\epsilon) > 0$ such that $\pi(\epsilon) \to 0$ as $\epsilon \to 0$ and such that $c_{V_0} + \pi(\epsilon)$ is not a critical level for $\mathcal{I}_\epsilon$. Together with Lemma 4.6, we see that $\mathcal{I}_\epsilon$ satisfies the Palais-Smale condition at level $c \in (c_{V_0}, c_{V_0} + \pi(\epsilon))$ on $\mathcal{N}_\epsilon$. Consequently, using the Ljusternik-Schnirelmann category theory of critical points (see [15, Theorem 2.1]), we can conclude that $\mathcal{I}_\epsilon$ has at least $\operatorname{cat}_{\Pi_\delta}(\Pi)$ critical points in $\mathcal{N}_\epsilon$. So, $\mathcal{I}_\epsilon$ has at least $\operatorname{cat}_{\Pi_\delta}(\Pi)$ critical points in E_ϵ.

Concentration: Let $\epsilon_n \to 0$ and $u_n = u_{\epsilon_n}$ be a solution of problem

$$\begin{cases} -\Delta_p u_n - \Delta_q u_n + V_n(x)(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n) = \left(\frac{1}{|x|^\mu} * G(u_n)\right) g(u_n), & \text{in } \mathbb{R}^N, \\ u_n \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u_n > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

with $V_n(x) = V(\epsilon_n x)$, then by Lemma 5.3, we can see that there is a sequence $\{\tilde{z}_n\} \subset \mathbb{R}^N$ such that

$$v_n(x) = u_n(x + \tilde{z}_n) \to v \text{ in } E \text{ and } \epsilon_n \tilde{z}_n \to z \in \Pi.$$

If p_n is a global maximum point of $v_n(x)$, then, by Lemma 5.6 we know that there exists $R > 0$ such that $p_n \in B_R(0)$. Therefore, $z_n = p_n + \tilde{z}_n$ is a global maximum point of $u_n(x)$. We deduce from the boundedness of $\{p_n\}$ and the continuity of V that

$$\lim_{n\to\infty} \epsilon_n z_n = z \in \Pi \text{ and } \lim_{n\to\infty} V(\epsilon_n z_n) = V_0.$$

From the proof Theorem 1.1, we find that if $u_\epsilon(x)$ is a positive solution of problem (2.1), then $\widehat{u}_\epsilon(x) = u_\epsilon(\frac{x}{\epsilon})$ is a positive solution of problem (1.1). Obviously, the maximum points x_ϵ and z_ϵ of $\widehat{u}_\epsilon$ and u_ϵ, respectively, satisfy $x_\epsilon = \epsilon z_\epsilon$. Consequently, according to the above conclusion, we have

$$\lim_{\epsilon\to 0} V(x_\epsilon) = V_0.$$

We finish the proof of Theorem 1.2. □

Data availability

No data was used for the research described in the article.

Acknowledgments

The research of Vicenţiu D. Rădulescu was supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III. The research of Wen Zhang and Jian Zhang was supported by the National Natural Science Foundation of China (12271152), the Natural Science Foundation of Hunan Province (2021JJ30189, 2022JJ30200), the Key project of Scientific Research Project of Department of Education of Hunan Province (21A0387), Funding scheme for Young Backbone Teachers of universities in Hunan Province (Hunan Education Notification (2020) no. 43), and the China Scholarship Council (Nos. 201908430218, 201908430219) for visiting the University of Craiova (Romania). Wen Zhang and Jian Zhang would like to thank the China Scholarship Council and the Embassy of the People's Republic of China in Romania.

References

[1] C.O. Alves, A. da Silva, Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz-Sobolev space, J. Math. Phys. 57 (2016) 111502.
[2] C.O. Alves, F. Gao, M. Squassina, M. Yang, Singularly perturbed critical Choquard equations, J. Differ. Equ. 263 (2017) 3943–3988.
[3] C.O. Alves, G. Figueiredo, Multiplicity and concentration of positive solutions for a class of quasilinear problems, Adv. Nonlinear Stud. 11 (2011) 265–294.
[4] C.O. Alves, M. Yang, Existence of semiclassical ground state solutions for a generalized Choquard equation, J. Differ. Equ. 257 (2014) 4133–4164.
[5] C.O. Alves, M. Yang, Multiplicity and concentration of solutions for a quasilinear Choquard equation, J. Math. Phys. 55 (2014) 061502.
[6] C.O. Alves, M. Yang, Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method, Proc. R. Soc. Edinb., Sect. A 146 (2016) 23–58.

[7] V. Ambrosio, D. Repovš, Multiplicity and concentration results for a (p, q)-Laplacian problem in $\mathbb{R}^N$, Z. Angew. Math. Phys. 72 (2021) 33.
[8] V. Ambrosio, V.D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, J. Math. Pures Appl. 142 (2020) 101–145.
[9] A. Bahrouni, V.D. Rădulescu, D. Repovš, Double phase transonic flow problems with variable growth: nonlinear patterns and stationary waves, Nonlinearity 32 (2019) 2481–2495.
[10] J.M. Ball, Discontinuous equilibrium solutions and cavitation in nonlinear elasticity, Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. 306 (1982) 557–611.
[11] R. Bartolo, A.M. Candela, A. Salvatore, On a class of superlinear (p, q)-Laplacian type equations on $\mathbb{R}^N$, J. Math. Anal. Appl. 438 (2016) 29–41.
[12] V. Benci, G. Cerami, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology, Calc. Var. 2 (1994) 29–48.
[13] V. Benci, P. d'Avenia, D. Fortunato, L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, Arch. Ration. Mech. Anal. 154 (2000) 297–324.
[14] D. Bonheure, P. d'Avenia, A. Pomponio, On the electrostatic Born-Infeld equation with extended charges, Commun. Math. Phys. 346 (2016) 877–906.
[15] S. Cingolani, M. Lazzo, Multiple positive solutions to nonlinear Schrödinger equations with competing potential functions, J. Differ. Equ. 160 (2000) 118–138.
[16] M. del Pino, P. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differ. Equ. 4 (1996) 121–137.
[17] G.M. Figueiredo, Existence of positive solutions for a class of p&q elliptic problems with critical growth on $\mathbb{R}^N$, J. Math. Anal. Appl. 378 (2011) 507–518.
[18] C. He, G. Li, The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians, Ann. Acad. Sci. Fenn., Math. 33 (2006) 337–371.
[19] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, 1968.
[20] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976/77) 93–105.
[21] P.L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part II, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984) 223–283.
[22] E. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, American Mathematical Society, 2001.
[23] C. Mercuri, M. Willem, A global compactness result for the p-Laplacian involving critical nonlinearities, Discrete Contin. Dyn. Syst. 2 (2010) 469–493.
[24] G. Mingione, V.D. Rădulescu, Recent developments in problems with nonstandard growth and nonuniform ellipticity, J. Math. Anal. Appl. 501 (2021) 125197.
[25] V. Moroz, J. Van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (2017) 773–813.
[26] V. Moroz, J. Van Schaftingen, Semi-classical states for the Choquard equation, Calc. Var. Partial Differ. Equ. 52 (2015) 199–235.
[27] N.S. Papageorgiou, A. Pudełko, V.D. Rădulescu, Non-autonomous (p, q)-equations with unbalanced growth, Math. Ann. (2022), https://doi.org/10.1007/s00208-022-02381-0.
[28] N.S. Papageorgiou, D. Qin, V.D. Rădulescu, Nonlinear eigenvalue problems for the (p, q)–Laplacian, Bull. Sci. Math. 172 (2021) 103038.
[29] N.S. Papageorgiou, V.D. Rădulescu, D. Repovš, Existence and multiplicity of solutions for double-phase Robin problems, Bull. Lond. Math. Soc. 52 (2020) 546–560.
[30] N.S. Papageorgiou, V.D. Rădulescu, J. Zhang, Ambrosetti-Prodi problems for the Robin (p, q)-Laplacian, Nonlinear Anal., Real World Appl. 67 (2022) 103640.
[31] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[32] S. Pekar, Untersuchungen über die Elektronentheorie der Kristalle, Akademie-Verlag, Berlin, 1954.
[33] Z. Tan, F. Fang, Orlicz-Sobolev versus Holder local minimizer and multiplicity results for quasilinear elliptic equations, J. Math. Anal. Appl. 402 (2013) 348–370.
[34] N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Commun. Pure Appl. Math. 20 (1967) 721–747.
[35] J. Wei, M. Winter, Strongly interacting bumps for the Schrödinger-Newton equations, J. Math. Phys. 50 (2009) 012905.
[36] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser, Boston, 1996.
[37] M. Yang, Y. Ding, Existence of solutions for singularly perturbed Schrödinger equations with nonlocal part, Commun. Pure Appl. Anal. 12 (2013) 771–783.

[38] Y. Zhang, X. Tang, V.D. Rădulescu, Concentration of solutions for fractional double-phase problems: critical and supercritical cases, J. Differ. Equ. 302 (2021) 139–184.
[39] W. Zhang, S. Yuan, L. Wen, Existence and concentration of ground-states for fractional Choquard equation with indefinite potential, Adv. Nonlinear Anal. 11 (2022) 1552–1578.
[40] H. Zhang, F. Zhang, Multiplicity and concentration of solutions for Choquard equations with critical growth, J. Math. Anal. Appl. 481 (2020) 123457.
[41] J. Zhang, W. Zhang, Semiclassical states for coupled nonlinear Schrödinger system with competing potentials, J. Geom. Anal. 32 (2022) 114.
[42] W. Zhang, J. Zhang, Multiplicity and concentration of positive solutions for fractional unbalanced double phase problems, J. Geom. Anal. 32 (2022) 235.
[43] J. Zhang, W. Zhang, V.D. Rădulescu, Double phase problems with competing potentials: concentration and multiplication of ground states, Math. Z. 301 (2022) 4037–4078.
[44] V.V. Zhikov, On Lavrentiev's phenomenon, Russ. J. Math. Phys. 3 (1995) 249–269.