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Construction of infinitely many solutions for two-component Bose-Einstein condensates with nonlocal critical interaction

Weiwei Ye [a,b], Fashun Gao [c,1], Vicentiu D. Rădulescu [d,e,f,2], Minbo Yang [g,*,3]

[a] Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, People's Republic of China
[b] Department of Mathematics, Fuyang Normal University, Fuyang, Anhui, 236037, People's Republic of China
[c] Department of Mathematics and Physics, Henan University of Urban Construction, Pingdingshan, Henan, 467044, People's Republic of China
[d] Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Krakow, Poland
[e] Department of Mathematics, University of Craiova, 200585 Craiova, Romania
[f] China-Romania Research Center in Applied Mathematics, Romania
[g] Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, People's Republic of China

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Abstract

In this paper, we deal with the coupled Hartree system with axisymmetric potentials,

$$\begin{cases} -\Delta u + P(|x'|, x'')u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^6, \\ -\Delta v + Q(|x'|, x'')v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^6, \end{cases}$$

* Corresponding author.
E-mail addresses: yeweiweime@163.com (W. Ye), fsgao@zjnu.edu.cn (F. Gao), radulescu@inf.ucv.ro (V.D. Rădulescu), mbyang@zjnu.edu.cn (M. Yang).

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where $(x', x'') \in \mathbb{R}^2 \times \mathbb{R}^4$, $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$, $P(|x'|, x'')$ and $Q(|x'|, x'')$ are bounded nonnegative functions in $\mathbb{R}^+ \times \mathbb{R}^4$. The system is critical in the sense of the Hardy-Littlewood-Sobolev inequality. When the functions $r^2 P(r, x'')$ and $r^2 Q(r, x'')$ have a common topologically nontrivial critical point, using a finite dimensional reduction argument and developing new local Pohožaev identities, we construct infinitely many solutions of synchronized type, whose energy can be made arbitrary large. The main difficulty is caused by the non-local terms, since little is known about the nondegeneracy of the positive solutions of the limit system and the error estimates of the nonlocal parts in applying the reduction arguments and establishing the local Pohožaev identities.

MSC: 35J20; 35J60; 35A15

Keywords: Critical Hartree system; Infinitely many solutions; Pohožaev identities; Finite dimensional reduction

Contents

1. Introduction and main results

The phenomenon of Bose-Einstein condensation (BEC) was observed in 1995. The two-component coupled Hartree system plays an important role in the study of the two-component Bose-Einstein condensation. We are interested in constructing infinitely many solutions for the two-component coupled Hartree system

$$\begin{cases} i\partial_t\varphi_1 = -\Delta\varphi_1 + W_1(x)\varphi_1 - \alpha_1\big(K(x) * |\varphi_1|^2\big)\varphi_1 - \beta\big(K(x) * |\varphi_2|^2\big)\varphi_1, \\ \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ i\partial_t\varphi_2 = -\Delta\varphi_2 + W_2(x)\varphi_2 - \alpha_2\big(K(x) * |\varphi_2|^2\big)\varphi_2 - \beta\big(K(x) * |\varphi_1|^2\big)\varphi_2, \\ \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\varphi_i : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, W_i are the external potentials, K is a nonnegative response function, α_i measures the strength of the self-interactions in each component and β is the coupling constant. It is obvious that $\varphi_1 = e^{-iE_1t}u(x)$ and $\varphi_2 = e^{-iE_2t}v(x)$ solve (1.1) if and only if $u(x)$ and $v(x)$ solve the system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(K(x) * |u|^2\big)u - \beta\big(K(x) * |v|^2\big)u, & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(K(x) * |v|^2\big)v - \beta\big(K(x) * |u|^2\big)v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $V_i(x) = W_i(x) - E_i,\ i = 1, 2$.

If the response function $K(x) = \delta(x)$, then the nonlinear response is local, system (1.1) is transformed into the coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

The system has been intensively studied in the past twenty years. Many authors have investigated existence, multiplicity and properties of the solutions of (1.3), for example [1,4,5,8,17,18,20,24–26] and references.

We now review some results that are closely related to infinitely many solutions. Clapp and Pistoia considered the functional constrained on a subset of the Nehari manifold consisting of functions invariant with respect to a subgroup of $O(N+1)$ in [6], they obtained infinitely many positive solutions which are not conformally equivalent. In [30], the author studied the existence and multiplicity of semi-positive solution for the systems. New variational techniques are developed to study the existence of this type solutions. The semi-positive solutions are given by making use of a symmetry $\sigma(u, v) = (u, -v)$. Pistoia, Soave and Tavares [29] constructed positive solutions for systems by the Lyapunov-Schmidt reduction argument, revealing concentration and blow-up features as well as a tower shape of the solutions.

We will use a reduction argument together with the novel local Pohožaev identities to construct bubbling solutions concentrating at some saddle points of some functions. The local Pohožaev identities will be used to find algebraic equations which determine the location of the bubbles. Here we need to recall some ideas applied in [3,9,19,22,33] for non-singularly perturbed elliptic problems. Wei and Yan [33] studied the prescribed scalar curvature problem on $\mathbb{S}^N$

$$\begin{cases} -\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2} u = \tilde{K} u^{\frac{N+2}{N-2}} \text{ on } \mathbb{S}^N, \\ u > 0\,, \end{cases} \tag{1.4}$$

where $\tilde{K} > 0$ is rotationally symmetric. If $\tilde{K}$ has a local maximum point between the poles, equation (1.5) has infinitely many solutions which are non-radial and positive. The authors reduced the prescribed scalar curvature problem into

$$\begin{cases} -\Delta u = K(x) u^{\frac{N+2}{N-2}},\ u > 0 \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N)\,, \end{cases} \tag{1.5}$$

by using the stero-graphic projection. They introduced new ways to use the number of the bubbles of the solutions as parameter in the construction of bubbles solutions whose energy can be made arbitrarily large. In [9], Deng, Lin and Yan obtained a local uniqueness result for bubbling solutions of the prescribed scalar curvature problem. Moreover, they proved that if $K(y)$ is periodic in y_1 with period 1 and the local maximum is 0, then a bubbling solution of blow-up set $\{(jL, 0, \dots, 0) : j = 0, 1, 2, \dots\}$ must be periodic in y_1 under the condition that the positive integer L is sufficiently large.

Guo, Peng, Yan [19] considered poly-harmonic equations with critical exponents. They proved the existence and local uniqueness of solutions with infinitely many bubbles under some conditions on the coefficient $K(y)$, and the conditions imposed are optimal. In [22], the authors considered the critical semi-linear elliptic equation

$$-\Delta u = K(x)u^{\frac{N+2}{N-2}},\ u > 0 \text{ in } \mathbb{R}^N,$$

where $n \geq 3$, $K > 0$ is periodic in $(x_1, \dots, x_k)$ with $1 \leq k < \frac{n-2}{2}$. In the paper, they showed the existence of multi-bump solutions where the centers of bumps can be placed in some lattices in $\mathbb{R}^k$, including infinite lattices. They also got that no such solution exists for $k \geq \frac{n-2}{2}$.

Chen, Wei and Yan [3] applied the reduction argument to consider the Schrödinger equation with critical exponent

$$-\Delta u + V(|x|)u = u^{\frac{N+2}{N-2}} \quad \text{in } \mathbb{R}^N.$$

If V is radially symmetric, $r^2V(r)$ has a local maximum point or a local minimum point $r_0 > 0$ with $V(r_0) > 0$, they proved the existence of infinitely many positive solutions. Do the same thing for the following equations [28]

$$-\Delta u + V(|y'|, y'')u = u^{\frac{N+2}{N-2}} \quad u \in H^1(\mathbb{R}^N),$$

where $(y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, $V(|y'|, y'') > 0$ is bounded. By using the Pohožaev identities, Wang and Yan overcomed the difficulty appearing in using the standard reduction method to find algebraic equations which determine the location of the bubbles.

For the system of equations, the authors discussed the following system in [27],

$$\begin{cases} -\Delta u + P(|x|)u = \mu u^3 + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v + Q(|x|)v = \nu v^3 + \beta u^2 v & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where $P(r)$ and $Q(r)$ are positive radial potential, $\mu > 0$, $\nu > 0$. They examined the effect of nonlinear coupling on the solution structure. The authors used a finite dimensional reduction argument and found the location of bubbles by using a maximization procedure. They constructed an unbounded sequence of non-radial positive vector solutions of segregated type and an unbounded sequence of non-radial positive vector solutions of synchronized type. In [16], Guo, Liu, and Peng considered the following critical elliptic systems

$$\begin{cases} -\Delta u_1 = K_1(y)u_2^p & \text{in } \mathbb{R}^N, \\ -\Delta u_2 = K_2(y)u_1^p & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $N \geq 5$, $p, q \in (0, \infty)$ with $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N+2}{N}$, $K_1(y) > 0$ and $K_2(y) > 0$ are radial potentials. The authors constructed an unbounded sequence of non-radial positive vector solutions, whose energy can be made arbitrarily large. Furthermore, by use of various Pohozaev identities the authors proved a type of non-degeneracy result. Other results for the existence of infinitely many solutions to elliptic problems can be found in [2,32,34,35] and the references therein.

If we consider (1.2) with a Riesz potential response function, i.e., $K(x) = |x|^{-\mu}$ in the purely attractive case $\alpha_i,\ \beta > 0$ where $\mu \in (0, N)$, then (1.2) becomes the following system

$$\begin{cases} -\Delta u + V_1(x)u = \alpha_1\big(|x|^{-\mu} * u^2\big)u + \beta\big(|x|^{-\mu} * v^2\big)u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \alpha_2\big(|x|^{-\mu} * v^2\big)v + \beta\big(|x|^{-\mu} * u^2\big)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.8}$$

Compared with (1.3), there are relatively few researches on (1.8). Yang, Wei and Ding [37], for the first time, consider a singular perturbation problem related to (1.8), and prove the existence of ground state solutions for large coupling constant β. In [31], Wang and Shi studied (1.8) with positive constant potentials. Under optimal parameters, they gave the existence and non-existence of normal ground state solutions. Moreover, various qualitative properties of ground state solutions were proved. Wang and Yang [36] proved the existence and nonexistence of $L^2(\mathbb{R}^N)$ normalized solutions of coupled Hartree equations. Under certain type trapping potentials, they gave a precise description on the concentration behavior of minimizer solutions of the system. In [14], they proved the existence bound and ground states solutions for (1.8). And the authors also found some new conditions to guarantee the existence of positive ground state solutions for general nonlinear coupled Hartree type system. However, little is know about the multiplicity of the solutions of (1.8).

In [13], Gao, Moroz, Yang and Zhao studied a class of critical Hartree equations with axisymmetric potentials,

$$-\Delta u + V(|x'|, x'')u = \Big(|x|^{-4} * |u|^2\Big)u \quad \text{in } \mathbb{R}^6,$$

where $(x', x'') \in \mathbb{R}^2 \times \mathbb{R}^4$, $V(|x'|, x'')$ is a bounded nonnegative function in $\mathbb{R}^+ \times \mathbb{R}^4$. By applying a finite dimensional reduction argument and developing novel local Pohožaev identities, they proved that if the function $r^2 V(r, x'')$ has a topologically nontrivial critical point then the problem admits infinitely many solutions with arbitrary large energies. Inspired by Reference [13], we are interested in the existence of infinitely many solutions for the critical Hartree system,

$$\begin{cases} -\Delta u + P(|x'|, x'')u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^6, \\ -\Delta v + Q(|x'|, x'')v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^6, \end{cases} \tag{1.9}$$

where $(x', x'') \in \mathbb{R}^2 \times \mathbb{R}^4$, the potentials $P, Q \geq 0$ are axisymmetric and bounded, belong to C^1 and $P, Q \not\equiv 0$. Firstly, we need to recall the well-known Hardy-Littlewood-Sobolev inequality (see [23, Theorem 4.3]).

Proposition 1.1. *Let $t, r > 1$ and $0 < \mu < N$ be such that $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$. Then there is a sharp constant $C(N, \mu, t)$ such that, for $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$,*

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} dxdy \right| \leq C(N, \mu, t)|f|_{L^t(\mathbb{R}^N)}|h|_{L^r(\mathbb{R}^N)}.$$

If $t = r = 2N/(2N-\mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-1+\frac{\mu}{N}}.$$

In this case there is equality is achieved if and only if $f \equiv Ch$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ *and* $a \in \mathbb{R}^N$.

According to Proposition 1.1, we call $2^*_\mu = \frac{2N-\mu}{N-2}$ the upper Hardy-Littlewood-Sobolev critical exponent and so 2 plays the role of critical exponent in equation (1.9).

In the following we will assume that the function $r^2 P(r, x'')$ and $r^2 Q(r, x'')$ have a stable topologically nontrivial critical point, which was introduced in [28]:

(P) The functions $r^2 P(r, x'')$ and $r^2 Q(r, x'')$ have a common critical point (r_0, x''_0) such that $r_0 > 0$, $P(r_0, x''_0) > 0$ and $Q(r_0, x''_0) > 0$.

(Q)

$$deg(\nabla(r^2(P(r, x'') + Q(r, x''))), (r_0, x''_0)) \neq 0.$$

We will prove the following main results of this paper.

Theorem 1.2. *Let* $(r, x') \in \mathbb{R}^+ \times \mathbb{R}^4$, *if* $r^2 P(r, x')$ *and* $r^2 Q(r, x')$ *have a stable topology nontrivial critical point as described by assuming that* (P) *and* (Q), *then problem* (1.9) *has infinitely many solutions and the energy of the solutions diverges to* $+\infty$.

Here we would like to point out the difficulties in the study of nonlocal Hartree system. One is the nonlocal convolution parts make the proof of the local Pohožaev identities and the estimate of error terms much more difficult. Another difficulty is that little is known for the uniqueness and nondegeneracy of the ground states of the following system

$$\begin{cases} -\Delta u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^N, \\ -\Delta v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $\beta > \max\{\alpha_1, \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} > 0$. For the single equation with critical exponent, Lei [21], Du and Yang [11], Guo et al. [15] classified independently the positive solutions of the critical Hartree equation

$$-\Delta u = \Big(|x|^{-\mu} * |u|^{2^*_\mu}\Big)|u|^{2^*_\mu - 2}u \ \text{ in } \mathbb{R}^N, \tag{1.11}$$

and proved that every positive solution of (1.11) must has the form

$$U_{z,\lambda}(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N, \mu)^{\frac{2-N}{2(N-\mu+2)}} [N(N-2)]^{\frac{N-2}{4}} \Big(\frac{\lambda}{1 + \lambda^2|x - z|^2}\Big)^{\frac{N-2}{2}}. \tag{1.12}$$

In reference [12], the authors employ moving sphere arguments in integral form to classify positive solutions and to prove the uniqueness of ground states solutions. They got the following conclusion.

Lemma 1.3. (Corollary 3.5, [12]) *Let* $\beta > \max\{\alpha_1, \alpha_2\}$. *If* $(u, v) \in H$ *is a nontrivial classical positive solution of* (1.10), *then we have*

$$(u, v) = (\sqrt{k_0}U_{z,\lambda}, \sqrt{l_0}U_{z,\lambda}),$$

for some $\lambda > 0$ and $z \in \mathbb{R}^N$, where $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}, l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$. Moreover, each nontrivial classical positive solution $(u, v) \in H$ of (1.10) is a ground state solution.

In this paper, for clarity, we let

$$(U^*_{z,\lambda}, V^*_{z,\lambda}) = (\sqrt{k_0}U_{z,\lambda}, \sqrt{l_0}U_{z,\lambda}),$$

where $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}, l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$.

We know that the following equation

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N, \tag{1.13}$$

has a family of solutions of the following form

$$U_{\xi,\lambda}(x) := [N(N-2)]^{\frac{N-2}{4}} \Big(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\Big)^{\frac{N-2}{2}}. \tag{1.14}$$

And it has an $(N+1)$-dimensional manifold of solutions given by

$$\mathcal{Z} = \Big\{z_{\lambda,\xi} = [N(N-2)]^{\frac{N-2}{4}} \Big(\frac{\lambda}{\lambda^2+|x-\xi|^2}\Big)^{\frac{N-2}{2}}, \xi \in \mathbb{R}^N, \lambda \in \mathbb{R}^+\Big\}.$$

If the linearized equation

$$-\Delta v = Z^{\frac{4}{N-2}} v \tag{1.15}$$

in $D^{1,2}(\mathbb{R}^N)$ only admits solutions of the form

$$\eta = aD_\lambda Z + \mathbf{b} \cdot \nabla Z,$$

where $a \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^N$ and $Z \in \mathcal{Z}$, we say that it is nondegenerate. For the critical case (1.11), the nondegeneracy property is mostly open. If μ is close to N, the limit equation of (1.11) is the critical Lane-Emden equation whose nondegeneracy property is well known. According to this result, the authors proved the nondegenerate property by approximation approach in [11]. Recently, Yang and Zhao got the nondegeneracy result in $N = 6, \mu = 4$ case. However, as far as we know, there are no relevant results for the Hartree system. In this paper, we will prove a nondegeneracy result for the critical Hartree system in dimension 6.

In order to prove Theorem 1.2, we define

$$\begin{aligned} H_s = \Big\{&u \in D^{1,2}(\mathbb{R}^6), u(x_1, -x_2, x'') = u(x_1, x_2, x''), \\ &u(r\cos\theta, r\sin\theta, x'') = u\Big(r\cos(\theta + \frac{2j\pi}{m}), r\sin(\theta + \frac{2j\pi}{m}), x''\Big)\Big\} \end{aligned}$$

and let

$$z_j = \Big(\bar{r}\cos\frac{2(j-1)\pi}{m}, \bar{r}\sin\frac{2(j-1)\pi}{m}, \bar{x}''\Big), \ j = 1, \cdots, m,$$

where $\overline{x}''$ is a vector in $\mathbb{R}^4$. By the weak symmetry of $P(x)$ and $Q(x)$, we have $P(z_j) = P(\overline{r}, \overline{x}'')$ and $Q(z_j) = Q(\overline{r}, \overline{x}'')$, $j = 1, \cdots, m$. We will use $(U^*_{z_j,\lambda}, V^*_{z_j,\lambda})$ as an approximate solution. Let $\delta > 0$ be a small constant, such that $r^2 P(r, x'') > 0$ and $r^2 Q(r, x'') > 0$ if $|(r, x'') - (r_0, x''_0)| \leq 10\delta$. Let $\xi(x) = \xi(|x'|, x'')$ be a smooth function satisfying $\xi = 1$ if $|(r, x'') - (r_0, x''_0)| \leq \delta$, $\xi = 0$ if $|(r, x'') - (r_0, x''_0)| \geq 2\delta$, and $0 \leq \xi \leq 1$. Denote

$$Z_{z_j,\lambda}(x) = \xi U^*_{z_j,\lambda}(x),\ Z_{\overline{r},\overline{x}'',\lambda}(x) = \sum_{j=1}^m Z_{z_j,\lambda}(x),\ Z^*_{\overline{r},\overline{x}'',\lambda}(x) = \sum_{j=1}^m U^*_{z_j,\lambda}(x),$$

$$Y_{z_j,\lambda}(x) = \xi V^*_{z_j,\lambda}(x),\ Y_{\overline{r},\overline{x}'',\lambda}(x) = \sum_{j=1}^m Y_{z_j,\lambda}(x),\ Y^*_{\overline{r},\overline{x}'',\lambda}(x) = \sum_{j=1}^m V^*_{z_j,\lambda}(x),$$

$$Z_{j,1} = \frac{\partial Z_{z_j,\lambda}}{\partial \lambda}, Z_{j,2} = \frac{\partial Z_{z_j,\lambda}}{\partial \overline{r}}, Z_{j,k} = \frac{\partial Z_{z_j,\lambda}}{\partial \overline{x}''_k},\ \text{for } k = 3, \cdots, N,\ \ j = 1, \cdots, m,$$

and

$$Y_{j,1} = \frac{\partial Y_{z_j,\lambda}}{\partial \lambda}, Y_{j,2} = \frac{\partial Y_{z_j,\lambda}}{\partial \overline{r}}, Y_{j,k} = \frac{\partial Y_{z_j,\lambda}}{\partial \overline{x}''_k},\ \text{for } k = 3, \cdots, N,\ \ j = 1, \cdots, m.$$

In this paper, we always assume that $m > 0$ is a large integer, $\lambda \in [L_0 m^2, L_1 m^2]$ for some constants $L_1 > L_0 > 0$ and

$$|(\overline{r}, \overline{x}'') - (r_0, x''_0)| \leq \vartheta < \delta,$$

where $\vartheta > 0$ is a small constant. We will prove the following result.

Theorem 1.4. *Under the assumptions of Theorem 1.2, there exists a positive integer $m_0 > 0$, such that for any integer $m \geq m_0$, (1.9) has a solution (u_m, v_m) of the form*

$$u_m = Z_{\overline{r}_m,\overline{x}''_m,\lambda_m} + \phi_{\overline{r}_m,\overline{x}''_m,\lambda_m} = \sum_{j=1}^m \xi U^*_{z_j,\lambda_m} + \phi_{\overline{r}_m,\overline{x}''_m,\lambda_m},$$

$$v_m = Y_{\overline{r}_m,\overline{x}''_m,\lambda_m} + \varphi_{\overline{r}_m,\overline{x}''_m,\lambda_m} = \sum_{j=1}^m \xi V^*_{z_j,\lambda_m} + \varphi_{\overline{r}_m,\overline{x}''_m,\lambda_m},$$

where $\phi_{\overline{r}_m,\overline{x}''_m,\lambda_m} \in H_s$, $\varphi_{\overline{r}_m,\overline{x}''_m,\lambda_m} \in H_s$ and $\lambda_m \in \left[L_0 m^2, L_1 m^2\right]$. Moreover, as $m \to \infty$, $(\overline{r}_m, \overline{x}''_m) \to (r_0, x''_0)$, $\lambda_m^{-2} \|\phi_{\overline{r}_m,\overline{x}''_m,\lambda_m}\|_{L^\infty} \to 0$ and $\lambda_m^{-2} \|\varphi_{\overline{r}_m,\overline{x}''_m,\lambda_m}\|_{L^\infty} \to 0$.

Remark 1.5. What we're talking about here is that $r^2 P(r, x'')$ and $r^2 Q(r, x'')$ have a common critical point, that is, we are constructing synchronized solutions. If $r^2 P(r, x'')$ and $r^2 Q(r, x'')$ have no common critical point, i.e., $r^2 P(r, x'')$ has a critical point (r_1, x''_2), while $r^2 Q(r, x'')$ has critical point (r_2, x''_2), then we can construct segregated peak solutions. In fact, the couple terms in (1.9) will play less role in this case.

The paper is organized as follows: In Section 2, we will prove a nondegeneracy result for the critical Hartree system (1.10), then introduce some preliminary results. In Section 3, we carry out the reduction procedure for the critical Hartree system (1.9). At last section, we will prove our main results by establishing local Pohožaev identities.

2. Preliminary results

To prove the main results in Theorem 1.2, the arguments depend a lot on the nondegeneracy property of the positive solutions of the following system

$$\begin{cases} -\Delta u = \alpha_1\big(|x|^{-4} * u^2\big)u + \beta\big(|x|^{-4} * v^2\big)u & \text{in } \mathbb{R}^6, \\ -\Delta v = \alpha_2\big(|x|^{-4} * v^2\big)v + \beta\big(|x|^{-4} * u^2\big)v & \text{in } \mathbb{R}^6. \end{cases}$$

The next part, we will prove the nondegeneracy of solution $(U^*_{z,\lambda}, V^*_{z,\lambda})$ for the critical Hartree system in dimension 6. Our proof is mainly inspired by reference [26]. It is a simple matter to get the linearized equation of system (1.10) around the solution $(U^*_{z,\lambda}, V^*_{z,\lambda})$.

$$\begin{cases} \hat{L}_1(\phi,\varphi) := & -\Delta\phi - (\alpha_1 k_0 + \beta l_0)\Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\phi - 2\alpha_1 k_0\Big(|x|^{-4} * (U_{z,\lambda}\phi)\Big)U_{z,\lambda} \\ & -2\beta\sqrt{k_0}\sqrt{l_0}\Big(|x|^{-4} * (U_{z,\lambda}\varphi)\Big)U_{z,\lambda} = 0 \quad \text{in} \quad \mathbb{R}^N, \\ \hat{L}_2(\phi,\varphi) := & -\Delta\varphi - (\alpha_2 l_0 + \beta k_0)\Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\varphi - 2\alpha_2 l_0\Big(|x|^{-4} * (U_{z,\lambda}\varphi)\Big)U_{z,\lambda} \\ & -2\beta\sqrt{k_0}\sqrt{l_0}\Big(|x|^{-4} * (U_{z,\lambda}\phi)\Big)U_{z,\lambda} = 0 \quad \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $k_0 = \frac{\beta-\alpha_2}{\beta^2-\alpha_1\alpha_2}$, $l_0 = \frac{\beta-\alpha_1}{\beta^2-\alpha_1\alpha_2}$. We see at once that $\alpha_1 k_0 + \beta l_0 = \alpha_2 l_0 + \beta k_0 = 1$, then

$$\begin{cases} -\Delta\phi = \Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\phi + 2\alpha_1 k_0\Big(|x|^{-4} * (U_{z,\lambda}\phi)\Big)U_{z,\lambda} \\ \quad +2\beta\sqrt{k_0}\sqrt{l_0}\Big(|x|^{-4} * (U_{z,\lambda}\varphi)\Big)U_{z,\lambda} \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta\varphi = \Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\varphi + 2\alpha_2 l_0\Big(|x|^{-4} * (U_{z,\lambda}\varphi)\Big)U_{z,\lambda} \\ \quad +2\beta\sqrt{k_0}\sqrt{l_0}\Big(|x|^{-4} * (U_{z,\lambda}\phi)\Big)U_{z,\lambda} \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{2.1}$$

Lemma 2.1. *For $N = 6$, the positive solution $(U^*_{z,\lambda}, V^*_{z,\lambda})$ is non-degenerate for the system (1.10) in $D^{1,2}(\mathbb{R}^6) \times D^{1,2}(\mathbb{R}^6)$ in the sense that the kernel of the linearised operator $(\hat{L}_1, \hat{L}_2)$ at $(U^*_{z,\lambda}, V^*_{z,\lambda})$ is given by span$\Big\{\Big(\overline{C}\Phi_j, \Phi_j\Big) \mid j = 2, 3, \dots, 8.\Big\}$ a 7 dimensional space, where $\overline{C} = \frac{1-\alpha_2 l_0}{\beta\sqrt{k_0}\sqrt{l_0}} > 0$ and*

$$\Phi_1 = U_{z,\lambda},\ \Phi_{i+1} = \partial_{x_i} U_{z,\lambda},\ i = 1, 2, \dots, 6,\ \Phi_8 = 2U_{z,\lambda} + x \cdot \nabla U_{z,\lambda}.$$

Proof. When $N=6$, for convenience, we rewrite system (2.1) as follows

$$\begin{cases} -\Delta\phi+\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)\phi=2\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)\phi+2\Big(|x|^{-4}*(U_{z,\lambda}(a\phi+b\varphi))\Big)U_{z,\lambda} \\ \quad \text{in } \mathbb{R}^6, \\ -\Delta\varphi+\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)\varphi=2\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)\varphi+2\Big(|x|^{-4}*(U_{z,\lambda}(c\varphi+b\phi))\Big)U_{z,\lambda} \\ \quad \text{in } \mathbb{R}^6, \end{cases} \tag{2.2}$$

where $a=\alpha_1k_0$, $b=\beta\sqrt{k_0}\sqrt{l_0}$, $c=\alpha_2l_0$. Set $\Upsilon=\frac{a-1}{b}=\frac{\alpha_1-\beta}{\sqrt{(\beta-\alpha_1)(\beta-\alpha_2)}}$, an easy computation shows that

$$\begin{aligned} &-\Delta(\phi-\Upsilon\varphi)+\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)(\phi-\Upsilon\varphi) \\ &=2\Big[\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)(\phi-\Upsilon\varphi)+\Big(|x|^{-4}*(U_{z,\lambda}(\phi-\Upsilon\varphi))\Big)U_{z,\lambda}\Big]. \end{aligned} \tag{2.3}$$

Now we are going to recall the following eigenvalue problem

$$\begin{aligned} &-\Delta v+\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v=\mu\Big[\Big(|x|^{-4}*(U_{z,\lambda}v)\Big)U_{z,\lambda}+\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v\Big], \\ &v\in\mathcal{D}^{1,2}(\mathbb{R}^6). \end{aligned} \tag{2.4}$$

According to reference [10], the first eigenvalue of problem (2.4) can be defined as

$$\mu_1:=\inf_{v\in\mathcal{D}^{1,2}(\mathbb{R}^6)\setminus\{0\}}\frac{\int_{\mathbb{R}^6}|\nabla v|^2dx+\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v^2dx}{\int_{\mathbb{R}^6}\Big(|x|^{-4}*(U_{z,\lambda}v)\Big)U_{z,\lambda}vdx+\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v^2dx}.$$

Moreover, for any $k\in\mathbb{N}^+$ the eigenvalue can be characterized as follows

$$\mu_{k+1}:=\inf_{v\in\mathbb{P}_{k+1}\setminus\{0\}}\frac{\int_{\mathbb{R}^6}|\nabla v|^2dx+\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v^2dx}{\int_{\mathbb{R}^6}\Big(|x|^{-4}*(U_{z,\lambda}v)\Big)U_{z,\lambda}vdx+\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z,\lambda}|^2\Big)v^2dx},$$

where

$$\mathbb{P}_{k+1}:=\Big\{v\in\mathcal{D}^{1,2}(\mathbb{R}^6):\int_{\mathbb{R}^6}\nabla v\cdot\nabla e_jdx=0,\quad for\ all\quad j=1,...,k.\Big\},$$

and e_j is the corresponding eigenfunction to λ_j. By Proposition 2.2 of [10], we know that $\mu_1=1$ is simple and the corresponding eigenfunction is $\zeta U_{z,\lambda}$ with $\zeta\in\mathbb{R}$, and there exists $k\in\mathbb{N}$ such that $\mu_{k+2}=\mu_{k+3}=\cdots=\mu_{k+8}=2$ with the corresponding 7-dimensional eigenfunction space spanned by

$$\Big\{2U_{z,\lambda}+x\cdot\nabla U_{z,\lambda},\partial_{x_1}U_{z,\lambda},\ldots,\partial_{x_6}U_{z,\lambda}\Big\}.$$

Furthermore, $\mu_{k+9}>\mu_{k+2}=2$.

By the above statement, we see that $\phi - \Upsilon\varphi = \Sigma_{j=2}^{8} A_j \Phi_j$. So the second equation of system (2.2) can be rewritten as

$$
\begin{aligned}
&-\Delta\varphi + \Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\varphi \\
&= 2\Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\varphi + 2\Big(|x|^{-4} * (U_{z,\lambda}(b(\phi - \Upsilon\varphi) + (b\Upsilon + c)\varphi))\Big)U_{z,\lambda} \\
&= 2\Big(|x|^{-4} * |U_{z,\lambda}|^2\Big)\varphi + 2\Big(|x|^{-4} * (U_{z,\lambda}(b\Sigma_{j=2}^{8} A_j\Phi_j + (b\Upsilon + c)\varphi))\Big)U_{z,\lambda}.
\end{aligned}
$$

Here $b\Upsilon + c = \frac{\beta(\alpha_1+\alpha_2)-\alpha_1\alpha_2-\beta^2}{\beta^2-\alpha_1\alpha_2}$ and $0 < b\Upsilon + c < 1$ because of $\beta > \max\{\alpha_1, \alpha_2\}$. Set $\varphi = \Sigma_{j=2}^{\infty} M_j \Phi_j$, we have $M_j = 0$ for $j \neq 2, 3, \dots, 8$ by orthogonality. From the second equation of system (2.2) we can directly get $M_j = \frac{b}{1-b\Upsilon-c} A_j$ for $j = 2, 3, \dots, 8$. Therefore $\phi = \Sigma_{j=2}^{8}(\Upsilon \frac{b}{1-b\Upsilon-c} A_j + A_j)\Phi_j$. We denote $\overline{C} = \Upsilon + \frac{1-b\Upsilon-c}{b} = \frac{1-c}{b} = \frac{1-\alpha_2 l_0}{\beta\sqrt{k_0}\sqrt{l_0}} > 0$, thus the kernel at $(U^*_{z,\lambda}, V^*_{z,\lambda})$ is given by span$\Big\{\Big(\overline{C}\Phi_j, \Phi_j\Big) \mid j = 2, 3, \dots, 8.\Big\}$ a 7 dimensional space. □

The rest of the section we will give the results of some estimates involving the convolution term.

Lemma 2.2. (Lemma B.1, [33]) *For each fixed k and j, $k \neq j$, let*

$$
g_{k,j}(x) = \frac{1}{(1 + |x - z_j|)^{\alpha}} \frac{1}{(1 + |x - z_k|)^{\beta}},
$$

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants. Then, for any constants $0 < \delta \leq \min\{\alpha, \beta\}$, there is a constant $C > 0$, such that

$$
g_{k,j}(x) \leq \frac{C}{|z_k - z_j|^{\delta}} \Big(\frac{1}{(1 + |x - z_j|)^{\alpha+\beta-\delta}} + \frac{1}{(1 + |x - z_k|)^{\alpha+\beta-\delta}} \Big).
$$

Lemma 2.3. (Lemma B.2, [33]) *For any constant $0 < \delta < N - 2$, $N \geq 5$, there is a constant $C > 0$, such that*

$$
\int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} \frac{1}{(1 + |y|)^{2+\delta}} dy \leq \frac{C}{(1 + |x|)^{\delta}}.
$$

Lemma 2.4. (Lemma 2.3, [12]) *For $N = 6$ and $1 \leq i \leq m$, there is a constant $C > 0$, such that*

$$
|x|^{-4} * \frac{\lambda^2}{(1 + \lambda|x - z_i|)^{6+\eta}} \leq \frac{C}{(1 + \lambda|x - z_i|)^4},
$$

where $\eta > 0$.

Lemma 2.5. (Lemma 2.4, [12]) *For $N = 6$ and $1 \leq i \leq m$, there is a constant $C > 0$, such that*

$$
|x|^{-4} * |U_{z_i,\lambda}(x)|^2 = C U_{z_i,\lambda}(x).
$$

Lemma 2.6. (Lemma 2.5, [12]) *For every $i \neq 1$, $\alpha > 3$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^6} \frac{1}{(1+|y-\lambda z_1|^2)^\alpha} \frac{1}{(1+|y-\lambda z_i|^2)^2} dy = \frac{C}{(\lambda |z_1 - z_i|)^4}. \tag{2.5}$$

3. Finite-dimensional reduction

In order to construct a reasonably good approximate solution, we perform the finite-dimensional reduction argument in a weighted space. Let

$$\|u\|_* = \sup_{x\in\mathbb{R}^6} \Big(\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}\Big)^{-1} \lambda^{-2} |u(x)|,$$

and

$$\|h\|_{**} = \sup_{x\in\mathbb{R}^6} \Big(\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{4+\tau}}\Big)^{-1} \lambda^{-4} |h(x)|,$$

where $\tau = \frac{1}{2}$. Set

$$\|(u,v)\|_* = \|u\|_* + \|v\|_* \ \text{ and } \ \|(h,g)\|_{**} = \|h\|_{**} + \|g\|_{**}.$$

Now let's think about the following system

$$\begin{cases} \mathcal{L}_m(\phi,\varphi) = (h,g) + \sum_{l=1}^6 c_l \sum_{j=1}^m \Big(\alpha_1\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\alpha_1\Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,l})\Big) Z_{z_j,\lambda} \\ \quad +\beta\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\beta\Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,l})\Big) Z_{z_j,\lambda}, \quad \alpha_2\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Y_{j,l} \\ \quad +2\alpha_2\Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,l})\Big) Y_{z_j,\lambda} + \beta\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Y_{j,l} + 2\beta\Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,l})\Big) Y_{z_j,\lambda}\Big), \\ (\phi,\varphi) \in H_s \times H_s, \\ \sum_{j=1}^m \Big\langle \Big(\alpha_1\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\alpha_1\Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,l})\Big) Z_{z_j,\lambda} + \beta\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,l} \\ \quad +2\beta\Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,l})\Big) Z_{z_j,\lambda}, \quad \alpha_2\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Y_{j,l} + 2\alpha_2\Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,l})\Big) Y_{z_j,\lambda} \\ \quad +\beta\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Y_{j,l} + 2\beta\Big(|x|^{-4} * (Z_{z_j,\lambda} \ Z_{j,l})\Big) Y_{z_j,\lambda}\Big), (\phi,\varphi)\Big\rangle = 0, \ l = 1,2,\ldots,6, \end{cases} \tag{3.1}$$

for some real numbers c_l, where

$$\begin{aligned} &\mathcal{L}_m(\phi,\varphi) \\ &= \Big(-\Delta\phi + P(r,x'')\phi - \alpha_1\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi - 2\alpha_1\Big(|x|^{-4} * (Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big) Z_{\overline{r},\overline{x}'',\lambda} \\ &- \beta\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi - 2\beta\Big(|x|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big) Z_{\overline{r},\overline{x}'',\lambda}, \\ &-\Delta\varphi + Q(r,x'')\varphi - \alpha_2\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi - 2\alpha_2\Big(|x|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big) Y_{\overline{r},\overline{x}'',\lambda} \\ &- \beta\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi - 2\beta\Big(|x|^{-4} * (Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big) Y_{\overline{r},\overline{x}'',\lambda}\Big) \end{aligned} \tag{3.2}$$

and $\langle (u_1,u_2),(v_1,v_2)\rangle = \int_{\mathbb{R}^N} (u_1 v_1 + u_2 v_2)$.

Lemma 3.1. *Suppose that* (ϕ_m, φ_m) *solves* (3.1) *for* $(h, g) = (h_m, g_m)$*. If* $\|(h_m, g_m)\|_{**} \to 0$ *as* $m \to +\infty$*, then* $\|(\phi_m, \varphi_m)\|_* \to 0$.

Proof. On the contrary, we suppose that there exist $m \to +\infty$, $\overline{r}_m \to r_0$, $\overline{y}''_m \to y''_0$, $\lambda_m \in [L_0 m^2, L_1 m^2]$ and (ϕ_m, φ_m) solving (3.1) for $(h, g) = (h_m, g_m)$, $\lambda = \lambda_m$, $\overline{r} = \overline{r}_m$, $\overline{y}'' = \overline{y}''_m$, with $\|(h_m, g_m)\|_{**} \to 0$ and $\|(\phi_m, \varphi_m)\|_* \ge c > 0$. For convenience, we assume that $\|(\phi_m, \varphi_m)\|_* = 1$.

According to (3.1), we get

$$
\begin{aligned}
&|\phi_m(x)| \\
&\le 2C\alpha_1 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{\overline{r},\overline{x}'',\lambda}|\phi_m|)\Big) Z_{\overline{r},\overline{x}'',\lambda}(y)dy \\
&+ C\alpha_1 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\phi_m(y)|dy \\
&+ 2C\beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}|\varphi_m|)\Big) Z_{\overline{r},\overline{x}'',\lambda}(y)dy \\
&+ C\beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\phi_m(y)|dy \\
&+ C\sum_{l=1}^{6}|c_l|\Bigg[\Big|\sum_{j=1}^{m} 2\alpha_1 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big) Z_{z_j,\lambda}(y)dy\Big| \\
&+ \Big|\sum_{j=1}^{m} \alpha_1 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,l}(y)dy\Big| \\
&+ \Big|\sum_{j=1}^{m} 2\beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big) Z_{z_j,\lambda}(y)dy\Big| \\
&+ \Big|\sum_{j=1}^{m} \beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,l}(y)dy\Big|\Bigg] \\
&+ C\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}|h_m(y)|dy,
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&|\varphi_m(x)| \\
&\le 2C\alpha_2 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}|\varphi_m|)\Big) Y_{\overline{r},\overline{x}'',\lambda}(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&+C\alpha_2 \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\varphi_m(y)|dy \\
&+2C\beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{\overline{r},\overline{x}'',\lambda}|\phi_m|)\Big)Y_{\overline{r},\overline{x}'',\lambda}(y)dy \\
&+C\beta \int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\varphi_m(y)|dy \\
&+C\sum_{l=1}^{6}|c_l|\Bigg[\Big|2\alpha_2\sum_{j=1}^{m}\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}(y)dy\Big| \\
&+\Big|\sum_{j=1}^{m}\alpha_2\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{z_j,\lambda}|^2\Big)Y_{j,l}(y)dy\Big| \\
&+\Big|\sum_{j=1}^{m}2\beta\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}(y)dy\Big| \\
&+\Big|\sum_{j=1}^{m}\beta\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{z_j,\lambda}|^2\Big)Y_{j,l}(y)dy\Big|\Bigg] \\
&+C\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}|g_m(y)|dy. \qquad (3.4)
\end{aligned}
$$

Now let's estimate the right terms of (3.3), and the same can be said for (3.4). Since

$$
Z_{\overline{r},\overline{y}'',\lambda} \le C\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^4}, \quad \text{and} \quad Y_{\overline{r},\overline{y}'',\lambda} \le C\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^4},
$$

then similar to Lemma 3.1 in reference [13], we have

$$
\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{\overline{r},\overline{x}'',\lambda}|\phi_m|)\Big)Z_{\overline{r},\overline{x}'',\lambda}(y)dy \le C\|\phi_m\|_*\lambda^2\sum_{j=1}^{m}\frac{1}{(1+\lambda|x-z_j|)^{2+\tau+\theta}}, \tag{3.5}
$$

$$
\int\limits_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}|\varphi_m|)\Big)Y_{\overline{r},\overline{x}'',\lambda}(y)dy \le C\|\varphi_m\|_*\lambda^2\sum_{j=1}^{m}\frac{1}{(1+\lambda|x-z_j|)^{2+\tau+\theta}},
$$

where $\theta = 2-2\tau$. The estimate for the second and the forth terms in the right side of (3.3) can be obtained in the same way

$$\int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\phi_m(y)|dy \le C\|\phi_m\|_*\lambda^2 \sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau+\theta}}, \tag{3.6}$$

$$\int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)|\varphi_m(y)|dy \le C\|\varphi_m\|_*\lambda^2 \sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau+\theta}}.$$

Since

$$\begin{cases} \Big|\int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{z_j,\lambda}Z_{j,1})\Big)Z_{z_j,\lambda}(y)dy\Big| \le C\lambda \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}, \ \text{if } l=1, \\ \Big|\int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}(y)dy\Big| \le C\lambda^3 \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}, \ \text{if } l\neq 1 \end{cases}$$

we can get

$$\left|\sum_{j=1}^m \int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}(y)dy\right| \le C\lambda^{2+n_l}\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}, \tag{3.7}$$

where $n_1=-1$, $n_l=1$, $l=2,\cdots,6$. Similarly,

$$\left|\sum_{j=1}^m \int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Z_{z_j,\lambda}|^2\Big)Z_{j,l}(y)dy\right| \le C\lambda^{2+n_l}\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}. \tag{3.8}$$

Of course

$$\Big|\sum_{j=1}^m \int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}(y)dy\Big| \le C\lambda^{2+n_l}\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}},$$

$$\Big|\sum_{j=1}^m \int_{\mathbb{R}^6} \frac{1}{|y-x|^4}\Big(|y|^{-4} * |Y_{z_j,\lambda}|^2\Big)Z_{j,l}(y)dy\Big| \le C\lambda^{2+n_l}\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}},$$

where $n_1=-1$, $n_l=1$, $l=2,\cdots,6$. By Lemma 2.3, we have

$$\int_{\mathbb{R}^6} \frac{1}{|y-x|^4}|h_m(y)|dy \le C\|h_m\|_{**}\lambda^2\sum_{j=1}^m \frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}. \tag{3.9}$$

We can claim that

$$c_l = \frac{1}{\lambda^{n_l}}\big(o(\|(\phi_m,\varphi_m)\|_*) + O(\|(h_m,g_m)\|_{**})\big), \ l=1,2,\cdots,6. \tag{3.10}$$

Next, we will prove the claim. Firstly, multiplying (3.1) by $(Z_{1,t},Y_{1,t})$ $(t=1,2,\cdots,6)$ and integrating, then

$$\sum_{l=1}^{6}\sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}$$
$$+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}$$
$$+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),\ (Z_{1,t},Y_{1,t})\Big\rangle c_l$$
$$=\big\langle\mathcal{L}_m(\phi,\varphi),(Z_{1,t},Y_{1,t})\big\rangle-\big\langle(h_m,g_m),(Z_{1,t},Y_{1,t})\big\rangle. \tag{3.11}$$

As Lemma 2.2, we deduce that

$$|\langle(h_m,g_m),(Z_{1,t},Y_{1,t})\rangle|$$
$$\le C\lambda^{n_t}\|(h_m,g_m)\|_{**}\int_{\mathbb{R}^6}\frac{\lambda^2}{(1+\lambda|x-z_1|)^4}\sum_{j=1}^{m}\frac{\lambda^4}{(1+\lambda|x-z_j|)^{4+\tau}}dx$$
$$\le C\lambda^{n_t}\|(h_m,g_m)\|_{**}. \tag{3.12}$$

By Lemma 2.1 in reference [28], we have

$$\Big|\Big\langle\Big(P(r,x'')\phi_m,Q(r,x'')\varphi_m\Big),(Z_{1,t},Y_{1,t})\Big\rangle\Big|$$
$$\le C\lambda^{n_t}\|(\phi,\varphi)\|_*\int_{\mathbb{R}^6}\frac{\xi\lambda^2}{(1+\lambda|x-z_1|)^4}\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}dx\le O(\frac{\lambda^{n_t}\|(\phi_m,\varphi_m)\|_*}{\lambda^{1+\varepsilon}}), \tag{3.13}$$

where $\varepsilon>0$ is small. Because of

$$||x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi_m)|$$
$$\le C\|\phi_m\|_*\int_{\mathbb{R}^6}\frac{1}{|y|^4}\sum_{j=1}^{m}\frac{\xi(x-y)\lambda^2}{(1+\lambda|x-y-z_j|)^4}\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-y-z_j|)^{2+\tau}}dy$$
$$=O(\frac{m^2\|\phi_m\|_*}{\lambda}),\quad x\ne z_i,\ i=1,2,\cdots,m,$$

we know that

$$\int_{\mathbb{R}^6}\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi_m)\Big)Z_{\overline{r},\overline{x}'',\lambda}Z_{1,t}dx=O(\frac{\lambda^{n_t}\|\phi_m\|_*}{\lambda^{1+\varepsilon}}),$$

for some $\varepsilon>0$. In the same way, we also obtain

$$\Big\langle\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi_m,Z_{1,t}\Big\rangle=O\Big(\frac{\lambda^{n_t}\|\phi_m\|_*}{\lambda^{1+\varepsilon}}\Big),$$

$$\Big\langle\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi_m, Z_{1,t}\Big\rangle = O\Big(\frac{\lambda^{n_t}\|\phi_m\|_*}{\lambda^{1+\varepsilon}}\Big),$$

and

$$\Big\langle\Big(|x|^{-4} * (Y_{\overline{r},\overline{x}'',\lambda}\varphi_m)\Big)Z_{\overline{r},\overline{x}'',\lambda}, Z_{1,t}\Big\rangle = O\Big(\frac{\lambda^{n_t}\|\varphi_m\|_*}{\lambda^{1+\varepsilon}}\Big).$$

Similar argument can be applied to the remaining four terms, the same estimate we can obtained. Furthermore, by Lemma 3.1 in [13] we have

$$\Big\langle(-\Delta\phi_m, -\Delta\varphi_m), (Z_{1,t}, Y_{1,t})\Big\rangle = O\Big(\frac{\lambda^{n_t}\|\phi_m\|_*}{\lambda^2}\Big).$$

Consequently,

$$\begin{aligned}&\Big\langle\mathcal{L}_m(\phi,\varphi), (Z_{1,t}, Y_{1,t})\Big\rangle - \Big\langle(h_m, g_m), (Z_{1,t}, Y_{1,t})\Big\rangle\\ &= O\Big(\frac{\lambda^{n_t}\|(\phi_m,\varphi_m)\|_*}{\lambda^{1+\varepsilon}} + \lambda^{n_t}\|(h_m, g_m)\|_{**}\Big).\end{aligned} \tag{3.14}$$

Furthermore,

$$\sum_{j=1}^{m}\Big\langle\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big)Z_{j,l}, Z_{1,t}\Big\rangle = (\overline{c}_1 + o(1))\delta_{tl}\lambda^{n_l}\lambda^{n_t},$$

$$\sum_{j=1}^{m}\Big\langle\Big(|x|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}, Z_{1,t}\Big\rangle = (\overline{c}_2 + o(1))\lambda^{n_l}\lambda^{n_t},$$

$$\sum_{j=1}^{m}\Big\langle\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big)Z_{j,l}, Z_{1,t}\Big\rangle = (\overline{c}_3 + o(1))\delta_{tl}\lambda^{n_l}\lambda^{n_t},$$

and

$$\sum_{j=1}^{m}\Big\langle\Big(|x|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}, Z_{1,t}\Big\rangle = (\overline{c}_4 + o(1))\lambda^{n_l}\lambda^{n_t},$$

for some constant $\overline{c}_1 > 0$, $\overline{c}_2 > 0$, $\overline{c}_3 > 0$ and $\overline{c}_4 > 0$. So we can have the similar estimates for the other four items. By substituting these and (3.14) into (3.11), the claim can be obtained.

According to the above analysis, we can conclude

$$\|(\phi_m,\varphi_m)\|_* \le o(1) + \|(h_m, g_m)\|_{**} + \frac{\sum_{j=1}^{6}\frac{1}{(1+\lambda|x-z_j|)^{2+\tau+\theta}}}{\sum_{j=1}^{6}\frac{1}{(1+\lambda|x-z_j|)^{2+\tau}}}. \tag{3.15}$$

Based on the assumption that $\|(\phi_m,\varphi_m)\|_* = 1$ at the beginning and (3.15), we have that there is $R > 0$ such that

$$\|\lambda^{-2}(\phi_m, \varphi_m)\|_{L^\infty(B_{\frac{R}{\lambda}}(z_j))} \geq a > 0, \tag{3.16}$$

for some j. Let $\Big(\overline{\phi}_m(x) = \lambda^{-2}\phi_m(\lambda(x - z_j)), \overline{\varphi}_m(x) = \lambda^{-2}\varphi_m(\lambda(x - z_j))\Big)$, then

$$\int_{\mathbb{R}^N} (|\nabla\overline{\phi}_m|^2 + P(r, x'')|\overline{\phi}_m|^2 + |\nabla\overline{\varphi}_m|^2 + Q(r, x'')|\overline{\varphi}_m|^2)dx \leq C.$$

We deduce that there are $(u, v) \in D^{1,2}(\mathbb{R}^6) \times D^{1,2}(\mathbb{R}^6)$, such that

$$(\overline{\phi}_m, \overline{\varphi}_m) \to (u, v), \quad \text{weakly in } D^{1,2}(\mathbb{R}^6)$$

and

$$(\overline{\phi}_m, \overline{\varphi}_m) \to (u, v), \quad \text{strongly in } L^2_{loc}(\mathbb{R}^6),$$

as $m \to +\infty$. It is easy to see that $(u, v) \in D^{1,2}(\mathbb{R}^6) \times D^{1,2}(\mathbb{R}^6)$ satisfies

$$\begin{cases} -\Delta u & = \alpha_1\big(|x|^{-4} * |U^*_{0,\Lambda}|^2\big)u + 2\alpha_1\big(|x|^{-4} * (U^*_{0,\Lambda}u)\big)U^*_{0,\Lambda} \\ & \quad +\beta\big(|x|^{-4} * |V^*_{0,\Lambda}|^2\big)u + 2\beta\big(|x|^{-4} * (V^*_{0,\Lambda}v)\big)U^*_{0,\Lambda} \quad \text{in} \quad \mathbb{R}^6, \\ -\Delta v & = \alpha_2\Big[\big(|x|^{-4} * |V^*_{0,\Lambda}|^2\big)v + 2\alpha_2\big(|x|^{-4} * (V^*_{0,\Lambda}v)\big)V^*_{0,\Lambda} \\ & \quad +\beta\big(|x|^{-4} * |U^*_{0,\Lambda}|^2\big)v + 2\beta\big(|x|^{-4} * (U^*_{0,\Lambda}u)\big)V^*_{0,\Lambda} \quad \text{in} \quad \mathbb{R}^6, \end{cases} \tag{3.17}$$

for some $\Lambda \in [\Lambda_1, \Lambda_2]$. Because (u, v) is perpendicular to the kernel of (3.17), we can conclude that $(u, v) = (0, 0)$ by the non-degeneracy of $(U^*_{0,1}, V^*_{0,1})$. From (3.16) we derive the contradiction. □

Let

$$\begin{aligned} E = \Big\{(\phi, \varphi) \in H_s \times H_s, \ & \sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big)Z_{j,l} \\ & + 2\alpha_1\Big(|x|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda} + \beta\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big)Z_{j,l} \\ & + 2\beta\Big(|x|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}, \\ & \alpha_2\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big)Y_{j,l} + 2\alpha_2\Big(|x|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda} \\ & + \beta\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big)Y_{j,l} \\ & +2\beta\Big(|x|^{-4} * (Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big), (\phi, \varphi)\Big\rangle = 0, \ l = 1, 2, \ldots, 6.\Big\} \end{aligned}$$

endowed with the usual inner product $[(\phi, \varphi), (u, v)] = \int_{\mathbb{R}^6} \nabla u\nabla\phi + \nabla v\nabla\varphi dx$. Thus Problem (3.1) translates to finding a $(\phi, \varphi) \in E$ such that

$$
\begin{aligned}
[(\phi,\varphi),(u,v)] = &\int_{\mathbb{R}^6}\Big[-P(r,x'')\phi u - Q(r,x'')\varphi v\Big]dx + \alpha_1\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi u \\
&+2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Z_{\overline{r},\overline{x}'',\lambda}u\Big]dx + \beta\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi u \\
&+2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Z_{\overline{r},\overline{x}'',\lambda}u\Big]dx \\
&+\alpha_2\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi v + 2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Y_{\overline{r},\overline{x}'',\lambda}v\Big]dx \\
&+\beta\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi v + 2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Y_{\overline{r},\overline{x}'',\lambda}u\Big]dx \\
&+\int_{\mathbb{R}^6}(hu+gv)dx,\quad \forall (u,v)\in E.
\end{aligned}
$$

The same argument as in the proof of Proposition 4.1 in [7], using Riesz's representation theorem and Fredholm's alternative theorem, we can get the existence of unique solution for any (h,g) provided that the following equation

$$
\begin{cases}
\mathcal{L}_m(\phi,\varphi) = \sum_{l=1}^6 c_l \sum_{j=1}^m \Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l} + 2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda} \\
\quad +\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l} + 2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l} \\
\quad +2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda} + \beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l} + 2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big), \\
(\phi,\varphi)\in H_s\times H_s, \\
\sum_{j=1}^m \Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l} + 2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda} + \beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l} \\
\quad +2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l} + 2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda} \\
\quad +\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l} + 2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),(\phi,\varphi)\Big\rangle = 0,\ l=1,2,\ldots,6,
\end{cases}
\tag{3.18}
$$

for certain constants c_l, has only trivial solution in E. We can also have the following Lemma.

Lemma 3.2. *There exist $m_0>0$ and a constant $C>0$, independent of m, such that for all $m\ge m_0$ and all $h,g\in L^\infty(\mathbb{R}^6)$, problem (3.1) has a unique solution $(\phi,\varphi)\equiv L_m(h,g)$. Besides,*

$$
\|L_m(h,g)\|_*\le C\|(h,g)\|_{**},\qquad |c_l|\le\frac{C}{\lambda^{n_l}}\|(h,g)\|_{**}. \tag{3.19}
$$

Now, we study the system

$$\begin{cases} -\Delta(Z_{\overline{r},\overline{x}'',\lambda}+\phi)+P(r,x'')(Z_{\overline{r},\overline{x}'',\lambda}+\phi)-\alpha_1\Big(|x|^{-4}*|(Z_{\overline{r},\overline{x}'',\lambda}+\phi)|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi)\\ -\beta\Big(|x|^{-4}*|(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi)\\ \quad =\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\Big\{\alpha_1\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}\Big]\\ \qquad +\beta\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\Big\} \quad \text{in } \mathbb{R}^6,\\ -\Delta(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)+Q(r,x'')(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)-\alpha_2\Big(|x|^{-4}*|(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)\\ -\beta\Big(|x|^{-4}*|(Z_{\overline{r},\overline{x}'',\lambda}+\phi)|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)\\ \quad =\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\Big\{\alpha_2\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\Big]\\ \qquad +\beta\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\Big\} \quad \text{in } \mathbb{R}^6,\\ \phi,\varphi\in H_s,\quad \sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}\\ \qquad +2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\\ \qquad +\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),(\phi,\varphi)\Big\rangle=0,\ l=1,2,\dots,6. \end{cases} \tag{3.20}$$

Rewrite (3.20) as

$$\begin{cases} -\Delta\phi+P(r,x'')\phi-\alpha_1\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi-2\alpha_1\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Z_{\overline{r},\overline{x}'',\lambda}\\ -\beta\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi-2\beta\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Z_{\overline{r},\overline{x}'',\lambda}\\ \quad =N_1(\phi,\varphi)+l_{m1}+\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}\\ \qquad +\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big] \quad \text{in } \mathbb{R}^6,\\ -\Delta\varphi+Q(r,x'')\varphi-\alpha_2\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi-2\alpha_2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Y_{\overline{r},\overline{x}'',\lambda}\\ -\beta\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi-2\beta\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Y_{\overline{r},\overline{x}'',\lambda}\\ \quad =N_2(\phi,\varphi)+l_{m2}+\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\\ \qquad +\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big] \quad \text{in } \mathbb{R}^6,\\ \phi,\varphi\in H_s,\quad \sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}\\ \qquad +2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\\ \qquad +\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),(\phi,\varphi)\Big\rangle=0,\ l=1,2,\dots,6, \end{cases} \tag{3.21}$$

where

$$\begin{aligned} N_1(\phi,\varphi)=&2\alpha_1\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)\phi+\alpha_1\Big(|x|^{-4}*|\phi|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}+\alpha_1\Big(|x|^{-4}*|\phi|^2\Big)\phi\\ &+2\beta\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)\phi+\beta\Big(|x|^{-4}*|\varphi|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}+\beta\Big(|x|^{-4}*|\varphi|^2\Big)\phi, \end{aligned}$$

$$
\begin{aligned}
l_{m1} =&\alpha_1\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}-\alpha_1\sum_{j=1}^{m}\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\\
&-\beta\sum_{j=1}^{m}\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{z_j,\lambda}-P(|x'|,x'')Z_{\overline{r},\overline{x}'',\lambda}+Z^*_{\overline{r},\overline{x}'',\lambda}\Delta\xi+2\nabla\xi\nabla Z^*_{\overline{r},\overline{x}'',\lambda},\\
N_2(\phi,\varphi) =&2\alpha_2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)\varphi+\alpha_2\Big(|x|^{-4}*|\varphi|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}+\alpha_2\Big(|x|^{-4}*|\varphi|^2\Big)\varphi\\
&+2\beta\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)\varphi+\beta\Big(|x|^{-4}*|\phi|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}+\beta\Big(|x|^{-4}*|\phi|^2\Big)\varphi,
\end{aligned}
$$

and

$$
\begin{aligned}
l_{m2} =&\alpha_2\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}-\alpha_2\sum_{j=1}^{m}\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\\
&-\beta\sum_{j=1}^{m}\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{z_j,\lambda}-Q(|x'|,x'')Y_{\overline{r},\overline{x}'',\lambda}+Y^*_{\overline{r},\overline{x}'',\lambda}\Delta\xi+2\nabla\xi\nabla Y^*_{\overline{r},\overline{x}'',\lambda}.
\end{aligned}
$$

At first we estimate $N_i(\phi,\varphi)$ and l_{mi} respectively, where $i=1,2$.

Lemma 3.3. *There is a constant $C>0$, such that*

$$\|N_1(\phi,\varphi)\|_{**}\le C\|(\phi,\varphi)\|_*^2, \tag{3.22}$$

and

$$\|N_2(\phi,\varphi)\|_{**}\le C\|(\phi,\varphi)\|_*^2. \tag{3.23}$$

Proof. For any $1\le j\le m$, there is a constant $C>0$ such that

$$|x|^{-4}*\frac{\lambda^2}{(1+\lambda|x-z_i|)^{4+2\tau}}\le\frac{C}{(1+\lambda|x-z_i|)^2},$$

so we get that

$$\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)|\phi|\le C\|\phi\|_*^2\lambda^4\sum_{j=1}^{m}\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}}.$$

Similarly, we also have

$$\Big(|x|^{-4}*|\phi|^2\Big)|Z_{\overline{r},\overline{x}'',\lambda}| \le C\|\phi\|_*^2\lambda^4\sum_{j=1}^m\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}},$$

$$\Big(|x|^{-4}*|\phi|^2\Big)|\phi| \le C\|\phi\|_*^3\lambda^4\sum_{j=1}^m\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}},$$

$$\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)|\phi| \le C\|\varphi\|_*\|\phi\|_*\lambda^4\sum_{j=1}^m\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}},$$

$$\Big(|x|^{-4}*|\varphi|^2\Big)|Z_{\overline{r},\overline{x}'',\lambda}| \le C\|\varphi\|_*^2\lambda^4\sum_{j=1}^m\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}},$$

and

$$\Big(|x|^{-4}*|\varphi|^2\Big)|\phi| \le C\|\varphi\|_*^2\|\phi\|_*\lambda^4\sum_{j=1}^m\frac{1}{(1+\lambda|x-z_j|)^{4+\tau}}.$$

By combining the above six equations together, we conclude

$$\|N_1(\phi,\varphi)\|_{**} \le C(\|\phi\|_*+\|\varphi\|_*)^2 = C\|(\phi,\varphi)\|_*^2.$$

And

$$\|N_2(\phi,\varphi)\|_{**} \le C\|(\phi,\varphi)\|_*^2,$$

can be estimated in the same way. □

Lemma 3.4. *There is a constant $\varepsilon>0$, such that*

$$\|l_{m1}\|_{**} \le C(\frac{1}{\lambda})^{1+\varepsilon}, \tag{3.24}$$

and

$$\|l_{m2}\|_{**} \le C(\frac{1}{\lambda})^{1+\varepsilon}. \tag{3.25}$$

Proof. Let

$$\begin{aligned} l_{m1} =\, &\alpha_1\Big(\sum_{i=1}^m\Big(\Big(|x|^{-4}*|Z_{z_i,\lambda}|^2\Big)\sum_{j\neq i}Z_{z_j,\lambda}\Big)+2\sum_{j=1}^m\sum_{i\neq j}\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{z_i,\lambda}|\Big)Z_{\overline{r},\overline{x}'',\lambda}\Big)\\ &+\beta\Big(\sum_{i=1}^m\Big(\Big(|x|^{-4}*|Y_{z_i,\lambda}|^2\Big)\sum_{j\neq i}Z_{z_j,\lambda}\Big)+2\sum_{j=1}^m\sum_{i\neq j}\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{z_i,\lambda}|\Big)Z_{\overline{r},\overline{x}'',\lambda}\Big)\\ &-P(|x'|,x'')Z_{\overline{r},\overline{x}'',\lambda}+(Z^*_{\overline{r},\overline{x}'',\lambda}\Delta\xi+2\nabla\xi\nabla Z^*_{\overline{r},\overline{x}'',\lambda})\\ :=\,&J_1+J_2+J_3+J_4. \end{aligned}$$

The estimate of J_1 can be directly obtained from the Lemma 3.4 of [13],

$$|J_1| \leq C(\frac{1}{\lambda})^{1+\varepsilon}\lambda^4 \sum_{i=1}^{m} \frac{1}{(1+\lambda|x-z_i|)^{4+\tau}}.$$

We know that $|x-z_j| \geq |x-z_1|,\ \forall x \in \Omega_1$, where

$$\Omega_j = \left\{ x=(x',x'') \in \mathbb{R}^2 \times \mathbb{R}^4 : \left\langle \frac{x'}{|x'|}, \frac{z_j'}{|z_j'|} \right\rangle \geq \cos\frac{\pi}{m} \right\},\ j=1,\cdots,m.$$

We apply the same argument again, to estimate J_2, only note that

$$\begin{aligned} |x|^{-4} * |Y_{z_i,\lambda}|^2 &= |x|^{-4} * |\xi V^*_{z_i,\lambda}|^2 = |x|^{-4} * |\xi \sqrt{l_0} U_{z_i,\lambda}|^2 \\ &\leq C|x|^{-4} * |\frac{\lambda^2}{(1+\lambda^2|x-z_i|^2)^2}|^2 \leq C\frac{\lambda^2}{(1+\lambda|x-z_i|)^4}, \end{aligned}$$

and taking $0<\alpha\leq 4$ we have that

$$|Y_{z_j,\lambda}Y_{z_i,\lambda}| \leq \frac{\lambda^2}{(1+\lambda|x-z_i|)^4}\frac{\lambda^2}{(1+\lambda|x-z_j|)^4} \leq C\frac{\lambda^4}{(1+\lambda|x-z_j|)^{8-\alpha}}\frac{1}{|\lambda(z_j-z_i)|^{\alpha}},$$

for any $x \in \Omega_j$ and $i \neq j$. Thus

$$|J_2| \leq C(\frac{1}{\lambda})^{1+\varepsilon}\lambda^4 \sum_{j=1}^{m} \frac{1}{(1+\lambda|x-z_j|)^{4+\tau}}.$$

As Lemma 2.5 in [28] that

$$|J_3| \leq C(\frac{1}{\lambda})^{1+\varepsilon}\lambda^4 \sum_{j=1}^{m} \frac{1}{(1+\lambda|x-z_j|)^{4+\tau}},\quad |J_4| \leq C(\frac{1}{\lambda})^{1+\varepsilon}\lambda^4 \sum_{j=1}^{m} \frac{1}{(1+\lambda|x-z_j|)^{4+\tau}}.$$

So

$$\|l_{m1}\|_{**} \leq C(\frac{1}{\lambda})^{1+\varepsilon}.$$

Likewise, we also have

$$\|l_{m2}\|_{**} \leq C(\frac{1}{\lambda})^{1+\varepsilon}. \quad \square$$

At the end of this section, we are prepared to make the following estimates for the constants c_l and the solutions ϕ and φ. Let us recall that $\lambda \in [L_0 m^2, L_1 m^2]$, set

$$\mathcal{N}=\Big\{(\phi,\varphi):\phi,\varphi\in C(\mathbb{R}^6)\cap H_s,\ \|(\phi,\varphi)\|_*\le\frac{1}{\lambda},$$
$$\sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}$$
$$+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),(\phi,\varphi)\Big\rangle=0,\ l=1,2,\dots,6.\Big\}$$

and

$$\mathcal{A}(\phi,\varphi):=L_m(N_1(\phi,\varphi),N_2(\phi,\varphi))+L_m(l_{m1},l_{m2}),$$

where L_m is defined in Lemma 3.2. In that case, (3.21) can be written as

$$(\phi,\varphi)=\mathcal{A}(\phi,\varphi). \tag{3.26}$$

Lemma 3.5. *There is an integer $m_0>0$, such that for each $m\ge m_0$, $\lambda\in[L_0m^2,L_1m^2]$, $\overline{r}\in[r_0-\theta,r_0+\theta]$, $\overline{x}''\in B_\theta(x_0'')$, where $\theta>0$ is a fixed small constant,* (3.20) *has a unique solution $(\phi,\varphi)=(\phi_{\overline{r},\overline{x}'',\lambda},\varphi_{\overline{r},\overline{x}'',\lambda})\in H_s\times H_s$, satisfying*

$$\|(\phi,\varphi)\|_*\le C(\frac{1}{\lambda})^{1+\varepsilon},\quad |c_l|\le C(\frac{1}{\lambda})^{1+n_l+\varepsilon}, \tag{3.27}$$

where $\varepsilon>0$ is a small constant.

Proof. According to the above, in order to proof (3.27), we need to show $\mathcal{A}$ is a contraction map from $\mathcal{N}$ to $\mathcal{N}$. First of all, $\mathcal{A}$ maps $\mathcal{N}$ to $\mathcal{N}$ because

$$\begin{aligned}\|\mathcal{A}\|_*&\le C(\|N_1(\phi,\varphi)\|_{**}+\|N_2(\phi,\varphi)\|_{**}+\|l_{m1}\|_{**}+\|l_{m2}\|_{**})\\&\le C(\|(\phi,\varphi)\|_*^2+(\frac{1}{\lambda})^{1+\varepsilon})\le\frac{1}{\lambda}.\end{aligned}$$

Secondly,

$$\begin{aligned}&N_1(\phi_1,\varphi_1)-N_1(\phi_2,\varphi_2)\\=&\alpha_1\Big[2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi_1)\Big)(\phi_1-\phi_2)+\Big(|x|^{-4}*(\phi_1+\phi_2)(\phi_1-\phi_2)\Big)Z_{\overline{r},\overline{x}'',\lambda}\\&+\Big(|x|^{-4}*|\phi_1|^2\Big)(\phi_1-\phi_2)+2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}(\phi_1-\phi_2))\Big)\phi_2\\&+\Big(|x|^{-4}*(\phi_1+\phi_2)(\phi_1-\phi_2)\Big)\phi_2\Big]\\&+\beta\Big[2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi_1)\Big)(\phi_1-\phi_2)+\Big(|x|^{-4}*(\varphi_1+\varphi_2)(\varphi_1-\varphi_2)\Big)Z_{\overline{r},\overline{x}'',\lambda}\\&+\Big(|x|^{-4}*|\varphi_1|^2\Big)(\phi_1-\phi_2)+2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}(\varphi_1-\varphi_2))\Big)\phi_2\end{aligned}$$

$$+\Big(|x|^{-4}*(\varphi_1+\varphi_2)(\varphi_1-\varphi_2)\Big)\phi_2\Big],$$

where (ϕ_1,φ_1) and (ϕ_2,φ_2) in $\mathcal{N}$. Similar to the estimates in Lemma 3.3, we have

$$\begin{aligned}
&\|N_1(\phi_1,\varphi_1)-N_1(\phi_2,\varphi_2)\|_{**}\\
\le& C\|\phi_1\|_*\|\phi_1-\phi_2\|_*+C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*+C\|\phi_1\|_*^2\|\phi_1-\phi_2\|_*\\
&+C\|\phi_2\|_*\|\phi_1-\phi_2\|_*+C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*\|\phi_2\|_*\\
&+C\|\varphi_1\|_*\|\phi_1-\phi_2\|_*+C(\|\varphi_1\|_*+\|\varphi_2\|_*)\|\varphi_1-\varphi_2\|_*+C\|\varphi_1\|_*^2\|\phi_1-\phi_2\|_*\\
&+C\|\phi_2\|_*\|\varphi_1-\varphi_2\|_*+C(\|\varphi_1\|_*+\|\varphi_2\|_*)\|\varphi_1-\varphi_2\|_*\|\phi_2\|_*,\\
&\|N_2(\phi_1,\varphi_1)-N_2(\phi_2,\varphi_2)\|_{**}\\
\le& C\|\varphi_1\|_*\|\varphi_1-\varphi_2\|_*+C(\|\varphi_1\|_*+\|\varphi_2\|_*)\|\varphi_1-\varphi_2\|_*+C\|\varphi_1\|_*^2\|\varphi_1-\varphi_2\|_*\\
&+C\|\varphi_2\|_*\|\varphi_1-\varphi_2\|_*+C(\|\varphi_1\|_*+\|\varphi_2\|_*)\|\varphi_1-\varphi_2\|_*\|\varphi_2\|_*\\
&+C\|\phi_1\|_*\|\varphi_1-\varphi_2\|_*+C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*+C\|\phi_1\|_*^2\|\varphi_1-\varphi_2\|_*\\
&+C\|\varphi_2\|_*\|\phi_1-\phi_2\|_*+C(\|\phi_1\|_*+\|\phi_2\|_*)\|\phi_1-\phi_2\|_*\|\varphi_2\|_*.
\end{aligned}$$

According to the above, we deduce that

$$\begin{aligned}
&\|\mathcal{A}((\phi_1,\varphi_1))-\mathcal{A}((\phi_2,\varphi_2))\|_*\\
&=\|L_m(N_1(\phi_1,\varphi_1),N_2(\phi_1,\varphi_1))-L_m(N_1(\phi_2,\varphi_2),N_2(\phi_2,\varphi_2))\|_*\\
&\le C\|N_1(\phi_1,\varphi_1)-N_1(\phi_2,\varphi_2)\|_{**}+C\|N_2(\phi_1,\varphi_1)-N_2(\phi_2,\varphi_2)\|_{**}\\
&\le \frac{1}{2}\|\phi_1-\phi_2\|_*+\frac{1}{2}\|\varphi_1-\varphi_2\|_*=\frac{1}{2}\|(\phi_1,\varphi_1)-(\phi_2,\varphi_2)\|_*.
\end{aligned}$$

So $\mathcal{A}$ is a contraction mapping from $\mathcal{N}$ into itself. By the Contraction Mapping Theorem, we have that there exists a unique $(\phi,\varphi)\in\mathcal{N}$ such that (3.26) holds. Moreover, we can get

$$\|(\phi,\varphi)\|_*\le C(\frac{1}{\lambda})^{1+\varepsilon},$$

by Lemmas 3.2, 3.3 and 3.4. From (3.19) we can get

$$|c_l|\le C(\frac{1}{\lambda})^{1+n_l+\varepsilon}.\quad\square$$

4. Local Pohožaev identities methods

In this section, we are going to look for suitable $(\overline{r},\overline{x}'',\lambda)$, so that the function $(Z_{\overline{r},\overline{x}'',\lambda}+\phi_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda}+\varphi_{\overline{r},\overline{x}'',\lambda})$ is a solution of (1.9). Inspired by references [28], we will use Pohožaev identities to find the location of the bubbles. First of all we need to establish the corresponding local Pohožaev identities for the critical Hartree system. Using these identities, one of our main tasks is to estimate the error term away from the point of concentration. The appearance of nonlocal interaction parts makes the process much more complicated, we need to calculate carefully.

Lemma 4.1. *Suppose that* $(\overline{r}, \overline{x}'', \lambda)$ *satisfies*

$$\begin{aligned}&\int\limits_{D_\rho}\Big(-\Delta u_m + P(|x'|,x'')u_m - \alpha_1\Big(|x|^{-4} * |u_m|^2\Big)u_m\Big) - \beta\Big(|x|^{-4} * |v_m|^2\Big)u_m\Big)\langle x, \nabla u_m\rangle dx\\&+\int\limits_{D_\rho}\Big(-\Delta v_m + Q(|x'|,x'')v_m - \alpha_2\Big(|x|^{-4} * |v_m|^2\Big)v_m\\&-\beta\Big(|x|^{-4} * |u_m|^2\Big)v_m\Big)\langle x, \nabla v_m\rangle dx = 0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\int\limits_{D_\rho}\Big(-\Delta u_m + P(|x'|,x'')u_m - \alpha_1\Big(|x|^{-4} * |u_m|^2\Big)u_m - \beta\Big(|x|^{-4} * |v_m|^2\Big)u_m\Big)\frac{\partial u_m}{\partial x_i}dx\\&+\int\limits_{D_\rho}\Big(-\Delta v_m + Q(|x'|,x'')v_m - \alpha_2\Big(|x|^{-4} * |v_m|^2\Big)v_m\\&-\beta\Big(|x|^{-4} * |u_m|^2\Big)v_m\Big)\frac{\partial v_m}{\partial x_i}dx = 0, i = 3,\cdots,6,\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\int\limits_{\mathbb{R}^6}\Big(-\Delta u_m + P(|x'|,x'')u_m - \alpha_1\Big(|x|^{-4} * |u_m|^2\Big)u_m - \beta\Big(|x|^{-4} * |v_m|^2\Big)u_m\Big)\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}dx\\&+\int\limits_{\mathbb{R}^6}\Big(-\Delta v_m + Q(|x'|,x'')v_m - \alpha_2\Big(|x|^{-4} * |v_m|^2\Big)v_m\\&-\beta\Big(|x|^{-4} * |u_m|^2\Big)v_m\Big)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}dx = 0,\end{aligned} \tag{4.3}$$

where $u_m = Z_{\overline{r},\overline{x}'',\lambda} + \phi_{\overline{r},\overline{x}'',\lambda}$, $v_m = Y_{\overline{r},\overline{x}'',\lambda} + \varphi_{\overline{r},\overline{x}'',\lambda}$ *and* $D_\rho = \{(r, x'') : |(r, x'') - (r_0, x_0'')| \le \rho\}$ *with* $\rho \in (2\delta, 5\delta)$. *Then* $c_i = 0, i = 1, \cdots, 6$.

Proof. We know that $Z_{\overline{r},\overline{x}'',\lambda} = 0$ and $Y_{\overline{r},\overline{x}'',\lambda} = 0$ in $\mathbb{R}^6\backslash D_\rho$, if (4.1), (4.2) and (4.3) hold, we can deduce

$$\begin{aligned}\sum_{l=1}^{6} c_l \sum_{j=1}^{m}\int\limits_{\mathbb{R}^6}\Big\{\Big[&\alpha_1\Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big)Z_{j,l} + 2\alpha_1\Big(|x|^{-4} * |Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}\\&+\beta\Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big)Z_{j,l} + 2\beta\Big(|x|^{-4} * (Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]f\end{aligned}$$

$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]g\Big\}dx=0, \quad (4.4)$$

by (3.20) for $f=\langle x,\nabla u_m\rangle$, $g=\langle x,\nabla v_m\rangle$, $\frac{\partial u_m}{\partial x_i}$, $\frac{\partial v_m}{\partial x_i}$, $i=3,\cdots,6$, $\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}$ and $\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}$.

It is easy to prove that,

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,i}\frac{\partial Z_{z_l,\lambda}}{\partial x_i}dx\right|$$
$$\le C\lambda^2\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{1}{(1+|x-\lambda z_j|^2)^4}\frac{1}{|x-y|^4}\frac{(y_i-\lambda\overline{x}_i)^2}{(1+|y-\lambda z_j|^2)^3}\frac{1}{(1+|y-\lambda z_l|^2)^3}dxdy,$$

where $(\overline{x}_3,\overline{x}_4,\cdots,\overline{x}_6)=\overline{x}$ and $i=3,\cdots,6$. Furthermore, we can estimate that

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{z_l,\lambda}|^2\Big)Z_{l,i}\frac{\partial Z_{z_l,\lambda}}{\partial x_i}dx\right|=O(\lambda^2),$$

and if $l\neq j$

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,i}\left(\sum_{l=1,\neq j}^{m}\frac{\partial Z_{z_l,\lambda}}{\partial x_i}\right)dx\right|=O(\lambda^{2-\varepsilon}),$$

for some $\varepsilon>0$. And by definitions of $Y_{z_j,\lambda}$ and $Y_{j,l}$, we immediately derive

$$\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\Big\}dx$$
$$=m(a_1+o(1))\lambda^2,\ i=3,\cdots,6, \quad (4.5)$$

for some constants $a_1\neq 0$. Similar considerations apply to

$$\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Z_{\overline{r},\overline{x}'',\lambda}\rangle$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Y_{\overline{r},\overline{x}'',\lambda}\rangle\Big\}dx$$
$$=m(a_2+o(1))\lambda^2, \tag{4.6}$$

and

$$\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big\}dx$$
$$=\frac{m}{\lambda^2}(a_3+o(1)), \tag{4.7}$$

for some constants $a_2\neq 0$ and $a_3>0$.

Claim 1:

$$\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]f_1$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]g_1\Big\}dx$$
$$=o(m\lambda^2)\sum_{l=2}^{6}|c_l|+o(m|c_1|), \tag{4.8}$$

holds for $f_1=\langle x,\nabla Z_{\overline{r},\overline{x}'',\lambda}\rangle$, $\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}$, $g_1=\langle x,\nabla Y_{\overline{r},\overline{x}'',\lambda}\rangle$, $\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}$, $i=3,\cdots,6$.

According to Lemma 4.1 in [13], there are

$$\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,1}\frac{\partial\phi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}dx=O(\frac{1}{\lambda^{\varepsilon}}),$$

and

$$\int_{\mathbb{R}^6} \Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,1})\Big) Z_{z_j,\lambda} \frac{\partial \phi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} dx = O(\frac{1}{\lambda^{\varepsilon}}).$$

Likewise,

$$\int_{\mathbb{R}^6} \Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,1} \frac{\partial \varphi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} dx = O(\frac{1}{\lambda^{\varepsilon}}),$$

$$\int_{\mathbb{R}^6} \Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,1})\Big) Z_{z_j,\lambda} \frac{\partial \varphi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} dx = O(\frac{1}{\lambda^{\varepsilon}}),$$

and the other terms have the same estimate. Therefore

$$\begin{aligned} c_1 \int_{\mathbb{R}^6} \Big\{ \Big[& \alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,1} + 2\alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda} Z_{j,1}|\Big) Z_{z_j,\lambda} \\ & + \beta \Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,1} + 2\beta \Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,1})\Big) Z_{z_j,\lambda} \Big] \frac{\partial \phi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} \\ & + \Big[\alpha_2 \Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Y_{j,1} + 2\alpha_2 \Big(|x|^{-4} * |Y_{z_j,\lambda} Y_{j,1}|\Big) Y_{z_j,\lambda} \\ & + \beta \Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Y_{j,1} + 2\beta \Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,1})\Big) Y_{z_j,\lambda} \Big] \frac{\partial \varphi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} \Big\} dx = o(m|c_1|). \end{aligned}$$

The proof for

$$\begin{aligned} \sum_{l=2}^{6} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^6} \Big\{ \Big[& \alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda} Z_{j,l}|\Big) Z_{z_j,\lambda} \\ & + \beta \Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\beta \Big(|x|^{-4} * (Y_{z_j,\lambda} Y_{j,l})\Big) Z_{z_j,\lambda} \Big] \frac{\partial \phi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} \\ & + \Big[\alpha_2 \Big(|x|^{-4} * |Y_{z_j,\lambda}|^2\Big) Y_{j,l} + 2\alpha_2 \Big(|x|^{-4} * |Y_{z_j,\lambda} Y_{j,l}|\Big) Y_{z_j,\lambda} \\ & + \beta \Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Y_{j,l} + 2\beta \Big(|x|^{-4} * (Z_{z_j,\lambda} Z_{j,l})\Big) Y_{z_j,\lambda} \Big] \frac{\partial \varphi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i} \Big\} dx \\ & = o(m\lambda^2) \sum_{l=2}^{6} |c_l|, \end{aligned}$$

and

$$\sum_{l=1}^{6} c_l \sum_{j=1}^{m} \int_{\mathbb{R}^6} \Big\{ \Big[\alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda}|^2\Big) Z_{j,l} + 2\alpha_1 \Big(|x|^{-4} * |Z_{z_j,\lambda} Z_{j,l}|\Big) Z_{z_j,\lambda}$$

$$
\begin{aligned}
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\frac{\partial\phi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\frac{\partial\varphi_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\Big\}dx\\
&=o(m\lambda^2)\sum_{l=2}^{6}|c_l|+o(m|c_1|),
\end{aligned}
$$

where $i=3,\cdots,6$, are similar. A slight change in the proof, we can get

$$
\begin{aligned}
\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int_{\mathbb{R}^6}&\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\langle x,\nabla\phi_{\overline{r},\overline{x}'',\lambda}\rangle\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\langle x,\nabla\varphi_{\overline{r},\overline{x}'',\lambda}\rangle\Big\}dx\\
&=o(m\lambda^2)\sum_{l=2}^{6}|c_l|+o(m|c_1|).
\end{aligned}
$$

Consequently, the claim can be proved by (4.4).

Claim 2:

$$
c_i=o(\frac{1}{\lambda^2})c_1,\ i=2,\cdots,6.
$$

We next prove Claim 2. As

$$
\langle x,\nabla Z_{\overline{r},\overline{x}'',\lambda}\rangle=\langle x',\nabla_{x'}Z_{\overline{r},\overline{x}'',\lambda}\rangle+\langle x'',\nabla_{x''}Z_{\overline{r},\overline{x}'',\lambda}\rangle,
$$

and

$$
\langle x,\nabla Y_{\overline{r},\overline{x}'',\lambda}\rangle=\langle x',\nabla_{x'}Y_{\overline{r},\overline{x}'',\lambda}\rangle+\langle x'',\nabla_{x''}Y_{\overline{r},\overline{x}'',\lambda}\rangle,
$$

we have

$$
\begin{aligned}
\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int_{\mathbb{R}^6}&\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\langle x,\nabla Z_{\overline{r},\overline{x}'',\lambda}\rangle\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\langle x,\nabla Y_{\overline{r},\overline{x}'',\lambda}\rangle\Big\}dx\\
=&c_2\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,2}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,2}|\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,2}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,2})\Big)Z_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Z_{\overline{r},\overline{x}'',\lambda}\rangle\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,2}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,2}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,2}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,2})\Big)Y_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Y_{\overline{r},\overline{x}'',\lambda}\rangle\Big\}dx\\
&+o(m\lambda^2)\sum_{l=3}^{6}|c_l|+o(m|c_1|),
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
&\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\Big\}dx\\
=&c_i\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,i}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,i}|\Big)Z_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,i}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,i})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,i}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,i}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,i}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,i})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\Big\}dx\\
&+o(m\lambda^2)\sum_{l\neq 1,i}^{6}|c_l|+o(m|c_1|),\ i=3,\cdots,6.
\end{aligned}
\tag{4.10}
$$

Combining (4.8), (4.9) and (4.10), yields

$$
c_2\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,2}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,2}|\Big)Z_{z_j,\lambda}
$$

$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,2}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,2})\Big)Z_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Z_{\overline{r},\overline{x}'',\lambda}\rangle$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,2}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,2}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,2}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,2})\Big)Y_{z_j,\lambda}\Big]\langle x',\nabla_{x'}Y_{\overline{r},\overline{x}'',\lambda}\rangle\Big\}dx$$
$$=o(m\lambda^2)\sum_{l=3}^{6}|c_l|+o(m|c_1|),$$

and

$$c_i\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,i}+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,i}|\Big)Z_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,i}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,i})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,i}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,i}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,i}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,i})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial x_i}\Big\}dx$$
$$=o(m\lambda^2)\sum_{l\neq 1,i}^{6}|c_l|+o(m|c_1|),\ i=3,\cdots,6.$$

Hence that, combine these with (4.5) and (4.6), show that

$$c_i=o(\frac{1}{\lambda^2})c_1,\ i=2,\cdots,6. \tag{4.11}$$

These are precisely the claim.

According to the claims above, we have

$$0=\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}$$
$$+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,l}|\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}$$
$$+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,l}|\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}\Big\}dx$$
$$=c_1\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,1}$$

$$
\begin{aligned}
&+2\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}Z_{j,1}|\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,1}\\
&+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,1})\Big)Z_{z_j,\lambda}\Big]\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}dx\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,1}+2\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}Y_{j,1}|\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,1}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,1})\Big)Y_{z_j,\lambda}\Big]\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\lambda}\Big\}dx\\
&+o(\frac{m}{\lambda^2})c_1\\
=&m(a_3+o(1))c_1+o(\frac{m}{\lambda^2})c_1,
\end{aligned}
$$

where $a_3>0$, then $c_1=0$. By Claim 2, we have $c_i=0$, $i=2,\cdots,6$. □

Lemma 4.2. *We have*

$$
\frac{\partial J(Z_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda})}{\partial\overline{x}''_t}=\frac{\partial J(Z^*_{\overline{r},\overline{x}'',\lambda},Y^*_{\overline{r},\overline{x}'',\lambda})}{\partial\overline{x}''_t}+O(\frac{1}{\lambda^2}),
$$

where

$$
\begin{aligned}
J(u,v)=&\frac{1}{2}\int\limits_{\mathbb{R}^6}(|\nabla u|^2+|\nabla v|^2)dx+\frac{1}{2}\int\limits_{\mathbb{R}^6}(P(x)u^2+Q(x)v^2)dx\\
&-\frac{\alpha_1}{4}\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|u(x)|^2|u(y)|^2}{|x-y|^4}dxdy\\
&-\frac{\beta}{2}\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|u(x)|^2|v(y)|^2}{|x-y|^4}dxdy-\frac{\alpha_2}{4}\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|v(x)|^2|v(y)|^2}{|x-y|^4}dxdy.
\end{aligned}
$$

Proof. The proof will be divided into three steps.

Step 1: We claim that

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|Z^*_{\overline{r},\overline{x}'',\lambda}(x)|^2Z^*_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_j}(y)}{|x-y|^4}dxdy\\
&-\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|Z_{\overline{r},\overline{x}'',\lambda}(x)|^2Z_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_j}(y)}{|x-y|^4}dxdy=O(\frac{1}{\lambda^2}),
\end{aligned}
$$

$$
\int\limits_{\mathbb{R}^6}\int\limits_{\mathbb{R}^6}\frac{|Y^*_{\overline{r},\overline{x}'',\lambda}(x)|^2Z^*_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_j}(y)}{|x-y|^4}dxdy
$$

$$-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Y_{\overline{r},\overline{x}'',\lambda}(x)|^2 Z_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_j}(y)}{|x-y|^4}dxdy = O(\frac{1}{\lambda^2}),$$

$$\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Y^*_{\overline{r},\overline{x}'',\lambda}(x)|^2 Y^*_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_j}(y)}{|x-y|^4}dxdy$$
$$-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Y_{\overline{r},\overline{x}'',\lambda}(x)|^2 Y_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_j}(y)}{|x-y|^4}dxdy = O(\frac{1}{\lambda^2}),$$

and

$$\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z^*_{\overline{r},\overline{x}'',\lambda}(x)|^2 Y^*_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_j}(y)}{|x-y|^4}dxdy$$
$$-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z_{\overline{r},\overline{x}'',\lambda}(x)|^2 Y_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_j}(y)}{|x-y|^4}dxdy = O(\frac{1}{\lambda^2}),$$

where $j=3,\cdots,6$. We only need to prove one of them, and the other three can done in the same way. Here we establish the last one, it is clear that

$$\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|U^*_{z_j,\lambda}(x)|^2 V^*_{z_j,\lambda}(y)\frac{\partial V^*_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy - \int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z_{z_j,\lambda}(x)|^2 Y_{z_j,\lambda}(y)\frac{\partial Y_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy$$
$$=\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{k_0 l_0|U_{z_j,\lambda}(x)|^2(1-\xi^2(y))U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy$$
$$+\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{k_0 l_0(1-\xi^2(x))|U_{z_j,\lambda}(x)|^2|\xi(y)|^2 U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy,$$

where $j=1,2,\cdots,m$. We know that

$$\int_{\mathbb{R}^6}\left[\frac{(1-\xi^2(y+z_j))\lambda^6|y|}{(1+\lambda^2|y|^2)^5}\right]^{\frac{3}{2}}dy \le C\lambda^9\int_{\delta-\vartheta}^{\infty}\frac{r^{\frac{13}{2}}}{(1+\lambda^2 r^2)^{\frac{15}{2}}}dr = O(\frac{1}{\lambda^6}).$$

And by the Hardy-Littlewood-Sobolev inequality, we can estimate

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|U_{z_j,\lambda}(x)|^2(1-\xi^2(y))U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy\\&\leq C\left(\int_{\mathbb{R}^6}\left[\frac{\lambda^4}{(1+\lambda^2|x|^2)^4}\right]^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\left(\int_{\mathbb{R}^6}\left[\frac{(1-\xi^2(y+z_j))\lambda^6|y|}{(1+\lambda^2|y|^2)^5}\right]^{\frac{3}{2}}dy\right)^{\frac{2}{3}}\\&=O(\frac{1}{\lambda^4}).\end{aligned}$$

Similarly

$$\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{(1-\xi^2(x))|\xi(y)|^2)|U_{z_j,\lambda}(x)|^2U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy=O(\frac{1}{\lambda^4}).$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|U^*_{z_j,\lambda}(x)|^2V^*_{z_j,\lambda}(y)\frac{\partial V^*_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy\\&-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z_{z_j,\lambda}(x)|^2Y_{z_j,\lambda}(y)\frac{\partial Y_{z_j,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy=O(\frac{1}{\lambda^4}),\end{aligned}$$

where $j=1,2,\cdots,m$. We continue in this fashion obtaining

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{U_{z_j,\lambda}(x)U_{z_i,\lambda}(x)U_{z_l,\lambda}(y)\frac{\partial U_{z_k,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy\\&-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{Z_{z_j,\lambda}(x)Z_{z_i,\lambda}(x)Y_{z_l,\lambda}(y)\frac{\partial Y_{z_k,\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy=O(\frac{1}{\lambda^4}),\end{aligned}$$

where $j,i,l,k=1,2,\cdots,m$. Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z^*_{\overline{r},\overline{x}'',\lambda}(x)|^2Y^*_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy\\&-\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\frac{|Z_{\overline{r},\overline{x}'',\lambda}(x)|^2Y_{\overline{r},\overline{x}'',\lambda}(y)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4}dxdy=O(\frac{m^4}{\lambda^4})=O(\frac{1}{\lambda^2}).\end{aligned}$$

Step 2: Because $P \geq 0$ is bounded, we see at once that

$$\int_{\mathbb{R}^6} P(x)U^*_{z_j,\lambda}(y)\frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy - \int_{\mathbb{R}^6} P(x)Z_{z_j,\lambda}(y)\frac{\partial Z_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy$$
$$\leq C\int_{\mathbb{R}^6} (1-\xi^2(y))U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy,$$

where $j = 1, 2, \cdots, m$. By direct calculation, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^6} (1-\xi^2(y))U_{z_j,\lambda}(y)\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy\right| &= C\left|\int_{\mathbb{R}^6} \frac{(1-\xi^2(y))\lambda^2}{(1+\lambda^2|y-z_j|^2)^2}\frac{\lambda^4(y_t-\overline{x}''_t)}{(1+\lambda^2|y-z_j|^2)^3}dy\right| \\ &\leq C\int_{\mathbb{R}^6} \frac{(1-\xi^2(y+z_j))\lambda^2}{(1+\lambda^2|y|^2)^2}\frac{\lambda^4|y|}{(1+\lambda^2|y|^2)^3}dy \\ &\leq C\int_{\mathbb{R}^6\setminus B_{\delta-\vartheta}(0)} \frac{\lambda^2}{(1+\lambda^2|y|^2)^2}\frac{\lambda^4|y|}{(1+\lambda^2|y|^2)^3}dy \\ &= O(\frac{1}{\lambda^4}).\end{aligned} \tag{4.12}$$

Likewise, for other cases, we can obtain

$$\int_{\mathbb{R}^6} P(x)U^*_{z_i,\lambda}(y)\frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy - \int_{\mathbb{R}^6} P(x)Z_{z_i,\lambda}(y)\frac{\partial Z_{z_j,\lambda}}{\partial \overline{x}''_t}(y)dy = O(\frac{1}{\lambda^4}),$$

where $j, i = 1, 2, \cdots, m$. Thus, we have

$$\int_{\mathbb{R}^6} P(x)\Big(Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} - Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}\Big)dx = O(\frac{m^2}{\lambda^4}). \tag{4.13}$$

Analogously,

$$\int_{\mathbb{R}^6} Q(x)\Big(Y^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} - Y_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}\Big)dx = O(\frac{m^2}{\lambda^4}). \tag{4.14}$$

Step 3: We observe that

$$\int_{\mathbb{R}^6} \Delta Z^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx - \int_{\mathbb{R}^6} \Delta Z_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx$$
$$= \int_{\mathbb{R}^6} \Delta Z^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx - \int_{\mathbb{R}^6} \xi(\xi \Delta Z^*_{\overline{r},\overline{x}'',\lambda} + Z^*_{\overline{r},\overline{x}'',\lambda} \Delta\xi + 2\nabla\xi\nabla Z^*_{\overline{r},\overline{x}'',\lambda}) \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx.$$

Obviously,

$$\int_{\mathbb{R}^6} (1-\xi^2)\Delta Z^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx$$
$$= \sum_{j=1}^m \Big[\int_{\mathbb{R}^6}\int_{\mathbb{R}^6} \frac{\alpha_1 (1-\xi^2(x))|U^*_{z_j,\lambda}(x)|^2 U^*_{z_j,\lambda}(y) \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4} dx dy$$
$$+ \int_{\mathbb{R}^6}\int_{\mathbb{R}^6} \frac{\beta (1-\xi^2(x))|U^*_{z_j,\lambda}(x)|^2 V^*_{z_j,\lambda}(y) \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}(y)}{|x-y|^4} dx dy \Big] = O(\frac{m^2}{\lambda^3}),$$

can be obtained by step 1. Combining this with

$$\int_{\mathbb{R}^6} \xi Z^*_{\overline{r},\overline{x}'',\lambda} \Delta\xi \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx = O(\frac{m^2}{\lambda^3}),$$

and

$$\int_{\mathbb{R}^6} \xi \nabla\xi \nabla Z^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx \le C \sum_{j=1}^m \int_{\mathbb{R}^6} \frac{\xi |\nabla\xi| \lambda^4 |y-z_j|}{(1+\lambda^2|y-z_j|^2)^3} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dy = O(\frac{m^2}{\lambda^3}),$$

we conclude

$$\int_{\mathbb{R}^6} \Delta Z^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx - \int_{\mathbb{R}^6} \Delta Z_{\overline{r},\overline{x}'',\lambda} \frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx = O(\frac{m^2}{\lambda^3}) = O(\frac{1}{\lambda^2}).$$

Similarly,

$$\int_{\mathbb{R}^6} \Delta Y^*_{\overline{r},\overline{x}'',\lambda} \frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx - \int_{\mathbb{R}^6} \Delta Y_{\overline{r},\overline{x}'',\lambda} \frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t} dx = O(\frac{m^2}{\lambda^3}) = O(\frac{1}{\lambda^2}).$$

This finishes the poof. □

Lemma 4.3. *We have*

$$\frac{\partial J(Z_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda})}{\partial \overline{x}''_t}=m\Big(\frac{B_1}{\lambda^2}\frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+\frac{B_2}{\lambda^2}\frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+O(\frac{1}{\lambda^{1+\varepsilon}})\Big),$$

where $B_1>0$ and $B_2>0$ are constants.

Proof. It is easily seen that

$$\begin{aligned}
&\frac{\partial J(Z^*_{\overline{r},\overline{x}'',\lambda},Y^*_{\overline{r},\overline{x}'',\lambda})}{\partial \overline{x}''_t}\\
&=\Big[\int\limits_{\mathbb{R}^6} P(x)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx+\int\limits_{\mathbb{R}^6} Q(x)Y^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]\\
&-\Big\{\alpha_1\Big[\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|Z^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx-\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|U^*_{z_j,\lambda}|^2\Big)U^*_{z_j,\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]\\
&+\beta\Big[\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|Y^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx-\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|V^*_{z_j,\lambda}|^2\Big)U^*_{z_j,\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]\\
&+\alpha_2\Big[\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|Y^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx-\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|V^*_{z_j,\lambda}|^2\Big)V^*_{z_j,\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]\\
&+\beta\Big[\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|Z^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx-\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big(|x|^{-4}*|U^*_{z_j,\lambda}|^2\Big)V^*_{z_j,\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]\Big\}\\
&:=\mathcal{P}_1-\mathcal{P}2. \qquad (4.15)
\end{aligned}$$

In order to estimate the first part $\mathcal{P}_1$ in (4.15), integrating by parts, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^6} P(x)U^*_{z_j,\lambda}(x)\frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}''_t}(x)dx\\
=&C\int\limits_{\mathbb{R}^6}|U_{0,\lambda}(x)|^2\frac{\partial P(x+z_j)}{\partial \overline{x}''_t}dx\\
=&C\int\limits_{\mathbb{R}^6}|U_{0,\lambda}(x)|^2\frac{\partial P(z_j)}{\partial \overline{x}''_t}dx+C\int\limits_{\mathbb{R}^6}|U_{0,\lambda}(x)|^2\frac{\partial (P(x+z_j)-P(z_j))}{\partial \overline{x}''_t}dx\\
=&\frac{C}{\lambda^2}\frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}\int\limits_{\mathbb{R}^6}|U_{0,1}(x)|^2dx+O(\frac{1}{\lambda^2}),
\end{aligned}$$

where $j=2,\dots,m$. As

$$\int_{\mathbb{R}^6} P(x)U^*_{z_1,\lambda}(x)\sum_{j=2}^m \frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}''_t}(x)dx \le \sum_{j=2}^m C\int_{\mathbb{R}^6} \frac{\lambda^2}{(1+\lambda^2|x-z_1|^2)^2}\frac{\lambda^4|x-z_j|}{(1+\lambda^2|x-z_j|^2)^3}dx$$
$$\le \frac{C}{\lambda}\sum_{j=2}^m \frac{1}{\lambda^2|z_1-z_j|^2} = O(\frac{1}{\lambda^2}),$$

we deduce that

$$\mathcal{P}_1 = \int_{\mathbb{R}^6} P(x)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx + \int_{\mathbb{R}^6} Q(x)Y^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx$$
$$= \Big(\frac{C_1 m}{\lambda^2}\frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}''_t} + \frac{C_2 m}{\lambda^2}\frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}\Big)\int_{\mathbb{R}^6}|U_{0,1}(x)|^2dx + O(\frac{1}{\lambda^{1+\varepsilon}}). \tag{4.16}$$

And then we estimate the second part $\mathcal{P}_2$ in (4.15), we have

$$\int_{\mathbb{R}^6}\Bigg[\Big(|x|^{-4} * |Z^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z^*_{\overline{r},\overline{x}'',\lambda} - \sum_{j=1}^m \Big(|x|^{-4} * |U^*_{z_j,\lambda}|^2\Big)U^*_{z_j,\lambda}\Bigg]\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx$$
$$= 2\sum_{j=1}^m\sum_{i\ne j}\int_{\mathbb{R}^6}\Big(|x|^{-4} * |U^*_{z_j,\lambda}U^*_{z_i,\lambda}|\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx$$
$$+ \sum_{i=1}^m\Big[\Big(|x|^{-4} * |U^*_{z_i,\lambda}|^2\Big)\sum_{j\ne i}U^*_{z_j,\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\Big]$$
$$= 2m\int_{\mathbb{R}^6}\Big(|x|^{-4} * |U^*_{z_1,\lambda}\sum_{i=2}^m U^*_{z_i,\lambda}|\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx$$
$$+ m\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4} * |U^*_{z_1,\lambda}|^2\Big)\sum_{i=2}^m U^*_{z_i,\lambda}\Big]\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx$$
$$:= 2m\mathcal{Q}_1 + m\mathcal{Q}_2. \tag{4.17}$$

Step 1: Let's think about $\mathcal{Q}_2$, because of Lemmas 2.5 and 2.6, we have

$$\left|\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4} * |U^*_{z_1,\lambda}|^2\Big)U^*_{z_i,\lambda}\Big]\frac{\partial U^*_{z_1,\lambda}}{\partial \overline{x}''_t}dx\right|$$
$$= \left|\int_{\mathbb{R}^6} k_0^2\Big[\Big(|x|^{-4} * |U_{z_1,\lambda}|^2\Big)U_{z_i,\lambda}\Big]\frac{\partial U_{z_1,\lambda}}{\partial \overline{x}''_t}dx\right|$$

$$
\begin{aligned}
&\leq \left| C \int_{\mathbb{R}^6} \frac{\lambda^2}{(1+\lambda^2|x-z_1|^2)^2} \frac{\lambda^2}{(1+\lambda^2|x-z_i|^2)^2} \frac{\lambda^4(x_t-\overline{x}_t'')}{(1+\lambda^2|x-z_1|^2)^3} dx \right| \\
&\leq C \int_{\mathbb{R}^6} \frac{\lambda}{(1+|x-\lambda z_1|)^8} \frac{1}{(1+|x-\lambda z_i|)^5} dx \\
&\leq \frac{C}{\lambda^4|z_1-z_i|^5} \int_{\mathbb{R}^6} \Big(\frac{1}{(1+|x-\lambda z_1|)^8} + \frac{1}{(1+|x-\lambda z_i|)^8} \Big) dx \\
&= \frac{C}{\lambda^4|z_1-z_i|^5},
\end{aligned}
$$

where $i = 2, \dots, m$. Thus

$$
\left| \sum_{i=2}^{m} \int_{\mathbb{R}^6} \Big[\Big(|x|^{-4} * |U^*_{z_1,\lambda}|^2 \Big) U^*_{z_i,\lambda} \Big] \frac{\partial U^*_{z_1,\lambda}}{\partial \overline{x}_t''} dx \right| \leq \sum_{i=2}^{m} \frac{C}{\lambda^4|z_1-z_i|^5} = \frac{Cm^5}{\lambda^4} = O(\frac{1}{\lambda^{1+\varepsilon}}). \tag{4.18}
$$

Similarly, we obtain

$$
\sum_{i=2}^{m} \int_{\mathbb{R}^6} \Big[\Big(|x|^{-4} * |U^*_{z_1,\lambda}|^2 \Big) U^*_{z_i,\lambda} \Big] \frac{\partial U^*_{z_i,\lambda}}{\partial \overline{x}_t''} dx = O(\frac{1}{\lambda^{1+\varepsilon}}). \tag{4.19}
$$

Step 2: For $j \neq 1, i$, by Lemma 2.2, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^6} \Big[\Big(|x|^{-4} * |U^*_{z_1,\lambda}|^2 \Big) U^*_{z_i,\lambda} \Big] \frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}_t''} dx \\
&\quad \leq C \int_{\mathbb{R}^6} \frac{\lambda^2}{(1+\lambda^2|x-z_1|^2)^2} \frac{\lambda^2}{(1+\lambda^2|x-z_i|^2)^2} \frac{\lambda^4|x-z_j|}{(1+\lambda^2|x-z_j|^2)^3} dx \\
&\quad \leq C\lambda \int_{\mathbb{R}^6} \frac{1}{(1+|x-\lambda z_1|)^4} \frac{1}{(1+|x-\lambda z_i|)^4} \frac{1}{(1+|x-\lambda z_j|)^5} dx \\
&\quad \leq \frac{C}{\lambda^4|z_1-z_i|^4} \frac{1}{\lambda|z_1-z_j|^2} + \frac{C}{\lambda^4|z_1-z_i|^4} \frac{1}{\lambda|z_i-z_j|^2}.
\end{aligned}
$$

So

$$\sum_{i=2}^{m}\sum_{j=2,\neq i}^{m}\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|U^*_{z_1,\lambda}|^2\Big)U^*_{z_i,\lambda}\Big]\frac{\partial U^*_{z_j,\lambda}}{\partial \overline{x}''_t}dx$$

$$\leq \sum_{i=2}^{m}\sum_{j=2,\neq i}^{m}\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_1-z_j|^2}+\sum_{i=2}^{m}\sum_{j=2,\neq i}^{m}\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_i-z_j|^2}$$

$$=\frac{Cm^6}{\lambda^5}=O(\frac{1}{\lambda^{1+\varepsilon}}).$$

Combining this and (4.18), (4.19), gives

$$\int_{\mathbb{R}^6}\Big[\Big(|x|^{-4}*|U^*_{z_1,\lambda}|^2\Big)\sum_{i=2}^{m}U^*_{z_i,\lambda}\Big]\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx=O(\frac{1}{\lambda^{1+\varepsilon}}). \tag{4.20}$$

Step 3: It is clear that

$$\begin{aligned}\mathcal{Q}_1=&\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U^*_{z_1,\lambda}\sum_{i=2}^{m}U^*_{z_i,\lambda}|\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\ =&\sum_{j=1}^{m}\int_{\mathbb{R}^6}k_0^2\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_j,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx\\ &+\int_{\mathbb{R}^6}k_0^2\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_1,\lambda}\sum_{j=2}^{m}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx\\ &+(m-1)\sum_{j=1,\neq 2}^{m}\int_{\mathbb{R}^6}k_0^2\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_2,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx.\end{aligned} \tag{4.21}$$

An direct calculation shows that

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}U_{z_i,\lambda}|\Big)U_{z_j,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx\right|$$

$$\leq C\left|\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}U_{z_1,\lambda}(x)U_{z_i,\lambda}(x)\frac{(x_t-y_t)}{|x-y|^6}|U_{z_j,\lambda}(y)|^2dxdy\right|$$

$$\leq\left|\int_{\mathbb{R}^6}\Big(|y|^{-4}*|U_{z_j,\lambda}|^2\Big)U_{z_1,\lambda}\frac{\lambda^4(x_t-\overline{x}''_t)}{(1+\lambda^2|y-z_i|^2)^3}dy\right|$$

$$+\left|\int_{\mathbb{R}^6}\Big(|y|^{-4}*|U_{z_j,\lambda}|^2\Big)U_{z_i,\lambda}\frac{\lambda^4(x_t-\overline{x}_t'')}{(1+\lambda^2|y-z_1|^2)^3}dy\right|,$$

where $i=2,\cdots,m$ and $j=1,\cdots,m$. If $j=1$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}U_{z_i,\lambda}|\Big)U_{z_1,\lambda}\frac{\partial U_{z_1,\lambda}}{\partial\overline{x}_t''}dx\right|\\ \le& C\left|\int_{\mathbb{R}^6}\frac{\lambda^2}{(1+\lambda|x-z_1|)^4}\frac{\lambda^2}{(1+\lambda^2|x-z_1|^2)^2}\frac{\lambda^3}{(1+\lambda|x-z_i|)^5}dx\right|\\ &+\left|\int_{\mathbb{R}^6}\Big(|y|^{-4}*|U_{z_1,\lambda}|^2\Big)U_{z_i,\lambda}\frac{\partial U_{z_1,\lambda}}{\partial\overline{x}_t''}dy\right|\\ \le& C\int_{\mathbb{R}^6}\frac{1}{(1+|x-\lambda z_1|)^8}\frac{\lambda}{(1+|x-\lambda z_i|)^5}dx+\frac{C}{\lambda^4|z_1-z_i|^5}\\ \le&\frac{C}{\lambda^4|z_1-z_i|^5}.\end{aligned}$$

If $j=i$, in the same way we have that

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}U_{z_i,\lambda}|\Big)U_{z_j,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial\overline{x}_t''}dx\right|\le\frac{C}{\lambda^4|z_1-z_i|^5}.$$

We now turn to the case $j\neq 1,i$

$$\begin{aligned}&\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}U_{z_i,\lambda}|\Big)U_{z_j,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial\overline{x}_t''}dx\right|\\ \le&\frac{C}{\lambda^4|z_1-z_i|^4}\int_{\mathbb{R}^6}\Big(\frac{1}{(1+|x-\lambda z_1|)^4}+\frac{1}{(1+|x-\lambda z_i|)^4}\Big)\frac{1}{(1+|x-\lambda z_j|)^4}dx\\ \le&\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_1-z_j|}\int_{\mathbb{R}^6}\Big(\frac{1}{(1+|x-\lambda z_1|)^7}+\frac{1}{(1+|x-\lambda z_j|)^7}\Big)dx\\ &+\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_i-z_j|}\int_{\mathbb{R}^6}\Big(\frac{1}{(1+|x-\lambda z_i|)^7}+\frac{1}{(1+|x-\lambda z_j|)^7}\Big)dx\end{aligned}$$

$$=\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_1-z_j|}+\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_i-z_j|}.$$

So,

$$\begin{aligned}
&\left|\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_j,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx\right|\\
\le&\sum_{i=2}^{m}\frac{C}{\lambda^4|z_1-z_i|^5}+\sum_{i=2}^{m}\sum_{j=2,\neq i}^{m}\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_1-z_j|}\\
&+\sum_{i=2}^{m}\sum_{j=2,\neq i}^{m}\frac{C}{\lambda^4|z_1-z_i|^4}\frac{1}{\lambda|z_i-z_j|}\\
=&\frac{Cm^5}{\lambda^4}+\sum_{i=2}^{m}\frac{C}{\lambda^4|z_1-z_i|^4}\frac{m}{\lambda}\\
=&\frac{C}{\lambda^{\frac{3}{2}}}+\frac{Cm}{\lambda}\frac{m^4}{\lambda^4}=O(\frac{1}{\lambda^{1+\varepsilon}}).
\end{aligned}\tag{4.22}$$

We continue in this fashion to have

$$\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_1,\lambda}\sum_{j=2}^{m}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx=O(\frac{1}{\lambda^{1+\varepsilon}}),\tag{4.23}$$

and

$$(m-1)\sum_{j=1,\neq 2}^{m}\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)U_{z_2,\lambda}\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}''_t}dx=O(\frac{1}{\lambda^{1+\varepsilon}}).\tag{4.24}$$

From the above estimates, we have

$$\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_1,\lambda}\sum_{i=2}^{m}U_{z_i,\lambda}|\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx=O(\frac{1}{\lambda^{1+\varepsilon}}).$$

Combining this and (4.20), we get

$$\begin{aligned}
&\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z^*_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z^*_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx-\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big(|x|^{-4}*|U_{z_j,\lambda}|^2\Big)U_{z_j,\lambda}\frac{\partial Z^*_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\
&=mO(\frac{1}{\lambda^{1+\varepsilon}}),
\end{aligned}$$

hence that

$$\frac{\partial J(Z_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda})}{\partial \overline{x}''_t}=m\Big(\frac{B_1}{\lambda^2}\frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+\frac{B_2}{\lambda^2}\frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+O(\frac{1}{\lambda^{1+\varepsilon}})\Big),$$

can be got by Lemma 4.2, where $B_1>0$ and $B_2>0$ are constants. □

Lemma 4.4. *We have*

$$\begin{aligned}&\int_{\mathbb{R}^6}\Big(-\Delta u_m+P(|x'|,x'')u_m-\alpha_1(|x|^{-4}*|u_m|^2)u_m-\beta(|x|^{-4}*|v_m|^2)u_m\Big)\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&+\int_{\mathbb{R}^6}\Big(-\Delta v_m+Q(|x'|,x'')v_m-\alpha_2(|x|^{-4}*|v_m|^2)v_m-\beta(|x|^{-4}*|u_m|^2)v_m\Big)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&=m\Big(\frac{B_1}{\lambda^2}\frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+\frac{B_2}{\lambda^2}\frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{x}''_t}+O(\frac{1}{\lambda^{1+\varepsilon}})\Big).\end{aligned}$$

Proof. It is easy to check that

$$\begin{aligned}&\int_{\mathbb{R}^6}\Big(-\Delta u_m+P(|x'|,x'')u_m-\alpha_1(|x|^{-4}*|u_m|^2)u_m-\beta(|x|^{-4}*|v_m|^2)u_m\Big)\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&+\int_{\mathbb{R}^6}\Big(-\Delta v_m+Q(|x'|,x'')v_m-\alpha_2(|x|^{-4}*|v_m|^2)v_m-\beta(|x|^{-4}*|u_m|^2)v_m\Big)\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&=\langle J'(Z_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda}),(\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t},\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t})\rangle\\&+m\Big\langle\mathcal{L}_m(\phi,\varphi),(\frac{\partial Z_{z_1,\lambda}}{\partial \overline{x}''_t},\frac{\partial Y_{z_1,\lambda}}{\partial \overline{x}''_t})\Big\rangle\\&-\alpha_1\int_{\mathbb{R}^6}\Big(|x|^{-4}*|u_m|^2\Big)u_m\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx+\alpha_1\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&+\alpha_1\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&+\alpha_1\int_{\mathbb{R}^6}2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&-\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|v_m|^2\Big)u_m\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx+\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\\&+\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}''_t}dx\end{aligned}$$

$$
\begin{aligned}
&+\beta\int_{\mathbb{R}^6}2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&-\alpha_2\int_{\mathbb{R}^6}\Big(|x|^{-4}*|v_m|^2\Big)v_m\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx+\alpha_2\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&+\alpha_2\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&+\alpha_2\int_{\mathbb{R}^6}2\Big(|x|^{-4}*(Y_{\overline{r},\overline{x}'',\lambda}\varphi)\Big)Y_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&-\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|u_m|^2\Big)v_m\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&+\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&+\beta\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\varphi\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&+\beta\int_{\mathbb{R}^6}2\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Y_{\overline{r},\overline{x}'',\lambda}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
:=&\langle J'(Z_{\overline{r},\overline{x}'',\lambda},Y_{\overline{r},\overline{x}'',\lambda}),(\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''},\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''})\rangle+mI_1-I_2.
\end{aligned}
$$

By Lemma 3.1, we deduce

$$
I_1=O(\frac{\|(\phi,\varphi)\|_*}{\lambda^{\varepsilon}})=O(\frac{1}{\lambda^{1+\varepsilon}}).
$$

Next we claim that

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\Big(|x|^{-4}*|u_m|^2\Big)u_m\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx-\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&-\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx-\int_{\mathbb{R}^6}2\Big(|x|^{-4}*Z_{\overline{r},\overline{x}'',\lambda}\phi\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{x}_t''}dx\\
&=O(\frac{m}{\lambda^{1+\varepsilon}}).
\end{aligned}
$$

As

$$
||x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)|\le C\|\phi\|_*\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^4},
$$

we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^6}\Big(|x|^{-4}*(Z_{\overline{r},\overline{x}'',\lambda}\phi)\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{z_1,\lambda}}{\partial \overline{x}_t''}dx\right|\\
\leq& C\|\phi\|_*\|Z_{\overline{r},\overline{x}'',\lambda}\|_*\left|\int_{\mathbb{R}^6}\sum_{j=1}^m\frac{\lambda^2}{(1+\lambda|x-z_j|)^4}\sum_{j=1}^m\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\xi\frac{\partial U_{z_1,\lambda}}{\partial \overline{x}_t''}dx\right|\\
\leq& C\|\phi\|_*\left|\int_{\mathbb{R}^6}\sum_{j=1}^m\frac{\lambda^4}{(1+\lambda|x-z_j|)^{6+\tau}}\xi\frac{\partial U_{z_1,\lambda}}{\partial \overline{x}_t''}dx\right|=O(\frac{1}{\lambda^{1+\varepsilon}}),
\end{aligned}
$$

Likewise,

$$
\Big\langle\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi,\frac{\partial Z_{z_1,\lambda}}{\partial \overline{x}_t''}\Big\rangle=O(\frac{1}{\lambda^{1+\varepsilon}}).
$$

By direct calculation, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\Big(|x|^{-4}*|u_m|^2\Big)u_m\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}dx-\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}dx\\
&-\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)\phi\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}dx-\int_{\mathbb{R}^6}2\Big(|x|^{-4}*Z_{\overline{r},\overline{x}'',\lambda}\phi\Big)Z_{\overline{r},\overline{x}'',\lambda}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}dx\\
\leq&\int_{\mathbb{R}^6}\Big(|x|^{-4}*|\phi|^2\Big)|\phi||\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}|dx+\int_{\mathbb{R}^6}\Big(|x|^{-4}*|\phi|^2\Big)|Z_{\overline{r},\overline{x}'',\lambda}||\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}|dx\\
&+2\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}\phi|\Big)|\phi||\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}|dx\\
\leq& C\int_{\mathbb{R}^6}\Big(|x|^{-4}*|\phi|^2\Big)|\phi|\xi\sum_{j=1}^m|\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}_t''}|dx+C\int_{\mathbb{R}^6}\Big(|x|^{-4}*|\phi|^2\Big)|Z_{\overline{r},\overline{x}'',\lambda}|\xi\sum_{j=1}^m|\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}_t''}|dx\\
&+C\int_{\mathbb{R}^6}\Big(|x|^{-4}*|Z_{\overline{r},\overline{x}'',\lambda}\phi|\Big)|\phi|\xi\sum_{j=1}^m|\frac{\partial U_{z_j,\lambda}}{\partial \overline{x}_t''}|dx\\
\leq&\frac{C\|\phi\|_*^3}{\lambda}\int_{\mathbb{R}^6}\Big(|x|^{-4}*\Big(\sum_{j=1}^m\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\Big)^2\Big)\sum_{j=1}^m\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\sum_{j=1}^m U_{z_j,\lambda}dx\\
&+\frac{C\|\phi\|_*^2\|Z_{\overline{r},\overline{x}'',\lambda}\|_*}{\lambda}\int_{\mathbb{R}^6}\Big(|x|^{-4}*\Big(\sum_{j=1}^m\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\Big)^2\Big)
\end{aligned}
$$

$$
\times \sum_{j=1}^{m} \frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}} \sum_{j=1}^{m} U_{z_j,\lambda} dx
$$

$$
+ \frac{C\|\phi\|_*^2 \|Z_{\overline{r},\overline{x}'',\lambda}\|_*}{\lambda} \int_{\mathbb{R}^6} \Big(|x|^{-4} * \Big(\sum_{j=1}^{m} \frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}} \Big)^2 \Big)
$$

$$
\times \sum_{j=1}^{m} \frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}} \sum_{j=1}^{m} U_{z_j,\lambda} dx.
$$

Since (3.27), we get the claim using the same method as the Lemma 4.3. The rest of I_2 run as before.

Note that we have actually proved that

$$
\begin{aligned}
&\langle J'(Z_{\overline{r},\overline{x}'',\lambda}+\phi, Y_{\overline{r},\overline{x}'',\lambda}+\varphi), (\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}, \frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''})\rangle \\
&= \langle J'(Z_{\overline{r},\overline{x}'',\lambda}, Y_{\overline{r},\overline{x}'',\lambda}), (\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''}, \frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{x}_t''})\rangle + O(\frac{m}{\lambda^{1+\varepsilon}}) \\
&= m\Big(\frac{B_1}{\lambda^2} \frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{x}_t''} + \frac{B_2}{\lambda^2} \frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{x}_t''} + O(\frac{1}{\lambda^{1+\varepsilon}}) \Big),
\end{aligned}
$$

by Lemma 4.3. □

A slight change in the proof, we can also obtain

Lemma 4.5. *We have*

$$
\begin{aligned}
&\Big\langle J'(Z_{\overline{r},\overline{x}'',\lambda}+\phi, Y_{\overline{r},\overline{x}'',\lambda}+\varphi), (\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \lambda}, \frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \lambda}) \Big\rangle \\
=& m\Big(-\frac{B_1}{\lambda^3} P(\overline{r},\overline{x}'') - \frac{B_2}{\lambda^3} Q(\overline{r},\overline{x}'') + \sum_{j=2}^{m} \frac{B_3}{\lambda^5 |z_1-z_j|^4} + O(\frac{1}{\lambda^{3+\varepsilon}}) \Big) \\
=& m\Big(-\frac{B_1}{\lambda^3} P(\overline{r},\overline{x}'') - \frac{B_2}{\lambda^3} Q(\overline{r},\overline{x}'') + \frac{B_4 m^4}{\lambda^5} + O(\frac{1}{\lambda^{3+\varepsilon}}) \Big),
\end{aligned} \tag{4.25}
$$

and

$$
\begin{aligned}
&\Big\langle J'(Z_{\overline{r},\overline{x}'',\lambda}+\phi, Y_{\overline{r},\overline{x}'',\lambda}+\varphi), (\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{r}}, \frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial \overline{r}}) \Big\rangle \\
=& m\Big(\Big(\frac{B_1}{\lambda^2} \frac{\partial P(\overline{r},\overline{x}'')}{\partial \overline{r}} + \frac{B_2}{\lambda^2} \frac{\partial Q(\overline{r},\overline{x}'')}{\partial \overline{r}} + \frac{B_4 m^4}{\lambda^4} + O(\frac{1}{\lambda^{1+\varepsilon}}) \Big),
\end{aligned} \tag{4.26}
$$

where $B_i > 0$, $i = 1, 2, 3, 4$ are constants.

Lemma 4.6. *It holds*

$$\int_{\mathbb{R}^6} |\nabla\phi|^2 dx + \int_{\mathbb{R}^6} P(r,x'')\phi^2 dx + \int_{\mathbb{R}^6} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^6} Q(r,x'')\varphi^2 dx = O(\frac{m}{\lambda^{2+\varepsilon}}). \tag{4.27}$$

Proof. Since (3.20), we have

$$\begin{aligned}
&\int_{\mathbb{R}^6} |\nabla\phi|^2 dx + \int_{\mathbb{R}^6} P(r,x'')\phi^2 dx + \int_{\mathbb{R}^6} |\nabla\varphi|^2 dx + \int_{\mathbb{R}^6} Q(r,x'')\varphi^2 dx \\
=&\int_{\mathbb{R}^6} \Big(\Delta Z_{\overline{r},\overline{x}'',\lambda} - P(r,x'')Z_{\overline{r},\overline{x}'',\lambda} + \alpha_1\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}+\phi|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi) \\
&+\beta\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}+\varphi|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi)\Big)\phi dx \\
&+\int_{\mathbb{R}^6} \Big(\Delta Y_{\overline{r},\overline{x}'',\lambda} - Q(r,x'')Y_{\overline{r},\overline{x}'',\lambda} + \alpha_2\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}+\varphi|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi) \\
&+\beta\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}+\phi|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi)\Big)\varphi dx \\
=&\Big[\int_{\mathbb{R}^6} \Big(\Delta Z_{\overline{r},\overline{x}'',\lambda} + \alpha_1\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda} + \beta\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\Big)\phi dx \\
&+\int_{\mathbb{R}^6} \Big(\Delta Y_{\overline{r},\overline{x}'',\lambda} + \alpha_2\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda} + \beta\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\Big)\varphi dx\Big] \\
&-\Big[\int_{\mathbb{R}^6} P(r,x'')Z_{\overline{r},\overline{x}'',\lambda}\phi dx + \int_{\mathbb{R}^6} Q(r,x'')Y_{\overline{r},\overline{x}'',\lambda}\varphi dx\Big] \\
&+\Big[\alpha_1\int_{\mathbb{R}^6} \Big(\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}+\phi|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi) - \Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\Big)\phi dx \\
&+\beta\int_{\mathbb{R}^6} \Big(\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}+\varphi|^2\Big)(Z_{\overline{r},\overline{x}'',\lambda}+\phi) - \Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\Big)\phi dx \\
&+\alpha_2\int_{\mathbb{R}^6} \Big(\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}+\varphi|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi) - \Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\Big)\varphi dx \\
&+\beta\int_{\mathbb{R}^6} \Big(\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}+\phi|^2\Big)(Y_{\overline{r},\overline{x}'',\lambda}+\varphi) - \Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\Big)\varphi dx\Big] \\
&:= I_1 - I_2 + I_3.
\end{aligned}$$

According to the estimates of J_3 and J_4 in Lemma 3.4, we have

$$\left|\int_{\mathbb{R}^6} \Delta Z_{\overline{r},\overline{x}'',\lambda}\phi dx\right| \leq C\frac{\|\phi\|_*}{\lambda^{1+\varepsilon}}\int_{\mathbb{R}^6}\sum_{j=1}^{m}\frac{\lambda^4}{(1+\lambda|x-z_j|)^{4+\tau}}\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}dx \leq \frac{Cm}{\lambda^{2+2\varepsilon}},$$

$$\left|\int_{\mathbb{R}^6} \Delta Y_{\overline{r},\overline{x}'',\lambda}\varphi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}}.$$

In addition,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^6}\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\phi dx\right| \\
\leq &C\frac{\|\phi\|_*}{\lambda^{1+\varepsilon}}\int_{\mathbb{R}^6}\sum_{j=1}^{m}\Big(\frac{\lambda^4}{(1+\lambda|x-z_j|)^{4+\tau}}+\frac{\lambda^4}{(1+\lambda|x-z_j|)^{8}}\Big)\sum_{j=1}^{m}\frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}dx \\
\leq &\frac{Cm}{\lambda^{2+2\varepsilon}},
\end{aligned}$$

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Z_{\overline{r},\overline{x}'',\lambda}\phi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}},$$

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4} * |Y_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\varphi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}},$$

and

$$\left|\int_{\mathbb{R}^6}\Big(|x|^{-4} * |Z_{\overline{r},\overline{x}'',\lambda}|^2\Big)Y_{\overline{r},\overline{x}'',\lambda}\varphi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}},$$

can be obtained from the estimate of J_1 in the Lemma 3.4. Hence that

$$|I_1| \leq \frac{Cm}{\lambda^{2+2\varepsilon}}.$$

Since (3.37) in [28], we thus get

$$\left|\int_{\mathbb{R}^6} P(r,x'')Z_{\overline{r},\overline{x}'',\lambda}\phi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}} \text{ and } \left|\int_{\mathbb{R}^6} Q(r,x'')Y_{\overline{r},\overline{x}'',\lambda}\varphi dx\right| \leq \frac{Cm}{\lambda^{2+2\varepsilon}}.$$

So,

$$|I_2| \le \frac{Cm}{\lambda^{2+2\varepsilon}}.$$

By direct calculations

$$\begin{aligned}|I_3| \le &C(\|\phi\|_*^4 + \|\phi\|_*^2 + \|\phi\|_*^2\|\varphi\|_*^2 + \|\varphi\|_*^4 + \|\varphi\|_*^2)\\ &\cdot \int_{\mathbb{R}^6} \Big(|x|^{-4} * \Big(\sum_{j=1}^m \frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\Big)^2\Big)\Big(\sum_{j=1}^m \frac{\lambda^2}{(1+\lambda|x-z_j|)^{2+\tau}}\Big)^2 dx \le \frac{Cm}{\lambda^{2+2\varepsilon}},\end{aligned}$$

and the proof is complete. □

Proof of Theorem 1.4. We will divide the proof into four steps.

Step 1: Integration by parts, it is easy to verify that (4.1) is equivalent to

$$\begin{aligned}
&-2\int_{D_\rho} (|\nabla u_m|^2 + |\nabla v_m|^2)dx\\
&-\frac{1}{2}\int_{D_\rho} [(6P(x) + \langle x, \nabla P(x)\rangle)u_m^2 + (6Q(x) + \langle x, \nabla Q(x)\rangle)v_m^2]dx\\
&+3\alpha_1 \int_{D_\rho}\int_{\mathbb{R}^6} \frac{|u_m(y)|^2|u_m(x)|^2}{|x-y|^4}dxdy + 3\beta \int_{D_\rho}\int_{\mathbb{R}^6} \frac{|v_m(y)|^2|u_m(x)|^2}{|x-y|^4}dxdy\\
&-2\alpha_1 \int_{D_\rho}\int_{\mathbb{R}^6} x(x-y)\frac{|u_m(y)|^2|u_m(x)|^2}{|x-y|^6}dxdy\\
&-2\beta \int_{D_\rho}\int_{\mathbb{R}^6} x(x-y)\frac{|v_m(y)|^2|u_m(x)|^2}{|x-y|^6}dxdy\\
&+3\alpha_2 \int_{D_\rho}\int_{\mathbb{R}^6} \frac{|v_m(y)|^2|v_m(x)|^2}{|x-y|^4}dxdy + 3\beta \int_{D_\rho}\int_{\mathbb{R}^6} \frac{|u_m(y)|^2|v_m(x)|^2}{|x-y|^4}dxdy\\
&-2\alpha_2 \int_{D_\rho}\int_{\mathbb{R}^6} x(x-y)\frac{|v_m(y)|^2|v_m(x)|^2}{|x-y|^6}dxdy\\
&-2\beta \int_{D_\rho}\int_{\mathbb{R}^6} x(x-y)\frac{|u_m(y)|^2|v_m(x)|^2}{|x-y|^6}dxdy\\
&= O\Big(\int_{\partial D_\rho} \big(|\nabla\phi|^2 + \phi^2\big)ds + \int_{\partial D_\rho} \Big(\int_{\mathbb{R}^6} \frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2 ds + \int_{\partial D_\rho} \Big(\int_{\mathbb{R}^6} \frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2 ds\Big)
\end{aligned}$$

$$+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big). \tag{4.28}$$

As

$$\sum_{j=1}^{m}\Big\langle\Big(\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}$$
$$+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda},\quad \alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}\ Z_{j,l})\Big)Y_{z_j,\lambda}\Big),(\phi,\varphi)\Big\rangle=0,$$

where $l=1,2,\dots,6$, and (3.20) we deduce that

$$\int\limits_{D_\rho}|\nabla u_m|^2dx+\int\limits_{D_\rho}P(x)u_m^2dx+\int\limits_{D_\rho}|\nabla v_m|^2dx+\int\limits_{D_\rho}Q(x)v_m^2dx$$
$$=\alpha_1\int\limits_{D_\rho}\Big(|x|^{-4}*|u_m|^2\Big)|u_m|^2dx+\beta\int\limits_{D_\rho}\Big(|x|^{-4}*|v_m|^2\Big)|u_m|^2dx$$
$$+\alpha_2\int\limits_{D_\rho}\Big(|x|^{-4}*|v_m|^2\Big)|v_m|^2dx+\beta\int\limits_{D_\rho}\Big(|x|^{-4}*|u_m|^2\Big)|v_m|^2dx$$
$$+\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int\limits_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda} \tag{4.29}$$
$$+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]Z_{\overline{r},\overline{x}'',\lambda}$$
$$+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}$$
$$+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]Y_{\overline{r},\overline{x}'',\lambda}\Big\}dx$$
$$+O\Big(\int\limits_{\partial D_\rho}(|\nabla\phi|^2+\phi^2)ds+\int\limits_{\partial D_\rho}(|\nabla\varphi|^2+\varphi^2)ds\Big).$$

Substituting (4.29) into (4.28), yields

$$\int\limits_{D_\rho}(P(x)+\frac{1}{2}\langle x,\nabla P(x)\rangle)u_m^2dx+\int\limits_{D_\rho}(Q(x)+\frac{1}{2}\langle x,\nabla Q(x)\rangle)v_m^2dx$$
$$=-2\sum_{l=1}^{6}c_l\sum_{j=1}^{m}\int\limits_{\mathbb{R}^6}\Big\{\Big[\alpha_1\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\alpha_1\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}$$

$$
\begin{aligned}
&+\beta\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\beta\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]Z_{\overline{r},\overline{x}'',\lambda}\\
&+\Big[\alpha_2\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\alpha_2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\\
&+\beta\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\beta\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]Y_{\overline{r},\overline{x}'',\lambda}\Big\}dx\\
&+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\Big)\\
&+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big)\\
&+O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\varphi(y)|^2|\phi(x)|^2}{|x-y|^6}dxdy\Big)\\
&+O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\phi(y)|^2|\varphi(x)|^2}{|x-y|^6}dxdy\Big)\\
&+O(\frac{1}{\lambda^{2+\varepsilon}}),
\end{aligned}
\tag{4.30}
$$

where $i=3,\cdots,6$.

Step 2: On the other hand, we have

$$
\begin{aligned}
&\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big\{\alpha_1\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Z_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Z_{z_j,\lambda}\Big]\Big\}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{r}}dx=O(m\lambda^2),\quad l=2,\cdots,6,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big\{\alpha_2\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,l})\Big)Y_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,l})\Big)Y_{z_j,\lambda}\Big]\Big\}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{r}}dx=O(m\lambda^2),\quad l=2,\cdots,6,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big\{\alpha_1\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,1}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,1})\Big)Z_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,1}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,1})\Big)Z_{z_j,\lambda}\Big]\Big\}\frac{\partial Z_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{r}}dx=O(m)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^m\int\limits_{\mathbb{R}^6}\Big\{\alpha_2\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,1}+2\Big(|x|^{-4}*(Y_{z_j,\lambda}Y_{j,1})\Big)Y_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,1}+2\Big(|x|^{-4}*(Z_{z_j,\lambda}Z_{j,1})\Big)Y_{z_j,\lambda}\Big]\Big\}\frac{\partial Y_{\overline{r},\overline{x}'',\lambda}}{\partial\overline{r}}dx=O(m)
\end{aligned}
$$

by direct calculations. Combining these and Lemma 4.4, (4.11), (4.25), (4.26) and (3.20), we have

$$c_i = O(\frac{1}{\lambda^{3+\varepsilon}}), i = 2, \cdots, 6, \tag{4.31}$$

and

$$c_1 = O(\frac{1}{\lambda^{1+\varepsilon}}). \tag{4.32}$$

Using a similar approach to Lemma 4.1, we have

$$\begin{aligned}
&\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\alpha_1\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*Z_{z_j,\lambda}Z_{j,l}\Big)Z_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,l}+2\Big(|x|^{-4}*Y_{z_j,\lambda}Y_{j,l}\Big)Z_{z_j,\lambda}\Big]\Big\}Z_{\overline{r},\overline{x}'',\lambda}dx=O(m\lambda),\\
&\quad l=2,3,4,5,6,\\
&\qquad\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\alpha_1\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Z_{j,1}+2\Big(|x|^{-4}*Z_{z_j,\lambda}Z_{j,1}\Big)Z_{z_j,\lambda}\Big]\\
&\qquad+\beta\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Z_{j,1}+2\Big(|x|^{-4}*Y_{z_j,\lambda}Y_{j,1}\Big)Z_{z_j,\lambda}\Big]\Big\}Z_{\overline{r},\overline{x}'',\lambda}dx\\
&\qquad=O(\frac{m}{\lambda}),\\
&\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\alpha_2\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*Y_{z_j,\lambda}Y_{j,l}\Big)Y_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,l}+2\Big(|x|^{-4}*Z_{z_j,\lambda}Z_{j,l}\Big)Y_{z_j,\lambda}\Big]\Big\}Y_{\overline{r},\overline{x}'',\lambda}dx=O(m\lambda),\\
&\quad l=2,3,4,5,6,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{m}\int_{\mathbb{R}^6}\Big\{\alpha_2\Big[\Big(|x|^{-4}*|Y_{z_j,\lambda}|^2\Big)Y_{j,1}+2\Big(|x|^{-4}*Y_{z_j,\lambda}Y_{j,1}\Big)Y_{z_j,\lambda}\Big]\\
&+\beta\Big[\Big(|x|^{-4}*|Z_{z_j,\lambda}|^2\Big)Y_{j,1}+2\Big(|x|^{-4}*Z_{z_j,\lambda}Z_{j,1}\Big)Y_{z_j,\lambda}\Big]\Big\}Y_{\overline{r},\overline{x}'',\lambda}dx=O(\frac{m}{\lambda}).
\end{aligned}$$

Combining these and (4.31), (4.32), we can rewrite (4.30) as

$$\int_{D_\rho}(P(x)+\frac{1}{2}\langle x,\nabla P(x)\rangle)u_m^2dx+\int_{D_\rho}(Q(x)+\frac{1}{2}\langle x,\nabla Q(x)\rangle)v_m^2dx$$

$$
\begin{aligned}
=&O(\frac{m}{\lambda^{2+\varepsilon}})+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\\
&+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\Big)\\
&+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big)\\
&+O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(y)|^2|\phi(x)|^2}{|x-y|^6}dxdy\Big)\\
&+O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(y)|^2|\varphi(x)|^2}{|x-y|^6}dxdy\Big)\\
&+O(\frac{1}{\lambda^{2+\varepsilon}}),
\end{aligned}
\tag{4.33}
$$

for some small $\varepsilon>0$.

Step 3: We integrate by parts again to find that (4.2) is equivalent to

$$
\begin{aligned}
&\int\limits_{D_\rho}(u_m^2\frac{\partial P(|x'|,x'')}{\partial x_i}+v_m^2\frac{\partial Q(|x'|,x'')}{\partial x_i})dx\\
=&O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\\
&+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\Big)\\
&+O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int\limits_{\partial D_\rho}\Big(\int\limits_{\mathbb{R}^6}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big)\\
&+O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\Big)
\end{aligned}
$$

$$+ O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy$$

$$+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\Big)$$

$$+ O(\frac{1}{\lambda^{2+\varepsilon}}), i=3,\cdots,6. \tag{4.34}$$

Therefore, we rewrite (4.33) as

$$\int\limits_{D_\rho}\Big(P(x)+Q(x)+\frac{r}{2}(\frac{\partial P(r,x'')}{\partial r}+\frac{\partial Q(r,x'')}{\partial r})\Big)u_m^2dx = o(\frac{m}{\lambda^2})$$

$$+ O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds$$

$$+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\Big)$$

$$+ O\Big(\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds$$

$$+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big)$$

$$+ O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy$$

$$+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\Big)$$

$$+ O\Big(\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy$$

$$+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\Big)$$

$$+ O(\frac{1}{\lambda^{2+\varepsilon}}),\ i=3,\cdots,6, \tag{4.35}$$

that is,

$$
\begin{aligned}
&\int_{D_\rho} \frac{1}{2r}\frac{\partial(r^2(P(r,x'')+Q(r,x'')))}{\partial r}u_m^2dx = o(\frac{m}{\lambda^2})\\
&+O\Big(\int_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int_{\partial D_\rho}\Big(\int_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\\
&+\int_{\partial D_\rho}\Big(\int_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\Big)\\
&+O\Big(\int_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int_{\partial D_\rho}\Big(\int_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int_{\partial D_\rho}\Big(\int_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\Big)\\
&+O\Big(\int_{D_\rho}\int_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\\
&+\int_{D_\rho}\int_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\Big)\\
&+O\Big(\int_{D_\rho}\int_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\\
&+\int_{D_\rho}\int_{\mathbb{R}^6\setminus D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\Big)\\
&+O(\frac{1}{\lambda^{2+\varepsilon}}),\ i=3,\cdots,6. \qquad (4.36)
\end{aligned}
$$

By Lemma 3.5 in [28], we have

$$
\int_{D_{4\delta}\setminus D_{3\delta}}|\nabla\phi|^2dx = O(\frac{m}{\lambda^{2+\varepsilon}}) \text{ and } \int_{D_{4\delta}\setminus D_{3\delta}}|\nabla\varphi|^2dx = O(\frac{m}{\lambda^{2+\varepsilon}}).
$$

Moreover, according to Lemma 4.6 we can get

$$
\begin{aligned}
&\int_{D_{4\delta}\setminus D_{3\delta}}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int_{D_{4\delta}\setminus D_{3\delta}}\Big(\int_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\\
&+\int_{D_{4\delta}\setminus D_{3\delta}}\Big(\int_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\Big(\int\limits_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\Big(\int\limits_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\\
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\\
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\\
&+\int\limits_{D_{4\delta}\backslash D_{3\delta}}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\\
&=O(\frac{m}{\lambda^{2+\varepsilon}}),
\end{aligned}
$$

where $i=3,\cdots,6$. So we can find a $\rho\in(3\delta,4\delta)$, such that

$$
\begin{aligned}
&\int\limits_{\partial D_\rho}\Big(|\nabla\phi|^2+\phi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\phi|^2ds\\
&+\int\limits_{\partial D_\rho}\Big(|\nabla\varphi|^2+\varphi^2\Big)ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\varphi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds+\int\limits_{\partial D_\rho}\Big(\int\limits_{D_\rho}\frac{|\phi(y)|^2}{|x-y|^4}dy\Big)|\varphi|^2ds\\
&+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\phi(y)|^2}{|x-y|^6}dxdy\\
&+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy+\int\limits_{D_\rho}\int\limits_{\mathbb{R}^6\backslash D_\rho}(x_i-y_i)\frac{|\phi(x)|^2|\varphi(y)|^2}{|x-y|^6}dxdy\\
&=O(\frac{m}{\lambda^{2+\varepsilon}}),
\end{aligned}
$$

where $i=3,\cdots,6$.

Step 4: For any C^1 function $g(r,x'')$, it holds

$$
\int\limits_{D_\rho}g(r,x'')u_m^2dx=m(\frac{1}{\lambda^2}g(\overline{r},\overline{x}'')\int\limits_{\mathbb{R}^6}U_{0,1}^2dx+o(\frac{1}{\lambda^2})),
$$

this was proved in Lemma 3.4 in [28]. Therefore, from (4.34) and (4.36) we deduce that

$$m\Big(\frac{1}{\lambda^2}\frac{\partial(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}''))}{\partial\overline{x}_i}\int_{\mathbb{R}^6}U_{0,1}^2dx+o(\frac{1}{\lambda^2})\Big)=o(\frac{m}{\lambda^2}),$$

and

$$m\Big(\frac{1}{\lambda^2}\frac{1}{2\overline{r}}\frac{\partial(\overline{r}^2P(\overline{r},\overline{x}'')+\overline{r}^2Q(\overline{r},\overline{x}''))}{\partial\overline{r}}\int_{\mathbb{R}^6}U_{0,1}^2dx+o(\frac{1}{\lambda^2})\Big)=o(\frac{m}{\lambda^2}).$$

Hence

$$\frac{\partial(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}''))}{\partial\overline{x}_i}=o(1),\, i=3,\cdots,6, \tag{4.37}$$

and

$$\frac{\partial(\overline{r}^2P(\overline{r},\overline{x}'')+\overline{r}^2Q(\overline{r},\overline{x}''))}{\partial\overline{r}}=o(1). \tag{4.38}$$

Combining these we have already proved that (4.1), (4.2) and (4.3) are equivalent to (4.37), (4.38) and

$$-\frac{B_1}{\lambda^3}\Big(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}'')\Big)+\frac{m^4B_3}{\lambda^5}=O(\frac{1}{\lambda^{3+\varepsilon}}).$$

Let $\lambda=tm^2$, by $\lambda\in[L_0m^2,L_1m^2]$ we have $t\in[L_0,L_1]$, hence that

$$-\frac{B_1}{t^3}\Big(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}'')\Big)+\frac{B_3}{t^5}=o(1),\, t\in[L_0,L_1]. \tag{4.39}$$

Set

$$\begin{aligned}&F(t,\overline{r},\overline{x}'')\\&=\Big(\nabla_{\overline{r},\overline{x}''}(\overline{r}^2(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}''))),-\frac{2B_1}{t^3}(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}''))+\frac{B_3}{t^5}\Big),\end{aligned}$$

then we can deduce

$$\begin{aligned}&\deg(F(t,\overline{r},\overline{x}''),[L_0,L_1]\times B_\theta((r_0,x_0'')))\\&=-\deg(\nabla_{\overline{r},\overline{x}''}(\overline{r}^2(P(\overline{r},\overline{x}'')+Q(\overline{r},\overline{x}''))),B_\theta((r_0,x_0'')))\neq0,\end{aligned}$$

and finally that (4.37), (4.38) and (4.39) have a solution $t_m\in[L_0,L_1]$, $(\overline{r}_m,\overline{x}_m'')\in B_\theta((r_0,x_0''))$. □

Data availability

No data was used for the research described in the article.

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