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Global small finite energy solutions for the incompressible magnetohydrodynamics equations in $\mathbb{R}^+ \times \mathbb{R}^2$

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Abstract

In this paper, we prove the global well-posedness for the incompressible magnetohydrodynamics (MHD) equations in the three-dimensional unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$. More precisely, we construct global small Sobolev regularity solutions with the initial data near 0 for the three-dimensional MHD equations in Ω. The key point of the proof is to find the suitable initial approximation function such that the linearized equations around it admitted a partial dissipative structure when we carry out the weighted energy estimate. Meanwhile, the asymptotic expansion of Sobolev regularity solutions is given.

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Keywords: MHD equations; Global Sobolev solution; Calderón-Zygmund theory

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1. Introduction and main results

We consider the global existence of Sobolev regularity solutions to the three-dimensional incompressible magnetohydrodynamics equations:

$$\begin{aligned}
&\partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P = (\mathbf{H}\cdot\nabla)\mathbf{H},\\
&\partial_t \mathbf{H} - \mu\Delta\mathbf{H} + (\mathbf{v}\cdot\nabla)\mathbf{H} = (\mathbf{H}\cdot\nabla)\mathbf{v},\\
&\nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{H} = 0,
\end{aligned} \tag{1.1}$$

with the initial data

$$\mathbf{v}(0,x) = \mathbf{v}_0(x), \quad \mathbf{H}(0,x) = \mathbf{H}_0(x), \tag{1.2}$$

where $t \in \mathbb{R}^+$, $x \in \Omega$ with an unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$, $\mathbf{v} := (v_1, v_2, v_3)$ denotes the 3D velocity field of the fluid, P stands for the pressure in the fluid, $\mathbf{H} := (H_1, H_2, H_3)$ is the magnetic field, and $\nu, \mu > 0$ denotes the viscosity constant and the magnetic diffusion constant, respectively. The divergence free condition in second equations of problem (1.1) guarantees the incompressibility of the fluid. The pressure takes the form

$$P = -\Delta^{-1}\mathrm{div}\Big(\mathbf{v}\cdot\nabla\mathbf{v} - (\mathbf{H}\cdot\nabla)\mathbf{H}\Big). \tag{1.3}$$

We supplement the 3D incompressible MHD equations (1.1) with the boundary condition

$$\mathbf{v}(t,x)|_{x\in\partial\Omega} = 0, \quad \mathbf{H}(t,x)|_{x\in\partial\Omega} = 0, \tag{1.4}$$

that is, in x_1 direction

$$\begin{aligned}
&v_i(t,x)|_{x_1=0} = 0, \quad H_i(t,x)|_{x_1=0} = 0, \quad i = 1,2,3,\\
&\lim_{x_1\to+\infty} v_i(t,x) = 0, \quad \lim_{x_1\to+\infty} H_i(t,x) = 0,
\end{aligned}$$

and the vanishing boundary condition in $\bar{x} := (x_2, x_3)$ direction

$$\lim_{|\bar{x}|\to+\infty} v_i(t,x) = 0, \quad \lim_{|\bar{x}|\to+\infty} H_i(t,x) = 0, \quad i = 1,2,3.$$

It is easy to check that solutions of 3D incompressible MHD equations (1.1) admit the scaling invariant property. More precisely, if $(\mathbf{v}, \mathbf{H}, P)$ is an arbitrary solution of problem (1.1), then for any constant $\lambda > 0$, the functions

$$\begin{aligned}
&\mathbf{v}_{\lambda,\alpha}(t,x) = \lambda\mathbf{v}(\lambda^2 t, \lambda x),\\
&\mathbf{H}_{\lambda,\alpha}(t,x) = \lambda\mathbf{H}(\lambda^2 t, \lambda x),\\
&P_{\lambda,\alpha}(t,x) = \lambda^2 P(\lambda^2 t, \lambda x),
\end{aligned}$$

are also solutions of 3D incompressible MHD equations (1.1). Here, the initial data $(\mathbf{v}_0(x), \mathbf{H}_0(x))$ is changed into $(\lambda\mathbf{v}_0(\lambda x), \lambda\mathbf{H}_0(\lambda x))$.

1.1. Introduction

The incompressible Magneto-hydrodynamics equations (MHD) describe the dynamics of electrically conducting fluids arising in plasmas or some other physical phenomena. These equations are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Califano & Chiuderi [5] conjectured that the energy of an incompressible Magnetic-hydrodynamical system is dissipated at a rate that is independent of the ohmic resistivity. This conjecture also implies that the three-dimensional non-resistive MHD equations (1.1) may be global well-posed in some functional space. As we know, the question of finite time singularity/global regularity for three-dimensional incompressible Navier-Stokes equations is the most important open problem in mathematical fluid mechanics [9], so it is also a natural important problem for the three-dimensional incompressible MHD equations. Sermange & Temam [24] established the local well-posedness of classical solutions for fully viscous MHD equations, in which the global well-posedness is also proved in two dimensions. Fefferman, McCormick, Robinson & Rodrigo [10] proved a local existence result for MHD equations (1.1) taking arbitrary initial data in $\mathbb{H}^s(\mathbb{R}^d)$ with $s > d/2$ for $d = 2, 3$. Chemin, McCormick, Robinson & Rodrigo [6] obtained the local existence of solutions to the viscous, non-resistive MHD equations with initial data $(\mathbf{v}_0, \mathbf{H}_0)$ in the Besov space $B^{\frac{d}{2}}_{2,1}(\mathbb{R}^n) \times B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^n)$ with $n = 2, 3$. Fefferman, McCormick, Robinson & Rodrigo [11] improved the initial data $(\mathbf{v}_0, \mathbf{H}_0) \in \mathbb{H}^{s-1+\varepsilon}(\mathbb{R}^n) \times \mathbb{H}^s(\mathbb{R}^n)$ for $s > d/2$ and $0 < \varepsilon < 1$. Next, Li, Tan & Yin [15] improved these results in homogeneous Besov spaces. For global existence results, Lin, Xu & Zhang [16] established the global well-posedness of a two-dimensional incompressible MHD system with smooth initial data close to some non-trivial steady state. We also refer to Lin & Zhang [17] who extended the global well-posedness of a two-dimensional incompressible MHD system to the three-dimensional case with small initial data. Abidi & Zhang [1] considered a more general initial data closed to the non-trivial equilibrium state $(x_3, 0)$. Cai & Lei [4] showed the global well-posedness for the incompressible MHD system with or without viscosity with the initial data near a constant vector (Alfvén waves) by means of the ghost weight technique. Their result depends on the inherent strong null structure of the system, which can make sure of the decay in time of the energy. Deng & Zhang [8] constructed the smooth solutions near the trivial equilibrium state $(e_3, 0)$ by using Nash-Moser iteration scheme, where e_3 is a constant. They did not use the strong null structure of the system. They reduced problem (1.1) into a dissipative system, which admits the same dissipative structure with the following equation

$$Y_{tt} - \Delta Y_t - \partial_3^2 Y = F,$$

where F is the nonlinear term. But we notice that if the equilibrium state is moved into $(0, 0, 0)$, then it is changed into a new dissipative equation

$$Y_{tt} - \Delta Y_t = \overline{F}.$$

Since there is loss of term $\partial_3^2 Y$, it is difficult to carry out the energy estimates. For this reason, we cannot follow the idea of Deng & Zhang in this paper. We will deal with the parabolic system (1.1) in a direct way. We remark that all of the known results deal with global infinite energy solutions for the incompressible MHD system. Recently, Yan [32] found a family of stable infinite energy blowup solutions for three-dimensional incompressible MHD systems.

It is meaningful to consider the well-posedness problem of partial differential equations in thin domains. Thin domains are widely studied in solid mechanics, fluid dynamics and magnetohydrodynamics, one can see [3,7,19,23] for more details. Raugel & Sell [22] showed that the existence of global strong solutions (H^2 space) and attractors of the three-dimensional Navier-Stokes equations with external force in a bounded thin domain with a periodic boundary condition; see also the monograph by Temam [26] for more details. After that, there are several results [13,14] on the three-dimensional Navier-Stokes equations. Recently, Xu [27] obtained the global well-posedness result for the ideal Magnetohydrodynamics in three-dimensional thin domains $\Omega_\delta := \mathbb{R}^2 \times (-\delta, \delta)$ with slip boundary conditions by the construction of a global solution with infinite energy near Alfvén waves, where the constant δ must be sufficient small. Moreover, he got that the $3D$ Alfvén waves converge to the $2D$ Alfvén waves in $\mathbb{R}^2$ as the constant $\delta \to 0$.

To the best of our knowledge, all of known results concerning with the global well-posedness of three-dimensional incompressible MHD equations in the unbounded domain are the global infinite energy solutions due to the solutions near the equilibrium state (constant vectors or non-constant vectors with infinite energy), there is very few result on the global finite energy solutions for the three-dimensional incompressible MHD equations. In this paper, we will construct Sobolev regular solutions for equations (1.1) in the unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$.

1.2. Main results

Assume the initial data (1.2) satisfies the following conditions

$$\begin{aligned} &\nabla \cdot \mathbf{v}_0(x) = 0, \quad \nabla \cdot \mathbf{H}_0(x) = 0, \\ &\mathbf{v}_0(x)|_{x\in\partial\Omega} = 0, \quad \mathbf{H}_0(x)|_{x\in\partial\Omega} = 0, \\ &\partial_{x_i}^l \mathbf{v}_0(x)|_{x\in\partial\Omega} = 0, \quad \partial_{x_i}^l \mathbf{H}_0(x)|_{x\in\partial\Omega} = 0, \quad \text{for } 1 \le l \le s. \end{aligned} \tag{1.5}$$

We now state the main result of this paper.

Theorem 1.1. *Let $\nu, \mu > 0$ and the unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$. For any fixed integer $s \ge 1$, there exists a positive small constant $\varepsilon \in (0, 1)$ such that if the initial data (1.2) of the 3D incompressible MHD equations (1.1) in the unbounded domain Ω with the boundary condition (1.4) satisfies the condition (1.5) and*

$$\|\boldsymbol{v}_0(x)\|_{H^{s+2}(\Omega)} + \|\boldsymbol{H}_0(x)\|_{H^{s+2}(\Omega)} \lesssim \varepsilon,$$

then it admits a global Sobolev solution

$$\Big(\boldsymbol{v}(t,x), \boldsymbol{H}(t,x)\Big) \in \mathbb{C}((0,+\infty); H^s(\Omega) \times H^s(\Omega)).$$

Moreover, the solution satisfies

$$\sup_{t\in(0,+\infty)} \|\boldsymbol{v}(t,x)\|_{H^s(\Omega)} \lesssim \varepsilon,$$

$$\sup_{t\in(0,+\infty)} \|\boldsymbol{H}(t,x)\|_{H^s(\Omega)} \lesssim \varepsilon,$$

$$\sup_{t\in(0,+\infty)} \|P(t,x)\|_{H^s(\Omega)} \lesssim \varepsilon,$$

for any $(t,x)\in(0,\infty)\times\Omega$. *Here the pressure is given by (1.3).*

In particularly, we have the following explicit representation formulas.

Corollary 1.1. *Let the integers* $p>0$ *and* $2<q<2(p+1)$, *and the parameter* $0<\varepsilon\ll 1$. *Assume that the small initial data (1.2) satisfies the conditions (1.5). The* 3*D incompressible MHD equations (1.1) admit an explicit expansion of global Sobolev regular solutions with finite energy as follows*

$$\begin{aligned}
\boldsymbol{v}^*(t,x) &= \Big(v_1^{(0)}(t,x), v_2^{(0)}(t,x), v_3^{(0)}(t,x)\Big) + \boldsymbol{v}_0(x) + \mathcal{R}_0(t,x),\\
\boldsymbol{H}^*(t,x) &= \Big(H_1^{(0)}(t,x), H_2^{(0)}(t,x), H_3^{(0)}(t,x)\Big) + \boldsymbol{H}_0(x) + \mathcal{R}_1(t,x),
\end{aligned}$$

where $\forall x=(x_1,x_2,x_3)\in\mathbb{R}^+\times\mathbb{R}^2$, *and*

$$\begin{aligned}
&v_1^{(0)}(t,x) = H_1^{(0)}(t,x) = -\varepsilon(1-e^{-t})x_1^q x_2^{2p+1}x_3^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},\\
&v_2^{(0)}(t,x) = H_2^{(0)}(t,x)\\
&\quad = -2^{-1}\varepsilon(1+p)^{-1}(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_3^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},\\
&v_3^{(0)}(t,x) = H_3^{(0)}(t,x)\\
&\quad = \varepsilon(1+p)^{-1}(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_2^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},
\end{aligned}$$

and the remainder terms $\mathcal{R}_0(t,x),\ \mathcal{R}_1(t,x)\in H^s(\Omega)$ *for any* $s\geq 1$ *satisfy*

$$\begin{aligned}
&\mathcal{R}_k(0,x)=0, \quad \nabla\cdot\mathcal{R}_k(t,x)=0, \quad \textit{for } k=0,1,\\
&\mathcal{R}_k(t,x)|_{x\in\partial\Omega}=0, \quad \sup_{t\in(0,+\infty)}\|\mathcal{R}_k(t,x)\|_{H^s(\Omega)}\sim\mathcal{O}(\varepsilon^2).
\end{aligned}$$

Moreover, the pressure is determined by

$$P^*(t,x) = -\Delta^{-1}\mathrm{div}\Big(\boldsymbol{v}^*\cdot\nabla\boldsymbol{v}^* - (\boldsymbol{H}^*\cdot\nabla)\boldsymbol{H}^*\Big).$$

1.3. Sketch of the proof

In order to solve the nonlinear equations (1.1) with the boundary condition (1.4), the main difficulty is to find suitable dissipative terms, which can cause the time decay of energy of solutions for the linearized equations. Assume we have chosen a suitable initial approximation function $\Big(\mathbf{v}^{(0)}(t,x),\mathbf{H}^0(t,x)\Big)^T$. Then we linearize the nonlinear equation around $\Big(\mathbf{v}^{(0)}(t,x),\mathbf{H}^0(t,x)\Big)^T$ to get the linearized operator

$$\begin{aligned}\mathcal{J}_1[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) :=\ & \mathbf{h}_t^{(1)} - \nu\lambda^2\Delta\mathbf{h}^{(1)} + \Pi_{N_1}\Big[\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)} \\ & + \nabla(\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)} - \Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} \\ & - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)}\Big], \\ \mathcal{J}_2[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) :=\ & \mathbf{w}_t^{(1)} - \mu\lambda^2\Delta\mathbf{w}^{(1)} + \Pi_{N_1}\Big[\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)} \\ & - \Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)}\Big],\end{aligned}$$

where $\mathcal{D}_{\mathbf{v}^{(0)}}$ denotes the Fréchet derivative at the function $\mathbf{v}^{(0)}$.

Furthermore, if we set $\Big(\mathbf{h}^{(1)}(t,\lambda x),\mathbf{w}^{(1)}(t,\lambda x)\Big)^T$ with a big constant (e.g. we can choose $\lambda > 2$), then the linearized operator is changed into

$$\begin{aligned}\mathcal{J}_1[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) :=\ & \mathbf{h}_t^{(1)} - \nu\lambda^2\Delta\mathbf{h}^{(1)} + \Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)} \\ & + \nabla(\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)} - \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} \\ & - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)}\Big], \\ \mathcal{J}_2[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) :=\ & \mathbf{w}_t^{(1)} - \mu\lambda^2\Delta\mathbf{w}^{(1)} + \Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)} \\ & - \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)}\Big].\end{aligned}$$

Here we observe that the terms $-\nu\lambda^2\Delta\mathbf{h}^{(1)}+\Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)}$ and $-\mu\lambda^2\Delta\mathbf{w}^{(1)}+\Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)}$ can be dissipative if we choose a suitable initial approximation function $\mathbf{v}^{(0)}(t,x)$ and a big positive constant λ. We denoted by Π_{N_1} the smoothing operator. Thus, the decay in time of the solution for the linearized equation

$$\begin{aligned}&\mathcal{J}_1[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) = \Pi_{N_1}E^{(0)}, \\ &\mathcal{J}_2[\mathbf{v}^0,\mathbf{H}^0](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) = \Pi_{N_1}\overline{E}^{(0)}, \\ &\nabla\cdot\mathbf{h}^{(1)} = 0, \quad \nabla\cdot\mathbf{w}^{(1)} = 0, \\ &\mathbf{h}^{(1)}(0,\lambda x) = \mathbf{h}_0^{(1)}(\lambda x), \quad \mathbf{w}^{(1)}(0,\lambda x) = \mathbf{w}_0^{(1)}(\lambda x),\end{aligned}$$

is as desired.

Consequently, the Nash-Moser iteration scheme can be used to construct global small solutions of (1.1) as follows

$$\begin{aligned}\mathbf{v}^{(\infty)}(t,x) &= \mathbf{v}^{(0)}(t,x) + \sum_{m=1}^{\infty}\mathbf{h}^{(m)}(t,\lambda x) + \mathbf{v}_0(x) \in H^s(\mathbb{R}^3), \\ \mathbf{H}^{(\infty)}(t,x) &= \mathbf{H}^{(0)}(t,x) + \sum_{m=1}^{\infty}\mathbf{w}^{(m)}(t,\lambda x) + \mathbf{H}_0(x) \in H^s(\mathbb{R}^3),\end{aligned}$$

where $\Big(\mathbf{h}^{(m)}(t,\lambda x),\mathbf{w}^{(m)}(t,\lambda x)\Big)^T$ is obtained by solving the linearized equations

$$\mathcal{J}_1[\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=E^{(m-1)}(t,x),$$
$$\mathcal{J}_2[\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=\overline{E}^{(m-1)}(t,x)$$

and $\Big(E^{(m-1)}(t,x),\overline{E}^{(m-1)}(t,x)\Big)^T$ denotes the error term at $(m-1)$-th step.

This method has been used in [28–31,33,35]. For the general Nash-Moser implicit function theorem, we refer to the seminal papers by Nash [20], Moser [18] and Hörmander [12].

Notations. Throughout this paper, let $\Omega:=\mathbb{R}^+\times\mathbb{R}^2$, we denote the usual norms of the Lebesgue space $\mathbb{L}^2(\Omega)$ and of Sobolev space $\mathbb{H}^s(\Omega)$ by $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$, respectively. The norm of the Sobolev space $H^s(\mathbb{R}^3):=(\mathbb{H}^s(\Omega))^3$ is denoted by $\|\cdot\|_{H^s}$. The symbol $a\lesssim b$ means that there exists a positive constant C such that $a\le Cb$. We denote by $(a,b,c)^T$ the column vector in $\mathbb{R}^3$. The letter C with subscripts to denote dependencies stands for a positive constant that might change its value at each occurrence.

The paper is organized as follows. In section 2, we show how to choose suitable initial approximation functions, which lead to the partial dissipative structure of the linearized MHD equations. Next, we give the existence of the global time-decay Sobolev solution for the linearized equations of first approximation step. In section 3, we establish the general approximation step for the construction of the Nash-Moser iteration scheme. In the last section we show how to construct a global Sobolev solution for the MHD equations (1.1) in the unbounded domain Ω by the proof of convergence for Nash-Moser iteration scheme.

2. The first approximation step

For $m=1,2,\dots$, by setting $N_m=2^m$, we introduce a family of smoothing operators (see [25] for more details) $\Pi_{N_m}:L^2\to C^\infty$ such that

$$\begin{aligned}&\|\Pi_{N_m}U\|_{H^{s_1}}\lesssim N_m^{s_1-s_2}\|U\|_{H^{s_2}},\quad\forall\, s_1\ge s_2\ge 0,\\&\|\Pi_{N_m}U-U\|_{H^{s_1}}\lesssim N_m^{s_1-s_2}\|U\|_{H^{s_2}},\quad\forall\, 0\le s_1\le s_2,\end{aligned}\tag{2.1}$$

where the smooth operator Π_θ is defined by

$$\Pi_\theta U=\sum_M\chi(\frac{2^M}{\theta})U_M,$$

where $\chi\in\mathbb{C}_0^\infty(\mathbb{R})$, $\chi=1$ in the neighbourhood of the origin. In what follows, we use the symbol Π_{N_m} for convenience.

We now consider the approximation problem of incompressible MHD equations (1.1) as follows

$$\begin{aligned}&\mathcal{L}_1(\mathbf{v},\mathbf{H}):=\partial_t\mathbf{v}-\nu\Delta\mathbf{v}+\Pi_{N_m}\Big(\mathbf{v}\cdot\nabla\mathbf{v}+\nabla P-(\mathbf{H}\cdot\nabla)\mathbf{H}\Big),\\&\mathcal{L}_2(\mathbf{v},\mathbf{H}):=\partial_t\mathbf{H}-\mu\Delta\mathbf{H}+\Pi_{N_m}\Big((\mathbf{v}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{v}\Big),\end{aligned}\tag{2.2}$$

with the initial data (1.2), the boundary condition (1.4) and the incompressible condition

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{H} = 0.$$

2.1. The initial approximation function

A family of explicit examples. We first give a family of exact examples of the initial approximation function in the unbounded domain $\mathbb{R}^+ \times \mathbb{R}^2$. Let the integers $p > 1$ and $2 < q < 2(p+1)$. We choose the initial approximation functions of the following form

$$\begin{aligned} \mathbf{v}^{(0)}(t,x) &= (v_1^{(0)}(t,x), v_2^{(0)}(t,x), v_3^{(0)}(t,x)), \quad \forall (t,x) \in \mathbb{R}^+ \times \Omega, \\ \mathbf{H}^{(0)}(t,x) &= (H_1^{(0)}(t,x), H_2^{(0)}(t,x), H_3^{(0)}(t,x)), \quad \forall (t,x) \in \mathbb{R}^+ \times \Omega, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} v_1^{(0)}(t,x) &= H_1^{(0)}(t,x) := \varepsilon_0(1-e^{-t}) x_1^q x_2^{2p+1} x_3^{2p+1} e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})}, \\ v_2^{(0)}(t,x) &= H_2^{(0)}(t,x) \\ &:= \varepsilon_0(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q}) x_1^{q-1} x_3^{2p+1} e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})}, \\ v_3^{(0)}(t,x) &= H_3^{(0)}(t,x) \\ &:= -2\varepsilon_0(1-e^{-t})(q-2(p+1)x_1^{2(p+1)-q}) x_1^{q-1} x_2^{2p+1} e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})}. \end{aligned}$$

By straightforward computations, we obtain

$$\nabla \cdot \mathbf{v}^{(0)}(t,x) = 0, \quad \nabla \cdot \mathbf{H}^{(0)}(t,x) = 0,$$

and

$$\begin{aligned} \mathbf{v}^{(0)}(t,x)|_{x\in\partial\Omega} &= 0, \\ \mathbf{H}^{(0)}(t,x)|_{x\in\partial\Omega} &= 0, \end{aligned}$$

and $\forall t \geq 0$, for $0 \leq l \leq s$, it holds

$$\begin{aligned} \partial_{x_i}^l \mathbf{v}^{(0)}(t,x)|_{x\in\partial\Omega} &= 0, \\ \partial_{x_i}^l \mathbf{H}^{(0)}(t,x)|_{x\in\partial\Omega} &= 0. \end{aligned}$$

Here, the initial approximation pressure satisfies

$$P^{(0)}(t,x) = -\Delta^{-1}\mathrm{div}\Big(\mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} - (\mathbf{H}^{(0)} \cdot \nabla)\mathbf{H}^{(0)}\Big).$$

The general condition of initial approximation functions. We now give an abstract condition on the initial approximation function. Let $s \geq 1$ be a fixed finite constant and $0 < \varepsilon_0 < \varepsilon^2 \ll 1$. For any $x \in \Omega$, we choose the initial approximation functions

$$\mathbf{v}^{(0)}(t,x) = \left(v_1^{(0)}(t,x), v_2^{(0)}(t,x), v_3^{(0)}(t,x)\right) \in H^s(\Omega),$$
$$\mathbf{H}^{(0)}(t,x) = \left(H_1^{(0)}(t,x), H_2^{(0)}(t,x), H_3^{(0)}(t,x)\right) \in H^s(\Omega),$$

where we require

$$\begin{cases} \nabla \cdot \mathbf{v}^{(0)}(t,x) = 0, \\ \mathbf{v}^{(0)}(0,x) = 0, \\ \|\mathbf{v}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ \partial_{x_i}^l \mathbf{v}^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \quad \text{for } 0 \le l \le s, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \nabla \cdot \mathbf{H}^{(0)}(t,x) = 0, \\ \mathbf{H}^{(0)}(0,x) = 0, \\ \|\mathbf{H}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ \partial_{x_i}^l \mathbf{H}^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \quad \text{for } 0 \le l \le s. \end{cases} \tag{2.5}$$

Moreover, for any fixed $s \ge 1$ and $(t,x) \in \Omega$ and $i, j = 1, 2, 3$, it also needs the conditions

$$\sum_{k=0}^{s} \|\partial_{x_i}^k \mathbf{v}_j^{(0)}(t,x)\|_{L^\infty} \lesssim \varepsilon_0, \tag{2.6}$$

and

$$\sum_{k=0}^{s} \|\partial_{x_i}^k \mathbf{H}_j^{(0)}(t,x)\|_{L^\infty} \lesssim \varepsilon_0, \tag{2.7}$$

and the initial error term

$$\begin{aligned} &\|E^{(0)}\|_{H^s} \lesssim \varepsilon_0, \quad \|\overline{E}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ &\partial_{x_i}^l E^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \quad \partial_{x_i}^l \overline{E}^{(0)}(t,x)|_{x\in\partial\Omega} = 0, \end{aligned} \tag{2.8}$$

where $E^{(0)}$ and $\overline{E}^{(0)}$ denote the error term taking the form

$$E^{(0)} := \mathcal{L}_1(\mathbf{v}^{(0)}, \mathbf{H}^{(0)}),$$
$$\overline{E}^{(0)} := \mathcal{L}_2(\mathbf{v}^{(0)}, \mathbf{H}^{(0)}),$$

with the vector form

$$E^{(0)} = (E_1^{(0)}, E_2^{(0)}, E_3^{(0)}),$$
$$\overline{E}^{(0)} = (\overline{E}_1^{(0)}, \overline{E}_2^{(0)}, \overline{E}_3^{(0)}).$$

One can see that for every fixed integers p and q, the exact functions (2.3) satisfy the assumptions (2.4)-(2.8).

2.2. The time-decay of first approximation step

We now construct the first approximation solution of problem (2.2). This solution is denoted by $\left(\mathbf{v}^{(1)}(t,\lambda x), \mathbf{H}^{(1)}(t,\lambda x)\right)^T$. The first approximation step between the initial approximation function and the first approximation solution is denoted by

$$\mathbf{h}^{(1)}(t,\lambda x) := \mathbf{v}^{(1)}(t,\lambda x) - \mathbf{v}^{(0)}(t,x),$$
$$\mathbf{w}^{(1)}(t,\lambda x) := \mathbf{H}^{(1)}(t,\lambda x) - \mathbf{H}^{(0)}(t,x).$$

Next, we linearize the nonlinear system (2.2) around $\left(\mathbf{v}^{(0)}, \mathbf{H}^{(0)}\right)^T$ and we obtain the linearized operators as follows

$$\begin{aligned}
\mathcal{J}_1[\mathbf{v}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) := \mathbf{h}_t^{(1)} - \nu\lambda^2\Delta \mathbf{h}^{(1)} + \Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)} \\
+ \nabla(\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)} - \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} \\
- \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)}\Big], \\
\mathcal{J}_2[\mathbf{v}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) := \mathbf{w}_t^{(1)} - \mu\lambda^2\Delta \mathbf{w}^{(1)} + \Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)} \\
- \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)}\Big],
\end{aligned} \tag{2.9}$$

where $\mathcal{D}_{\mathbf{v}^{(0)}}$ denotes the Fréchet derivatives on $\mathbf{v}^{(0)}$.

We now consider the linear system

$$\begin{aligned}
&\mathcal{J}_1[\mathbf{v}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \Pi_{N_1} E^{(0)}, \\
&\mathcal{J}_2[\mathbf{v}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \Pi_{N_1} \overline{E}^{(0)}, \\
&\nabla\cdot\mathbf{h}^{(1)} = 0, \quad \nabla\cdot\mathbf{w}^{(1)} = 0, \\
&\mathbf{h}^{(1)}(0,\lambda x) = \mathbf{h}_0^{(1)}(\lambda x), \quad \mathbf{w}^{(1)}(0,\lambda x) = \mathbf{w}_0^{(1)}(\lambda x),
\end{aligned} \tag{2.10}$$

and the boundary condition

$$\mathbf{h}^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0, \quad \mathbf{w}^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0. \tag{2.11}$$

The solution of this problem gives the first approximation step of MHD equations (1.1).

Before we carry out some *a priori* estimates, we rewrite equations of (2.10) into a coupled system as follows

$$\begin{aligned}
&\partial_t h_j^{(1)} - \nu\lambda^2 \Delta h_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} v_i^{(0)} \partial_{x_i} h_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} h_i^{(1)} \partial_{x_i} v_j^{(0)} \\
&\quad + \Pi_{N_1} \partial_{x_j} \Big((\mathcal{D}_{\mathbf{v}^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}} P)\mathbf{w}^{(1)} \Big) \\
&\quad - \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_j} w_j^{(1)} - \Pi_{N_1} \sum_{i=1}^{3} w_i^{(1)} \partial_{x_i} H_j^{(0)} = \Pi_{N_1} E_j^{(0)}, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\partial_t w_j^{(1)} - \nu\lambda^2 \Delta w_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} v_i^{(0)} \partial_{x_i} w_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} h_i^{(1)} \partial_{x_i} H_j^{(0)} \\
&\quad - \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_i} h_j^{(1)} - \Pi_{N_1} \sum_{i=1}^{3} w_i^{(1)} \partial_{x_i} v_j^{(0)} = \Pi_{N_1} \overline{E}_j^{(0)}, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.13}$$

with the initial data

$$h_j^{(1)}(0, \lambda x) = h_{0j}^{(1)}(\lambda x), \quad w_j^{(1)}(0, \lambda x) = w_{0j}^{(1)}(\lambda x), \tag{2.14}$$

and the boundary condition (2.11), where

$$\begin{aligned}
\partial_{x_j} \Big((\mathcal{D}_{\mathbf{v}^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}} P)\mathbf{w}^{(1)} \Big) &= -\partial_{x_j} \Delta^{-1} \sum_{i,j=1}^{3} \Big(\partial_{x_j} h_i^{(1)} \partial_{x_i} v_j^{(0)} + \partial_{x_j} v_i^{(0)} \partial_{x_i} h_j^{(1)} \\
&\quad + \partial_{x_j} w_i^{(1)} \partial_{x_i} H_j^{(0)} + \partial_{x_j} H_i^{(0)} \partial_{x_i} w_j^{(1)} \Big).
\end{aligned} \tag{2.15}$$

Moreover, for $0 \le l \le s$, we require the initial data of (2.10) satisfying

$$\begin{aligned}
\partial_{x_i}^l \mathbf{h}_0^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0, \\
\partial_{x_i}^l \mathbf{w}_0^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0.
\end{aligned} \tag{2.16}$$

2.3. *A priori estimate*

We now derive a $\mathbb{L}^2$ weighted estimate of the solution for the linear system (2.12)-(2.13). Let $\phi(x_1)$ be a function defined in $(0, +\infty)$ such that

$$0 < \kappa \le \phi''(x_1) - (\phi'(x_1))^2 < +\infty, \tag{2.17}$$

and $e^{-\phi(x_1)}$ is bounded in $(0, +\infty)$. The condition (2.17) implies $\phi''(x_1) \ge \kappa > 1$. In fact, there are many functions can satisfy above conditions. For a simple example, we take the function as the form

$$\phi(x_1) = -\ln|\cos(\sqrt{\kappa} x_1)|, \quad x_1 \ne 2i\pi + \frac{\pi}{2}, \text{ for } i \in \mathbb{Z}.$$

Lemma 2.1. *Let the parameter $\lambda > 1$. Assume that the initial approximation function $\Big(\boldsymbol{v}^{(0)}, \boldsymbol{H}^{(0)}\Big)$ satisfies conditions (2.4)-(2.8). Then the solution $\Big(\boldsymbol{h}^{(1)}(t, \lambda x), \boldsymbol{w}^{(1)}(t, \lambda x)\Big)$ of the linear system (2.12)-(2.13) satisfies*

$$\begin{aligned}\sum_{j=1}^{3}\int_{\Omega}\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)dx \le e^{-C_{\nu,\mu,\lambda,\varepsilon}t}\sum_{j=1}^{3}\Big[\int_{\Omega}\Big((h_{0j}^{(1)})^2+(w_{0j}^{(1)})^2\Big)dx \\ +\Pi_{N_1}\int_0^{+\infty}\int_{\Omega}\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)dxdt\Big], \quad \forall t>0,\end{aligned} \tag{2.18}$$

where $C_{\nu,\mu,\lambda,\varepsilon}$ denotes a positive constant depending on $\nu, \mu, \lambda, \varepsilon$.

Proof. Multiplying both sides of the six equations in (2.12)-(2.13) by $h_j^{(1)}e^{-\phi(x_1)}$ and $w_j^{(1)}e^{-\phi(x_1)}$, respectively, integrating over Ω, and using the boundary condition (2.11), for $j = 1, 2, 3$, it holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega}(h_j^{(1)})^2e^{-\phi(x_1)}dx+\nu\lambda^2\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_i}h_j^{(1)})^2e^{-\phi(x_1)}dx \\ &+\frac{\nu\lambda^2}{2}\int_{\Omega}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)(h_j^{(1)})^2e^{-\phi(x_1)}dx+\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\Big(v_i^{(0)}\partial_{x_i}h_j^{(1)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx \\ &+\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\Big(h_i^{(1)}\partial_{x_i}v_j^{(0)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx \\ &+\lambda\Pi_{N_1}\int_{\Omega}\partial_{x_j}\Big((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx \\ &-\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\Big(H_i^{(0)}\partial_{x_i}w_j^{(1)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx-\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\Big(w_i^{(1)}\partial_{x_i}H_j^{(0)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx \\ =&\,\Pi_{N_1}\int_{\Omega}E_j^{(0)}h_j^{(1)}e^{-\phi(x_1)}dx,\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega}(w_j^{(1)})^2e^{-\phi(x_1)}dx+\mu\lambda^2\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_i}w_j^{(1)})^2e^{-\phi(x_1)}dx \\ &+\frac{\mu\lambda^2}{2}\int_{\Omega}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)(w_j^{(1)})^2e^{-\phi(x_1)}dx+\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\Big(v_i^{(0)}\partial_{x_i}w_j^{(1)}\Big)w_j^{(1)}e^{-\phi(x_1)}dx\end{aligned}$$

$$+\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\left(h_i^{(1)}\partial_{x_i}H_j^{(0)}\right)w_j^{(1)}e^{-\phi(x_1)}dx-\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\left(H_i^{(0)}\partial_{x_i}h_j^{(1)}\right)w_j^{(1)}e^{-\phi(x_1)}dx$$

$$-\Pi_{N_1}\sum_{i=1}^{3}\int_{\Omega}\left(w_i^{(1)}\partial_{x_i}v_j^{(0)}\right)w_j^{(1)}e^{-\phi(x_1)}dx=\Pi_{N_1}\int_{\Omega}\overline{E}_j^{(0)}w_j^{(1)}e^{-\phi(x_1)}dx. \tag{2.20}$$

We sum up (2.19)-(2.20) from $j=1$ to $j=3$. It follows that

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}\int_{\Omega}\left((h_j^{(1)})^2+(w_j^{(1)})^2\right)e^{-\phi(x_1)}dx$$

$$+\lambda^2\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(\nu(\partial_{x_i}h_j^{(1)})^2+\mu(\partial_{x_i}w_j^{(1)})^2\right)e^{-\phi(x_1)}dx$$

$$+\frac{\lambda^2}{2}\sum_{j=1}^{3}\int_{\Omega}\left(\phi''(x_1)-(\phi'(x_1))^2\right)\left(\nu(h_j^{(1)})^2+\mu(w_j^{(1)})^2\right)e^{-\phi(x_1)}dx$$

$$+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(v_i^{(0)}\partial_{x_i}h_j^{(1)}\right)h_j^{(1)}e^{-\phi(x_1)}dx$$

$$+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(h_i^{(1)}\partial_{x_i}v_j^{(0)}\right)h_j^{(1)}e^{-\phi(x_1)}dx$$

$$+\lambda\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}\left((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\right)h_j^{(1)}e^{-\phi(x_1)}dx$$

$$-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(H_i^{(0)}\partial_{x_i}w_j^{(1)}\right)h_j^{(1)}e^{-\phi(x_1)}dx$$

$$+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(v_i^{(0)}\partial_{x_i}w_j^{(1)}\right)w_j^{(1)}e^{-\phi(x_1)}dx$$

$$-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(H_i^{(0)}\partial_{x_i}h_j^{(1)}\right)w_j^{(1)}e^{-\phi(x_1)}dx$$

$$-\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(w_i^{(1)}\partial_{x_i}v_j^{(0)}\right)w_j^{(1)}e^{-\phi(x_1)}dx$$

$$=\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}\left(E_j^{(0)}h_j^{(1)}+\overline{E}_j^{(0)}w_j^{(1)}\right)e^{-\phi(x_1)}dx. \tag{2.21}$$

In what follows, we estimate each term in equality (2.21). Note that $\nabla \cdot \mathbf{v}^{(0)} = 0$, and we have chosen the initial approximation function $\left(\mathbf{v}^{(0)}, \mathbf{H}^{(0)}\right)^T$ satisfying (2.4)-(2.8). We integrate by parts and we find

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(v_i^{(0)}\partial_{x_i}h_j^{(1)}\right)h_j^{(1)}e^{-\phi(x_1)}dx &= -\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\partial_{x_i}v_i^{(0)}(h_j^{(1)})^2e^{-\phi(x_1)}dx \\ &\quad+\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)v_1^{(0)}(h_j^{(1)})^2e^{-\phi(x_1)}dx \\ &= \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)v_1^{(0)}(h_j^{(1)})^2e^{-\phi(x_1)}dx.\end{aligned}\tag{2.22}$$

By direct computation we obtain

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}h_i^{(1)}\partial_{x_i}v_j^{(0)}h_j^{(1)}e^{-\phi(x_1)}dx &= \sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}v_j^{(0)}(h_j^{(1)})^2e^{-\phi(x_1)}dx \\ &\quad+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}v_j^{(0)}h_j^{(1)}e^{-\phi(x_1)}dx.\end{aligned}\tag{2.23}$$

Noticing the incompressible condition

$$\nabla\cdot\mathbf{h}^{(1)} = 0, \quad \nabla\cdot\mathbf{w}^{(1)} = 0,$$

it follows that

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}\left((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\right)h_j^{(1)}e^{-\phi(x_1)}dx \\ &= -\sum_{j=1}^{3}\int_{\Omega}\left((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\right)\partial_{x_j}h_j^{(1)}e^{-\phi(x_1)}dx \\ &\quad+\int_{\Omega}\psi'(x_1)\left((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\right)h_1^{(1)}e^{-\phi(x_1)}dx \\ &= \int_{\Omega}\psi'(x_1)\left((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\right)h_1^{(1)}e^{-\phi(x_1)}dx.\end{aligned}\tag{2.24}$$

Furthermore, from (2.15), using the standard Calderón-Zygmund theory, i.e.

$$\|Zg\|_{L^p} \lesssim \|g\|_{L^p},$$

for Riesz operator Z and $p \in (1, +\infty)$, and Young's inequality, it follows that

$$\begin{aligned}
&\Big|\int_\Omega \psi'(x_1)\Big((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\Big)h_1^{(1)}e^{-\phi(x_1)}dx\Big| \\
&\le \Big|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \psi'(x_1)\Delta^{-1}\Big(\partial_{x_j}h_i^{(1)}\partial_{x_i}v_j^{(0)} + \partial_{x_j}v_i^{(0)}\partial_{x_i}h_j^{(1)}\Big)h_1^{(1)}e^{-\phi(x_1)}dx\Big| \\
&\quad + \Big|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \psi'(x_1)\Delta^{-1}\Big(\partial_{x_j}w_i^{(1)}\partial_{x_i}H_j^{(0)} + \partial_{x_j}H_i^{(0)}\partial_{x_i}w_j^{(1)}\Big)h_1^{(1)}e^{-\phi(x_1)}dx\Big| \\
&\lesssim \sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega \Big|\psi'(x_1)(\partial_{x_i}v_j^{(0)} + \partial_{x_j}v_i^{(0)})\Big|(h_1^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\psi'(x_1)|\Big(|\partial_{x_i}v_j^{(0)}|(\partial_{x_j}h_i^{(1)})^2 + |\partial_{x_j}v_i^{(0)}|(\partial_{x_i}h_j^{(1)})^2\Big)e^{-\phi(x_1)}dx \\
&\quad + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega |\psi'(x_1)|\Big(|\partial_{x_i}H_j^{(0)}|(\partial_{x_j}w_i^{(1)})^2 + |\partial_{x_j}H_i^{(0)}|(\partial_{x_i}w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx.
\end{aligned} \tag{2.25}$$

Using Young's inequality, we integrate by parts and we find

$$\sum_{j=1}^{3}\sum_{i\neq j}\int_\Omega h_i^{(1)}\partial_{x_i}v_j^{(0)}h_j^{(1)}e^{-\phi(x_1)}dx \tag{2.26}$$

$$\begin{aligned}
&\le \frac{1}{2}\int_\Omega \Big|\partial_{x_2}v_1^{(0)} + \partial_{x_1}v_2^{(0)} + \partial_{x_3}v_1^{(0)} + \partial_{x_1}v_3^{(0)}\Big|(h_1^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad + \frac{1}{2}\int_\Omega \Big|\partial_{x_2}v_1^{(0)} + \partial_{x_1}v_2^{(0)} + \partial_{x_3}v_2^{(0)} + \partial_{x_2}v_3^{(0)}\Big|(h_2^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad + \frac{1}{2}\int_\Omega \Big|\partial_{x_2}v_3^{(0)} + \partial_{x_3}v_2^{(0)} + \partial_{x_3}v_1^{(0)} + \partial_{x_1}v_3^{(0)}\Big|(h_3^{(1)})^2e^{-\phi(x_1)}dx,
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\Big|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big(H_i^{(0)}\partial_{x_i}w_j^{(1)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx\Big| \\
&\le \Big|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big(w_j^{(1)}\partial_{x_i}H_i^{(0)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx\Big|
\end{aligned}$$

$$
\begin{aligned}
&+\Big|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(w_j^{(1)}H_i^{(0)}\right)\partial_{x_i}h_j^{(1)}e^{-\phi(x_1)}dx\Big| \\
&\lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left((h_j^{(1)})^2+2(w_j^{(1)})^2+(\partial_{x_i}h_j^{(1)})^2\right)e^{-\phi(x_1)}dx,
\end{aligned}
\tag{2.28}
$$

and

$$
\Big|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(H_i^{(0)}\partial_{x_i}h_j^{(1)}\right)w_j^{(1)}e^{-\phi(x_1)}dx\Big| \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left((w_j^{(1)})^2+(\partial_{x_i}h_j^{(1)})^2\right)e^{-\phi(x_1)}dx.
\tag{2.29}
$$

Meanwhile, similar to (2.22)-(2.23), we derive

$$
\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(v_i^{(0)}\partial_{x_i}w_j^{(1)}\right)w_j^{(1)}e^{-\phi(x_1)}dx = \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)v_1^{(0)}(w_j^{(1)})^2e^{-\phi(x_1)}dx,
\tag{2.30}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\left(w_i^{(1)}\partial_{x_i}v_j^{(0)}\right)w_j^{(1)}e^{-\phi(x_1)}dx \\
&=\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}v_j^{(0)}(w_j^{(1)})^2e^{-\phi(x_1)}dx+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}w_i^{(1)}\partial_{x_i}v_j^{(0)}w_j^{(1)}e^{-\phi(x_1)}dx \\
&\lesssim \sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}v_j^{(0)}(w_j^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad+\frac{1}{2}\int_{\Omega}\left|\partial_{x_2}v_1^{(0)}+\partial_{x_1}v_2^{(0)}+\partial_{x_3}v_1^{(0)}+\partial_{x_1}v_3^{(0)}\right|(w_1^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad+\frac{1}{2}\int_{\Omega}\left|\partial_{x_2}v_1^{(0)}+\partial_{x_1}v_2^{(0)}+\partial_{x_3}v_2^{(0)}+\partial_{x_2}v_3^{(0)}\right|(w_2^{(1)})^2e^{-\phi(x_1)}dx \\
&\quad+\frac{1}{2}\int_{\Omega}\left|\partial_{x_2}v_3^{(0)}+\partial_{x_3}v_2^{(0)}+\partial_{x_3}v_1^{(0)}+\partial_{x_1}v_3^{(0)}\right|(w_3^{(1)})^2e^{-\phi(x_1)}dx,
\end{aligned}
\tag{2.31}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_{\Omega}\Big(E_j^{(0)}h_j^{(1)}+\overline{E}_j^{(0)}w_j^{(1)}\Big)e^{-\phi(x_1)}dx \\
&\leq \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\Big[(E_j^{(0)})^2+(\overline{E}_j^{(0)})^2+|\psi''(x_1)|\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)\Big]e^{-\phi(x_1)}dx.
\end{aligned}
\tag{2.32}
$$

Next, we substitute (2.22)-(2.32) into (2.21) and we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}\int_{\Omega}\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx \\
&+\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(\nu\lambda-\frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}v_j^{(0)}|+|\partial_{x_j}v_i^{(0)}|)\Big)(\partial_{x_i}h_j^{(1)})^2e^{-\phi(x_1)}dx \\
&+\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(\mu\lambda-\frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}v_j^{(0)}|+|\partial_{x_j}v_i^{(0)}|)\Big)(\partial_{x_i}w_j^{(1)})^2e^{-\phi(x_1)}dx \\
&+\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}A_j(t,x)\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx \\
&\lesssim \Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}\Big((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2\Big)e^{-\phi(x_1)}dx,
\end{aligned}
\tag{2.33}
$$

with the coefficients

$$
\begin{aligned}
A_1(t,x):=&\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)+\frac{\lambda}{2}v_1^{(0)}\phi'(x_1)+\partial_{x_1}v_1^{(0)} \\
&-\frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\Big|\phi'(x_1)(\partial_{x_i}v_j^{(0)}+\partial_{x_j}v_i^{(0)})\Big| \\
&-\frac{1}{2}\Big|\partial_{x_2}v_1^{(0)}+\partial_{x_1}v_2^{(0)}+\partial_{x_3}v_1^{(0)}+\partial_{x_1}v_3^{(0)}\Big|-\frac{1}{2}|\phi''(x_1)|-\varepsilon, \\
A_2(t,x):=&\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)+\frac{\lambda}{2}v_1^{(0)}\phi'(x_1)+\partial_{x_2}v_2^{(0)} \\
&-\frac{1}{2}\Big|\partial_{x_2}v_1^{(0)}+\partial_{x_1}v_2^{(0)}+\partial_{x_3}v_2^{(0)}+\partial_{x_2}v_3^{(0)}\Big|-\frac{1}{2}|\phi''(x_1)|-\varepsilon, \\
A_3(t,x):=&\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)+\frac{\lambda}{2}v_1^{(0)}\phi'(x_1)+\partial_{x_3}v_3^{(0)} \\
&-\Big|\partial_{x_2}v_3^{(0)}+\partial_{x_3}v_2^{(0)}+\partial_{x_3}v_1^{(0)}+\partial_{x_1}v_3^{(0)}\Big|-\frac{1}{2}|\phi''(x_1)|-\varepsilon.
\end{aligned}
$$

Since the weighted function $\phi(x_1)$ satisfies (2.17), the main term of $A_i(t,x)$ $(i=1,2,3)$ is

$$\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)-\frac{1}{2}|\phi''(x_1)|.$$

Thus, there is a suitable constant $\lambda>1$ such that

$$\begin{aligned}
A_1(t,x)\geq{}&\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)-\frac{1}{2}|\phi''(x_1)|-\frac{\lambda}{2}|\phi'(x_1)|\|v_1^{(0)}\|_{L^\infty(\Omega)}-\|\partial_{x_1}v_1^{(0)}\|_{L^\infty(\Omega)}\\
&-\frac{\lambda}{2}\sum_{i=1}^3\sum_{j=1}^3|\phi'(x_1)|(\|\partial_{x_i}v_j^{(0)}\|_{L^\infty}+\|\partial_{x_j}v_i^{(0)}\|_{L^\infty})\\
&-\frac{1}{2}(\|\partial_{x_2}v_1^{(0)}\|_{L^\infty}+\|\partial_{x_1}v_2^{(0)}\|_{L^\infty}+\|\partial_{x_3}v_1^{(0)}\|_{L^\infty}+\|\partial_{x_1}v_3^{(0)}\|_{L^\infty})-\varepsilon\\
\gtrsim{}&\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)-\frac{1}{2}|\phi''(x_1)|-\lambda\varepsilon|\phi'(x_1)|-\varepsilon.
\end{aligned}$$

By this relation and (2.17), we deduce that there exists a positive constant $C_{\nu,\lambda,\varepsilon}$ depending on ν,λ,ε such that

$$A_1(t,x)\gtrsim\frac{\nu\lambda^2}{2}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)-\varepsilon\geq\frac{\nu\kappa\lambda^2}{2}-\varepsilon\geq C_{\nu,\lambda,\varepsilon}>0,$$

where $\kappa\in(0,\frac{1}{4})$. With similar arguments we deduce that

$$A_2(t,x),\ A_3(t,x)\gtrsim C_{\nu,\lambda,\varepsilon},$$

and

$$\begin{aligned}
&\nu\lambda-\frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}v_j^{(0)}|+|\partial_{x_j}v_i^{(0)}|)\gtrsim C_{\nu,\lambda},\\
&\mu\lambda-\frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}v_j^{(0)}|+|\partial_{x_j}v_i^{(0)}|)\gtrsim C_{\nu,\lambda}.
\end{aligned}$$

Integrating (2.33) over $(0,t)$ we find

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^3\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&\quad+C_{\lambda,\nu,\mu}\sum_{j=1}^3\sum_{i=1}^3\int_0^t\int_\Omega\Big((\partial_{x_i}h_j^{(1)})^2+(\partial_{x_i}w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt\\
&\quad+C_{\lambda,\nu,\mu,\varepsilon}\Pi_{N_1}\sum_{j=1}^3\int_0^t\int_\Omega\Big((h_j^{(1)})^2+(w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt
\end{aligned}$$

$$\lesssim \Pi_{N_1} \sum_{j=1}^{3} \int_0^t \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2 \Big) e^{-\phi(x_1)} dx dt, \tag{2.34}$$

which gives the following inequality

$$\begin{aligned} &\sum_{j=1}^{3} \sum_{i=1}^{3} \int_0^t \int_\Omega \Big((\partial_{x_i} h_j^{(1)})^2 + (\partial_{x_i} w_j^{(1)})^2 \Big) e^{-\phi(x_1)} dx dt \\ &\lesssim \Pi_{N_1} \sum_{j=1}^{3} \int_0^t \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2 \Big) e^{-\phi(x_1)} dx dt. \end{aligned}$$

Therefore, we apply Gronwall's inequality to inequality (2.34) to obtain

$$\begin{aligned} \sum_{j=1}^{3} \int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2 \Big) e^{-\phi(x_1)} dx \le e^{-C_{\nu,\mu,\lambda,\varepsilon} t} \sum_{j=1}^{3} \Big[\int_\Omega \Big((h_{0j}^{(1)})^2 + (w_{0j}^{(1)})^2 \Big) e^{-\phi(x_1)} dx \\ + \Pi_{N_1} \int_0^t \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2 \Big) e^{-\phi(x_1)} dx dt \Big], \end{aligned}$$

furthermore, since $e^{-\phi(x_1)}$ is a bounded smooth function in $(0, +\infty)$, it follows that

$$\begin{aligned} \sum_{j=1}^{3} \int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2 \Big) dx \le e^{-C_{\nu,\mu,\lambda,\varepsilon} t} \sum_{j=1}^{3} \Big[\int_\Omega \Big((h_{0j}^{(1)})^2 + (w_{0j}^{(1)})^2 \Big) dx \\ + \Pi_{N_1} \int_0^t \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2 \Big) dx dt \Big]. \end{aligned}$$

The proof is now complete. □

Furthermore, we derive higher order derivatives estimates. For a fixed integer $s \ge 1$, we apply $D_i^s := \partial_{x_i}^s$ $(\forall k = 1, 2, 3)$ to both sides of (2.12). It follows that

$$\begin{aligned} &\partial_t D_i^s h_j^{(1)} - \nu\lambda^2 \Delta D_i^s h_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} v_i^{(0)} \partial_{x_i} D_i^s h_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} D_i^s h_i^{(1)} \partial_{x_i} v_j^{(0)} \\ &+ \Pi_{N_1} \partial_{x_j} D_i^s \Big((\mathcal{D}_{\mathbf{v}^{(0)}} P) \mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}} P) \mathbf{w}^{(1)} \Big) \\ &- \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_j} D_i^s w_j^{(1)} - \Pi_{N_1} \sum_{i=1}^{3} D_i^s w_i^{(1)} \partial_{x_i} H_j^{(0)} = F_j, \quad \text{for } j = 1, 2, 3, \end{aligned} \tag{2.35}$$

and

$$\partial_t D_i^s w_j^{(1)} - \mu\lambda^2 \Delta D_i^s w_j^{(1)} + \lambda \Pi_{N_1} \sum_{i=1}^{3} v_i^{(0)} \partial_{x_i} D_i^s w_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} D_i^s h_i^{(1)} \partial_{x_i} H_j^{(0)}$$
$$- \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_i} D_i^s h_j^{(1)} - \Pi_{N_1} \sum_{i=1}^{3} D_i^s w_i^{(1)} \partial_{x_i} v_j^{(0)} = \overline{F}_j, \quad \text{for } j = 1, 2, 3, \tag{2.36}$$

with the boundary condition

$$D_i^l h_j^{(1)}(t, \lambda x)|_{x\in\partial\Omega} = 0, \quad D_i^l w_j^{(1)}(t, \lambda x)|_{x\in\partial\Omega} = 0, \tag{2.37}$$

where the integer l satisfies $1 \le l \le s$, and

$$F_j := \Pi_{N_1} D_i^s E_j^{(0)} - \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} D_i^{s_1} v_i^{(0)} \partial_{x_i} D_i^{s_2} h_j^{(1)}$$
$$- \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(D_i^{s_1} \partial_{x_i} v_j^{(0)})$$
$$+ \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_1} H_i^{(0)})(\partial_{x_j} D_i^{s_2} w_j^{(1)})$$
$$+ \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} w_i^{(1)})(\partial_{x_i} D_i^{s_1} H_j^{(0)}),$$

$$\overline{F}_j := \Pi_{N_1} D_i^s \overline{E}_j^{(0)} - \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_1} v_i^{(0)})(\partial_{x_i} D_i^{s_2} w_j^{(1)})$$
$$- \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(\partial_{x_i} D_i^{s_1} H_j^{(0)})$$
$$+ \lambda \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_1} H_i^{(0)})(\partial_{x_i} D_i^{s_2} h_j^{(1)})$$
$$+ \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} (D_i^{s_2} w_i^{(1)})(\partial_{x_i} D_i^{s_1} v_j^{(0)}).$$

It should be noticed that we will construct the first approximation step $h_j^{(1)}(t, \lambda x)$, $w_j^{(1)}(t, \lambda x)$ to satisfy the boundary condition (2.37). It depends on the initial approximation function $\mathbf{v}^{(0)}(t, x)$ and $\mathbf{H}^{(0)}(t, x)$ satisfying

$$\partial_{x_i}^l \mathbf{v}^{(0)}(t, x)|_{x\in\partial\Omega} = 0,$$
$$\partial_{x_i}^l \mathbf{H}^{(0)}(t, x)|_{x\in\partial\Omega} = 0,$$

for any $0 \le l \le s$. A family of explicit functions is given in (2.3).

Next, we derive higher derivative estimates of solutions to problem (2.12)-(2.13).

Lemma 2.2. *Let the parameter $\lambda > 1$. Assume that the initial approximation function $\left(\boldsymbol{v}^{(0)}, \boldsymbol{H}^{(0)}\right)$ satisfies conditions (2.4)-(2.8). Then the solution $\left(\boldsymbol{h}^{(1)}(t, \lambda x), \boldsymbol{w}^{(1)}(t, \lambda x)\right)$ of the linear system (2.12)-(2.13) satisfies*

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^s h_j^{(1)})^2 + (D_i^s w_j^{(1)})^2\Big)dx \\
&\lesssim e^{-C_{\nu,\mu,\lambda,\varepsilon} t}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{s}\Big[\int_{\Omega}\Big((D_i^k h_{0j}^{(1)})^2 + (D_i^k w_{0j}^{(1)})^2\Big)dx \\
&\quad + \int_0^{\infty}\int_{\Omega}\Big((D_i^k E_j^{(0)})^2 + (D_i^k \overline{E}_j^{(0)})^2\Big)dxdt\Big], \quad \forall t > 0,
\end{aligned} \tag{2.38}$$

where $C_{\nu,\mu,\lambda,\varepsilon}$ denotes a positive constant depending on $\nu, \mu, \lambda, \varepsilon$.

Proof. This proof is based on an induction argument.

Let $s = 1$. By (2.35)-(2.36), we have

$$\begin{aligned}
&\partial_t D_i^1 h_j^{(1)} - \nu\lambda^2 \Delta D_i^1 h_j^{(1)} + \lambda \Pi_{N_1}\sum_{i=1}^{3} v_i^{(0)}\partial_{x_i} D_i^1 h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3} D_i^1 h_i^{(1)}\partial_{x_i} v_j^{(0)} \\
&+ \Pi_{N_1}\partial_{x_j} D_i^1\Big((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\Big) \\
&+ \lambda \Pi_{N_1}\sum_{i=1}^{3} D_i^1 v_i^{(0)}\partial_{x_i} h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3} h_i^{(1)} D_i^1 \partial_{x_i} v_j^{(0)} \\
&- \lambda \Pi_{N_1}\sum_{i=1}^{3} H_i^{(0)}\partial_{x_j} D_i^1 w_j^{(1)} - \Pi_{N_1}\sum_{i=1}^{3} D_i^1 w_i^{(1)}\partial_{x_i} H_j^{(0)} \\
&- \lambda \Pi_{N_1}\sum_{i=1}^{3} D_i^1 H_i^{(0)}\partial_{x_j} w_j^{(1)} - \Pi_{N_1}\sum_{i=1}^{3} w_i^{(1)}\partial_{x_i} D_i^1 H_j^{(0)} = \Pi_{N_1} D_i^1 E_j^{(0)}, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
&\partial_t D_i^1 w_j^{(1)} - \mu\lambda^2 \Delta D_i^1 w_j^{(1)} + \lambda \Pi_{N_1}\sum_{i=1}^{3} v_i^{(0)}\partial_{x_i} D_i^1 w_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3} D_i^1 h_i^{(1)}\partial_{x_i} H_j^{(0)} \\
&+ \lambda \Pi_{N_1}\sum_{i=1}^{3} D_i^1 v_i^{(0)}\partial_{x_i} w_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3} h_i^{(1)}\partial_{x_i} D_i^1 H_j^{(0)} - \lambda \Pi_{N_1}\sum_{i=1}^{3} H_i^{(0)}\partial_{x_i} D_i^1 h_j^{(1)}
\end{aligned}$$

$$
- \Pi_{N_1} \sum_{i=1}^{3} D_i^1 w_i^{(1)} \partial_{x_i} v_j^{(0)} - \lambda \Pi_{N_1} \sum_{i=1}^{3} D_i^1 H_i^{(0)} \partial_{x_i} h_j^{(1)}
- \Pi_{N_1} \sum_{i=1}^{3} w_i^{(1)} \partial_{x_i} D_i^1 v_j^{(0)} = \Pi_{N_1} D_i^1 \overline{E}_j, \quad \text{for } j = 1, 2, 3, \tag{2.40}
$$

with the boundary condition

$$
D_i^1 h_j^{(1)}(t, \lambda x)|_{x \in \partial\Omega} = 0, \quad D_i^1 w_j^{(1)}(t, \lambda x)|_{x \in \partial\Omega} = 0. \tag{2.41}
$$

Multiplying both sides of (2.39)-(2.37) by $D_i^1 h_j^{(1)} e^{-\phi(x_1)}$ and $D_i^1 w_j^{(1)} e^{-\phi(x_1)}$, respectively, then integrating over Ω by noticing (2.41), and summing up those equalities from $j = 1$ to $j = 3$, we deduce that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{3} \int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2 \Big) e^{-\phi(x_1)} dx \\
&+ \lambda^2 \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega \Big(\nu (\partial_{x_k} D_i^1 h_j^{(1)})^2 + \mu (\partial_{x_k} D_i^1 w_j^{(1)})^2 \Big) e^{-\phi(x_1)} dx \\
&+ \frac{\lambda^2}{2} \sum_{j=1}^{3} \int_\Omega \Big(\phi''(x_1) - (\phi'(x_1))^2 \Big) \Big(\nu (D_i^1 h_j^{(1)})^2 + \mu (D_i^1 w_j^{(1)})^2 \Big) e^{-\phi(x_1)} dx \\
&+ \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega v_i^{(0)} (\partial_{x_i} D_i^1 h_j^{(1)}) (D_i^1 h_j^{(1)}) e^{-\phi(x_1)} dx \\
&+ \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (D_i^1 h_i^{(1)}) (\partial_{x_i} v_j^{(0)}) (D_i^1 h_j^{(1)}) e^{-\phi(x_1)} dx \\
&+ \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (v_i^{(0)}) (\partial_{x_i} D_i^1 w_j^{(1)}) (D_i^1 w_j^{(1)}) e^{-\phi(x_1)} dx \\
&- \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (D_i^1 w_i^{(1)}) (\partial_{x_i} v_j^{(0)}) (D_i^1 w_j^{(1)}) e^{-\phi(x_1)} dx \\
&+ \Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \partial_{x_j} D_i^1 \Big((\mathcal{D}_{\mathbf{v}^{(0)}} P) \mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}} P) \mathbf{w}^{(1)} \Big) D_i^1 h_j^{(1)} e^{-\phi(x_1)} dx + \Pi_{N_1} \sum_{k=1}^{11} I_k = 0,
\end{aligned} \tag{2.42}
$$

where

$$
I_1 := \lambda \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (D_i^1 v_i^{(0)}) (\partial_{x_i} h_j^{(1)}) (D_i^1 h_j^{(1)}) e^{-\phi(x_1)} dx,
$$

$$I_2 := \sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} h_i^{(1)}(D_i^1\partial_{x_i}v_j^{(0)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_3 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} H_i^{(0)}(\partial_{x_j}D_i^1w_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_4 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} D_i^1H_i^{(0)}(\partial_{x_j}w_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_5 := -\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} w_i^{(1)}(\partial_{x_i}D_i^1H_j^{(0)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_6 := \lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1v_i^{(0)})(\partial_{x_i}w_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_7 := \sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} h_i^{(1)}(\partial_{x_i}D_i^1H_j^{(0)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_8 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} H_i^{(0)}(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_9 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1H_i^{(0)})(\partial_{x_i}h_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_{10} := -\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} w_i^{(1)}(\partial_{x_i}D_i^1v_j^{(0)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_{11} := \sum_{j=1}^{3}\int_{\Omega}\Big((D_i^1E_j^{(0)})(D_i^1h_j^{(1)}) + (D_i^1\overline{E}_j^{(0)})(D_i^1w_j^{(1)})\Big)dx.$$

We now estimate each term in (2.42). On the one hand, since we have chosen the initial approximation function $\left(\mathbf{v}^{(0)}, \mathbf{H}^{(0)}\right)^T$ satisfying (2.4)-(2.8), using the similar method of getting (2.22)-(2.24), we get

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega} v_i^{(0)}(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx\\ &\lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(|\psi'(x_1)|+1\Big)(D_i^1h_j^{(1)})^2e^{-\phi(x_1)}dx,\end{aligned} \tag{2.43}$$

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_i^{(1)})(\partial_{x_i} v_j^{(0)})(D_i^1 h_j^{(1)})e^{-\phi(x_1)}dx$$
$$=\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_i} v_i^{(0)})(D_i^1 h_i^{(1)})^2 e^{-\phi(x_1)}dx$$
$$+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}(\partial_{x_i} v_j^{(0)})(D_i^1 h_i^{(1)})(D_i^1 h_j^{(1)})e^{-\phi(x_1)}dx$$
$$\lesssim \varepsilon\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_j^{(1)})^2 e^{-\phi(x_1)}dx, \tag{2.44}$$

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(v_i^{(0)})(\partial_{x_i} D_i^1 w_j^{(1)})(D_i^1 w_j^{(1)})e^{-\phi(x_1)}dx$$
$$\lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(|\psi'(x_1)|+1\Big)(D_i^1 w_j^{(1)})^2 e^{-\phi(x_1)}dx, \tag{2.45}$$

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 w_i^{(1)})(\partial_{x_i} v_j^{(0)})(D_i^1 w_j^{(1)})e^{-\phi(x_1)}dx \lesssim \varepsilon\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 w_j^{(1)})^2 e^{-\phi(x_1)}dx. \tag{2.46}$$

By the incompressible condition, we have

$$\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j} D_i^1\Big((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\Big)D_i^1 h_j^{(1)} e^{-\phi(x_1)}dx$$
$$\lesssim \varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\psi'(x_1)|\Big((D_i h_j^{(1)})^2+(\partial_{x_j} D_i^1 h_j^{(1)})^2+(\partial_{x_j} D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx. \tag{2.47}$$

Moreover, we use Young's inequality to derive

$$I_1 \lesssim \varepsilon\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_j^{(1)})^2 e^{-\phi(x_1)}dx,$$
$$I_2 \lesssim \frac{\varepsilon}{2}\int_{\Omega}\Big(\sum_{i=1}^{3}(h_i^{(1)})^2+\sum_{j=1}^{3}(D_i^1 h_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$
$$I_3 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((\partial_{x_j} D_i^1 w_j^{(1)})^2+(D_i^1 h_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_4 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((\partial_{x_j} w_j^{(1)})^2 + (D_i^1 h_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_5 \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((w_i^{(1)})^2 + (D_i^1 h_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_6 \lesssim \varepsilon\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_i} w_j^{(1)})^2 e^{-\phi(x_1)}dx,$$

$$I_7 \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((h_i^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_8 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((\partial_{x_i} D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_9 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((\partial_{x_i} h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_{10} \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big((w_i^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx,$$

$$I_{11} \lesssim \frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\Big[(D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2 + |\psi''(x_1)|\Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)\Big]e^{-\phi(x_1)}dx, \tag{2.48}$$

thus, by noticing the weighted function $\phi(x_1)$ satisfying (2.17), and using (2.43)-(2.48), we integrate inequality (2.42) over $(0,t)$ to get

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx \\
&+C_{\nu,\mu,\lambda}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((\partial_{x_k} D_i^1 h_j^{(1)})^2 + (\partial_{x_k} D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt \\
&+\Pi_{N_1} C_{\nu,\mu,\lambda,\varepsilon}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt \\
\lesssim\ &\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2\Big)e^{-\phi(x_1)}dxdt
\end{aligned}$$

$$+\varepsilon \sum_{j=1}^{3} \int_0^t \int_\Omega \Big((h_j^{(1)})^2 + (w_j^{(1)})^2\Big) e^{-\phi(x_1)} dx dt. \tag{2.49}$$

We observe that the last term in the right-hand side of (2.49) can be controlled by using (2.18). Therefore

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3} \int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big) e^{-\phi(x_1)} dx \\
&\lesssim \Pi_{N_1} \sum_{j=1}^{3}\sum_{i=1}^{3} \int_0^t \int_\Omega \Big((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2\Big) e^{-\phi(x_1)} dx dt \\
&\quad + \sum_{j=1}^{3} \Big(\int_\Omega \Big((h_{0j}^{(1)})^2 + (w_{0j}^{(1)})^2\Big) e^{-\phi(x_1)} dx + \Pi_{N_1} \int_0^t \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2\Big) e^{-\phi(x_1)} dx dt \Big).
\end{aligned} \tag{2.50}$$

Hence, by noticing $e^{-\phi(x_1)}$ is a bounded smooth function in $(0, +\infty)$, we apply Gronwall's inequality to (2.49) to obtain

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3} \int_\Omega \Big((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2\Big) e^{-\phi(x_1)} dx \\
&\lesssim e^{-C_{\nu,\mu,\lambda,\varepsilon} t} \Pi_{N_1} \sum_{j=1}^{3} \Big[\int_\Omega \Big((h_{0j}^{(1)})^2 + (w_{0j}^{(1)})^2\Big) e^{-\phi(x_1)} dx \\
&\quad + \sum_{i=1}^{3} \int_\Omega \Big((D_i^1 h_{0j}^{(1)})^2 + (D_i^1 w_{0j}^{(1)})^2\Big) e^{-\phi(x_1)} dx + \int_0^{+\infty} \int_\Omega \Big((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2\Big) e^{-\phi(x_1)} dx dt \\
&\quad + \sum_{i=1}^{3} \int_0^{\infty} \int_\Omega \Big((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2\Big) e^{-\phi(x_1)} dx dt \Big].
\end{aligned} \tag{2.51}$$

Assume that the $2 \le l \le s-1$ derivative case holds, that is,

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3} \int_\Omega \Big((D_i^l h_j^{(1)})^2 + (D_i^l w_j^{(1)})^2\Big) e^{-\phi(x_1)} dx \\
&\lesssim e^{-C_{\nu,\mu,\lambda,\varepsilon} t} \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{l} \Big[\int_\Omega \Big((D_i^k h_{0j}^{(1)})^2 + (D_i^k w_{0j}^{(1)})^2\Big) e^{-\phi(x_1)} dx \\
&\quad + \int_0^{\infty} \int_\Omega \Big((D_i^k E_j^{(0)})^2 + (D_i^k \overline{E}_j^{(0)})^2\Big) e^{-\phi(x_1)} dx dt \Big].
\end{aligned} \tag{2.52}$$

We now prove that the sth derivative case holds. Multiplying both sides of equations (2.35)-(2.36) by $D_i^s h_j^{(1)} e^{-\phi(x_1)}$ and $D_i^s w_j^{(1)} e^{-\phi(x_1)}$, respectively, then integrating over Ω by using the boundary condition (2.37), and summing up these equalities from $j=1$ to $j=3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&+\lambda^2\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(\nu(\partial_{x_k}D_i^s h_j^{(1)})^2+\mu(\partial_{x_k}D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&+\frac{\lambda^2}{2}\sum_{j=1}^{3}\int_{\Omega}\Big(\phi''(x_1)-(\phi'(x_1))^2\Big)\Big(\nu(D_i^s h_j^{(1)})^2+\mu(D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(v_i^{(0)}\partial_{x_i}D_i^s h_j^{(1)}\Big)D_i^s h_j^{(1)}e^{-\phi(x_1)}dx\\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(D_i^s h_i^{(1)}\partial_{x_i}v_j^{(0)}\Big)D_i^s h_j^{(1)}e^{-\phi(x_1)}dx\\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\partial_{x_j}D_i^s\Big((\mathcal{D}_{\mathbf{v}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}\Big)h_j^{(1)}e^{-\phi(x_1)}dx\\
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(v_i^{(0)}\partial_{x_i}D_i^s w_j^{(1)}\Big)D_i^s w_j^{(1)}e^{-\phi(x_1)}dx\\
&-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(H_i^{(0)}\partial_{x_i}D_i^s h_j^{(1)}\Big)D_i^s w_j^{(1)}e^{-\phi(x_1)}dx\\
&-\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(D_i^s w_i^{(1)}\partial_{x_i}v_j^{(0)}\Big)D_i^s w_j^{(1)}e^{-\phi(x_1)}dx\\
&=\sum_{j=1}^{3}\int_{\Omega}\Big(F_j D_i^s h_j^{(1)}+\overline{F}_j D_i^s w_j^{(1)}\Big)e^{-\phi(x_1)}dx. \qquad (2.53)
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\sum_{s_1+s_2=s,\ 0\le s_2\le s-1}\ \sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}v_i^{(0)})(\partial_{x_i}D_i^{s_2}h_j^{(1)})(D_i^s h_j^{(1)})e^{-\phi(x_1)}dx\\
&=\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}v_i^{(0)})(D_i^s h_j^{(1)})^2 e^{-\phi(x_1)}dx
\end{aligned}
$$

$$+\sum_{s_1+s_2=s,\ 0\le s_2\le s-2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}v_i^{(0)})(\partial_{x_i}D_i^{s_2}h_j^{(1)})(D_i^{s}h_j^{(1)})e^{-\phi(x_1)}dx. \tag{2.54}$$

Next, with the same arguments as for obtaining (2.49), we find

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&+C_{\nu,\mu,\lambda}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((\partial_{x_k}D_i^s h_j^{(1)})^2+(\partial_{x_k}D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt\\
&+C_{\nu,\mu,\lambda,\varepsilon}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt\\
\lesssim{}&\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}\Big((D_i^s E_j^{(0)})^2+(D_i^s\overline{E}_j^{(0)})^2\Big)e^{-\phi(x_1)}dxdt\\
&+\varepsilon\sum_{j=1}^{3}\sum_{l=0}^{s-1}\int_0^t\int_{\Omega}\Big((D_i^l h_j^{(1)})^2+(D_i^l w_j^{(1)})^2\Big)e^{-\phi(x_1)}dxdt.
\end{aligned} \tag{2.55}$$

Hence, by (2.4)-(2.8) and (2.52), applying Gronwall's inequality to (2.55), we deduce that

$$\begin{aligned}
&\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\Big((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
\lesssim{}& e^{-C_{\nu,\mu,\lambda,\varepsilon}t}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{s}\Big[\int_{\Omega}\Big((D_i^k h_{0j}^{(1)})^2+(D_i^k w_{0j}^{(1)})^2\Big)e^{-\phi(x_1)}dx\\
&+\int_0^\infty\int_{\Omega}\Big((D_i^k E_j^{(0)})^2+(D_i^k\overline{E}_j^{(0)})^2\Big)e^{-\phi(x_1)}dxdt\Big],
\end{aligned}$$

which combining with $e^{-\phi(x_1)}$ being a bounded smooth function in $(0,+\infty)$ gives (2.38). The proof is now complete. □

2.4. The existence of the first approximation step

Based on the above *a priori* estimates of solutions, we are ready to prove the existence property for the first approximation step. The proof relies on semigroup theory arguments, see [2,21].

Proposition 2.1. *Let the parameter $\lambda > 1$. Assume that the initial approximation function $\left(\boldsymbol{v}^{(0)}, \boldsymbol{H}^{(0)}\right)$ satisfies conditions (2.4)-(2.8). Then the linearized system (2.10) with the boundary condition (2.11) admits a global solution*

$$\left(\boldsymbol{h}^{(1)}(t,\lambda x), \boldsymbol{w}^{(1)}(t,\lambda x)\right)^T \in \mathbb{C}((0,+\infty); H^s(\Omega)\times H^s(\Omega)).$$

Moreover, this solution satisfies

$$\|\boldsymbol{h}^{(1)}\|_{H^s}^2 + \|\boldsymbol{w}^{(1)}\|_{H^s}^2 \lesssim \|\boldsymbol{h}_0^{(1)}\|_{H^s}^2 + \|\boldsymbol{w}_0^{(1)}\|_{H^s}^2 + \|\Pi_{N_1} E^{(0)}\|_{H^s}^2 + \|\Pi_{N_1} \overline{E}^{(0)}\|_{H^s}^2, \quad \forall t > 0, \tag{2.56}$$

and the boundary condition

$$D_i^1 h_j^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0, \quad D_i^1 w_j^{(1)}(t,\lambda x)|_{x\in\partial\Omega} = 0. \tag{2.57}$$

Proof. Let $\mathbb{P}$ be the Leray projector onto the space of divergence free functions. We apply the Leray projector to equations (2.10), hence

$$\begin{aligned} &\mathbf{h}_t^{(1)} - \nu\lambda^2 \mathbb{P}\Delta\mathbf{h} + \mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \mathbb{P}E_0, \\ &\mathbf{w}_t^{(1)} - \mu\lambda^2 \mathbb{P}\Delta\mathbf{w} + \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \mathbb{P}\overline{E}_0, \end{aligned} \tag{2.58}$$

where

$$\begin{aligned} &\mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\ &:= \mathbb{P}\Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)} - \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)}\Big], \\ &\mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\ &:= \mathbb{P}\Pi_{N_1}\Big[\lambda\Big(\mathbf{v}^{(0)}\cdot\nabla\Big)\mathbf{w}^{(1)} + \Big(\mathbf{h}^{(1)}\cdot\nabla\Big)\mathbf{H}^{(0)} - \lambda\Big(\mathbf{H}^{(0)}\cdot\nabla\Big)\mathbf{h}^{(1)} - \Big(\mathbf{w}^{(1)}\cdot\nabla\Big)\mathbf{v}^{(0)}\Big]. \end{aligned}$$

For convenience, we rewrite equations (2.58) as an abstract evolution equation as follows

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix} = \mathbb{Z}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix} + \begin{pmatrix}\mathbb{P}E_0\\ \mathbb{P}\overline{E}_0\end{pmatrix}, \tag{2.59}$$

where

$$\mathbb{Z}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix} := -\begin{pmatrix}-\nu\lambda^2\mathbb{P}\Delta & 0\\ 0 & -\mu\lambda^2\mathbb{P}\end{pmatrix}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix} - \begin{pmatrix}\mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)})\\ \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)})\end{pmatrix}.$$

We follow the idea of [2,34] to show that the linear operator $\mathbb{Z}$ generates a strongly continuous semigroup $e^{\mathbb{Z}\tau}$ in the Sobolev space $H^s(\Omega)\times H^s(\Omega)$. For this purpose, by the same arguments as those used to get (2.38), we deduce that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\left(\partial_i^s h_j^{(1)}, \partial_i^s w_j^{(1)}\right)\cdot \partial_i^s\left(\mathbb{Z}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix}\right)dx \lesssim 0. \tag{2.60}$$

Hence the linear operator $\mathbb{Z}$ is a linear dissipative operator in $H^s(\Omega)\times H^s(\Omega)$. Moreover, if we set

$$\mathbb{Z}\begin{pmatrix}\mathbf{h}^{(1)}\\ \mathbf{w}^{(1)}\end{pmatrix}=0,$$

then by (2.60), we know that the linear operator $\mathbb{Z}$ is injective. Furthermore, we can verify that this linear operator is surjective by using the standard theory of elliptic-type equations of the general order. Thus, by the Lumer-Phillips theorem [21], the linear operator $\mathbb{Z}$ generates a strongly continuous semigroup $\mathcal{S}_0(\tau) := e^{\mathbb{Z}\tau}$ in the Sobolev space $H^s(\Omega)\times H^s(\Omega)$. Therefore, the linear system (2.58) has a global solution in $H^s(\Omega)\times H^s(\Omega)$. Furthermore, it follows from Lemma 2.1-2.2 that (2.56) holds, and the boundary condition (2.57) can be derived from the expression of solution by noticing the external force $(E^{(0)}, \overline{E}^{(0)})$ satisfying the boundary condition. □

3. The mth approximation step

Let $\varepsilon \in (0, 1)$ be a fixed constant. We define

$$\mathcal{B}_\varepsilon := \{u^{(k)}(t, \lambda x) : \|\mathbf{v}^{(k)}\|_{H^s} + \|\mathbf{w}^{(k)}\|_{H^s} \lesssim \varepsilon < 1\} \tag{3.1}$$

with the integers $2 \le k \le m-1$ and $s \ge 1$.

Assume that the m-th approximation solution of (2.2) is denoted by $\left(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x)\right)^T$ with $m = 2, 3, \ldots$. Let

$$\mathbf{h}^{(m)}(t, \lambda x) := \mathbf{v}^{(m)}(t, \lambda x) - \mathbf{v}^{(m-1)}(t, \lambda x),$$
$$\mathbf{w}^{(m)}(t, \lambda x) := \mathbf{H}^{(m)}(t, \lambda x) - \mathbf{H}^{(m-1)}(t, \lambda x).$$

It follows that

$$\mathbf{v}^{(m)}(t, \lambda x) = \mathbf{v}^{(0)}(t, x) + \sum_{i=1}^{m}\mathbf{h}^{(i)}(t, \lambda x),$$
$$\mathbf{H}^{(m)}(t, \lambda x) = \mathbf{H}^{(0)}(t, x) + \sum_{i=1}^{m}\mathbf{w}^{(i)}(t, \lambda x).$$

We linearize the nonlinear system (2.2) around $\left(\mathbf{v}^{(m-1)}(t, \lambda x), \mathbf{H}^{(m-1)}(t, \lambda x)\right)^T$ to get the following initial value problem

$$
\begin{aligned}
&\mathcal{J}_1[\mathbf{v}^{m-1}, \mathbf{H}^{m-1}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = \Pi_{N_m} E^{(m-1)}, \\
&\mathcal{J}_2[\mathbf{v}^{m-1}, \mathbf{H}^{m-1}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = \Pi_{N_m} \overline{E}^{(m-1)}, \\
&\nabla \cdot \mathbf{h}^{(m)} = 0, \quad \nabla \cdot \mathbf{w}^{(m)} = 0, \\
&\mathbf{h}^{(m)}(0, \lambda x) = \mathbf{h}_0^{(m)}(\lambda x), \quad \mathbf{w}^{(m)}(0, \lambda x) = \mathbf{w}_0^{(m)}(\lambda x),
\end{aligned} \tag{3.2}
$$

with the boundary conditions

$$
\mathbf{h}^{(m)}(t, \lambda x)|_{x\in\partial\Omega} = 0, \quad \mathbf{w}^{(m)}(t, \lambda x)|_{x\in\partial\Omega} = 0. \tag{3.3}
$$

The error terms are given by

$$
\begin{aligned}
E^{(m-1)} &:= \mathcal{L}_1[\mathbf{v}^{m-1}(t, \lambda x), \mathbf{H}^{m-1}(t, \lambda x)] = \mathcal{R}_1(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x)), \\
\overline{E}^{(m-1)} &:= \mathcal{L}_2[\mathbf{v}^{m-1}(t, \lambda x), \mathbf{H}^{m-1}(t, \lambda x)] = \mathcal{R}_2(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x)),
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
\mathcal{R}_1(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x)) :=& \mathcal{L}_1(\mathbf{v}^{(m-1)} + \mathbf{h}^{(m)}, \mathbf{H}^{(m-1)} + \mathbf{w}^{(m)}) - \mathcal{L}_1(\mathbf{v}^{(m-1)}, \mathbf{H}^{(m-1)}) \\
&- \mathcal{L}_1[\mathbf{v}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) \\
=& \lambda\Big(- (\mathbf{h}^{(m)} \cdot \nabla)\mathbf{h}^{(m)} + (\mathbf{w}^{(m)} \cdot \nabla)\mathbf{w}^{(m)} + \nabla P^{(m)}\Big), \\
\mathcal{R}_2(\mathbf{h}^{(m)}(t, \lambda x), \mathbf{w}^{(m)}(t, \lambda x)) :=& \mathcal{L}_2(\mathbf{v}^{(m-1)} + \mathbf{h}^{(m)}, \mathbf{H}^{(m-1)} + \mathbf{w}^{(m)}) - \mathcal{L}_2(\mathbf{v}^{(m-1)}, \mathbf{H}^{(m-1)}) \\
&- \mathcal{L}_2[\mathbf{v}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) \\
=& \lambda\Big(- (\mathbf{h}^{(m)} \cdot \nabla)\mathbf{w}^{(m)} + (\mathbf{w}^{(m)} \cdot \nabla)\mathbf{h}^{(m)}\Big),
\end{aligned} \tag{3.5}
$$

where the approximation pressure takes the form

$$
P^{(m)}(t, x) = -\Delta^{-1}\mathrm{div}\Big(\mathbf{v}^{(m)} \cdot \nabla\mathbf{v}^{(m)} - (\mathbf{H}^{(m)} \cdot \nabla)\mathbf{H}^{(m)}\Big),
$$

which are also the nonlinear term in approximation problem (2.2) at $\Big(\mathbf{v}^{(m-1)}(t, \lambda x), \mathbf{H}^{(m-1)}(t, \lambda x)\Big)^T$.

The following result establishes how to construct the m-th approximation solution.

Proposition 3.1. *Let the parameter $\lambda > 1$ and $s \geq 1$. Assume that the initial approximation function $\Big(\boldsymbol{v}^{(0)}, \boldsymbol{H}^{(0)}\Big)$ satisfies conditions (2.4)-(2.8), and $\Big(\boldsymbol{v}^{(m-1)}(t, \lambda x), \boldsymbol{H}^{(m-1)}(t, \lambda x)\Big)^T \in \mathcal{B}_\varepsilon$, and*

$$
\sum_{i=1}^{m-1} \Big(\|\boldsymbol{h}^{(i)}\|_{H^s}^2 + \|\boldsymbol{w}^{(i)}\|_{H^s}^2\Big) \lesssim \varepsilon^2.
$$

Then the linearized problem (3.2) with the boundary condition (3.3) admits a global solution

$$\Big(\boldsymbol{h}^{(m)}(t,\lambda x), \boldsymbol{w}^{(m)}(t,\lambda x)\Big) \in \mathbb{C}((0,+\infty); H^s(\Omega)\times H^s(\Omega)),$$

which satisfies

$$\|\boldsymbol{h}^{(m)}\|_{H^s}^2 + \|\boldsymbol{w}^{(m)}\|_{H^s}^2 \lesssim \|\boldsymbol{h}_0^{(m)}\|_{H^s}^2 + \|\boldsymbol{w}_0^{(m)}\|_{H^s}^2 + \|\Pi_{N_m}E^{(m-1)}\|_{H^s}^2 + \|\Pi_{N_m}\overline{E}^{(m-1)}\|_{H^s}^2, \quad \forall t>0, \tag{3.6}$$

and

$$\partial_{x_i}^l \boldsymbol{h}^{(m)}(t,x)|_{x\in\partial\Omega} = 0, \quad \partial_{x_i}^l \boldsymbol{w}^{(m)}(t,x)|_{x\in\partial\Omega} = 0, \quad 0\le l\le s,$$

where the error term satisfies

$$\|E^{(m)}\|_{H^s} + \|\overline{E}^{(m)}\|_{H^s} \lesssim \lambda N_m^2 \Big(\|\boldsymbol{h}^{(m)}\|_{H^s}^2 + \|\boldsymbol{w}^{(m)}\|_{H^s}^2\Big). \tag{3.7}$$

Proof. By the assumption $\sum_{i=1}^{m-1}\Big(\|\mathbf{h}^{(i)}\|_{H^s}^2 + \|\mathbf{w}^{(i)}\|_{H^s}^2\Big) \lesssim \varepsilon^2$ for any $s\ge 1$, we find

$$\begin{aligned}\sum_{k=0}^{s}\partial_{x_i}^k \mathbf{v}_j^{(m-1)} &= \sum_{k=0}^{s}\partial_{x_i}^k \mathbf{v}_j^{(0)}(t,x) + \sum_{i=1}^{m-1}\sum_{k=0}^{s}\partial_{x_i}^k \mathbf{h}^{(i)}(t,\lambda x)\\ &:= \sum_{k=0}^{s}\partial_{x_i}^k \mathbf{v}_j^{(0)}(t,x) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{s}\partial_{x_i}^k \mathbf{H}_j^{(m-1)} &= \sum_{k=0}^{s}\partial_{x_i}^k \mathbf{H}_j^{(0)}(t,x) + \sum_{i=1}^{m-1}\sum_{k=0}^{s}\partial_{x_i}^k \mathbf{w}^{(i)}(t,\lambda x)\\ &:= \sum_{k=0}^{s}\partial_{x_i}^k \mathbf{H}_j^{(0)}(t,x) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Thus, noticing that $\Big(\mathbf{v}^{(0)}(t,x), \mathbf{H}^{(0)}(t,x)\Big)^T$ satisfies (2.4)-(2.8), then it holds

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k \mathbf{v}_j^{(m-1)}(t,x)\|_{L^\infty} \lesssim \varepsilon^2, \tag{3.8}$$

and

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k \mathbf{H}_j^{(m-1)}(t,x)\|_{L^\infty} \lesssim \varepsilon^2. \tag{3.9}$$

The $(m-1)$-th approximation solution is

$$\mathbf{v}^{(m-1)}(t,\varepsilon x)=\mathbf{v}^{(0)}(t,x)+\sum_{i=1}^{m-1}\mathbf{h}^{(i)}(t,\lambda x),$$

$$\mathbf{H}^{(m-1)}(t,\varepsilon x)=\mathbf{H}^{(0)}(t,x)+\sum_{i=1}^{m-1}\mathbf{w}^{(i)}(t,\lambda x).$$

Moreover, it holds

$$\begin{cases}\nabla\cdot\mathbf{v}^{(m-1)}(t,x)=0,\\ \mathbf{v}^{(m-1)}(0,x)=0,\\ \|\mathbf{v}^{(m-1)}\|_{H^s}\lesssim\varepsilon^2,\\ \partial_{x_i}^l\mathbf{v}^{(m-1)}(t,x)|_{x\in\partial\Omega}=0,\quad 0\le l\le s,\end{cases}\tag{3.10}$$

and

$$\begin{cases}\nabla\cdot\mathbf{H}^{(m-1)}(t,x)=0,\\ \mathbf{H}^{(m-1)}(0,x)=0,\\ \|\mathbf{H}^{(m-1)}\|_{H^s}\lesssim\varepsilon^2,\\ \partial_{x_i}^l\mathbf{H}^{(m-1)}(t,x)|_{x\in\partial\Omega}=0,\quad 0\le l\le s.\end{cases}\tag{3.11}$$

Next, we find the m-th approximation solution $\left(\mathbf{v}^{(m)}(t,\lambda x),\mathbf{H}^{(m)}(t,\lambda x)\right)^T$, which is equivalent to find $\left(\mathbf{h}^{(m)}(t,\lambda x),\mathbf{w}^{(m)}(t,\lambda x)\right)^T$ such that

$$\begin{aligned}\mathbf{v}^{(m)}(t,\lambda x)&=\mathbf{v}^{(m-1)}(t,\lambda x)+\mathbf{h}^{(m)}(t,\lambda x),\\ \mathbf{H}^{(m)}(t,\lambda x)&=\mathbf{H}^{(m-1)}(t,\lambda x)+\mathbf{w}^{(m)}(t,\lambda x).\end{aligned}\tag{3.12}$$

Substituting (3.12) into (2.2), we obtain

$$\mathcal{L}_1(\mathbf{v}^{(m)},\mathbf{H}^{(m)})=\mathcal{L}_1(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})+\mathcal{L}_1[(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})+\mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}),$$
$$\mathcal{L}_2(\mathbf{v}^{(m)},\mathbf{H}^{(m)})=\mathcal{L}_2(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})+\mathcal{L}_2[(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})+\mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}).$$

Set

$$\mathcal{L}_1[(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=-\mathcal{L}_1(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})=-E^{(m-1)},$$
$$\mathcal{L}_2[(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=-\mathcal{L}_2(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)})=-\overline{E}^{(m-1)},$$

which we supplement it with the boundary conditions (3.3).

Since we assume $\left(\mathbf{v}^{(m-1)}(t,\lambda x),\mathbf{H}^{(m-1)}(t,\lambda x)\right)^T \in \mathcal{B}_\varepsilon$, then there is the same structure between the linear system (2.10) and the linear system of mth approximation solutions. Thus, by means of the same proof process as in Proposition 2.1, we can show that the above problem admits a solution $\left(\mathbf{h}^{(m)}(t,\lambda x),\mathbf{w}^{(m)}(t,\lambda x)\right)^T \in H^s(\Omega)\times H^s(\Omega)$. Here we should use (2.1). Furthermore, similar to (2.56), we can use (3.8)-(3.11) to derive

$$\|\mathbf{h}^{(m)}\|_{H^s}^2+\|\mathbf{w}^{(m)}\|_{H^s}^2 \lesssim \|\mathbf{h}_0^{(m)}\|_{H^s}^2+\|\mathbf{w}_0^{(m)}\|_{H^s}^2+\|E^{(m-1)}\|_{H^s}^2+\|\overline{E}^{(m-1)}\|_{H^s}^2, \quad \forall t>0,$$

where one can see the $(m-1)$-th error term $\left(E^{(m-1)},\overline{E}^{(m-1)}\right)^T$ such that

$$\begin{aligned} E^{(m-1)} &:= \mathcal{L}_1(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)}) = \mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}),\\ \overline{E}^{(m-1)} &:= \mathcal{L}_2(\mathbf{v}^{(m-1)},\mathbf{H}^{(m-1)}) = \mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}). \end{aligned}$$

Furthermore, by (3.5) and the Calderón-Zygmund theory, it holds

$$\begin{aligned} \|E^{(m)}\|_{H^s}+\|\overline{E}^{(m)}\|_{\mathbb{H}^s} &= \lambda\|\Pi_{N_m}\left(\mathbf{h}^{(m)}\cdot\nabla\mathbf{h}^{(m)}\right)\|_{H^s}+\lambda\|\Pi_{N_m}\left(\mathbf{w}^{(m)}\cdot\nabla\mathbf{w}^{(m)}\right)\|_{H^s}\\ &\quad+\lambda\|\Pi_{N_m}\left(\mathbf{h}^{(m)}\cdot\nabla\mathbf{w}^{(m)}\right)\|_{H^s}+\lambda\|\Pi_{N_m}\left(\mathbf{w}^{(m)}\cdot\nabla\mathbf{h}^{(m)}\right)\|_{H^s}\\ &\quad+\lambda\|\Pi_{N_m}\nabla P^{(m)}\|_{H^s}\\ &\lesssim \lambda N_m^2\left(\|\mathbf{h}^{(m)}\|_{H^s}^2+\|\mathbf{w}^{(m)}\|_{H^s}^2\right). \end{aligned}$$

The proof is now complete. □

4. Convergence of the approximation scheme

Our target is to prove that $\left(\mathbf{v}^{(\infty)}(t,\lambda x),\mathbf{H}^{\infty}(t,\lambda x)\right)^T$ is a global solution of the nonlinear equations (1.1). This is equivalent to show that the series $\sum_{i=1}^{m}\mathbf{h}^{(i)}(t,\lambda x)$ and $\sum_{i=1}^{m}\mathbf{w}^{(i)}(t,\lambda x)$ are convergent.

For a fixed integer $s\geq 1$, let $1\leq s=\bar{k}<k_0\leq k$ and

$$\begin{aligned} k_m &:= \bar{k}+\frac{k-\bar{k}}{2^m}, \quad k_{+\infty}=\bar{k},\\ \alpha_{m+1} &:= k_m-k_{m+1}=\frac{k-\bar{k}}{2^{m+1}}. \end{aligned}$$

Therefore

$$k_0>k_1>\ldots>k_m>k_{m+1}>\ldots. \tag{4.1}$$

Proposition 4.1. *Let the parameter $\lambda > 1$ and $s \geq 1$. Assume that the initial approximation function $\left(\boldsymbol{v}^{(0)}, \boldsymbol{H}^{(0)}\right)$ satisfies the conditions (2.4)-(2.8). Then the MHD equations (1.1) with the small initial data (1.2) and the boundary condition (1.4) admit global Sobolev solutions*

$$\boldsymbol{v}^{(\infty)}(t,x) = \boldsymbol{v}^{(0)}(t,x) + \sum_{m=1}^{\infty} \boldsymbol{h}^{(m)}(t,\lambda x) + \boldsymbol{v}_0(x) \in H^s(\Omega),$$

$$\boldsymbol{H}^{(\infty)}(t,x) = \boldsymbol{H}^{(0)}(t,x) + \sum_{m=1}^{\infty} \boldsymbol{w}^{(m)}(t,\lambda x) + \boldsymbol{H}_0(x) \in H^s(\Omega).$$

Moreover, we have

$$\|\boldsymbol{v}^{(\infty)}\|_{H^s} + \|\boldsymbol{H}^{(\infty)}\|_{H^s} \lesssim \varepsilon.$$

Proof. The proof is based on an induction argument. For convenience, we first deal with the case of zero initial data, that is, $\mathbf{v}_0(x) = \mathbf{H}_0(x) = \left(0,0,0\right)^T$. Next, we discuss the case $\mathbf{v}_0(x) \not\equiv 0$ and $\mathbf{H}_0(x) \not\equiv 0$. Note that $N_m = N_0^m$ with $N_0 > 1$. For all $m = 1, 2, \dots$, we claim that there exists a small positive constant ε such that

$$\begin{aligned} &\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{w}^{(m)}\|_{H^{k_{m-1}}} < \varepsilon^{2^{m-1}}, \\ &\|E^{(m)}\|_{H^{k_{m-1}}} + \|\overline{E}^{(m)}\|_{H^{k_{m-1}}} < \varepsilon^{2^m}, \\ &\left(\mathbf{v}^{(m)}, \mathbf{H}^{(m)}\right)^T \in \mathcal{B}_\varepsilon. \end{aligned} \tag{4.2}$$

For the case of $m = 1$, we recall the assumptions (2.4)-(2.8) on the initial approximation function $\left(\mathbf{v}^{(0)}(t,x), \mathbf{H}^{(0)}(t,x)\right)^T$. By (2.56), let $0 < \varepsilon_0 < N_0^{-(8+k-\bar{k})}\varepsilon^2 < \frac{\varepsilon}{2} \ll 1$, hence

$$\|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \|E^{(0)}\|_{H^{k_0}} + \|\overline{E}^{(0)}\|_{H^{k_0}} < \varepsilon.$$

Moreover, by (3.7) and the above estimate, we deduce that

$$\begin{aligned} \|E^{(1)}\|_{H^{k_0}} + \|\overline{E}^{(1)}\|_{H^{k_0}} &\lesssim \|\mathcal{R}_1(\mathbf{h}^{1)}, \mathbf{w}^{1)})\|_{H^{k_0}} + \|\mathcal{R}_2(\mathbf{h}^{1)}, \mathbf{w}^{1)})\|_{H^{k_0}} \\ &\lesssim \lambda N_1^2\left(\|\mathbf{h}^{(1)}\|_{H^{k_0}}^2 + \|\mathbf{w}^{(1)}\|_{H^{k_0}}^2\right) \\ &< \varepsilon^2, \end{aligned}$$

and

$$\|\mathbf{v}^{(1)}\|_{H^{k_0}} + \|\mathbf{H}^{(1)}\|_{H^{k_0}} \lesssim \|\mathbf{v}^{(0)}\|_{H^{k_0}} + |\mathbf{H}^{(0)}\|_{H^{k_0}} + \|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \varepsilon,$$

which means that $\left(\mathbf{v}^{(1)}, \mathbf{H}^{(1)}\right)^T \in \mathcal{B}_\varepsilon$.

Assume that the case of $m-1$ holds, that is,

$$\begin{aligned}
&\|\mathbf{h}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{w}^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-2}}, \\
&\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-1}}, \\
&\left(\mathbf{v}^{(m-1)}, \mathbf{H}^{(m-1)}\right)^T \in \mathcal{B}_\varepsilon.
\end{aligned} \tag{4.3}$$

We now prove that the case of m holds. Using (2.1), (3.6) and the second inequality of (4.3), we derive

$$\begin{aligned}
\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{w}^{(m)}\|_{H^{k_{m-1}}} &\lesssim \|\Pi_{N_m} E^{(m-1)}\|_{H^{k_{m-1}}} + \|\Pi_{N_m} \overline{E}^{(m-1)}\|_{H^{k_{m-1}}} \\
&\lesssim N_m^{\alpha_m}\left(\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}}\right) \\
&< N_m^{\alpha_m}\varepsilon^{2^{m-1}} < \varepsilon^{2^{m-2}},
\end{aligned} \tag{4.4}$$

which combined with (2.1), (3.7) and (4.1) yields

$$\begin{aligned}
&\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} \\
&\lesssim \lambda N_m^2\left(\|\mathbf{h}^{(m)}\|_{H^{k_m}}^2 + \|\mathbf{w}^{(m)}\|_{H^{k_m}}^2\right) \\
&\lesssim \lambda N_m^{2+\alpha_{m+1}}\left(\|E^{(m-1)}\|_{H^{k_{m+1}}} + \|\overline{E}^{(m-1)}\|_{H^{k_{m+1}}}\right)^2 \\
&\lesssim (\lambda N_0)^{(2+\alpha_{m+1})m+2(2+\alpha_{m+2})(m-1)}\left(\|E^{(m-2)}\|_{H^{k_{m+2}}} + \|\overline{E}^{(m-2)}\|_{H^{k_{m+2}}}\right)^{2^2} \\
&\lesssim \dots, \\
&\lesssim \left[\lambda N_0^{8+k-\bar{k}}\left(\|E^{(0)}\|_{H^{k_{2m}}} + \|\overline{E}^{(0)}\|_{H^{k_{2m}}}\right)\right]^{2^m}.
\end{aligned} \tag{4.5}$$

We choose a sufficient small positive constant ε_0 such that

$$0 < \lambda N_0^{8+k-\bar{k}}\left(\|E^{(0)}\|_{H^{\bar{k}}} + \|\overline{E}^{(0)}\|_{H^{\bar{k}}}\right) < 2N_0^4\varepsilon_0 < \varepsilon^2.$$

Thus, by (4.5) we have

$$\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} < \varepsilon^{2^m},$$

and

$$0 \le \lim_{m\to+\infty}\left(\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}\right) \lesssim \left[N_0^{8+k-\bar{k}}\left(\|E^{(0)}\|_{H^{k_{+\infty}}} + \|\overline{E}^{(0)}\|_{H^{k_{+\infty}}}\right)\right]^{2^{+\infty}} \longrightarrow 0.$$

So the error term goes to 0 as $m \to \infty$, that is,

$$\lim_{m\to\infty}\left(\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}\right) = 0.$$

On the other hand, note that $N_m = N_0^m$, by (4.3)-(4.4). Therefore

$$\|\mathbf{v}^{(m)}\|_{H^{k_m}} + \|\mathbf{H}^{(m)}\|_{H^{k_m}} \lesssim \|\mathbf{v}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{H}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{h}^{(m)}\|_{H^{k_m}} + \|\mathbf{w}^{(m)}\|_{H^{k_m}}$$
$$\lesssim \varepsilon + N_m^3 \varepsilon^{2^{m-1}} \lesssim \varepsilon.$$

This means that $\left(\mathbf{v}^{(m)}, \mathbf{H}^{(m)}\right)^T \in \mathcal{B}_\varepsilon$. Hence we conclude that (4.2) holds.

Therefore, the MHD equations (1.1) with the zero initial data admit a global solution

$$\mathbf{v}^{(\infty)}(t,x) = \mathbf{v}^{(0)}(t,x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(t, \lambda x) = \mathbf{v}^{(0)}(t,x) + \mathcal{O}(\varepsilon),$$
$$\mathbf{H}^{(\infty)}(t,x) = \mathbf{H}^{(0)}(t,x) + \sum_{m=1}^{\infty} \mathbf{w}^{(m)}(t, \lambda x) = \mathbf{H}^{(0)}(t,x) + \mathcal{O}(\varepsilon),$$

from which, one can see that the solution depends on the initial approximation function

$$(\mathbf{v}^{(0)}(t,x), \mathbf{H}^{(0)}(t,x)).$$

Next, we discuss the case of small non-zero initial data. We introduce the auxiliary functions

$$\overline{\mathbf{v}}(t,x) = \mathbf{v}(t,x) - \mathbf{v}_0(x), \quad x \in \Omega,$$
$$\overline{\mathbf{H}}(t,x) = \mathbf{H}(t,x) - \mathbf{H}_0(x), \quad x \in \Omega,$$

where the initial data $(\mathbf{v}_0, \mathbf{H}_0)^T \in H^{s+2} \times H^{s+1}$. So the non-autonomous term containing $(\mathbf{v}_0, \mathbf{H}_0)^T$ belongs to $H^s \times H^s$. Then the initial data reduces into

$$\overline{\mathbf{v}}(0,x) = \left(0,0,0\right)^T, \quad \overline{\mathbf{H}}(0,x) = \left(0,0,0\right)^T,$$

and equations (1.1) are transformed into equations of $\left(\overline{\mathbf{v}}, \overline{\mathbf{H}}\right)^T$. Thus, we can follow above iteration scheme to construct global Sobolev solution $\left(\overline{\mathbf{v}}, \overline{\mathbf{H}}\right)^T$. Furthermore, the global Sobolev solution of equations (1.1) with a small non-zero initial data takes the form

$$\left(\overline{\mathbf{v}}(t,x) + \mathbf{v}_0(x), \overline{\mathbf{H}}(t,x) + \mathbf{H}_0(x)\right)^T.$$

Finally, we use (1.3) and apply the Calderón-Zygmund theory. Thus, for the Riesz operator $\mathcal{R}$, there exists $\|\mathcal{R}w\|_{\mathbb{L}^{s_0}} \leq \|w\|_{\mathbb{L}^{s_0}}$ with $1 < s_0 < \infty$ such that

$$\|P\|_{H^s} \lesssim \varepsilon.$$

This completes the proof. □

CRediT authorship contribution statement

Both authors contributed to the analysis developed in this paper. The first draft of the manuscript was written by W. Yan and all authors commented on several versions of the manuscript. Both authors read and approved the final manuscript.

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