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Bound states of fractional Choquard equations with Hardy-Littlewood-Sobolev critical exponent

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Abstract

We deal with the following fractional Choquard equation

$$(-\Delta)^s u + V(x)u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u, x \in \mathbb{R}^N,$$

where $I_\mu(x)$ is the Riesz potential, $s \in (0,1)$, $2s < N \neq 4s$, $0 < \mu < \min\{N, 4s\}$ and $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ is the fractional critical Hardy-Littlewood-Sobolev exponent. By combining variational methods and the Brouwer degree theory, we investigate the existence and multiplicity of positive bound solutions to this equation when $V(x)$ is a positive potential bounded from below. The results obtained in this paper extend and improve some recent works in the case where the coefficient $V(x)$ vanishes at infinity.

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1. Introduction and main results

In this article, we are interested in the following fractional Choquard equation

$$(-\Delta)^s u + V(x)u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u, x \in \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $2s < N \neq 4s$, $0 < \mu < \min\{N, 4s\}$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ is the fractional critical Hardy-Littlewood-Sobolev exponent, $I_\mu(x) : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\mu = \frac{A_\mu}{|x|^\mu}, A_\mu = \frac{\Gamma(\frac{\mu}{2})}{\pi^{N/2} 2^{N-\mu} \Gamma(\frac{N-\mu}{2})}$$

and

$$(-\Delta)^s v(x) = C_{N,s} P.V. \int\limits_{\mathbb{R}^N} \frac{v(x)-v(y)}{|x-y|^{N+2s}} \mathrm{d}y, \;\; v \in \mathbb{S}(\mathbb{R}^N),$$

where $P.V.$ represent the Cauchy principal value, $C_{N,s}$ is a normalized constant, and $\mathbb{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions. We notice that the fractional Laplace operator was first introduced in the pioneering work by Laskin [24,25]. For more details about the fractional Laplacian and fractional Sobolev spaces we refer the interested reader to the monograph [37].

When $s = 1$, $\mu = 1$, $N = 3$, $V = 1$, equation (1.1) stems from the following Choquard equation

$$-\Delta u + u = (I_1 * |u|^2)u, x \in \mathbb{R}^3. \tag{1.2}$$

For studying the quantum theory of quantum polaron, equation (1.2) was introduced by Fröhlich [14] and Pekar [41]. As noticed by Lieb [30], Choquard used equation (1.2) as approximation to Hartree-Fock theory of one-component plasma. It remarked that, as a model of self gravitating matter and is known in that context as the Schrödinger-Newton equation, this equation was studied by Penrose [42,43]. The existence and uniqueness of positive solutions to equation (1.2) was investigated by Lieb and Lions in [30,33]. In [33,49], Lenzmann, Wei and Winter studied the nondegeneracy and uniqueness of the ground state. Classification of solutions of generalized nonlinear Choquard problem was investigated by Ma and Zhao in [34]. Moroz and Van Schaftingen [38] completely studied the qualitative properties of solutions of generalized nonlinear Choquard problem. In [39], Moroz and Van Schaftingen gave a broad survey about Choquard equations. For more results on classical Choquard equations, we refer to [1,3,4,6,10,16,29,36,44–46] and the references therein. In order to be consistent with the theme of this article, in the following we shall recall some previous results for this case. In [18], when $|V|_{\frac{N}{2}}$ is suitable small, Guo et al. studied the positive high-energy solutions for Choquard equation

$$-\Delta u + V(x)u = (I_\mu * |u|^{2_\mu^*})|u|^{2_\mu^*-2}u, u \in D^{1,2}(\mathbb{R}^N), \tag{1.3}$$

where $0 < \mu < N$ if $N = 3$ or $N = 4$, and $N - 4 \leq \mu < N$ if $N \geq 5$, $2_\mu^* = \frac{2N-\mu}{N-2}$ is the upper Hardy-Littlewood-Sobolev critical exponent and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap C^\gamma(\mathbb{R}^N)$ is nonnegative for some $\gamma \in (0, 1)$. It is remarked that, under different conditions about V, Gao et al. [15] also studied high-energy solutions of the Choquard equation (1.3) by different methods.

Recently, Alves, Figueiredo and Molle [2] considered the following Choquard equation

$$\begin{cases} -\Delta u + V_\lambda(x)u = (I_\mu * |u|^{2_\mu^*})|u|^{2_\mu^*-2}u, \text{ in } \mathbb{R}^N, \\ u > 0, \text{ in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $V_\lambda = \lambda + V_0$ with $\lambda \geq 0$, $V_0 \in L^{\frac{N}{2}}(\mathbb{R}^N)$, $0 < \mu < \min\{N, 4\}$ and $N \geq 3$. Under V_0 and λ are suitable small, they obtained the existence of two positive solutions to equation (1.4).

We notice that the motivation of papers [2,15,18] is due to Benci and Cerami in the seminal paper [5]. In fact, the results obtained in [2,15,18] extended the results about the classical Schrödinger equation [5] to the Choquard equation. There are other papers similar to [5], see [2,7,11,15,18,20,21] and the references therein.

Compared with classical Choquard equations, there are few papers considering fractional Choquard equations. For instance, Frank et al. [13] studied the following equation

$$\sqrt{-\Delta}u + u = (|x|^{-1} * |u|^2)u, u \in H^{\frac{1}{2}}(\mathbb{R}^3). \tag{1.5}$$

Authors investigated analyticity and radial symmetry of ground state solutions to equation (1.5).

Next, d'Avenia, Siciliano and Squassina [12] considered the following fractional Choquard equation

$$(-\Delta)^s u + \omega u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^N, \tag{1.6}$$

where ω is a positive constant, $s \in (0, 1)$, $2s < N$, $0 < \mu < \min\{N, 4s\}$, $\frac{2N-\mu}{N} < p < 2_{\mu,s}^*$. The regularity, existence, nonexistence, symmetry as well as decay properties of weak solutions to equation (1.6) were obtained in [12].

Under general source terms, Shen, Gao and Yang [47] studied the following fractional Choquard equation

$$(-\Delta)^s u + u = (|x|^{-\mu} * F(u))f(u) \text{ in } \mathbb{R}^N. \tag{1.7}$$

They obtained the existence of ground state solutions to equation (1.7) when f satisfies Berestycki-Lions-type assumptions. Other details about fractional Choquard equation (1.7) with subcritical nonlinearity $f(u) = |u|^{p-2}u$, $p < 2_{\mu,s}^*$, we refer to [9,17,28,27,31] and the references therein.

On the other hand, there are some results on fractional Choquard equation with critical exponent $p = 2_{\mu,s}^*$. In [40], Mukherjee and Sreenadh studied the existence of weak solutions of the following doubly nonlocal fractional elliptic problem:

$$\begin{cases} (-\Delta)^s u = \left(\int_\Omega \frac{|u|^{2^*_{\mu,s}}}{|x-y|^\mu} dy \right) |u|^{2^*_{\mu,s}-2} u + \lambda u, \text{ in } \Omega, \\ u = 0, \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, λ is a real parameter, $0 < \mu < N$ and $N > 2s$. They obtained some existence, nonexistence and regularity results for weak solution of the above problem using variational methods. By using the mountain pass lemma and the Lusternik-Schnirelmann theory, Ma and Zhang [35] proved that the existence and multiplicity of ground state solutions to equation (1.1) with $V(x) = \lambda a(x) - \beta$.

In [21], He and Rădulescu were concerned with the qualitative analysis of positive solutions to the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + V(x)u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u, x \in \mathbb{R}^N, \\ u \in D^{s,2}(\mathbb{R}^N), \ u(x) > 0, \ x \in \mathbb{R}^N, \end{cases} \tag{1.9}$$

where $s \in (0, 1)$, $2s < N$, $0 < \mu < \min\{N, 4s\}$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ and $V(x)$ satisfies the following conditions:

(1) The function V is positive on a set of positive measure.

(2) $V \in L^q(\mathbb{R}^N)$ for all $q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N-\mu}{4s-\mu} < p_2$ with $p_2 < \frac{N}{4s-N}$ if $2s < N < 4s$.

(3) We have

$$|V|_{\frac{N}{2s}} < \left(2^{\frac{4s-\mu}{2N-\mu}} - 1\right) S_s^{\frac{(2s-N)[(N-\mu)(1-s)+2s]+(2N-\mu)2s}{2s(N-\mu+2s)}}.$$

By proving a version of the global compactness result of Struwe [48] for the case of fractional operators in $\mathbb{R}^N$, they showed that equation (1.9) has at least one bound state solution. Some similar results as in [21] were also obtained in [22,51]. We point out that the results obtained in [21,22,51] are strongly dependent on the condition $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, which means that $V(x)$ may vanish at the infinity. For other details about fractional Choquard equation with critical exponent, we refer to [19,23] and the references therein.

Inspired by the works mentioned above, in this paper, we are interested in the existence and multiplicity of positive bound state solutions to Choquard equation (1.1) in which $V(x)$ is positive bounded from below. Throughout this paper, we always suppose $V(x)$ satisfies:

$$\begin{cases} \lim_{|x|\to+\infty} V(x) = V_\infty > 0 & (i) \\ V(x) \geq V_\infty, x \in \mathbb{R}^N & (ii) \\ (V(x) - V_\infty) \in L^{\frac{N}{2s}}(\mathbb{R}^N) & (iii) \end{cases} \tag{V_1}$$

For $\Omega \subset \mathbb{R}^N$, the norm of u in $L^r(\Omega)$ and $L^r(\mathbb{R}^N)$ are denoted by $|u|_{r,\Omega}$ and $|u|_r$, $1 \leq r < \infty$. For any $s \in (0, 1)$, defined

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy < \infty \right\}$$

with the Gagliardo seminorm

$$\|u\|_s^2 = (u,u)_s = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

Let S_s be the best Sobolev constant for the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$, that is,

$$S_s = \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_s^2}{|u|_{2_s^*}^2}.$$

Denote by $H^s(\mathbb{R}^N)$ the fractional Sobolev space endowed with the norm

$$\|u\|^2 = (u,u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V_\infty u^2 \mathrm{d}x = \|u\|_s^2 + \int_{\mathbb{R}^N} V_\infty u^2 \mathrm{d}x.$$

The main result in this paper establishes the following existence property of bound states. In the case of small perturbations from infinity of the indefinite potential we also obtain a multiplicity property of positive bound states.

Theorem 1.1. *Suppose that* (V_1) *is satisfied.*
(1) If $|V - V_\infty|_{\frac{N}{2s}} \neq 0$*, then there exists* $V^* > 0$ *such that for* $V_\infty \in (0, V^*)$*, problem* (1.1) *has at least one positive bound state solution.*
(2) Moreover, suppose that

$$0 < |V - V_\infty|_{\frac{N}{2s}} < \left(2^{\frac{4s-\mu}{2N-\mu}} - 1\right) S_s. \tag{V_2}$$

Then there is $V_* > 0$ *such that for* $V_\infty \in (0, V_*)$*, the equation* (1.1) *has at least two distinct positive bound state solutions.*

Remark 1.1. Obviously, it follows from the condition (V_1) that $V \notin L^{\frac{N}{2s}}(\mathbb{R}^N)$ and it is positive bounded from below. However, the results obtained in [21,22,51] are strongly dependent on the condition $V(x) \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, which means that $V(x)$ may vanish at infinity. So, the methods used in [21,22,51] seem to be not valid for our case.

Remark 1.2. The proof of our results is inspired from the paper due to Cerami, Molle and Passaseo [7,8], in which the authors deal with the Schrödinger-Poisson system and Schrödinger equation with Neumann boundary respectively. Since there are double nonlocal characteristics in our equation which come from the nonlocal operator $(-\Delta)^s$ and the fractional Choquard nonlinear term, some refined estimates for our problem are very necessary and delicate. Especially, the most important thing we need to do is that we must extend the global compactness results in [7,21,22,51] to our equation when $V(x)$ is positive and bounded from below.

2. Preliminary results

Proposition 2.1. *([32]) Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t, N, \mu, r)$ independent of f, h such that*

$$\Big| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\mu} dxdy \Big| \le C(t, N, \mu, r)|f|_t \cdot |g|_r. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{\pi-\mu}{2})}{\Gamma(\frac{2N-\mu}{2})} \Big(\frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)} \Big)^{-1+\frac{\mu}{N}}.$$

In this case, the equality in (2.1) is achieved if and only if $f \equiv (const.)g$ and

$$g(x) = A(\gamma^2 + |x-a|^2)^{-\frac{2N-\mu}{2}}$$

for some $A \in \mathbb{C}$, $0 \ne \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Proposition 2.2. *([21,38]) Let $N > 2s$ and $\mu \in (0, N)$. If $\{u_n\}$ is a bounded sequence in $L^{2_s^*}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then*

$$\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}} dx - \int_{\mathbb{R}^N} (I_\mu * |u_n - u|^{2^*_{\mu,s}})|u_n - u|^{2^*_{\mu,s}} dx \to \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} dx$$

and

$$\int_{\mathbb{R}^N} (I_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}-2} u_n dx - \int_{\mathbb{R}^N} (I_\mu * |u_n - u|^{2^*_{\mu,s}})|u_n - u|^{2^*_{\mu,s}-2}(u_n - u) dx \to \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2} u dx, \text{ in } (D^{s,2}(\mathbb{R}^N))',$$

where $(D^{s,2}(\mathbb{R}^N))'$ is the dual space of $D^{s,2}(\mathbb{R}^N)$.

Lemma 2.1. *([50]) If $N \ge 3$ and $W \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, $\psi : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}, u \mapsto \int_{\mathbb{R}^N} W(x)u^2 dx$ is weakly continuous.*

Let $f = g = |u|^q$, then by the Hardy-Littlewood-Sobolev inequality we deduce that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)u(y)|^q}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y$$

is well defined if $|u|^q \in L^t(\mathbb{R})$ for some $t > 1$ with $\frac{2}{t} + \frac{\mu}{N} = 2$. Therefore, for $u \in D^{s,2}(\mathbb{R}^N)$, it follows from Sobolev embedding theorems that

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-2s}. \tag{2.2}$$

Hence, for $u \in D^{s,2}(\mathbb{R}^N)$, we get

$$\Big(\int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x \Big)^{\frac{1}{2^*_{\mu,s}}} \leq (A_\mu C(N,\mu))^{\frac{1}{2^*_{\mu,s}}} |u|^2_{2^*_{\mu,s}}.$$

From above arguments, the energy functional associated with equation (1.1) is defined by

$$\mathcal{J}(u) = \frac{1}{2}\|u\|_s^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x, \; u \in H^s(\mathbb{R}^N).$$

Furthermore, $\mathcal{J}(u) \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and

$$\langle \mathcal{J}'(u), v\rangle = (u,v)_s + \int_{\mathbb{R}^N} V(x)uv\mathrm{d}x - \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}uv\mathrm{d}x$$

for $u, v \in H^s(\mathbb{R}^N)$.

Define the Nehari manifold as

$$\mathcal{N} := \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : \langle \mathcal{J}'(u), u\rangle = 0\}.$$

It is easy to show that, for each $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, there is a unique $\kappa_u > 0$ satisfying $\kappa_u u \in \mathcal{N}$ and $\Phi(\kappa_u u) = \max_{\kappa>0} \Phi(\kappa u)$. Here, $\kappa_u u$ is called the projection of u on $\mathcal{N}$.

Firstly, we introduce the following equation

$$(-\Delta)^s u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u, \text{ in } \mathbb{R}^N, \tag{2.3}$$

and its energy functional $\mathcal{J}_\infty : D^{s,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}_\infty(u) = \frac{1}{2}\|u\|_s^2 - \frac{1}{2\cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x.$$

It follows from [26] that the positive solutions of equation (2.3) are unique, up to translations and scalings, and must be of the form

$$U_{\delta,y}(x) = \frac{C\delta^{\frac{N-2s}{2}}}{(\delta^2 + |x-y|^2)^{\frac{N-2s}{2}}}, \; y \in \mathbb{R}^N, \; \delta > 0, \tag{2.4}$$

where C is a positive constant. Let

$$S_{\mu,s} := \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_s^2}{\left(\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\mu,s}}}},$$

then $S_{\mu,s}$ is achieved if and only if u is of the form (2.4), and furthermore, one has

$$S_{\mu,s} = \frac{S_s}{(A_\mu C(N,\mu))^{\frac{1}{2^*_{\mu,s}}}}, \|U_{\delta,y}\|_s^2 = \int\limits_{\mathbb{R}^N}(I_\mu * |U_{\delta,y}|^{2^*_{\mu,s}})|U_{\delta,y}|^{2^*_{\mu,s}}\mathrm{d}x = S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}},$$

where $C(N,\mu)$ is defined in Proposition 2.1.

Let

$$\mathcal{N}_\infty := \{u \in D^{s,2}(\mathbb{R}^N)\setminus\{0\} : \langle \mathcal{J}'_\infty(u), u\rangle = 0\}.$$

Then

$$\mathcal{J}_\infty(U_{\delta,y}) = \min_{\mathcal{N}_\infty}\mathcal{J}_\infty(u) = \frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.$$

It is easy to prove that, for each $u \in H^s(\mathbb{R}^3)\setminus\{0\}$, there is a unique $\iota_u > 0$ satisfying $\iota_u u \in \mathcal{N}_\infty$. Furthermore, we can easily obtain the following two results from [21,22].

Lemma 2.2. *If $u \in D^{s,2}(\mathbb{R}^N)$ is a nodal solution of equation (2.3), then*

$$\mathcal{J}_\infty(u) \geq 2^{\frac{4s-\mu}{N-\mu+2s}}\frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.$$

Lemma 2.3. *If $\{u_n\} \subset D^{s,2}(\mathbb{R}^N)$ satisfies*

$$\frac{\|u_n\|_s^2}{\left(\int_{\mathbb{R}^N}(I_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}dx\right)^{\frac{1}{2^*_{\mu,s}}}} \to S_{\mu,s},$$

then there exist $\delta_n > 0$ and $y_n \in \mathbb{R}^N$ satisfying

$$\frac{u_n}{|u_n|_{2^*_s}} \to \frac{U_{\delta_n,y_n}}{|U_{\delta_n,y_n}|_{2^*_s}} + o_n(1) \ \ in\ D^{s,2}(\mathbb{R}^N).$$

Lemma 2.4. *Assume that $u \in H^s(\mathbb{R}^N)\setminus\{0\}$, $\kappa_u u$ and $\iota_u u$ are the projections of u on $\mathcal{N}$ and $\mathcal{N}_\infty$ respectively, then we have that $\iota_u \leq \kappa_u$.*

Proof. It follows from (V_1) that

$$\iota_u^{\frac{2N-2\mu+4s}{N-2s}} = \frac{\|u\|_s^2}{\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x}$$

$$= \frac{\kappa_u^{\frac{2N-2\mu+4s}{N-2s}} \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x - \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x}{\int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x}$$

$$\leq \frac{\kappa_u^{\frac{2N-2\mu+4s}{N-2s}} \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x}{\int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x} = \kappa_u^{\frac{2N-2\mu+4s}{N-2s}},$$

which shows that $\iota_u \leq \kappa_u$. □

Proposition 2.3. *Suppose that* (V_1) *holds, then* $m := \min_{\mathcal{N}} \mathcal{J}(u) = \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$ *and* m *is not achieved.*

Proof. For any $u \in \mathcal{N}$, we have that

$$\|u\|_s^2 \leq \|u\|_s^2 + \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x = \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x \leq S_{\mu,s}^{-\frac{2N-\mu}{N-2s}} \left(\|u\|_s^2\right)^{\frac{2N-\mu}{N-2s}},$$

which shows that $\|u\|_s^2 \geq S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$. Thanks to $u \in \mathcal{N}$, we obtain that

$$\mathcal{J}(u) = \mathcal{J}(u) - \frac{1}{2 \cdot 2^*_{\mu,s}} \langle \mathcal{J}'(u), u \rangle = \frac{N-\mu+2s}{2(2N-\mu)} \|u\|_s^2 + \frac{N-\mu+2s}{2(2N-\mu)} \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x$$

$$\geq \frac{N-\mu+2s}{2(2N-\mu)} \|u\|_s^2 \geq \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.$$

So, we can conclude that $m \geq \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$.

In the following, we prove that $m \leq \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$. Let

$$\widetilde{U}_n(x) = \chi(|x|) U_{\frac{1}{n},0}(x),$$

where $\chi \in C_0^\infty([0, \infty), [0, 1])$ satisfying $\chi(t) = 1, t \in [0, \frac{1}{2}]$ and $\chi(t) = 0, t \geq 1$.

It follows from estimates obtained in [37] that

$$\|\widetilde{U}_n\|_s^2 = \|U_{\frac{1}{n},0}\|_s^2 + o_n(1). \tag{2.5}$$

$$\int_{\mathbb{R}^N} (I_\mu * |\widetilde{U}_n|^{2^*_{\mu,s}})|\widetilde{U}_n|^{2^*_{\mu,s}} \mathrm{d}x = \int_{\mathbb{R}^N} (I_\mu * |U_{\frac{1}{n},0}|^{2^*_{\mu,s}})|U_{\frac{1}{n},0}|^{2^*_{\mu,s}} \mathrm{d}x + o_n(1). \tag{2.6}$$

By using arguments as in [19], we have that

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)\widetilde{U}_n^2(x) \mathrm{d}x = o_n(1). \tag{2.7}$$

In the following, we prove that

$$\int_{\mathbb{R}^3} \widetilde{U}_n^2(x)\mathrm{d}x = o_n(1). \tag{2.8}$$

On the one hand, if $4s > N$, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} |\widetilde{U}_n(x)|^2\mathrm{d}x &= \int_{\mathbb{R}^N} |\chi(|x|)U_{\frac{1}{n},0}(x)|^2\mathrm{d}x \\
&\le C_1 \int_{|x|\le 1} \frac{(\frac{1}{n})^{N-2s}}{(\frac{1}{n_2}+|x|^2)^{N-2s}}\mathrm{d}x \\
&\le C_2(\frac{1}{n})^{N-2s}\int_0^1 \frac{r^{N-1}}{r^{2N-4s}}\mathrm{d}r = C_2(\frac{1}{n})^{3-2s}\int_0^1 \frac{1}{r^{N-4s+1}}\mathrm{d}r \\
&\le C_3(\frac{1}{n})^{N-2s}.
\end{aligned}$$

On the other hand, if $4s < N$, let $\lambda > 0$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\widetilde{U}_n(x)|^2\mathrm{d}x &= \int_{\mathbb{R}^N} |\chi(|x|)U_{\frac{1}{n},0}(x)|^2\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} |U_{\frac{1}{n},0}(x)|^2\mathrm{d}x = Cn^{-2s}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{N-2s}}\mathrm{d}y \\
&= (\frac{1}{n})^{2s}\Big(C\int_{B_0(\lambda)} \frac{1}{(1+|nx|^2)^{N-2s}}\mathrm{d}x + C\int_{\mathbb{R}^N\setminus B_0(\lambda)} \frac{1}{(1+|y|^2)^{N-2s}}\mathrm{d}y\Big) \\
&\le (\frac{1}{n})^{2s}\Big(C_4 + C_5\int_\lambda^\infty \frac{1}{r^{N-4s+1}}\mathrm{d}x\Big) \\
&\le C_6(\frac{1}{n})^{2s}.
\end{aligned}$$

Hence, by above argument, we have that (2.8) holds.
Combining with (2.5), (2.6), (2.7) and (2.8), we can conclude that

$$\kappa_{\widetilde{U}_n} = 1 + o_n(1). \tag{2.9}$$

Let $\widehat{U}_n(x) = \kappa_{\widetilde{U}_n}\widetilde{U}_n(x)$, it follows from $\widehat{U}_n \in \mathcal{N}$, (2.5), (2.6), (2.7), (2.8) and (2.9) that

$$
\begin{aligned}
m &\le \lim_{n\to\infty} \mathcal{J}(\widehat{U}_n)\\
&= \lim_{n\to\infty}\Big(\frac{\kappa_{\widetilde{U}_n}^2}{2}\|\widetilde{U}_n\|_s^2+\frac{\kappa_{\widetilde{U}_n}^2}{2}\int_{\mathbb{R}^3} V(x)\widetilde{U}_n^2\mathrm{d}x-\frac{\kappa_{\widetilde{U}_n}^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu*|\widetilde{U}_n|^{2^*_{\mu,s}})|\widetilde{U}_n|^{2^*_{\mu,s}}\mathrm{d}x\Big)\\
&= \frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.
\end{aligned}
$$

Consequently, we obtain $m=\frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$.

Now, we prove that m is not achieved. Suppose that, by contradiction, there exists $u_\star\in\mathcal{N}$ such that $\mathcal{J}(u_\star)=\frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$. Thanks to $\kappa_{u_\star}=1$, it follows from Lemma 2.4 that $\iota_{u_\star}\le 1$. Then we deduce that

$$
\begin{aligned}
\frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}} = \mathcal{J}(u_\star) &= \frac{N-\mu+2s}{2(2N-\mu)}\|u_\star\|_s^2+\frac{N-\mu+2s}{2(2N-\mu)}\int_{\mathbb{R}^N} V(x)u_\star^2\mathrm{d}x\\
&> \frac{N-\mu+2s}{2(2N-\mu)}\|u_\star\|_s^2 \ge \frac{N-\mu+2s}{2(2N-\mu)}\|\iota_{u_\star}u_\star\|_s^2\\
&\ge \frac{N-\mu+2s}{2(2N-\mu)}S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}},
\end{aligned}
$$

from which we obtain a contradiction. □

From Proposition 2.3, we know that the equation (1.1) does not have any ground state solution. So, we intend to find a bound state solution. For this purpose, we first establish the following global compactness result.

Lemma 2.5. *Suppose that $\{u_n\}$ is a sequence of (P.S.)c sequence for $\mathcal{J}$ and $u_n \rightharpoonup u_0$ in $H^s(\mathbb{R}^N)$. Then, up to a subsequence, $\{u_n\}$ satisfies either*
(a) $u_n\to u_0$ in $H^s(\mathbb{R}^N)$ or
(b) there are $k\in\mathbb{N}$ and nontrivial solutions $u_1,u_2,\dots,u_k$ for the equation (2.3), satisfying

$$
\|u_n\|^2\to\|u_0\|^2+\sum_{j=1}^k\|u_j\|_s^2 \textit{ and } \mathcal{J}(u_n)\to\mathcal{J}(u_0)+\sum_{j=1}^k\mathcal{J}_\infty(u_j).
$$

Proof. For any $\psi\in C_0^\infty(\mathbb{R})$, by Proposition 2.2 and Lemma 2.1 we have that

$$
\begin{aligned}
\langle\mathcal{J}'(u_n),\psi\rangle =&(u_n,\psi)_s+\int_{\mathbb{R}^N} V(x)u_n\psi\mathrm{d}x-\int_{\mathbb{R}^N}(I_\mu*|u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}-2}u_n\psi\mathrm{d}x\\
=&(u_n,\psi)+\int_{\mathbb{R}^N}(V(x)-V_\infty)u_n\psi\mathrm{d}x-\int_{\mathbb{R}^N}(I_\mu*|u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}-2}u_n\psi\mathrm{d}x
\end{aligned}
$$

$$=(u_0,\psi)+\int_{\mathbb{R}^N}(V(x)-V_\infty)u_0\psi\mathrm{d}x-\int_{\mathbb{R}^N}(I_\mu*|u_0|^{2^*_{\mu,s}})|u_0|^{2^*_{\mu,s}-2}u_0\psi\mathrm{d}x+o_n(1)$$
$$=\langle\mathcal{J}'(u_0),\psi\rangle+o_n(1),$$

which shows $\langle\mathcal{J}'(u_0),\psi\rangle=0$. That is, u_0 satisfies

$$(-\Delta)^s u_0+V(x)u_0=(I_\mu*|u_0|^{2^*_{\mu,s}})|u_0|^{2^*_{\mu,s}-2}u,\ \ x\in\mathbb{R}^N.$$

Since $u_n\rightharpoonup u_0$ in $H^s(\mathbb{R}^N)$, then we obtain that

$$u_n\to u_0 \text{ in } L^2_{loc}(\mathbb{R}^N);\ u_n\rightharpoonup u_0 \text{ a.e. on } \mathbb{R}^N.$$

Let $v_n^1=u_n-u_0$, it follows from Proposition 2.2, Lemma 2.1 and the Brezis-Lieb lemma that

$$\mathcal{J}_{V_\infty}(v_n^1)=\mathcal{J}(u_n)-\mathcal{J}(u_0)+o_n(1);\ \ \mathcal{J}'_{V_\infty}(v_n^1)=\mathcal{J}'(u_n)-\mathcal{J}'(u_0)+o_n(1)=o_n(1), \tag{2.10}$$

where

$$\mathcal{J}_{V_\infty}(u):=\frac{1}{2}\|u\|_s^2+\frac{1}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x-\frac{1}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x$$
$$=\mathcal{J}_\infty(u)+\frac{1}{2}\int_{\mathbb{R}^N}V_\infty u^2\mathrm{d}x.$$

It follows from (2.10) that v_n^1 is a $(P.S.)$ sequence for $\mathcal{J}_{V_\infty}$. If $v_n^1\to 0$ in $H^s(\mathbb{R}^N)$, we have done. If not, we suppose that $v_n^1\nrightarrow 0$ in $H^s(\mathbb{R}^N)$. Hence, there are $C_1,C_2>0$ satisfying

$$\|v_n^1\|>C_1,\ |v_n^1|_{2_s^*}>C_2. \tag{2.11}$$

Let $\mathbb{R}^N=\Sigma_{i\in\mathbb{N}}Q_i$, where Q_i are hypercubes with disjoint interior and unitary sides. Set $l_n^1:=\max_{i\in\mathbb{N}}|v_n^1|_{2_s^*,Q_i}$, we have that

$$C_2^{2_s^*}<|v_n^1|_{2_s^*}^{2_s^*}=\sum_{i=1}^{\infty}|v_n^1|_{2_s^*,Q_i}^{2_s^*}\le(l_n^1)^{2_s^*-2}|v_n^1|_{2_s^*,Q_i}^2\le C_3(l_n^1)^{2_s^*-2}\|v_n^1\|^2\le C_4(l_n^1)^{2_s^*-2}.$$

So, we obtain that $l_n^1>0$.

Let z_n^1 be the center of a hypercube so that l_n^1 is attained. Define $\widehat{v}_n^1(x)=v_n^1(x+z_n^1)$. Obviously, $\{\widehat{v}_n^1\}$ is a (P.S.) sequence for $\mathcal{J}_{V_\infty}$, and then $\{\widehat{v}_n^1\}$ bounded in $H^s(\mathbb{R}^N)$. So we assume that, up to a subsequence, there is $v\in H^s(\mathbb{R}^N)$ such that $\widehat{v}_n^1\rightharpoonup v$ in $H^s(\mathbb{R}^N)$. Then for any $\psi\in C_0^\infty(\mathbb{R})$, one has that $\langle\mathcal{J}'_{V_\infty}(v),\psi\rangle=0$. That is, v satisfies

$$(-\Delta)^s u+V_\infty u=(I_\mu*|u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u,\ \ x\in\mathbb{R}^N.$$

According to the Pohozaev identity [12,47], we obtain that $v = 0$. If $\widehat{v}_n^1 \to 0$ in $H^s(\mathbb{R}^N)$, we have done. If not, that is, $\widehat{v}_n^1 \rightharpoonup 0$ but $\widehat{v}_n^1 \nrightarrow 0$ in $H^s(\mathbb{R}^N)$. From fact that v_n^1 is a (P.S.) sequence for $\mathcal{J}_{V_\infty}$, it is easy to conclude that $\{\widehat{v}_n^1\}$ is a (P.S.) sequence for $\mathcal{J}_\infty$ in $D^{s,2}(\mathbb{R}^N)$ and with $\widehat{v}_n^1 \rightharpoonup 0$ and $\widehat{v}_n^1 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. Then, from the results obtained in [11], there exists nontrivial solution u_1 of equation (2.3). Now, we claim that there exist $\{y_n\} \subset \mathbb{R}^N$, $\{\delta_n\} \subset \mathbb{R}$ with $\delta_n \to 0$ satisfying

$$v_n^2 := \widehat{v}_n^1(x) - \psi\Big(\frac{x-y_n}{\delta_n^{\frac{1}{2}}}\Big)\delta_n^{-\frac{N-2s}{2}} u_1\Big(\frac{x-y_n}{\delta_n}\Big) \rightharpoonup 0 \text{ in } H^s(\mathbb{R}^N), \tag{2.12}$$

where $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying $\psi \equiv 1, x \in B_1(0)$ and $\psi \equiv 0, x \in \mathbb{R}^N \backslash B_2(0)$. In fact, thanks to $|\widehat{v}_n^1|_{2_s^*, Q_i} > 0$, we used a similar argument as in Lemma 3.3 [48] (or Theorem 3.2 in [50]) to find sequences of $\{y_n\}$, $\{\delta_n\}$ such that v_n^2 exists.

Next, we prove that $v_n^2 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Let $\psi_n(x) = \psi(\delta_n^{\frac{1}{2}} x)$, then it is easy to see that

$$\begin{aligned} \Big|\psi\Big(\frac{x-y_n}{\delta_n^{\frac{1}{2}}}\Big)\delta_n^{-\frac{N-2s}{2}} u_1\Big(\frac{x-y_n}{\delta_n}\Big)\Big|_2^2 &= \delta_n^{2s} \int_{\mathbb{R}^N} \psi_n^2 |u_1|^2 \mathrm{d}x \\ &\le C\delta_n^s \Big(\int_{\mathbb{R}^N} |u_1|^{2_s^*} \mathrm{d}x\Big)^{\frac{1}{2_s^*}} \\ &= o_n(1), \end{aligned} \tag{2.13}$$

which together with $\widehat{v}_n^1 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ show that $v_n^2 \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$.

To our goal, we just prove $v_n^2 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$. It follows from result of (3.23) obtained in [11] that

$$\Big\|\psi\Big(\frac{x-y_n}{\delta_n^{\frac{1}{2}}}\Big)\delta_n^{-\frac{N-2s}{2}} u_1\Big(\frac{x-y_n}{\delta_n}\Big) - \delta_n^{-\frac{N-2s}{2}} u_1\Big(\frac{x-y_n}{\delta_n}\Big)\Big\|_s = \|\psi_n u_1 - u_1\|_s = o_n(1). \tag{2.14}$$

Let $\widehat{u}_n^1 = \delta_n^{-\frac{N-2s}{2}} u_1\Big(\frac{x-y_n}{\delta_n}\Big)$, then for any $\xi \in C_0^\infty(\mathbb{R})$ with $\|\xi\|_s = C > 0$, we conclude that $(\widehat{u}_n^1, \xi)_s = (u_1, \widehat{\xi}_n)_s$, where $\widehat{\xi}_n = \delta_n^{\frac{N-2s}{2}} \xi(\delta_n x + y_n)$. Thanks to $\|\widehat{\xi}_n\|_s = \|\xi\|_s$ and $\widehat{\xi}_n \to 0$ a.e. on $\mathbb{R}^N$, we get $\widehat{\xi}_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$. So, $(\widehat{u}_n^1, \xi)_s = (u_1, \widehat{\xi}_n)_s = o_n(1)$. As ξ is arbitrarily chosen, then $\widehat{u}_n^1 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$. Hence, by (2.14), we get that $v_n^2 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$. Consequently, we conclude that $v_n^2 \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. Combining with (2.12), (2.13) and (2.14), it is easy to obtain that

$$\mathcal{J}_{V_\infty}(v_n^2) = \mathcal{J}_{V_\infty}(\widehat{v}_n^1) - \mathcal{J}_\infty(u_1) + o_n(1), \tag{2.15}$$

$$\|u_n\|^2 = \|u_0\|^2 + \|v_n^1\|^2 + o_n(1) = \|u_0\|^2 + \|\widehat{v}_n^1\|^2 + o_n(1) = \|u_0\|^2 + \|u_1\|_s^2 + \|v_n^2\|^2 + o_n(1).$$

It follows from (2.10) and (2.15) that

$$\begin{aligned}\mathcal{J}(u_n) &= \mathcal{J}(u_0) + \mathcal{J}_{V_\infty}(v_n^1) + o_n(1) = \mathcal{J}(u_0) + \mathcal{J}_{V_\infty}(\widehat{v}_n^1) + o_n(1) \\ &= \mathcal{J}(u_0) + \mathcal{J}_\infty(u_1) + \mathcal{J}_{V_\infty}(v_n^2) + o_n(1).\end{aligned}$$

By virtue of (2.12) and (2.15), we easily obtain that $\{v_n^2\}$ is a (P.S.) sequence for $\mathcal{J}_{V_\infty}$. If $v_n^2 \to 0$ in $H^s(\mathbb{R}^N)$, we have done. If not, then we can iterate the above procedure. That is, there exist $u_1, u_2, \ldots, u_k$ nontrivial solutions for equation (2.1) such that

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|u_j\|_s^2 + \|v_{k+1}^2\|^2$$

and

$$\mathcal{J}(u_n) \to \mathcal{J}(u_0) + \sum_{j=1}^{k} \mathcal{J}_\infty(u_j) + \mathcal{J}_{V_\infty}(v_{k+1}).$$

Thanks to

$$0 = \langle \mathcal{J}_\infty'(u_j), u_j\rangle = \|u_j\|_s^2 - \int_{\mathbb{R}^N} (I_\mu * |u_j|^{2^*_{\mu,s}})|u_j|^{2^*_{\mu,s}} \mathrm{d}x$$

and the definition of $S_{\mu,s}$, we obtain that $\|u_j\|_s \geq S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}$. Then, we conclude that the iteration must terminate at a finite index $k \geq 1$, that is, $v_n^{k+1} \to 0$ in $H^s(\mathbb{R}^N)$. □

Corollary 2.1. *Let $\{u_n\}$ be a sequence of $(P.S.)_c$ sequence for $\mathcal{J}$ with $c \in (0, m)$, then, up to a subsequence, $\{u_n\}$ converges strongly in $H^s(\mathbb{R}^N)$.*

From Lemma 2.2 and Lemma 2.5, we can east to obtain the following result.

Corollary 2.2. *If $c \in (m, 2^{\frac{4s-\mu}{N-\mu+2s}} m)$, then the functional $\mathcal{J}$ satisfying the $(P.S.)_c$ condition.*

In the sequel, we consider the functional $I : H^s(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \|u\|_s^2 + \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x.$$

Let

$$\mathcal{M} = \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}} \mathrm{d}x = 1\Big\}.$$

The next results are direct consequence of the Corollaries above.

Lemma 2.6. *If* $\{u_n\} \subset \mathcal{M}$ *is a sequence satisfying*

$$I(u_n) \to c \text{ and } I'|_{\mathcal{M}}(u_n) \to 0.$$

Then, the sequence $c_n = c^{\frac{N-2s}{2N+4s-2\mu}} u_n$ *satisfies*

$$\mathcal{J}(v_n) \to \frac{N-\mu+2s}{2(2N-\mu)} c^{\frac{2N-\mu}{N-\mu+2s}} \text{ and } \mathcal{J}'(u_n) \to 0.$$

Lemma 2.7. *Suppose that there are a sequence* $\{u_n\} \subset \mathcal{M}$ *and* $c \in (S_{\mu,s}, 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s})$ *satisfying*

$$I(u_n) \to c \text{ and } I'|_{\mathcal{M}}(u_n) \to 0.$$

Then
(i) there is $u_0 \in \mathcal{M}$ *such that, up to a subsequence,* $u_n \to u_0$ *in* $D^{s,2}(\mathbb{R}^N)$ *and* u_0 *is a critical point for* I *constrained on* $\mathcal{M}$*;*
(ii) $\mathcal{J}$ *has a critical point* $v_0 \in H^s(\mathbb{R}^N)$ *with* $\mathcal{J}(v_0) = \frac{N-\mu+2s}{2(2N-\mu)} c^{\frac{2N-\mu}{N-\mu+2s}}$.

3. Main technique and some basic estimates

Inspired by the idea from [7,8], we introduce a barycenter type map $\beta : H^s(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}^N$ and a functional $\gamma : H^s(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}$ defined as

$$\beta(u) = \frac{1}{|u|_{2_s^*}^{2_s^*}} \int_{\mathbb{R}^N} \frac{x}{1+|x|} |u|^{2_s^*} \mathrm{d}x, \gamma(u) = \frac{1}{|u|_{2_s^*}^{2_s^*}} \int_{\mathbb{R}^N} \left| \frac{x}{1+|x|} - \beta(u) \right| |u|^{2_s^*} \mathrm{d}x.$$

Obviously, $\beta(u)$ and $\gamma(u)$ are continuous. Furthermore, we have

$$\beta(\rho u) = \beta(u),\ \gamma(\rho u) = \gamma(u),\ \forall \rho \in \mathbb{R},\ u \in H^s(\mathbb{R}^N)\backslash\{0\}.$$

Proposition 3.1. *Suppose that* (V_1) *holds, then* $m^* := \min_{\mathcal{M}} I(u) = S_{\mu,s}$ *and* m^* *is not achieved.*

Proof. Since the proof is similar to that of Proposition 2.3, we omit the details here. □

Proposition 3.2. $\vartheta := \inf\{I(u) :\ u \in \mathcal{M},\ \beta(u) = 0,\ \gamma(u) = \frac{1}{2}\} > S_{\mu,s}$.

Proof. By Proposition 3.1, we have that

$$\vartheta \geq S_{\mu,s}.$$

Suppose that $\vartheta = S_{\mu,s}$. Then, there is a sequence of $\{u_n\}$ satisfying

$$u_n \in \mathcal{M},\ \beta(u_n) = 0,\ \gamma(u_n) = \frac{1}{2},\ \lim_{n\to\infty} I(u_n) = S_{\mu,s}. \tag{3.1}$$

Thanks to $V(x) > 0$, one has

$$\begin{aligned} S_{\mu,s} &= \lim_{n\to\infty}\Big(\|u_n\|_s^2+\int_{\mathbb{R}^N} V(x)u_n^2\mathrm{d}x\Big)\\ &\geq \lim_{n\to\infty}\|u_n\|_s^2\\ &\geq S_{\mu,s}. \end{aligned}$$

So, we conclude that

$$\lim_{n\to\infty}\|u_n\|_s^2=S_{\mu,s}. \tag{3.2}$$

It follows from Lemma 2.3 and Theorem 2.5 in [5], we obtain that

$$u_n(x)=\lambda U_{\delta_n,y_n}(x)+\varepsilon_n(x),$$

where $\lambda>0$ is constant, $\delta_n>0,\ y_n\in\mathbb{R}^3$, $\varepsilon_n\to 0$ in $D^{s,2}(\mathbb{R}^{\mathbb{N}})$.

We claim that, passing to a subsequence if necessary,

$$\begin{cases} \lim_{n\to\infty}\delta_n=\delta_0>0 & (i)\\ \lim_{n\to\infty}y_n=y_0\in\mathbb{R}^N & (ii) \end{cases} \tag{3.3}$$

To prove (i) of (3.3), we firstly prove that $\{\delta_n\}$ is bounded. Arguing by contradiction, we suppose that $\{\delta_n\}$ is unbounded. Then, passing to a subsequence if necessary, we have $\lim_{n\to\infty}\delta_n=\infty$. Then, for each $\sigma>0$, we get

$$\lim_{n\to\infty}\int_{B_\sigma(0)}|u_n|^{2_s^*}\mathrm{d}x=\lambda^{2_s^*}\lim_{n\to\infty}\int_{B_\sigma(0)}|U_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x=0.$$

Thanks to $\beta(u_n)=0$, for any $\mu>0$, one has

$$\begin{aligned} \gamma(u_n) &= \frac{1}{|u_n|_{2_s^*}^{2_s^*}}\int_{\mathbb{R}^N}\frac{|x|}{1+|x|}|u_n|^{2_s^*}\mathrm{d}x\\ &= \frac{1}{|u_n|_{2_s^*}^{2_s^*}}\Big(\int_{B_\sigma(0)}\frac{|x|}{1+|x|}|u_n|^{2_s^*}\mathrm{d}x+\int_{\mathbb{R}^N\setminus B_\sigma(0)}\frac{|x|}{1+|x|}|u_n|^{2_s^*}\mathrm{d}x\Big)\\ &= \frac{1}{|u_n|_{2_s^*,\mathbb{R}^N\setminus B_\sigma(0)}^{2_s^*}+o_n(1)}\Big(\int_{\mathbb{R}^N\setminus B_\sigma(0)}\frac{|x|}{1+|x|}|u_n|^{2_s^*}\mathrm{d}x+o_n(1)\Big)\\ &\geq \frac{\sigma}{1+\sigma}+o_n(1). \end{aligned}$$

Hence, we get $\liminf_{n\to\infty}\gamma(u_n)\geq 1$. Thanks to (3.1), we obtain a contradiction. That is, $\{\delta_n\}$ is bounded. Then we suppose that, in subsequence sense, $\lim_{n\to\infty}\delta_n=\delta_0\geq 0$.

If $\delta_0 = 0$, then for any $\sigma > 0$, we have

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N\setminus B_\sigma(y_n)} |u_n|^{2_s^*}\mathrm{d}x = \lambda^{2_s^*} \lim_{n\to\infty} \int\limits_{\mathbb{R}^N\setminus B_\sigma(y_n)} |U_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x = 0.$$

Thanks to $\beta(u_n) = 0$, for arbitrary $\sigma > 0$, one has

$$\begin{aligned}
\frac{|y_n|}{1+|y_n|} &= \Big|\frac{y_n}{1+|y_n|} - \beta(u_n)\Big| \\
&= \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \Big| \int\limits_{\mathbb{R}^N} \Big(\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\Big)|u_n|^{2_s^*}\mathrm{d}x \Big| \\
&\le \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{B_\sigma(y_n)} \Big|\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\Big||u_n|^{2_s^*}\mathrm{d}x \\
&+ \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{\mathbb{R}^N\setminus B_\sigma(y_n)} \Big|\frac{y_n}{1+|y_n|} - \frac{x}{1+|x|}\Big||u_n|^{2_s^*}\mathrm{d}x \\
&\le 2\sigma + o_n(1),
\end{aligned}$$

from which we can conclude that $\lim_{n\to\infty} |y_n| = 0$.

On the other hand, for each $\sigma > 0$, we have that

$$0 \le \gamma(u_n) = \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{\mathbb{R}^N} \Big|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\Big||u_n|^{2_s^*}\mathrm{d}x + o_n(1) \le 2\sigma + o_n(1),$$

which shows that $\lim_{n\to\infty} \gamma(u_n) = 0$. Due to (3.1), we obtain a contradiction. That is, we prove that (i) of (3.3) holds.

Now, we will prove that (ii) of (3.3) holds. In fact, we just prove that $\{y_n\}$ is bounded. Arguing by contradiction, we suppose that there is a sequence of $\{y_n\}$ satisfying $\lim_{n\to\infty} |y_n| = \infty$. Then for each $\varepsilon > 0$ and $L > 0$, there is $n^\star \in \mathbb{N}$ satisfying

$$|x - y_n| < L \Longrightarrow \Big|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\Big| < \varepsilon, \forall n > n^\star, \tag{3.4}$$

and

$$\int\limits_{\mathbb{R}^N\setminus B_L(y_n)} |u_n|^{2_s^*}\mathrm{d}x = \lambda^{2_s^*} \int\limits_{\mathbb{R}^N\setminus B_L(y_n)} |U_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x + o_n(1) < \varepsilon. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\Big|\beta(u_n) - \frac{y_n}{1+|y_n|}\Big| \le \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{\mathbb{R}^N} \Big|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\Big||u_n|^{2_s^*}\mathrm{d}x$$

$$
\begin{aligned}
&\le \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{B_L(y_n)} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2_s^*} \mathrm{d}x \\
&+ \frac{1}{|u_n|_{2_s^*}^{2_s^*}} \int\limits_{\mathbb{R}^N \setminus B_L(y_n)} \left|\frac{x}{1+|x|} - \frac{y_n}{1+|y_n|}\right| |u_n|^{2_s^*} \mathrm{d}x \\
&\le \varepsilon + \frac{2\varepsilon}{|u_n|_{2_s^*}^{2_s^*}} + o_n(1).
\end{aligned}
$$

Then, we conclude that $\lim_{n\to\infty} |\beta(u_n)| = 1$. Thanks to (3.1), we also obtain a contradiction. That is, (ii) of (3.3) is satisfied.

Consequently, we have

$$
\begin{aligned}
S_{\mu,s} &= \lim_{n\to\infty} \Big(\|u_n\|_s^2 + \int\limits_{\mathbb{R}^N} V(x) u_n^2 \mathrm{d}x \Big) \\
&= \lambda^2 \Big(\|U_{\delta_0,y_0}\|_s^2 + \int\limits_{\mathbb{R}^N} V(x) U_{\delta_0,y_0}^2 \mathrm{d}x \Big) \\
&> \lambda^2 \|U_{\delta_0,y_0}\|_s^2 = S_{\mu,s}.
\end{aligned}
$$

So, we get a contradiction. □

Proposition 3.3. $\nu := \inf\{I(u) : u \in \mathcal{M},\ \beta(u) = 0,\ \gamma(u) \ge \frac{1}{2}\} > S_{\mu,s}$.

Proof. By Proposition 3.1, $\nu \ge S_{\mu,s}$. If $\nu = S_{\mu,s}$, Then, there is a sequence of $\{u_n\}$ satisfying

$$
u_n \in \mathcal{M},\ \beta(u_n) = 0,\ \gamma(u_n) \ge \frac{1}{2},\ \lim_{n\to\infty} I(u_n) = S_{\mu,s}. \tag{3.6}
$$

By the same argument as in Proposition 3.2, we can obtain that

$$
u_n(x) = \lambda U_{\delta_n,y_n}(x) + \varepsilon_n(x),
$$

where $\lambda > 0$, $\delta_n > 0$, $y_n \in \mathbb{R}^3$, $\varepsilon_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$. Furthermore, we can obtain $\lim_{n\to\infty} \delta_n = \delta_0 \in (0, \infty]$ and $\lim_{n\to\infty} y_n = y_0$ in $\mathbb{R}^N$. In the following, we prove $\delta_0 \in (0, \infty)$. Otherwise, one has

$$
\begin{aligned}
S_{\mu,s} &= \lim_{n\to\infty} \Big(\|u_n\|_s^2 + \int\limits_{\mathbb{R}^N} V(x) u_n^2 \mathrm{d}x \Big) \\
&\ge \liminf_{n\to\infty} \Big(\|u_n\|_s^2 + \int\limits_{B_{\sqrt{\delta_n}}(y_n)} V_\infty u_n^2 \mathrm{d}x \Big) \\
&\ge \Big(S_{\mu,s} + \lambda^2 V_\infty \liminf_{n\to\infty} \delta_n \int\limits_{B_1(0)} U_{1,0}^2 \mathrm{d}x \Big)
\end{aligned}
$$

$$= \infty.$$

That is, we obtained that $\delta_0 \in (0, \infty)$. Hence, one has

$$\begin{aligned} S_{\mu,s} &= \lim_{n\to\infty}\Big(\|u_n\|_s^2 + \int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x\Big) \\ &\geq \lambda^2\Big(\|U_{\delta_0,y_0}\|_s^2 + V_\infty\delta_0 \int_{B_{\delta_0}(y_0)} V(x)U_{\delta_0,y_0}^2 \mathrm{d}x\Big) \\ &> \lambda^2\|U_{\delta_0,y_0}\|_s^2 = S_{\mu,s}, \end{aligned}$$

which is a contradiction. □

Let $\alpha \in (0, 1)$ be such that

$$|V - V_\infty|_{\frac{N}{2s}} < \Big(2^{\alpha\frac{4s-\mu}{2N-\mu}} - 1\Big)S_s \tag{3.7}$$

and $c^\star$ satisfying

$$S_{\mu,s} < c^\star < \min\Big(\frac{S_{\mu,s} + \vartheta}{2}, 2^{(1-\alpha)\frac{4s-\mu}{2N-\mu}} S_{\mu,s}\Big). \tag{3.8}$$

Let $\zeta(x)$ be a function satisfying:

$$\begin{cases} (i)\ \zeta \in C_0^\infty(B_1(0)); \\ (ii)\ \zeta(x) \geq 0, \forall x \in B_1(0); \\ (iii)\ \zeta \in \mathcal{M} \text{ and } \|\zeta\|_s^2 = \Lambda \in (S_{\mu,s}, c^\star); \\ (iv)\ \zeta(x) = \zeta(|x|) \text{ and } |x_1| < |x_2| \Rightarrow \zeta(x_1) > \zeta(x_2). \end{cases} \tag{3.9}$$

For every $\delta > 0$ and $y \in \mathbb{R}^N$, let $\zeta_{\delta,y}(x) = 0$ if $x \notin B_\delta(y)$ and $\zeta_{\delta,y}(x) = \delta^{-\frac{N-2s}{2}}\zeta(\frac{x-y}{\delta})$ if $x \in B_\delta(y)$. Obviously, one has

$$\int_{\mathbb{R}^N} |\zeta_{\delta,y}|^{2_s^*}\mathrm{d}x = \int_{B_\delta(y)} |\zeta_{\delta,y}|^{2_s^*}\mathrm{d}x = \int_{B_1(0)} |\zeta|^{2_s^*}\mathrm{d}x;$$

$$\int_{\mathbb{R}^N} (I_\mu * |\zeta_{\delta,y}|^{2^*_{\mu,s}})|\zeta_{\delta,y}|^{2^*_{\mu,s}}\mathrm{d}x = \int_{\mathbb{R}^N} (I_\mu * |\zeta|^{2^*_{\mu,s}})|\zeta|^{2^*_{\mu,s}}\mathrm{d}x = 1.$$

Furthermore, we have

$$\zeta_{\delta,y} \in \mathcal{M}, \text{ and } \|\zeta_{\delta,y}\|_s^2 = \Lambda \in (S_{\mu,s}, c^\star)\ \ \forall\delta > 0 \text{ and } \forall y \in \mathbb{R}^N.$$

Lemma 3.1. *The following equalities hold*
(a) $\lim_{\delta\to 0}\sup\left\{\int_{\mathbb{R}^N}(V(x)-V_\infty)|\zeta_{\delta,y}|^2dx : y\in\mathbb{R}^N\right\}=0$;
(b) $\lim_{\delta\to\infty}\sup\left\{\int_{\mathbb{R}^N}(V(x)-V_\infty)|\zeta_{\delta,y}|^2dx : y\in\mathbb{R}^N\right\}=0$;
(c) $\lim_{r\to\infty}\sup\left\{\int_{\mathbb{R}^N}(V(x)-V_\infty)|\zeta_{\delta,y}|^2dx : |y|=r, \delta>0, y\in\mathbb{R}^N\right\}=0$.

Proof. Let $W(x)=V(x)-V_\infty$. For any $y\in\mathbb{R}^N$ and $\delta>0$, it follows from Hölder inequality that

$$\int_{\mathbb{R}^N} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x=\int_{B_\delta(y)} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x\le|W|_{\frac{N}{2s},B_\delta(y)}|\zeta|^2_{2_s^*,B_1(0)}\le C|W|_{\frac{N}{2s},B_\delta(y)},$$

where positive constant C is independent of δ.

Hence, we have that

$$\sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x\le C\sup\left\{|W|_{\frac{N}{2s},B_\delta(y)} : y\in\mathbb{R}^N\right\}. \tag{3.10}$$

It follows from

$$\lim_{\delta\to 0}|W|_{\frac{N}{2s},B_\delta(y)}=0 \text{ uniformly in } y\in\mathbb{R}^N$$

that (a) hold.

Now, we will prove (b). Fixed arbitrarily $y\in\mathbb{R}^N$, for each $\sigma>0$ and $\delta>0$ one has

$$\begin{aligned}\int_{\mathbb{R}^N} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x&=\int_{B_\sigma(0)} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x+\int_{\mathbb{R}^N\setminus B_\sigma(0)} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x\\&\le|W|_{\frac{N}{2s},B_\sigma(0)}|\zeta_{\delta,y}|^2_{2_s^*,B_\sigma(0)}+|W|_{\frac{N}{2s},\mathbb{R}^N\setminus B_\sigma(0)}|\zeta_{\delta,y}|^2_{2_s^*,\mathbb{R}^N\setminus B_\sigma(0)}\\&\le|W|_{\frac{N}{2s},B_\sigma(0)}\sup_{y\in\mathbb{R}^N}|\zeta_{\delta,y}|^2_{2_s^*,B_\sigma(0)}+C|W|_{\frac{3}{2s},\mathbb{R}^3\setminus B_\sigma(0)}\end{aligned}$$

where positive constant C is independent of δ and σ.

Thanks to

$$\lim_{\delta\to\infty}|\zeta_{\delta,y}|^2_{2_s^*,B_\sigma(0)}=0 \text{ uniformly in } y\in\mathbb{R}^N,$$

then for each $\sigma>0$, we get

$$\lim_{\delta\to\infty}\sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N} W(x)|\zeta_{\delta,y}|^2\mathrm{d}x\le C|W|_{\frac{N}{2s},\mathbb{R}^N\setminus B_\sigma(0)}.$$

Then, let $\sigma\to\infty$ in the inequality above, (b) is verified.

Lastly, we prove (c) by an indirect procedure, that is, suppose that there exist sequences of $\{y_n\} \subset \mathbb{R}^N$ and $\{\delta_n\} \subset \mathbb{R}^+\backslash\{0\}$ such that

$$\lim_{n\to\infty} |y_n| \to \infty, \tag{3.11}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x)|\zeta_{\delta_n,y_n}|^2 \mathrm{d}x > 0. \tag{3.12}$$

Combining with (a) and (b), we obtain that $\lim_{n\to\infty} \delta_n = \widetilde{\delta} > 0$. Due to $W \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, it follows from (3.11) that

$$\lim_{n\to\infty} |W|_{\frac{N}{2s}, B_{\delta_n}(y_n)} = 0.$$

Consequently, we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W(x)|\zeta_{\delta_n,y_n}|^2 \mathrm{d}x \le \left(|W|_{\frac{N}{2s}, B_{\delta_n}(y_n)} \cdot |\zeta_{\delta,y}|^2_{2_s^*, B_{\delta_n}(y_n)}\right) = 0$$

contradicting (3.12). □

Lemma 3.2. *The following relations hold*
(a) $\lim_{\delta\to 0} \sup\{\gamma(\zeta_{\delta,y}) : y \in \mathbb{R}^N\} = 0$*;*
(b) $\lim_{\delta\to\infty} \inf\{\gamma(\zeta_{\delta,y}) : y \in \mathbb{R}^N, |y| \le r\} = 1, \forall r > 0$*;*
(c) $(\beta(\zeta_{\delta,y}), y)_{\mathbb{R}^N} > 0, \forall y \in \mathbb{R}^N, \forall \delta > 0$*, where* $(x, y)_{\mathbb{R}^N}$ *denotes the inner product of* $x, y \in \mathbb{R}^N$*.*

Proof. For any $\delta > 0$ and $y \in \mathbb{R}^N$, one has

$$\begin{aligned} 0 \le \gamma(\zeta_{\delta,y}) &= \frac{1}{|\zeta_{\delta,y}|^{2_s^*}_{2_s^*, B_\delta(y)}} \int_{B_\delta(y)} \left|\frac{x}{1+|x|} - \beta(\zeta_{\delta,y})\right| |\zeta_{\delta,y}|^{2_s^*} \mathrm{d}x \\ &\le \frac{1}{|\zeta_{\delta,y}|^{2_s^*}_{2_s^*, B_\delta(y)}} \int_{B_\delta(y)} \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| |\zeta_{\delta,y}|^{2_s^*} \mathrm{d}x + \left|\frac{y}{1+|y|} - \beta(\zeta_{\delta,y})\right| \\ &\le \frac{1}{|\zeta_{\delta,y}|^{2_s^*}_{2_s^*, B_\delta(y)}} \left(2 \int_{B_\delta(y)} \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| |\zeta_{\delta,y}|^{2_s^*} \mathrm{d}x\right) \\ &\le 4\delta. \end{aligned}$$

Hence $0 \le \sup\{\gamma(\zeta_{\delta,y}) : y \in \mathbb{R}^N\} \le 4\delta$. It follows that (a) holds.

To prove (b), we first show that, for each $r > 0$ and $y \in \mathbb{R}^N$ with $|y| \le r$,

$$\lim_{\delta\to\infty} \sup_{|y|\le r} \beta(\zeta_{\delta,y}) = 0. \tag{3.13}$$

It follows from $\beta(\zeta_{\delta,0}) = 0$ and the definition of $\zeta_{\delta,y}$ that

$$\begin{aligned}|\beta(\zeta_{\delta,y})| &= \frac{1}{|\zeta_{\delta,y}|_{2_s^*}^{2_s^*}}\Big|\int_{\mathbb{R}^N} \frac{x}{1+|x|}|\zeta_{\delta,y}|^{2_s^*}\mathrm{d}x\Big| \\ &= \frac{1}{|\zeta_{\delta,0}|_{2_s^*}^{2_s^*}}\Big|\int_{\mathbb{R}^N} \frac{x}{1+|x|}(|\zeta_{\delta,y}|^{2_s^*}-|\zeta_{\delta,0}|^{2_s^*})\mathrm{d}x\Big| \\ &\le \frac{1}{|\zeta_{\delta,0}|_{2_s^*}^{2_s^*}}\int_{\mathbb{R}^N} \frac{|x|}{1+|x|}\Big||\zeta_{\delta,y}|^{2_s^*}-|\zeta_{\delta,0}|^{2_s^*}\Big|\mathrm{d}x \\ &\le C\int_{\mathbb{R}^N} \Big||\zeta_{1,\frac{y}{\delta}}|^{2_s^*}-|\zeta_{1,0}|^{2_s^*}\Big|\mathrm{d}x\end{aligned}$$

which shows that (3.13) holds.

For each $\delta > 0$,

$$\gamma(\zeta_{\delta,y}) = \frac{1}{|\zeta_{\delta,y}|_{2_s^*}^{2_s^*}}\int_{\mathbb{R}^N} \Big|\frac{x}{1+|x|}-\beta(\zeta_{\delta,y})\Big||\zeta_{\delta,y}|^{2_s^*}\mathrm{d}x \le 1+|\beta(\zeta_{\delta,y})|.$$

Together with (3.13), we deduce that

$$\limsup_{\delta\to\infty}\inf\{\gamma(\zeta_{\delta,y}) : y\in\mathbb{R}^N, |y|\le r\} \le 1.$$

If the following holds

$$\limsup_{\delta\to\infty}\inf\{\gamma(\zeta_{\delta,y}) : y\in\mathbb{R}^N, |y|\le r\} < 1. \tag{3.14}$$

Choosing $\{y_n\}$ and $\{\delta_n\}$ satisfying $|y_n| \le r$, $\delta_n \to \infty$ and

$$\lim_{n\to\infty}\gamma(\zeta_{\delta_n,y_n}) < 1. \tag{3.15}$$

Thanks to (3.13) and fact that $|\zeta_{\delta,y}|_{2_s^*,B_\sigma(0)} \to 0$ as $\delta\to\infty$, for each $\sigma > 0$ we have that

$$\begin{aligned}\gamma(\zeta_{\delta_n,y_n}) &= \frac{1}{|\zeta_{\delta_n,y_n}|_{2_s^*}^{2_s^*}}\int_{\mathbb{R}^N} \Big|\frac{x}{1+|x|}-\beta(\zeta_{\delta_n,y_n})\Big||\zeta_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x \\ &\ge \frac{1}{|\zeta_{\delta_n,y_n}|_{2_s^*,\mathbb{R}^N\setminus B_\sigma(0)}^{2_s^*}+o_n(1)}\int_{\mathbb{R}^N\setminus B_\sigma(0)} \frac{|x|}{1+|x|}|\zeta_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x - |\beta(\zeta_{\delta_n,y_n})| \\ &\ge \frac{1}{|\zeta_{\delta_n,y_n}|_{2_s^*,\mathbb{R}^N\setminus B_\sigma(0)}^{2_s^*}+o_n(1)}\int_{\mathbb{R}^N\setminus B_\sigma(0)} \frac{|x|}{1+|x|}|\zeta_{\delta_n,y_n}|^{2_s^*}\mathrm{d}x - o_n(1) \\ &\ge \frac{\sigma}{1+\sigma} - o_n(1).\end{aligned}$$

Let $\sigma \to \infty$, we can conclude that

$$\lim_{n\to\infty} \gamma(\zeta_{\delta_n,y_n}) \geq 1$$

which is an absurd due to (3.15). Hence, the proof of (b) has finished.

At last, we show that (c) holds. (c) obviously holds if $0 \notin B_\delta(y)$. If $0 \in B_\delta(y)$, for each $x \in B_\delta(y)$ satisfying $(x, y)_{\mathbb{R}^N} < 0$, then $-x \in B_\delta(y)$ so that $(-x, y)_{\mathbb{R}^N} > 0$ and $\zeta_{\delta,y}(-x) > \zeta_{\delta,y}(x)$. Hence, (c) holds. □

4. The proof of main results

Lemma 4.1. *There are* $\overline{r} > 0$ *and* $0 < \delta_1 < \frac{1}{2} < \delta_2$ *satisfying*

$$\gamma(\zeta_{\delta_1,y}) < \frac{1}{2},\ \forall y \in \mathbb{R}^N;\ \ \gamma(\zeta_{\delta_2,y}) > \frac{1}{2},\ \forall y \in \mathbb{R}^N,\ |y| < \overline{r} \tag{4.1}$$

and

$$\sup\{I_0(\zeta_{\delta,y}) : (\delta, y) \in \partial\Pi\} < c^\star, \tag{4.2}$$

where $\Pi := \{(\delta, y) \in \mathbb{R}^+ \times \mathbb{R}^N : \delta \in [\delta_1, \delta_2], |y| < \overline{r}\}$ *and* $I_0 : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ *be defined by*

$$I_0(u) = \|u\|_s^2 + \int_{\mathbb{R}^N} (V(x) - V_\infty)u^2 dx, u \in D^{s,2}(\mathbb{R}^N).$$

Proof. It follows from (a) and (b) of Lemma 3.2 that there are $\overline{r} > 0$ and $0 < \delta_1 < \frac{1}{2} < \delta_2$ such that (4.1) holds. On the other hand, by Lemma 3.1 and the characteristic of $\zeta_{\delta,y}$, we could conclude that (4.2) is satisfied. □

Lemma 4.2. *Let* $\delta_1, \delta_2, \overline{r}$ *and* Π *be defined as in Lemma 4.1. Then there is* $(\widetilde{\delta}, \widetilde{y}) \in \partial\Pi$ *and* $(\overline{\delta}, \overline{y}) \in \mathring{\Pi}$ *such that*

$$\beta(\zeta_{\widetilde{\delta},\widetilde{y}}) = 0, \gamma(\zeta_{\widetilde{\delta},\widetilde{y}}) > \frac{1}{2}; \tag{4.3}$$

$$\beta(\zeta_{\overline{\delta},\overline{y}}) = 0, \gamma(\zeta_{\overline{\delta},\overline{y}}) = \frac{1}{2}. \tag{4.4}$$

Proof. Thanks to Lemma 4.1, choosing $(\widetilde{\delta}, \widetilde{y}) = (\delta_2, 0)$, then (4.3) holds.

For any $(\delta, y) \in \Pi$ and $\varsigma \in [0, 1]$, we define $\theta(\delta, y) := (\gamma(\zeta_{\delta,y}), \beta(\zeta_{\delta,y}))$ and $\omega : [0, 1] \times \partial\Pi \to \mathbb{R} \times \mathbb{R}^3$ by

$$\omega(\delta, y, \varsigma) := (1 - \varsigma)(\delta, y) + \varsigma\theta(\delta, y). \tag{4.5}$$

To prove the (4.4), we just prove that

$$deg(\theta, \mathring{\Pi}, (\frac{1}{2}, 0)) = 1. \tag{4.6}$$

Indeed, thanks to $deg(id, \mathring{\Pi}, (\frac{1}{2}, 0)) = 1$, if we prove that for every $(\delta, y) \in \partial\Pi$ and $\varsigma \in [0, 1]$, $\omega(\delta, y, \varsigma) \neq (\frac{1}{2}, 0)$, then it follows from the topological degree theory that $deg(\theta, \mathring{\Pi}, (\frac{1}{2}, 0)) = 1$ is satisfied. Hence, we only to show that

$$((1-\varsigma)\delta + s\gamma(\zeta_{\delta,y}), (1-\varsigma)y + \varsigma\beta(\zeta_{\delta,y})) \neq (\frac{1}{2}, 0),\ \forall(\delta, y) \in \partial\Pi,\ \forall\varsigma \in [0, 1].$$

Set

$$\Pi_i = \{(\delta, y) \in \partial\Pi : |y| \leq \overline{r}, \delta = \delta_i\}, i = 1, 2;$$
$$\Pi_3 = \{(\delta, y) \in \partial\Pi : |y| = \overline{r}, \delta \in [\delta_1, \delta_2]\}.$$

Obviously, $\partial\Pi = \Pi_1 \cup \Pi_2 \cup \Pi_3$. It follows from (4.1) that, for any $(\delta, y) \in \Pi_1$,

$$(1-\varsigma)\delta_1 + \varsigma\gamma(\zeta_{\delta_1,y}) < \frac{1}{2}(1-\varsigma) + \frac{\varsigma}{2} < \frac{1}{2}.$$

By virtue of (4.1), for any $(\delta, y) \in \Pi_2$ we conclude

$$(1-\varsigma)\delta_2 + \varsigma\gamma(\zeta_{\delta_2,y}) > \frac{1}{2}(1-\varsigma) + \frac{\varsigma}{2} > \frac{1}{2}.$$

For any $(\delta, y) \in \Pi_3$, we used (c) of Lemma 3.2 to get that

$$((1-\varsigma)y + \varsigma\beta(\zeta_{\delta,y}), y)_{\mathbb{R}^3} = (1-\varsigma)|y|^2 + \varsigma(\beta(\zeta_{\delta,y}), y)_{\mathbb{R}^3} > 0,$$

which shows that $(1-\varsigma)y + \varsigma\beta(\zeta_{\delta,y}) \neq 0$. □

Lemma 4.3. *Let $\delta_1, \delta_2, \overline{r}$ and Π be defined as in Lemma 4.1. Suppose that (V_2) holds, then we have*

$$\sup\{I_0(\zeta_{\delta,y}) : (\delta, y) \in \Pi\} < 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s}. \tag{4.7}$$

Proof. For any $(\delta, y) \in \Pi$, it follows from (3.7) and (3.8) that

$$\begin{aligned} I_0(\zeta_{\delta,y}) &= \|\zeta_{\delta,y}\|_s^2 + \int_{\mathbb{R}^N} (V(x) - V_\infty)\zeta_{\delta,y}^2 \mathrm{d}x \\ &\leq \|\zeta_{\delta,y}\|_s^2 + |V - V_\infty|_{\frac{N}{2s}} |\zeta_{\delta,y}|_{2_s^*}^2 \\ &\leq \|\zeta_{\delta,y}\|_s^2 + \frac{1}{S_s}|V - V_\infty|_{\frac{N}{2s}} \|\zeta_{\delta,y}\|_s^2 \\ &= \Big(1 + \frac{1}{S_s}|V - V_\infty|_{\frac{N}{2s}}\Big)\Lambda \end{aligned} \tag{4.8}$$

$$< \left(1 + \frac{1}{S_s}|V - V_\infty|_{\frac{N}{2s}}\right)c^\star$$
$$< 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s}.$$

The proof is now complete. □

Lemma 4.4. *Let $\delta_1, \delta_2, \overline{r}$ and Π be defined as in Lemma 4.1. There is a constant $V^* > 0$ satisfying if $V_\infty \in (0, V^*)$ then*

$$\gamma(\zeta_{\delta_1,y}) < \frac{1}{2}, \gamma(\zeta_{\delta_2,y}) > \frac{1}{2}, \forall y \in \mathbb{R}^N, |y| < \overline{r} \tag{4.9}$$

$$\mathcal{L} := \sup\{I(\zeta_{\delta,y}) : (\delta, y) \in \partial\Pi\} < c^\star. \tag{4.10}$$

Proof. According to definition of $\zeta_{\delta,y}$,

$$\int\limits_{\mathbb{R}^3} V_\infty \zeta_{\delta,y}^2 \mathrm{d}x = V_\infty \delta^{2s} \int\limits_{B_1(0)} \zeta^2 \mathrm{d}x.$$

Then, we have

$$I(\zeta_{\delta,y}) = I_0(\zeta_{\delta,y}) + V_\infty \delta^{2s} \int\limits_{B_1(0)} \zeta^2 \mathrm{d}x,$$

which together with (4.2) we can conclude that if V_∞ small enough, then (4.9) and (4.10) are satisfied. □

By the same argument as in Lemma 4.4 and Lemma 4.3, we can obtain the following result.

Lemma 4.5. *Let $\delta_1, \delta_2, \overline{r}$ and Π be defined as in Lemma 4.1. Suppose that (V_2) holds, then there is a constant $V_*^1 > 0$ satisfying if $V_\infty \in (0, V_*^1)$ we have*

$$\mathcal{A} := \sup\{I(\zeta_{\delta,y}) : (\delta, y) \in \Pi\} < 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s}. \tag{4.11}$$

4.1. Proof of Theorem 1.1

Let

$$I^c = \{u \in \mathcal{M} : I(u) \le c\},$$

where $c \in \mathbb{R}$.

Firstly, we will show that I restricted on $\mathcal{M}$ has a critical level in $(S_{\mu,s}, c^*)$. Let $V_\infty \in (0, V^*)$, it follows from the definition of $c^\star$, Proposition 3.3, Lemma 4.4 that

$$S_{\mu,s} < \nu \le I(\zeta_{\widetilde{\delta},\widetilde{y}}) \le \mathcal{L} < c^\star < \vartheta.$$

In what follows, we prove that I constrained on $\mathcal{M}$ has a critical level in the interval $(\nu, \mathcal{L})$. Argue by contradiction that is not true. Thanks to Lemma 2.7, I satisfies PS condition in $(\nu, \mathcal{L})$. Therefore, according to Lemma 2.3 in [50], there is $\tau_1 > 0$ such that

$$S_{\mu,s} < \nu - \tau_1, c^\star > \mathcal{L} + \tau_1$$

and a continuous function $\psi : [0, 1] \times I^{\mathcal{L}+\tau_1} \to I^{\mathcal{L}+\tau_1}$ satisfying

$$I \circ \psi(\varsigma, u) \le I(u), \forall \varsigma \in [0, 1], \forall u \in I^{\mathcal{L}+\tau_1}, \tag{4.12}$$

$$\psi(1, I^{\mathcal{L}+\tau_1}) \subset I^{\nu-\tau_1}. \tag{4.13}$$

By virtue of (4.10) and (4.13), we obtain

$$(\delta, y) \in \partial\Pi \Rightarrow I(\zeta_{\delta,y}) \le \mathcal{L} \Rightarrow I \circ \psi(1, \zeta_{\delta,y}) \le \nu - \tau_1. \tag{4.14}$$

For $\varsigma \in [0, 1]$ and $(\delta, y) \in \Pi$, set

$$\Upsilon(\delta, y, \varsigma) = \begin{cases} \omega(\delta, y, 2\varsigma), \ \ \varsigma \in [0, \frac{1}{2}], \\ (\gamma \circ \psi(2\varsigma - 1, \zeta_{\delta,y}), \beta \circ \psi(2\varsigma - 1, \zeta_{\delta,y})), \ \ \varsigma \in [\frac{1}{2}, 1], \end{cases} \tag{4.15}$$

where ω is defined as (4.5). Via Lemma 4.2, we have

$$\Upsilon(\delta, y, \varsigma) \ne (\frac{1}{2}, 0), \forall \varsigma \in [0, \frac{1}{2}], \forall (\delta, y) \in \partial\Pi. \tag{4.16}$$

It follows from (4.12) and (4.14) that

$$I \circ \psi(2\varsigma - 1, \zeta_{\delta,y}) \le I(\zeta_{\delta,y}) \le \mathcal{L} < c^\star < \vartheta, \forall \varsigma \in [\frac{1}{2}, 1], \forall (\delta, y) \in \partial\Pi,$$

from which we get that

$$\Upsilon(\delta, y, \varsigma) \ne (\frac{1}{2}, 0), \forall \varsigma \in [\frac{1}{2}, 1], \forall (\delta, y) \in \partial\Pi. \tag{4.17}$$

Combining with (4.16), (4.17) and continuity of Υ we conclude that there exists $(\delta^\star, y^*) \in \partial\Pi$ satisfying

$$\beta \circ \psi(1, \zeta_{\delta^\star,y^*}) = 0, \gamma \circ \psi(1, \zeta_{\delta^\star,y^*}) \ge \frac{1}{2}.$$

Together with Proposition 3.3, we obtain

$$I \circ \psi(1, \zeta_{\delta^\star,y^*}) \ge \nu,$$

which contradicts to (4.14). That is, for each $V_\infty \in (0, V^*)$, Φ has at least a critical point $\widetilde{u} \in \mathcal{M}$ satisfying $\nu < I(\widetilde{u}) < \mathcal{L}$. Moreover, it follows from strong maximum principle that $\widetilde{u} > 0$.

Next, we intend prove that there is the critical level in $(c^*, 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s})$. It follows from the definition of $c^\star$, Proposition 3.2 and (4.11) that if $V_\infty \in (0, V_*^1)$, then

$$c^\star < \vartheta \le I(\zeta_{\overline{\delta},\overline{y}}) \le \mathcal{A} < 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s}.$$

We assert that I constrained on $\mathcal{M}$ has a critical level in the interval $(\vartheta, \mathcal{A})$. If not, thanks to Lemma 2.7 and Lemma 2.3 [50], I satisfies PS condition in $(\vartheta, \mathcal{A})$ and there is $\tau_2 > 0$ satisfying

$$c^\star < \vartheta - \tau_2,\ 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s} > \mathcal{A} + \tau_2$$

and a continuous function $\psi : [0, 1] \times I^{\mathcal{A}+\tau_2} \to I^{\vartheta-\tau_2}$ such that $\psi(u) = u,\ \forall u \in I^{\vartheta-\tau_2}$. Noticed that $\psi(\delta, y)$ is well defined on Π and $I \circ \psi(\zeta_{\delta,y}) \le \vartheta - \tau_2,\ \forall(\delta, y) \in \Pi$. Then, we obtain

$$\mathcal{E}(\delta, y) := (\gamma \circ \psi(\zeta_{\delta,y}), \beta \circ \psi(\zeta_{\delta,y})) \ne (\frac{1}{2}, 0), \forall(\delta, y) \in \Pi. \tag{4.18}$$

Then, together with (4.10), for each $V_\infty \in (0, V_*)$ where $V_* = \min\{V_*^1, V^*\}$, we conclude

$$I(\zeta_{\delta,y}) < c^\star < \vartheta - \tau_2,\ \forall(\delta, y) \in \partial\Pi,$$

which shows $\psi(\zeta_{\delta,y}) = \zeta_{\delta,y},\ \forall(\delta, y) \in \partial\Pi$. Then we have

$$\mathcal{E}(\delta, y) = \theta(\delta, y) = (\gamma(\zeta_{\delta,y}), \beta(\zeta_{\delta,y})),\ \forall(\delta, y) \in \partial\Pi.$$

It follows from proof of Lemma 4.2 and degree theory that

$$deg(\mathcal{E}, \mathring{\Pi}, (\frac{1}{2}, 0)) = deg(\theta, \mathring{\Pi}, (\frac{1}{2}, 0)) = 1.$$

So, there exists $(\delta_\star, y_*) \in \Pi$ satisfying

$$\mathcal{E}(\delta_\star, y_*) = (\frac{1}{2}, 0).$$

Via (4.18), we get a contradiction. That is, I has at least a positive critical point $\widehat{u} \in \mathcal{M}$ satisfying

$$c^\star < I(\widehat{u}) < 2^{\frac{4s-\mu}{2N-\mu}} S_{\mu,s}.$$

The proof is now complete. □

Data availability

No data was used for the research described in the article.

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