The Journal of Geometric Analysis (2022) 32:252
https://doi.org/10.1007/s12220-022-00980-6

Normalized Ground States for the Critical Fractional Choquard Equation with a Local Perturbation

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Received: 17 March 2022 / Accepted: 19 June 2022

Abstract

In this paper, we study the critical fractional Choquard equation with a local perturbation $(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \quad x \in \mathbb{R}^N$, having prescribed mass $\int_{\mathbb{R}^N} u^2 \mathrm{d}x = a^2$, where $I_\alpha(x)$ is the Riesz potential, $s \in (0, 1)$, $N > 2s$, $0 < \alpha < \min\{N, 4s\}$, $2 < q < 2^*_s = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, and $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the fractional Hardy–Littlewood–Sobolev critical exponent, $a > 0,\ \mu \in \mathbb{R}$.. Under some L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu|u|^{q-2}u$, respectively, we prove several existence and non-existence results. The qualitative behavior of the ground states as $\mu \to 0^+$ is also studied. The mathematical analysis carried out in this paper can be considered as a counterpart of the Brezis-Nirenberg problem in the context of normalized solutions for fractional Choquard equation. In this framework, several related results are extended and improved.

Keywords Fractional Choquard equation · Normalized ground state · Critical exponent · Local perturbation

Mathematics Subject Classification 35A15 · 35B33 · 35J20 · 35Q40 · 35Q55 · 46N50 · 81Q05

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1 Introduction and Main Results

In this paper, we study standing waves of prescribed mass to the fractional Choquard equation with combined power nonlinearities

$$i\partial_t\psi = (-\Delta)^s\psi - \mu|\psi|^{q-2}\psi - (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi \quad \text{in } \mathbb{R}\times\mathbb{R}^N, \tag{1.1}$$

where $N > 2s$, $\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $\mu > 0$, $\alpha \in (0, N)$. Here, $I_\alpha : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}$ is the Riesz potential, which is defined by

$$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^\alpha},$$

with $A_{N,\alpha} = \frac{\Gamma\frac{\alpha}{2}}{2^{N-\alpha}\pi^{N/2}\Gamma(\frac{N-\alpha}{2})}$, and $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u(x) := C_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where P.V. means the Cauchy principal value on the integral and $C_{N,s}$ is some positive normalization constant, see [19].

Problem (1.1) has the characteristics of double nonlocalities and it has important applications arising in the study of exotic stars. For instance, minimization properties related to problem (1.1) play a fundamental role in the mathematical description of the gravitational collapse of boson stars [24, 41] and white dwarfs stars [28]. Actually, the study of the ground states to (1.1) gives information on the size of the critical initial conditions for the solutions to the corresponding pseudo-relativistic equation [35, 43]. Particularly, when $s = 1/2$, $N = 3$, $\alpha = 1$, we have

$$\sqrt{-\Delta}u + \lambda u = \left(\frac{1}{2\pi|x|} * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^3,$$

related to the well-known massless boson stars equation [22, 30, 36], where the pseudo-relativistic operator $\sqrt{-\Delta + m}$ collapses to the square root of the Laplacian. For other applications in relativistic physics and quantum chemistry, we refer to [4, 27]; see also [45] for the study of graphene, where the nonlocal nonlinearity represents the short time interactions between particles.

In the limiting local case $s = 1$, when $N = 3$, $\alpha = 1$ and $p = 2$, Eq. (1.1) has been introduced in 1954 by Pekar in [57] to describe the quantum theory of a polaron at rest. Successively, in 1976 it was arisen in the work [40] suggested by Choquard on the modeling of an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one-component plasma (see, e.g., [22, 23]). In 1996 the same equation was derived by Penrose in his discussion on the self-gravitational collapse of a quantum mechanical wave function [54–56]; see also [52] and in that context it is referred as Schrödinger-Newton system. After that, variational methods were used to derive existence and qualitative results of standing wave solutions for more generic values of $\alpha \in (0, N)$ and of power type nonlinearities $|u|^{p-2}u$, see [1, 12, 34, 47, 48, 50]. The case of general functions F, almost optimal in the sense of Berestycki-Lions [8], has been treated in [15, 51].

The fractional power of the Laplacian appearing in (1.1), when $s \in (0, 1)$, has been introduced by Laskin [33] as an extension of the classical local Laplacian in the study of nonlinear Schrödinger ezquations, replacing the path integral over Brownian motions with Lévy flights [2]. This operator arises naturally in many contexts and concrete applications in a wide range of fields, such as optimization, finance, crystal dislocations, charge transport in biopolymers, flame propagation, minimal surfaces, water waves, geo-hydrology, anomalous diffusion, neural systems, phase transition and Bose–Einstein condensation, we refer to [19, 24, 44] and the references therein. Equations involving the fractional Laplacian together with local nonlinearities have been investigated extensively, and some fundamental contributions can be found in [11, 21]. Existence and qualitative properties of the solutions for general classes of fractional NLS equations with local sources have been studied in [9, 13, 20] and the references therein. For the existence results on the fractional critical problems, we refer to [7, 58] and references therein.

Mathematically, doubly nonlocal equations have been treated in [14, 18, 29, 48] in the case of pure power nonlinearities, obtaining existence and qualitative properties of the solutions. Other results can be found in [45, 59] for superlinear nonlinearities, in [26] for L^2-supercritical Cauchy problems, and in [63] for concentration phenomena with strictly noncritical and monotone sources.

When searching for stationary waves of problem (1.1) with the form $\psi(t, x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is the chemical potential and $u(x) : \mathbb{R}^N \to \mathbb{C}$ is a time-independent function, and u satisfies the equation

$$(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.2}$$

When looking for solutions to (1.2) one choice is to fix $\lambda < 0$ and to search for solutions to (1.2) as critical points of the action functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}u|^2 - \lambda u^2\right)\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x.$$

see for example [18, 59] and the references therein. Another choice is to prescribe the L^2-norm of the unknown u, that is to consider the problem

$$\begin{cases}(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,\\ u \in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}|u|^2\mathrm{d}x = a^2,\end{cases} \tag{1.3}$$

for fixed $a > 0$ and unknown $\lambda \in \mathbb{R}$. In this direction, define on $H^s(\mathbb{R}^N)$ the energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x.$$

It is standard to check that $E \in C^1$ under some assumptions on p and q, and a critical point of E constrained to

$$S_a := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a^2 \right\}$$

give rise to a solution of (1.3). Such solution is usually called normalized solution of (1.3) on S_a, which is the main purpose of this paper.

When studying normalized solutions to the fractional Choquard equation

$$(-\Delta)^s u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

the number $\widetilde{p} := 2 + \frac{2s-\alpha}{N}$ is the L^2-critical exponent or mass-critical exponent with respect to p. The Hardy-Littlewood-Sobolev upper critical exponent $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ and the lower critical exponent $\underline{p} := \frac{2N-\alpha}{N}$ play an important role. For $p \in (\max\{2, \widetilde{p}\}, 2^*_{\alpha,s})$, the existence of normalized ground state to (1.4) was studied by Li and Luo [42] by considering the minimizer of E constrained on S_a. In [62] Yang considered the existence and asymptotic properties of normalized solutions to the fractional Choquard equation

$$(-\Delta)^s u = \lambda u + |u|^{q-2}u + \mu(I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

with conditions $p \in (\widetilde{p}, 2^*_{\alpha,s})$, and $q \in (2 + \frac{4s}{N}, 2^*_s]$. Using a refined version of the min–max principle, the author showed that (1.5) admits a a mountain pass type solutions under suitable assumptions on the related parameters. In [15], Cingolani et al. prove the existence of a symmetric ground state solution for (1.5) with a general nonlinearity. For more results on the ground state solutions for the nonlinear Choquard equation with prescribed mass, we refer to Bartsch et al. [5], Li and Ye [41], Ye [64] and the references therein.

We note that the number $\overline{p} := 2 + \frac{4s}{N}$ is the L^2-critical exponent in studying normalized solutions to the fractional Schrödinger equation

$$(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.6}$$

which satisfies the prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \tag{1.7}$$

where $2 < q < p \leq 2^*_s$ and $2^*_s = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent. We refer to [3, 16, 46, 65] for more details about the existence of normalized solutions of (1.6–1.7); to [5, 6, 32, 61] for the results on normalized solutions of classical Schrodinger equations.

Motivated by the above mentioned works, in this paper, we aim to study the existence of normalized solutions for the following critical fractional Choquard equation involving local perturbation

$$\begin{cases}(-\Delta)^s u=\lambda u+\mu|u|^{q-2}u+(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & x\in\mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N), \quad \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a^2, \end{cases} \tag{1.8}$$

where $s\in(0,1)$, $N>2s$, $0<\alpha<\min\{N,4s\}$, a is a positive constant. We shall pprove several existence and non-existence results by distinguishing the three cases: (i) L^2-subcritical case: $2<q<\overline{p}<2^*_{\alpha,s}$; (ii) L^2-critical case: $q=\overline{p}$, and L^2-supercritical case: $\overline{p}<q<2^*_{\alpha,s}$. The qualitative behavior of the ground states above as $\mu\to 0^+$ is also studied.

Before we state our main results, we first introduce some notations. Let $H^s(\mathbb{R}^N)$ be the Hilbert space of function in $\mathbb{R}^N$ endowed with the standard inner product and norm

$$\langle u,v\rangle:=\int_{\mathbb{R}^N}((-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}u+uv)\mathrm{d}x, \quad \|u\|^2_{H^s(\mathbb{R}^N)}=\langle u,u\rangle.$$

The work space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N)=\left\{u\in L^{2^*_s}(\mathbb{R}^N):\ \iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y<+\infty\right\},$$

endowed with the norm

$$\|u\|^2:=\|u\|_{D^{s,2}(\mathbb{R}^N)}=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

According to Propositions 3.4 and 3.6 of [19], we have that,

$$\|u\|^2=|(-\Delta)^{\frac{s}{2}}u|_2^2=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \tag{1.9}$$

by omitting the normalization $C_{N,s}$. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces, see [19, 49].

The energy functional associated to problem (1.8) and the constraint is given

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\\ &\quad -\frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x-\frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x \end{aligned} \tag{1.10}$$

and

$$S_a=\left\{u\in H^s(\mathbb{R}^N):\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=a^2\right\}.$$

In the sequel we give some preliminary materials that will be useful in our approach. To begin with, we recall that the key point to apply variational method for problem (1.8) is the following standard estimates for the Riesz potential (see Theorem 4.3 [42]).

Proposition 1.1 *(Hardy-Littlewood-Sobolev inequality, [42]) Let $r, h > 1$ and $0 < \alpha < N$ be such that $\frac{1}{r} - \frac{1}{h} = \frac{N-\alpha}{N}$. Then the map*

$$f \in L^r(\mathbb{R}^N) \mapsto I_\alpha * f \in L^h(\mathbb{R}^N)$$

is continuous. In particular, if $r, t \in (1, +\infty)$ verify $\frac{1}{r} + \frac{1}{t} = \frac{2N-\alpha}{N}$, then there exists a constant $C = C(N, \alpha, r, t) > 0$ such that

$$\left| \int_{\mathbb{R}^N} (I_\alpha * g) h dx \right| \le C \|g\|_r \|h\|_t$$

for all $g \in L^r(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$.

Remark 1.1 As a direct consequence of this inequality, we have

$$\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u(x)|^{2^*_{\alpha,s}} \mathrm{d}x \right)^{\frac{1}{2^*_{\alpha,s}}} \le \mathcal{C}_{N,\alpha,s}^{\frac{1}{2^*_{\alpha,s}}} \|u\|^2_{2^*_s}, \quad \forall\, u \in D^s(\mathbb{R}^N), \tag{1.11}$$

where $\mathcal{C}_{N,\alpha,s} > 0$ is a constant depending on N, α and s.

From Proposition 1.1 we can define the best constant

$$S_{h,l} = \inf_{u \in D^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}} \mathrm{d}x \right)^{\frac{1}{2^*_{\alpha,s}}}}, \tag{1.12}$$

and from [29, 53] we know that $S_{h,l}$ is attained in $\mathbb{R}^N$ by the function

$$\widetilde{U}_{\varepsilon,y}(x) = S^{\frac{(N-\alpha)(2s-N)}{4s(N-\alpha+2s)}} \mathcal{C}_{N,\alpha,s}^{\frac{2s-N}{2(N-\alpha+2s)}} U_{\varepsilon,y}(x) := \widetilde{\mathcal{C}}_{N,\alpha,s} U_{\varepsilon,y}(x), \quad x, y \in \mathbb{R}^N, \tag{1.13}$$

and for any fixed $y \in \mathbb{R}^N$ and $\varepsilon > 0$, $\widetilde{U}_{\varepsilon,y}(x)$ satisfies the equation

$$(-\Delta)^s u = (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u, \quad x \in \mathbb{R}^N,$$

with

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{U}_{\varepsilon,y}|^2 \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |\widetilde{U}_{\varepsilon,y}|^{2^*_{\alpha,s}}) |\widetilde{U}_{\varepsilon,y}|^{2^*_{\alpha,s}} \mathrm{d}x = S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}}, \tag{1.14}$$

where the function

$$U_{\varepsilon,y}(x) = \kappa (\varepsilon^2 + |x-y|^2)^{-\frac{N-2s}{2}}, \tag{1.15}$$

solves the equation $(-\Delta)^s u = |u|^{2^*_s - 2} u,\ x \in \mathbb{R}^N$, and achieves the infimum of the problem

$$S := \inf_{u\in D^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2_s^*}^2}. \tag{1.16}$$

The constant κ is given by

$$\kappa = \left(\frac{S^{N/(2s)}\Gamma(N)}{\pi^{N/2}\Gamma(N/2)}\right)^{\frac{N-2s}{2N}},$$

see [17, 29]. In addition, one has the relationship for the constants $S_{h,l}$, S and $\mathcal{C}_{N,\alpha,s}$:

$$S_{h,l} = S\mathcal{C}_{N,\alpha,s}^{-\frac{1}{2_{\alpha,s}^*}}. \tag{1.17}$$

To enumerate our main results, we introduce the following three constants:

$$\gamma_{q,s} = \frac{N(q-2)}{2qs}, \tag{1.18}$$

$$K_1 = \frac{q(2_{\alpha,s}^*-1)}{C_{N,q,s}(22_{\alpha,s}^*-q\gamma_{q,s})}\left(\frac{2-q\gamma_{q,s}}{22_{\alpha,s}^*-q\gamma_{q,s}}2_{\alpha,s}^* S_{h,l}^{2_{\alpha,s}^*}\right)^{\frac{2-q\gamma_{q,s}}{22_{\alpha,s}^*-2}}, \tag{1.19}$$

$$K_2 := \frac{N+2s-\alpha}{2N-\alpha}\frac{22_{\alpha,s}^*q}{(22_{\alpha,s}^*-q\gamma_{q,s})C_{N,q,s}}S_{h,l}^{\frac{(2N-\alpha)(2-q\gamma_{q,s})}{2(N+2s-\alpha)}}, \tag{1.20}$$

and

$$K_3 := S^{\frac{(N-\alpha)(2_s^*-2)+2s(2_s^*-q\gamma_{q,s})}{2s(22_{\alpha,s}^*-2)}}C_{N,\alpha,s}^{\frac{q\gamma_{q,s}-2}{22_{\alpha,s}^*-2}}, \tag{1.21}$$

where $C_{N,q,s}$ is the fractional Gagliardo-Nirenberg-Sobolev constant from (2.1) below.

Theorem 1.1 *Assume that $N > 2s$, $a, \mu > 0$ and $2 < q < \overline{p} := 2 + 4s/N$. If there exists a constant $\widetilde{k} = \widetilde{k}(N, q, s) > 0$ such that*

$$\mu a^{q(1-\gamma_{q,s})} < \widetilde{k} := \min\{K_1, K_2\}, \tag{1.22}$$

then $I_\mu|_{S_a}$ has a ground state u which is a positive, radially symmetric function and solves (1.8) for some $\widetilde{\lambda} < 0$. Moreover, $m_{a,\mu} < 0$ and u is an interior local minimizer of $I_\mu(u)$ on the set $A_k = \{u \in S_a : ||u|| < k\}$, for suitable k small enough; and any other ground state solution of I_μ on S_a is a local minimizer of I_μ on A_k.

Theorem 1.2 *Assume that $N \geq 2\sqrt{2}s$, $a, \mu > 0$ and $2 < q = \overline{p} := 2 + 4s/N$. If*

$$\mu a^{\overline{p}(1-\gamma_{\overline{p},s})} < \bar{p}(2C_{N,\bar{p},s})^{-1}, \tag{1.23}$$

then $I_\mu|_{S_a}$ has a ground state $\widetilde{u}$ which is a positive, radially symmetric function and solves (1.8) for some $\widetilde{\lambda} < 0$. Moreover, $0 < m_{a,\mu} < \frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$ and u is a Mountain Pass type solution.

Theorem 1.3 *Assume that $N > 2s$, $a, \mu > 0$ and $\overline{p} < q < 2_s^*$. If one of the following conditions holds:*

(1) $N > 4s$ and $\mu a^{q(1-\gamma_{q,s})} < \frac{K_3}{\gamma_{q,s}}$;
(2) $N = 4s$ or $\frac{q}{q-2}2s < N < 4s$ or $2s < N \leq \frac{q}{q-2}2s$,

then $I_\mu|_{S_a}$ has a ground state $\widetilde{u}$ which is a positive, radially symmetric function and solves (1.8) for some $\widetilde{\lambda} < 0$. Moreover, $0 < m_{a,\mu} < \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$ and u is a Mountain Pass type solution.

Theorem 1.4 *Let $a > 0$ and $\mu = 0$. Then we have the following assertions:*

(1) If $N > 4s$, then I_0 on Sa has a unique positive radial ground state $\widetilde{U}_{\varepsilon,0}$ defined in (1.13) for the unique choice of $\varepsilon > 0$ which gives $\|\widetilde{U}_{\varepsilon,0}\|_{L^2(\mathbb{R}^N)} = a$.
(2) If $2s < N \leq 4s$, then (1.8) has no positive solutions in S_a for any $\lambda \in \mathbb{R}$.

Theorem 1.5 *Let u_μ be the corresponding positive ground state solution obtained in Theorems 1.1–1.3 with energy level $m_{a,\mu}$. Then the following conclusions hold:*

(1) *If $2 < q < \overline{p}$, then $m_{a,\mu} \to 0$, and $\|u_\mu\| \to 0$ in $D^{s,2}(\mathbb{R}^N)$ as $\mu \to 0^+$.*
(2) *If $\overline{p} < q < 2_s^*$, then $m_{a,\mu} \to \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$, and $\|u_\mu\| \to 0$ in $D^{s,2}(\mathbb{R}^N)$ as $\mu \to 0^+$.*

Remark 1.2 We notice that, there are only few papers dealing with the existence of normalized solutions of the fractional Choquard equation. Recently, Cao et al. [10], Cingolani et al. [15], Li and Luo [38], and Li et al. [39], considered the subcritical fractional Choquard equation with combined nonlinearities and proved the existence and nonexistence of normalized solutions. However, in this paper we consider the existence and nonexistence of normalized solutions for the critical fractional Choquard equation with combined nonlinearities. Compared with the subcritical case, the critical case is more complicated and needs to overcome the lack of compactness.

The paper is organized as follows. In Sect. 2, we give some preliminary results which will be used to prove Theorems 1.1–1.3. In Sect. 3, we show some lemmas for L^2-subcritical perturbation. In Sect. 4, we present some preliminaries for L^2-critical perturbation. In Sect. 5, we give some lemmas for L^2-supercritical perturbation. In Sect. 6, we prove Theorem 1.1. In Sect. 7, we prove Theorems 1.2, 1.3. In Sect. 8, we prove Theorem 1.4. Finally, Theorem 1.5 will be proven in Sect. 9.
Notation. Throughout this paper, $\|\cdot\|_q$ denotes for the norm in $L^q(\mathbb{R}^N)$; $B_r(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius r; The letters $C, C_i, i = 1, 2, \cdots$, denote various positive constants whose exact values are irrelevant, and $u^\pm = \max\{\pm u, 0\}$.

2 Preliminaries

We recall that, for $N > 2s$, $p \in (2, 2_s^*)$, there exists a constant $C_{N,p,s} > 0$ depending on N, p, s such that the fractional Gagliardo-Nirenberg-Sobolev inequality holds, see [21].

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \leq C_{N,p,s} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^{\frac{N(p-2)}{4s}} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\frac{p}{2} - \frac{N(p-2)}{4s}} \tag{2.1}$$

for all $u \in H^s(\mathbb{R}^N)$. Record the number

$$\gamma_{p,s} := \frac{N(p-2)}{2ps},$$

it is easy to see that

$$p\gamma_{p,s} \begin{cases} < 2, \text{ if } \ 2 < p < \bar{p}, \\ = 2, \text{ if } \ p = \bar{p}, \quad \text{and that } \gamma_{2_s^*,s} = 1, \\ > 2, \text{ if } \ \bar{p} < p < 2_s^*, \end{cases} \tag{2.2}$$

and

$$\|u\|_p \leq C_{N,p,s} \|(-\Delta)^{\frac{s}{2}} u\|_2^{\gamma_{p,s}} \|u\|_2^{1-\gamma_{p,s}}, \quad \forall\, u \in H^s(\mathbb{R}^N). \tag{2.3}$$

The following Pohozaev identity can be derived from [14, 37].

Proposition 2.1 *Let $u \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a positive weak solution of (1.8), then u satisfies the equality*

$$\begin{aligned} \frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx &= \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 dx \\ &+ \frac{2N-\alpha}{22_{\alpha,s}^*} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*}) |u|^{2_{\alpha,s}^*} dx + \frac{N\mu}{q} \int_{\mathbb{R}^N} |u|^q dx. \end{aligned} \tag{2.4}$$

Lemma 2.1 *Let $u \in H^s(\mathbb{R}^N)$ be a weak solution of (1.8), then we have the Pohozaev manifold*

$$\mathcal{N}_{a,\mu} = \{u \in S_a : \ P_\mu(u) = 0\}, \tag{2.5}$$

where

$$P_\mu(u) = s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - s\mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q dx - s \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*}) |u|^{2_{\alpha,s}^*} dx.$$

Proof By Proposition 2.1, u satisfies the Pohozaev identity

$$\begin{aligned} &\frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \\ &\quad = \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x + \frac{N\mu}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\ &\quad + \frac{2N-\alpha}{22_{\alpha,s}^*} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*}) |u|^{2_{\alpha,s}^*} \mathrm{d}x. \end{aligned} \tag{2.6}$$

Since u is a solution of (1.8), we get

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x &= \lambda \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x + \mu \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we infer to

$$s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = s\mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + s \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x,$$

and the conclusion follows. □

We introduce the transformation:

$$(t\star u)(x) = e^{\frac{Nt}{2}} u(e^t x), \quad \forall\, x \in \mathbb{R}^N, t \in \mathbb{R}, \tag{2.8}$$

it is easy to check that $t\star u \in S_a$. We define the fiber map as follows

$$\begin{aligned}\Psi_u^\mu(t) &= I_\mu(t\star u) \\ &= \frac{e^{2st}}{2}\|u\|^2 - \mu \frac{e^{q\gamma_{q,s}st}}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x. \end{aligned} \tag{2.9}$$

Clearly, $(\Psi_u^\mu)'(t) = P_\mu(t\star u)$, hence t is a critical point of $\Psi_u^\mu(t)$ if and only if $t\star u \in \mathcal{N}_{a,\mu}$ and in particular $u \in \mathcal{N}_{a,\mu}$ if 0 is a critical point of $\Psi_u^\mu(t)$. Now we split the manifold $\mathcal{N}_{a,\mu}$ into three parts.

$$\begin{aligned}\mathcal{N}^+_{a,\mu} &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) > 0\} \\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 > \mu q\gamma_{q,s}^2 s^2 \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + 22^*_{\alpha,s}s^2 \\ &\quad \times \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x\} \\ \mathcal{N}^0_{a,\mu} &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) = 0\} \\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 = \mu q\gamma_{q,s}^2 s^2 \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + 22^*_{\alpha,s}s^2 \\ &\quad \times \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x\} \\ \mathcal{N}^-_{a,\mu} &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) < 0\} \\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 < \mu q\gamma_{q,s}^2 s^2 \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + 22^*_{\alpha,s}s^2 \\ &\quad \times \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x\} \end{aligned} \tag{2.10}$$

Therefore, we have the decomposition

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^{+} \cup \mathcal{N}_{a,\mu}^{0} \cup \mathcal{N}_{a,\mu}^{-}.$$

Lemma 2.2 *Let $N > 2s, 2 < q < 2_s^*$ and $a, \mu > 0$. Let $\{u_n\} \subset S_{a,r} = S_a \cap H_r^s(\mathbb{R}^N)$ be a Palais-Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$, where $H_r^s(\mathbb{R}^N)$ is the subspace of $H_r^s(\mathbb{R}^N)$ consisting of radially symmetric functions. Then $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.*

Proof We divide the proof into three cases.

Case 1: $q < \bar{p}$. This implies that $q\gamma_{q,s} < 2$. From $P_\mu(u_n) \to 0$, we have

$$\|u_n\|^2 - \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x = o_n(1). \quad (2.11)$$

Using fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) we get

$$\begin{aligned} I_\mu(u_n) &= \frac{N+2s-\alpha}{2(2N-\alpha)}\|u_n\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right)\int_{\mathbb{R}^N} |u|^q \mathrm{d}x + o_n(1) \\ &\geq \frac{N+2s-\alpha}{2(2N-\alpha)}\|u_n\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right) C_{N,q,s}\|u_n\|^{q\gamma_{q,s}} a^{q(1-\gamma_{q,s})} + o_n(1). \end{aligned}$$

Since $\{u_n\}$ is a Palais-Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$, we have that $I_\mu(u_n) \leq m_{a,\mu} + 1$ for n large. Thus,

$$\frac{N+2s-\alpha}{2(2N-\alpha)}\|u_n\|^2 \leq \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right) C_{N,q,s}\|u_n\|^{q\gamma_{q,s}} \alpha^{q(1-\gamma_{q,s})} + m_{a,\mu} + 2,$$

which yields that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

Case 2: $q = \bar{p}$. In this case, we get $\bar{p}\gamma_{\bar{p},s} = 2$. By $P_\mu(u_n) \to 0$, we have

$$\|u_n\|^2 - \mu\gamma_{\bar{p},s} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x = o_n(1). \quad (2.12)$$

Hence,

$$I_\mu(u_n) = \frac{N+2s-\alpha}{2(2N-\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x + o_n(1) \leq m_{a,\mu} + 1,$$

which implies that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x \leq C.$$

Note that $q = \bar{p} \in (2, 2_s^*)$, we have $q = \bar{p} = \tau 2 + (1-\tau)2_s^*$ for some $\tau \in (0,1)$, and by Hölder inequality, we have

$$\int_{\mathbb{R}^N} |u_n|^{\bar{p}} \mathrm{d}x \le \left(\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x\right)^{\tau} \left(\int_{\mathbb{R}^N} |u_n|^{2_s^*} \mathrm{d}x\right)^{1-\tau} \le C.$$

Consequently, from (2.12), we see that

$$\|u_n\|^2 = \mu\gamma_{\bar{p},s} \int_{\mathbb{R}^N} |u|^{\bar{p}} \mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x + o_n(1) \le C,$$

which implies $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$.

Case 3: $\bar{p} < q < 2_s^*$. In this case, one has $q\gamma_{q,s} > 2$. Using $P_\mu(u_n) \to 0$ we have

$$\|u_n\|^2 - \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x = o_n(1).$$

So,

$$\begin{aligned} I_\mu(u_n) &= \frac{\mu}{q}\left(\frac{\gamma_{q,s}q}{2} - 1\right) \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x + \frac{N+2s-\alpha}{2(2N-\alpha)} \\ &\quad \times \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x + o_n(1) \\ &\le m_{a,\mu} + 1, \end{aligned}$$

which implies that $\int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x$ and $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x$ are both bounded. Hence

$$\|u_n\|^2 = \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x + o_n(1) \le C.$$

The proof is completed. □

Proposition 2.2 *Assume that $N > 2s$, $2 < q < 2_s^*$ and $a, \mu > 0$. Let $\{u_n\} \subset S_{a,r} = S_a \cap H_r^s(\mathbb{R}^N)$ be a Palais-Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$ with*

$$m_{a,\mu} < \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} \quad \text{and} \quad m_{a,\mu} \ne 0.$$

Suppose in addition that $P_\mu(u_n) \to 0$ as $n \to +\infty$. Then, we have the following alternatives:

(i) *either up to a subsequence $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ but not strongly, with u being a solution of (1.8) for some $\lambda < 0$, and*

$$I_\mu(u) < m_{a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}};$$

(ii) *or up to a subsequence* $u_n \to u$ *in* $H^s(\mathbb{R}^N)$, $I_\mu(u) = m_{a,\mu}$ *and* u *solves (1.8) for some* $\lambda < 0$.

Proof By Lemma 2.3, the sequence $\{u_n\}$ is a bounded sequence of radial functions in $H^s(\mathbb{R}^N)$, and hence, by compactness of $H^s_{rad}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, there exists $u \in H^s_{rad}(\mathbb{R}^N)$ such that up to a subsequence $u_n \rightharpoonup u$ in $H^s_{rad}(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$. Since $\{u_n\}$ is a bounded PS sequence for $I_\mu|_{S_a}$, by Lagrange multipliers rule, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for each $\varphi \in H^s(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi - \left(\lambda_n u_n + \mu|u_n|^{q-2}u_n + (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n\right)\varphi\right] \mathrm{d}x \\ = o_n(1)\|\varphi\| \tag{2.13}$$

as $n \to \infty$. Choosing $\varphi = u_n$, then from (2.13) and the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$, it is easy to obtain, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$. By virtue of $P_\mu(u_n) \to 0$ and $\gamma_{q,s} < 1$, we derive that

$$\begin{aligned} \lambda a^2 &= \lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^N} u_n^2 \mathrm{d}x = \lim_{n\to\infty} \left(\|u_n\|^2 - \int_{\mathbb{R}^N} (\mu|u_n|^q + (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x\right) \\ &= \lim_{n\to\infty} \mu(\gamma_{q,s} - 1) \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x = \mu(\gamma_{q,s} - 1) \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \le 0. \end{aligned} \tag{2.14}$$

with $\lambda = 0$ if and only if $u \equiv 0$. We claim that $u \not\equiv 0$. Assume by contradiction that $u \equiv 0$. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, we may assume that $\|u_n\|^2 \to \ell \ge 0$. By $P_\mu(u_n) \to 0$ and $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x = \|u_n\|^2 - \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x \to \ell.$$

Therefore, by the definition of $S_{h,l}$ in (1.12), we have $\ell \ge S_{h,l}\ell^{\frac{1}{2^*_{\alpha,s}}}$, hence

$$\ell = 0 \quad \text{or} \quad \ell \ge S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Case 1: If $\ell = 0$, then we have $|u_n|_q \to 0$, $\|u_n\| \to 0$ and $\int_{\mathbb{R}^N} (I_\alpha *)|u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x \to 0$, consequently, $I_\mu(u_n) \to 0$ which gives a contradiction to the fact that $I_\mu(u_n) \to m_{a,\mu} \ne 0$.

Case 2: If $\ell \ge S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$, from $I_\mu(u_n) \to m_{a,\mu}$ and $P_\mu(u_n) \to 0$, we have that

$$\begin{aligned} &m_{a,\mu} + o_n(1) \\ &\quad = I_\mu(u_n) = \frac{N+2s-\alpha}{2(2N-\alpha)}\|u_n\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right)\int_{\mathbb{R}^N} |u_n|^q \mathrm{d}x + o_n(1) \\ &\quad = \frac{N+2s-\alpha}{2(2N-\alpha)}\|u_n\|^2 + o_n(1) = \frac{N+2s-\alpha}{2(2N-\alpha)}\ell + o_n(1), \end{aligned}$$

which implies that

$$m_{a,\mu} = \frac{N+2s-\alpha}{2(2N-\alpha)}\ell \geq \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},$$

and this contradicts to our assumptions. Therefore, $u \not\equiv 0$. From (2.14), we see that $\lambda < 0$. By a Sobolev embedding, we know that $u_n \rightharpoonup u$ weakly in $L^{2_s^*}(\mathbb{R}^N)$, $u_n \to u$ a.e. in $\mathbb{R}^N$. Then

$$|u_n|^{2^*_{\alpha,s}} \rightharpoonup |u|^{2^*_{\alpha,s}} \text{ weakly in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$$

as $n \to \infty$. By the Hardy-Littlewood-Sobolev inequality, we have that

$$I_\alpha * |u_n|^{2^*_{\alpha,s}} \rightharpoonup I_\alpha * |u|^{2^*_{\alpha,s}} \text{ weakly in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$$

as $n \to \infty$. Combining this and the fact that

$$|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup |u|^{2^*_{\alpha,s}-2}u \text{ weakly in } L^{\frac{2N}{N+2s-\alpha}}(\mathbb{R}^N)$$

as $n \to \infty$, we arrive at that

$$(I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u \text{ weakly in } L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$$

as $n \to \infty$. Therefore, we have for any $\varphi \in H^s(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n\varphi \mathrm{d}x \to \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\varphi \mathrm{d}x. \quad (2.15)$$

Therefore, passing to the limit in (2.13) by the weak convergence, we infer that

$$(-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \quad x \in \mathbb{R}^N. \quad (2.16)$$

Hence by the Pohozaev identity we infer to $P_\mu(u) = 0$. Set $v_n = u_n - u$. Then $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, and by the well-known Brézis-Lieb lemma and [29], we get

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1), \quad (2.17)$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}}\mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x + o_n(1). \quad (2.18)$$

Hence, from $P_\mu(u_n) \to 0$ and $u_n \to u$ in $L^q(\mathbb{R}^N)$, we infer by (2.17) and (2.18) that

$$\begin{aligned}\|v_n\|^2 + \|u\|^2 &= \mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + \int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x + o_n(1).\end{aligned}$$

Combining this with $P_\mu(u) = 0$, we have $\|v_n\|^2 = \int_{\mathbb{R}^N}(I_\alpha*|v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x+o_n(1)$. Thus, by the definition of $S_{h,l}$, we have for some $l \geq 0$ such that

$$\lim_{n\to\infty}\|v_n\|^2 = \lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x = l \geq 0 \Rightarrow l \geq S_{h,l}\ell^{\frac{1}{2^*_{\alpha,s}}}.$$

Hence we can deduce

$$l = 0 \text{ or } l \geq S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$$

If $l \geq S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$, then by (2.17) and (2.18), we obtain

$$\begin{aligned}m_{a,\mu} &= \lim_{n\to\infty} I_\mu(u_n)\\ &= \lim_{n\to\infty}\left(I_\mu(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x\right)\\ &= I_\mu(u) + \frac{N+2s-\alpha}{2(2N-\alpha)}\ell \geq I_\mu(u) + \frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.\end{aligned}$$

Thus, the conclusion (i) holds. If instead $\ell = 0$, then we can show that $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$. In fact, $\|v_n\| = \|u_n - u\| \to 0$ establishes that $u_n \to u$ strongly in $D^s(\mathbb{R}^N)$ and hence in $L^{2^*_s}(\mathbb{R}^N)$ by the Sobolev inequality. We also have $\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}\mathrm{d}x \to 0$ by the definition of $S_{h,l}$ in (1.12). In order to show that $u_n \to u$ in $L^2(\mathbb{R}^N)$, we test (2.13) with $\varphi = u_n - u$, and multiply $u_n - u$ on both sides of (2.16) to get

$$\begin{aligned}&\|u_n - u\|^2 - \int_{\mathbb{R}^N}(\lambda_n u_n - \lambda u)(u_n - u)\mathrm{d}x\\ &= \int_{\mathbb{R}^N}(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^N}\left[(I_\alpha*|u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n-(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\right](u_n-u)\mathrm{d}x+o_n(1).\end{aligned}$$

Now the first, the third integrals tends to zero by convergence of u_n to u in $D^s(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$; while the fourth integral tends to zero by using the Hölder inequality and the convergence in $L^{2_s^*}(\mathbb{R}^N)$. As a consequence

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}(\lambda_n u_n - \lambda u)(u_n - u)\mathrm{d}x = \lim_{n\to\infty}\lambda\int_{\mathbb{R}^N}(u_n - u)^2\mathrm{d}x,$$

which implies that $u_n \to u$ strongly in $L^2(\mathbb{R}^N)$ by $\lambda < 0$ and this completes the assertion (ii). $\square$

We conclude this section stating the following variant of Proposition 2.2.

Proposition 2.3 *Assume that $N > 2s, 2 < q < 2_s^*$ and $a, \mu > 0$. Let $\{u_n\} \subset S_{a,r} = S_a \cap H^s(\mathbb{R}^N)$ be a Palais-Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$ such that*

$$m_{a,\mu} < \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} \quad and \quad m_{a,\mu} \neq 0.$$

Suppose in addition that $P_\mu(u_n) \to 0$ as $n \to +\infty$. and that there exists $\{v_n\} \subset S_a$ and v_n is a radially symmetric for every n satisfying $\|u_n - v_n\| \to 0$ as $n \to \infty$. Then one of the alternatives (i) and (ii) in Proposition 2.2 holds.

The proof is analogous to the previous one: as in Lemma 2.2, we show that $\{u_n\}$ is bounded. Then also $\{v_n\}$ is bounded, and, since each v_n is radial, we deduce that $v_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$, $v_n \to u$ strongly in $L^q(\mathbb{R}^N)$, and a.e. in $\mathbb{R}^N$, up to a subsequence. Since $\|u_n - v_n\| \to 0$, the same convergence is inherited by $\{u_n\}$, and we can proceed as in the proof of Proposition 2.2.

3 L^2-Subcritical Perturbation

For $N > 2s$ and $2 < q < 2 + 4s/N$, we recall that

$$K_1 := \frac{q(2_{\alpha,s}^* - 1)}{C_{N,q,s}(22_{\alpha,s}^* - q\gamma_{q,s})}\left(\frac{2 - q\gamma_{q,s}}{22_{\alpha,s}^* - q\gamma_{q,s}} 2_{\alpha,s}^* S_{h,l}^{2_{\alpha,s}^*}\right)^{\frac{2-q\gamma_{q,s}}{22_{\alpha,s}^*-2}}.$$

We consider the constrained functional $I_\mu|_{S_a}$. For each $u \in S_a$, by the fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) and inequality (1.13), one has

$$I_\mu(u) \geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q}C_{N,q,s}\|u_n\|^{q\gamma_{q,s}}a^{q(1-\gamma_{q,s})} - \frac{1}{22_{\alpha,s}^*}S_{h,l}^{-2_{\alpha,s}^*}\|u\|^{22_{\alpha,s}^*}. \quad (3.1)$$

Now, we introduce the function $h : \mathbb{R}^+ \to \mathbb{R}$

$$h(t) = \frac{1}{2}t^2 - \frac{\mu}{q}C_{N,q,s}a^{q(1-\gamma_{q,s})}t^{q\gamma_{q,s}} - \frac{1}{22_{\alpha,s}^*}S_{h,l}^{-2_{\alpha,s}^*}t^{22_{\alpha,s}^*}. \quad (3.2)$$

Since $\mu > 0$ and $q\gamma_{q,s} < 2 < 2_s^*$, we see that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$.

Lemma 3.1 *Suppose that the inequality $\mu a^{(1-\gamma_{q,s})q} < K_1$ holds, then the function h has a local strict minimum at negative level, a global maximum at positive level, and no other critical points, and there exists a R_0 and R_1 both depending on a and μ, such that $h(R_0) = 0 = h(R_1)$ and $h(t) \geq 0$, if and only if $t \in (R_0, R_1)$.*

Proof For $t > 0$, we have $h(t) > 0$ if and only if

$$\varphi(t) > \frac{\mu}{q} C_{N,q,s} a^{q(1-\gamma_{q,s})}, \quad \text{with } \varphi(t) = \frac{1}{2} t^{2-q\gamma_{q,s}} - \frac{1}{22^*_{\alpha,s}} S_{h,l}^{-2^*_{\alpha,s}} t^{22^*_{\alpha,s}-q\gamma_{q,s}}.$$

Notice that

$$\varphi'(t) = \frac{2-q\gamma_{q,s}}{2} t^{1-q\gamma_{q,s}} - \frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}} S_{h,l}^{-2^*_{\alpha,s}} t^{22^*_{\alpha,s}-1-q\gamma_{q,s}}.$$

It is easy to see that $\varphi(t)$ has a unique critical point at

$$\bar{t} = \left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}} 2^*_{\alpha,s} S_{h,l}^{2^*_{\alpha,s}} \right)^{\frac{1}{22^*_{\alpha,s}-2}},$$

and $\varphi(t)$ is increasing on $(0, \bar{t})$ and decreasing on $(\bar{t}, +\infty)$. Moreover, the maximum level is

$$\varphi(\bar{t}) = \frac{2^*_{\alpha,s}-1}{22^*_{\alpha,s}-q\gamma_{q,s}} \left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}} 2^*_{\alpha,s} S_{h,l}^{2^*_{\alpha,s}} \right)^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}.$$

Therefore, h is positive on a open interval (R_0, R_1) if and only if $\varphi(\bar{t}) > \frac{\mu}{q} C_{N,q,s} a^{q(1-\gamma_{q,s})}$, which implies that

$$\mu a^{q(1-\gamma_{q,s})} < \frac{q(2^*_{\alpha,s}-1)}{C_{N,q,s}(22^*_{\alpha,s}-q\gamma_{q,s})} \left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}} 2^*_{\alpha,s} S_{h,l}^{2^*_{\alpha,s}} \right)^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}.$$

By virtue of $h(0^+) = 0^-$ and $h(+\infty) = -\infty$ and h being positive on open interval (R_0, R_1), we see that h has a global maximum at positive level in (R_0, R_1), and has a local minimum point at negative level in $(0, R_0)$. Since

$$h'(t) = t^{q\gamma_{q,s}-1} \left[t^{2-q\gamma_{q,s}} - \mu\gamma_{q,s} C_{N,q,s} a^{q(1-\gamma_{q,s})} - S_{h,l}^{-2^*_{\alpha,s}} t^{22^*_{\alpha,s}-q\gamma_{,s}} \right] = 0$$

if and only if

$$\psi(t) = \mu\gamma_{q,s} C_{N,q,s} a^{q(1-\gamma_{q,s})} \quad \text{with} \quad \psi(t) = t^{2-q\gamma_{q,s}} - S_{h,l}^{-2^*_{\alpha,s}} t^{22^*_{\alpha,s}-q\gamma_{q,s}}.$$

Clearly, $\psi(t)$ has only one critical point, which is a strict maximum. So, if $\max_{t>0}\psi(t) \le \mu\gamma_{q,s}C_{N,q,s}a^{q(1-\gamma_{q,s})}$, then we get a contradiction to the fact that h is positive on the open interval (R_0, R_1). Hence, $\max_{t>0}\psi(t) > \mu\gamma_{q,s}C_{N,q,s}a^{q(1-\gamma_{q,s})}$, this implies that h only has a local strict minimum at negative level, a global strict maximum at positive level, and no other critical points. □

Lemma 3.2 *Assume that $\mu a^{(1-\gamma_{q,s})q} < K_1$, then $\mathcal{N}^0_{a,\mu} = \emptyset$ and $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 1 in S_a.*

Proof Suppose by contradiction that there is a $u \in \mathcal{N}^0_{a,\mu}$ such that

$$\|u\|^2 = \mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + \int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x, \tag{3.3}$$

and

$$2\|u\|^2 = \mu q\gamma_{q,s}^2\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + 22^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x. \tag{3.4}$$

Therefore, from (1.12), (2.1), (3.3) and (3.4), we have

$$\begin{aligned}
\mu\gamma_{q,s}(2-q\gamma_{q,s})\int_{\mathbb{R}^N}|u|^q\mathrm{d}x &= (22^*_{\alpha,s}-2)\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x;\\
\|u\|^2 &= \frac{22^*_{\alpha,s}-q\gamma_{q,s}}{2-q\gamma_{q,s}}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\\
&\le \frac{22^*_{\alpha,s}-q\gamma_{q,s}}{2-q\gamma_{q,s}}S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}};
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\|u\|^2 &= \mu\gamma_{q,s}\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x\\
&\le \mu\gamma_{q,s}\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}C_{N,q,s}\|u_n\|^{q\gamma_{q,s}}a^{q(1-\gamma_{q,s})}.
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we deduce that

$$\left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}}S_{h,l}^{2^*_{\alpha,s}}\right)^{\frac{1}{22^*_{\alpha,s}-2}} \le \left(\mu\gamma_{q,s}\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}C_{N,q,s}a^{q(1-\gamma_{q,s})}\right)^{\frac{1}{2-q\gamma_{q,s}}},$$

which implies that

$$\mu\alpha^{q(1-\gamma_{q,s})} \ge \frac{22^*_{\alpha,s}-2}{\gamma_{q,s}C_{N,q,s}(22^*_{\alpha,s}-q\gamma_{q,s})}\left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}}S_{h,l}^{2^*_{\alpha,s}}\right)^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}. \tag{3.7}$$

Next, we show that the right hand of (3.7) is greater than or equal to K_1, and this would lead to a contradiction to our assumption. To show:

$$\begin{aligned}&\frac{22^*_{\alpha,s}-2}{\gamma_{q,s}C_{N,q,s}(22^*_{\alpha,s}-q\gamma_{q,s})}\left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}}S_{h,l}^{2^*_{\alpha,s}}\right)^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}\\&\geq\frac{q(2^*_{\alpha,s}-1)}{C_{N,q,s}(22^*_{\alpha,s}-q\gamma_{q,s})}\left(\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-q\gamma_{q,s}}2^*_{\alpha,s}S_{h,l}^{2^*_{\alpha,s}}\right)^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}.\end{aligned}\tag{3.8}$$

To this aim, we only need to show that

$$\frac{q\gamma_{q,s}}{2}(2^*_{\alpha,s})^{\frac{2-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}\leq 1.\tag{3.9}$$

Set $q\gamma_{q,s}=x\in(0,2)$, and define the function

$$f(x)=\left(\frac{x}{2}\right)^{22^*_{\alpha,s}-2}(2^*_{\alpha,s})^{2-x}.$$

We intend to prove that $f(x)\leq 1$. Indeed, it is easy to see that $f(x)$ is increasing on $(0,(22^*_{\alpha,s}-2)/\ln 2^*_{\alpha,s})$ and decreasing on $((22^*_{\alpha,s}-2)/\ln 2^*_{\alpha,s},+\infty)$. Thus, when $x\in(0,2)$, $f(x)\leq f(2)=1$, which implies (3.9). From (3.9) and (3.8) we have $\mu\alpha^{q(1-\gamma_{q,s})}\geq K_1$, which contradicts to our assumption. Thus, $\mathcal{N}^0_{a,\mu}=\emptyset$.

Next, we show that $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 1 on S_a. To see this, note that $\mathcal{N}_{a,\mu}=\{u\in H^s(\mathbb{R}^N):\ P_\mu(u)=0,G(u)=0\}$, for $G(u)=\int_{\mathbb{R}^N}u^2\mathrm{d}x-a^2$, with P_μ and G being of class C^1 in $H^s(\mathbb{R}^N)$. Thus, we have to show that the differential $(dG(u),dP_\mu(u)):\ H^s(\mathbb{R}^N)\to\mathbb{R}^2$ is surjective, for every $u\in\mathcal{N}_{a,\mu}$. If this is not true, then $dP_\mu(u)$ must be linearly dependent from $dG(u)$, that is, there exists some $\lambda\in\mathbb{R}$ such that for every $v\in H^s(\mathbb{R}^N)$,

$$\begin{aligned}&2s\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v\mathrm{d}x-s\mu q\gamma_{q,s}\int_{\mathbb{R}^N}|u|^{q-2}uv\mathrm{d}x\\&\quad-s22^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}uv\mathrm{d}x\\&=\lambda\int_{\mathbb{R}^N}uv\mathrm{d}x,\end{aligned}$$

which derives to

$$2s(-\Delta)^su=\lambda u+s\mu q\gamma_{q,s}|u|^{q-2}u+s22^*_{\alpha,s}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\quad\text{in}\quad\mathbb{R}^N.$$

Using the Pohozaev identity for the last equation, we obtain

$$2s^2\|u\|^2=\mu q\gamma^2_{q,s}s^2\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+22^*_{\alpha,s}s^2\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x,$$

that is $u \in \mathcal{N}_{a,\mu}^0$, which leads to a contradiction. Thus, $u \in \mathcal{N}_{a,\mu}$ is a natural constraint. $\square$

Lemma 3.3 *For each $u \in S_a$, the function $\Psi_u^\mu(t)$ has exactly two critical points $\alpha_u < t_u \in \mathbb{R}$ and two zero points $c_u < d_u \in \mathbb{R}$ satisfying $\alpha_u < c_u < t_u < d_u$. Furthermore,*

(i) $\alpha_u \star u \in \mathcal{N}_{a,\mu}^+$, $t_u \star u \in \mathcal{N}_{a,\mu}^-$ and if $t \star u \in \mathcal{N}_{a,\mu}$, then either $t = \alpha_u$ or $t = t_u$.
(ii) $\|t \star u\| \leq R_0$ for each $t \leq c_u$, and

$$I_\mu(\alpha_u \star u) = \min\{I_\mu(t \star u) : \ t \in \mathbb{R} \ \ with \ \ \|t \star u\| < R_0\} < 0.$$

(iii) $I_\mu(t_u \star u) = \max\{I_\mu(t \star u) : \ t \in \mathbb{R}\} > 0$ and $\Psi_u^\mu(t)$ is strictly decreasing and concave on $(t_u, +\infty)$. In particular, if $t_u < 0$, then $P_\mu(u) < 0$.
(iv) The maps: $u \to \alpha_u, u \to t_u$, $\forall u \in S_a$, are of class C^1.

Proof Let $u \in S_a$, then $t \star u \in \mathcal{N}_{a,\mu}$ if and only if $(\Psi_u^\mu)'(t) = 0$. Firstly, we show that $\Psi_u^\mu(t)$ has at least two critical points. From (3.1), one has

$$\Psi_u^\mu(t) = I_\mu(t \star u) \geq h(\|t \star u\|) = h(e^{st}\|u\|).$$

Thus, the C^2 function $\Psi_u^\mu(t)$ is positive on $(s^{-1}\ln(R_0\|u\|^{-1}), s^{-1}\ln(R_1\|u\|^{-1}))$ and $\Psi_u^\mu(-\infty) = 0^-$, $\Psi_u^\mu(+\infty) = -\infty$, therefore, it is easy to see that $\Psi_u^\mu(t)$ has a local minimum point α_u at level in $(0, s^{-1}\ln(R_0\|u\|^{-1}))$ and has a global maximum point t_u at positive level in $(s^{-1}\ln(R_0\|u\|^{-1}), s^{-1}\ln(R_1\|u\|^{-1}))$. Next, we show that $\Psi_u^\mu(t)$ has no other critical points. In fact, $(\Psi_u^\mu)'(t) = 0$ implies that

$$g(t) = s\mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x$$

with

$$g(t) = se^{(2-q\gamma_{q,s})st}\|u\|^2 - se^{(22_{\alpha,s}^*-q\gamma_{q,s})st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*}\mathrm{d}x.$$

It is easy to see that $g(t)$ has a unique maximum point, thus the above equation has at most two solutions. From $u \in S_a$, $t \in \mathbb{R}$ is a critical point of $\Psi_u^\mu(t)$ if and only if $t \star u \in \mathcal{N}_{a,\mu}$, we have $\alpha_u, t_u \in \mathcal{N}_{a,\mu}$; Conversely, $t \star u \in \mathcal{N}_{a,\mu}$ if and only if $t = \alpha_u$, or $t = t_u$. In view of α_u being a local minimum point of $\Psi_u^\mu(t)$, we see that $(\Psi_{\alpha_u \star u}^\mu)''(0) = (\Psi_u^\mu)''(\alpha_u) \geq 0$. As $\mathcal{N}_{a,\mu}^0 = \emptyset$, we get that $(\Psi_{\alpha_u \star u}^\mu)''(0) = (\Psi_u^\mu)''(\alpha_u) > 0$. which implies that $\alpha_u \star u \in \mathcal{N}_{a,\mu}^+$. Similarly, we deduce that $t_u \star u \in \mathcal{N}_{a,\mu}^-$. By the monotonicity and the behavior at infinity of $\Psi_u^\mu(t)$ we see that $\Psi_u^\mu(t)$ has exactly two zero points $c_u < d_u$ with $\alpha_u < c_u < t_u < d_u$ and $\Psi_u^\mu(t)$ has exactly two inflection points, especially, $\Psi_u^\mu(t)$ is concave on $(t_u, +\infty)$ and so, if $t_u < 0$, then $P_\mu(u) = (\Psi_u^\mu)'(0) < 0$. Finally, we show that $u \in S_a \ : \ \alpha_n \in \mathbb{R}$ and $u \in S_a \ : \ t_n \in \mathbb{R}$ are of class C^1. In fact, we can apply the implicit function theorem on the C^1 function $\Phi(t, u) = (\Psi_u^\mu)'(t)$, therefore, $\Phi(\alpha_u, u)(\Psi_u^\mu)'(\alpha_u) = 0$, $\partial_t\Phi(\alpha_u, u) = (\Psi_u^\mu)''(\alpha_u) > 0$, by the implicit function theorem, we have that $u \to \alpha_u$, $\forall u \in S_a$, is of class C^1. Similarly, we can prove that $u \to t_u$, $\forall u \in S_a$, is of class C^1. $\square$

For $k > 0$, we define

$$A_k = \{u \in S_a : \|u\|^2 < k\} \quad \text{and} \quad m_{a,\mu} = \inf_{u \in A_{R_0}} I_\mu(u).$$

By Lemma 3.3, we can deduce the following conclusion.

Corollary 3.1 *The set $\mathcal{N}^+_{a,\mu}$ is contained in*

$$A_{R_0} = \{u \in S_a : \|u\|^2 < R_0\}, \quad \textit{and} \quad \sup_{u \in \mathcal{N}^+_{a,\mu}} I_\mu(u) \le 0 \le \inf_{u \in \mathcal{N}^-_{a,\mu}} I_\mu(u).$$

Lemma 3.4 *The level $m_{a,\mu} \in (-\infty, 0)$, and satisfies*

$$m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu = \inf_{\mathcal{N}^+_{a,\mu}} I_\mu \quad \textit{and} \quad m_{a,\mu} < \inf_{\overline{A_{R_0}} \setminus A_{R_0 - r}} I_\mu$$

for $r > 0$ small enough.

Proof For any $u \in A_{R_0}$, we have

$$I_\mu(u) \ge h(\|u\|) \ge \min_{t \in [0, R_0]} h(t) > -\infty.$$

Thus, $m_{a,\mu} > -\infty$. Furthermore, for any $u \in S_a$, we have $\|t\star u\| < R_0$ and $I_\mu(t\star u) < 0$ for $t \ll -1$ and so $m_{a,\mu} < 0$. Since $\mathcal{N}^+_{a,\mu} \subset A_{R_0}$, we have that $m_{a,\mu} \le \inf_{\mathcal{N}^+_{a,\mu}} I_\mu$. On the other hand, if $u \in A_{R_0}$, then $\alpha_u \star u \in \mathcal{N}^+_{a,\mu} \subset A_{R_0}$ and

$$I_\mu(\alpha_u \star u) = \min\{I_\mu(t\star u) : t \in \mathbb{R} \text{ and } \|t\star u\| < R_0\} \le I_\mu(u),$$

which implies that $\inf_{\mathcal{N}^+_{a,\mu}} I_\mu \le m_{a,\mu}$. By virtue of $I_\mu > 0$ on $\mathcal{N}^-_{a,\mu}$, we see that $\inf_{\mathcal{N}^+_{a,\mu}} I_\mu = \inf_{\mathcal{N}_{a,\mu}} I_\mu$. Finally, by the continuity of h there is $r > 0$ such that $h(t) \ge \frac{m_{a,\mu}}{2}$ if $t \in [R_0 - r, R_0]$. Consequently,

$$I_\mu(u) \ge h(\|u\|) \ge \frac{m_{a,\mu}}{2} > m_{a,\mu}$$

for any $u \in u \in S_a$ with $R_0 - r \le \|u\| \le R_0$. This completes the proof. □

4 L^2-Critical Perturbation

In this section, we deal with the case $N \ge 2\sqrt{2}s$, $2 < q = \bar{p}$ and a, μ satisfy the the inequality

$$\mu a^{\frac{4s}{N}} < \bar{p}(2C_{N,\bar{p},s})^{-1}. \tag{4.1}$$

We recall the decomposition of

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^{+} \cup \mathcal{N}_{a,\mu}^{0} \cup \mathcal{N}_{a,\mu}^{-}.$$

Lemma 4.1 *$\mathcal{N}_{a,\mu}^{0} = \emptyset$ and $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 1 in S_a.*

Proof Suppose by contradiction that, there exists a $u \in \mathcal{N}_{a,\mu}^{0}$. Then

$$\|u\|^2 = \mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + \int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x, \tag{4.2}$$

and

$$2\|u\|^2 = \mu q\gamma_{q,s}^2\int_{\mathbb{R}^N}|u|^q\mathrm{d}x + 22^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x, \tag{4.3}$$

from which, we get $\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x = 0$, this is not possible since $u \in S_a$, here we used the fact $q\gamma_{q,s} = 2$. The remainder parts of the proof is similar to that of Lemma 3.2, and so we omit the details. □

Lemma 4.2 *Under the condition (4.1), then for each $u \in S_a$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{N}_{a,\mu}$, where t_u is the unique critical point of the function of Ψ_u^μ and is a strict maximum point at positive level. Moreover,*

- (i) *$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^{-}$;*
- (ii) *$\Psi_u^\mu(t)$ is strict decreasing and concave on $(t_u, +\infty)$ and $t_u < 0$ implies that $P_\mu(u) < 0$;*
- (iii) *The map $u \in S_a : t_n \in \mathbb{R}$ is of C^1;*
- (iv) *If $P_\mu(u) < 0$, then $t_u < 0$.*

Proof Note that

$$\begin{aligned}\Psi_u^\mu(t) &= I_\mu(t\star u) = \left[\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x\right]e^{2st} \\ &\quad - \frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x,\end{aligned} \tag{4.4}$$

and the fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) implies that

$$\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x \geq \left(\frac{1}{2} - \frac{\mu}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\right)\|u\|^2.$$

By the condition $\mu a^{\frac{4s}{N}} < \bar{p}(2C_{N,\bar{p},s})^{-1}$, we infer to $\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x > 0$. From (4.4) we see that Ψ_u^μ has a unique critical point t_u, which is a strict maximum point at positive level. Moreover, if $u \in \mathcal{N}_{a,\mu}$, then $t_u = 0$ is a maximum point, and

$(\Psi_u^\mu)''(0) \leq 0$. In view of $\mathcal{N}_{a,\mu}^0 = \emptyset$, we have $(\Psi_u^\mu)''(0) < 0$. Thus, $\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^-$. To see that the map $u \in S_a : t_n \in \mathbb{R}$ is of C^1, we can apply the implicit function theorem as in Lemma 3.3. Finally, since $(\Psi_u^\mu)'(t) < 0$ if and only if $t > t_u$, so $P_\mu(u) = (\Psi_u^\mu)'(0) < 0$ if and only if $t_u < 0$. □

Lemma 4.3 $m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu > 0$.

Proof Let $u \in \mathcal{N}_{a,\mu}$, then $P_\mu(u) = 0$, and by the fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) and Hardy-Littlewood-Sobolev inequality (1.12), we get

$$\begin{aligned}
\|u\|^2 &= \mu\frac{2}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x + \int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x \\
&\leq \mu\frac{2}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\|u\|^2 + S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}}.
\end{aligned}$$

Combining (4.1) and the last inequality, we get

$$\|u\|^{22^*_{\alpha,s}} \geq S_{h,l}^{2^*_{\alpha,s}}\left(1-\mu\frac{2}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\right)\|u\|^2 \Rightarrow \inf_{\mathcal{N}_{a,\mu}}\|u\|^2 > 0. \tag{4.5}$$

Therefore, from $P_\mu(u) = 0$ and (4.5), we have

$$\begin{aligned}
I_\mu(u) &= \frac{N+2s-\alpha}{2(2N-\alpha)}\left(\|u\|^2 - \frac{2\mu}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x\right) \\
&\geq \frac{N+2s-\alpha}{2(2N-\alpha)}\left(1-\mu\frac{2}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\right)\|u\|^2 > 0.
\end{aligned}$$

Hence,

$$m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu > 0.$$

□

Lemma 4.4 *For $k > 0$ sufficiently small, we have $0 < \sup_{\overline{A_k}} I_\mu < m_{a,\mu}$. Moreover,*

$$u \in \overline{A_k} \Rightarrow I_\mu(u) > 0,\ P_\mu(u) > 0,$$

where $A_k = \{u \in S_a : \|u\|^2 < k\}$.

Proof By fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) and Sobolev inequality (1.12), we infer that

$$I_\mu(u) \geq \left(\frac{1}{2} - \frac{\mu}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\right)\|u\|^2 - \frac{1}{22^*_{\alpha,s}}S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}} > 0,$$

and

$$P_\mu(u) = s\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x - s\mu\gamma_{\bar{p},s}\int_{\mathbb{R}^N} |u|^{\bar{p}}\mathrm{d}x - s\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x$$
$$\geq s\left(1 - \frac{2\mu}{\bar{p}}C_{N,\bar{p},s}a^{\frac{4s}{N}}\right)\|u\|^2 - sS_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}} > 0,$$

provided that $u \in \overline{A_k}$ for k small enough. By Lemma 4.3 we see that $m_{a,\mu} > 0$, thus if k is small enough, we also have

$$I_\mu(u) \leq \frac{1}{2}\|u\|^2 < m_{a,\mu}.$$

The proof is complete. □

We shall employ Proposition 2.2 to recover compactness. To this aim, we first estimate from above for the value $m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu$, where $S_{r,a}$ is the subset of the radial functions in S_a.

Lemma 4.5 *Assume that condition (4.1) holds, and $N \geq 2\sqrt{2}s$, then we have $m_{r,a,\mu} < \frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$.*

Proof Let $\eta(x) \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_\delta(0)$ and $\eta = 0$ on $\mathbb{R}^N\backslash B_{2\delta}(0)$. Set

$$\widetilde{u}_\varepsilon(x) = \eta(x)\widetilde{U}_\varepsilon(x), \quad \widetilde{v}_\varepsilon = a\frac{\widetilde{u}_\varepsilon}{\|\widetilde{u}_\varepsilon\|_2} = a\frac{u_\varepsilon}{\|u_\varepsilon\|_2}$$

where $u_\varepsilon(x) = \eta(x)U_\varepsilon(x)$, $\widetilde{U}_\varepsilon$ and U_ε are given in (1.13) by taking $y = 0$, the origin point. By [58], we have the following estimations:

$$\|u_\varepsilon\|^2 = \int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \leq S^{\frac{N}{2s}} + O(\varepsilon^{N-2s}). \tag{4.6}$$

$$\int_{\mathbb{R}^N} u_\varepsilon^2 \mathrm{d}x = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}), & \text{if } N > 4s; \\ C\varepsilon^{2s}\log(1/\varepsilon), & \text{if } N = 4s; \\ C\varepsilon^{N-2s} + O(\varepsilon^{2s}), & \text{if } N < 4s. \end{cases} \tag{4.7}$$

$$\int_{\mathbb{R}^N} |u|_\varepsilon^{2^*_s}\mathrm{d}x = S^{\frac{N}{2s}} + O(\varepsilon^N). \tag{4.8}$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{\bar{p}}\mathrm{d}x = \begin{cases} C\varepsilon^{N-\frac{N-2s}{2}\bar{p}} + O(\varepsilon^{\frac{N-2s}{2}\bar{p}}), & \text{if } N > \frac{\bar{p}}{\bar{p}-1}2s; \\ C\varepsilon^{\frac{N}{2}}\log(1/\varepsilon) + O(\varepsilon^{\frac{N}{2}}), & \text{if } N = \frac{\bar{p}}{\bar{p}-1}2s; \\ C\varepsilon^{\frac{N-2s}{2}\bar{p}} + O(\varepsilon^{N-\frac{N-2s}{2}\bar{p}}), & \text{if } N < \frac{\bar{p}}{\bar{p}-1}2s \end{cases}$$
$$= \begin{cases} O(\varepsilon^{N-\frac{N-2s}{2}\bar{p}}), & \text{if } N > \frac{\bar{p}}{\bar{p}-1}2s; \\ O(\varepsilon^{\frac{N}{2}}|\log\varepsilon|), & \text{if } N = \frac{\bar{p}}{\bar{p}-1}2s; \\ O(\varepsilon^{\frac{N-2s}{2}\bar{p}}), & \text{if } N < \frac{\bar{p}}{\bar{p}-1}2s. \end{cases} \tag{4.9}$$

Using the Hardy-Littlewood-Sobolev inequality, on one hand, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}}|u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y &\leq \frac{\mathcal{C}_{N,\alpha,s}}{A_{N,\alpha}}\|u_\varepsilon\|_{2^*_s}^{22^*_{\alpha,s}}\\ =\frac{\mathcal{C}_{N,\alpha,s}}{A_{N,\alpha}}\left(S^{\frac{N}{2s}}+O(\varepsilon^N)\right)^{\frac{2N-\alpha}{N}} &= \frac{\mathcal{C}_{N,\alpha,s}}{A_{N,\alpha}}S^{\frac{2N-\alpha}{2s}}+O(\varepsilon^N).\end{aligned} \tag{4.10}$$

On the other hand, by the definition of u_ε, we have the estimate

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}}|u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \geq \int_{B_\delta}\int_{B_\delta}\frac{|u_\varepsilon(x)|^{2^*_{\alpha,s}}|u_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\ &=\int_{B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y-2\int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\ &\quad-\int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\ &=\frac{1}{\widetilde{\mathcal{C}}_{N,\alpha,s}^{22^*_{\alpha,s}}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{U}_\varepsilon(x)|^{2^*_{\alpha,s}}|\widetilde{U}_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y-2D-E\\ &=\frac{1}{A_{N,\alpha}\widetilde{\mathcal{C}}_{N,\alpha,s}^{22^*_{\alpha,s}}}S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}}-2D-E,\end{aligned} \tag{4.11}$$

where

$$D=\int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y$$

and

$$E=\int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta}\frac{|U_\varepsilon(x)|^{2^*_{\alpha,s}}|U_\varepsilon(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y.$$

By the Hardy-Littlewood-Sobolev inequality and a direct computation, we know

$$\begin{aligned}D&=\varepsilon^{2N-\alpha}\int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta}\frac{\beta_1^{22^*_{\alpha,s}}}{(\varepsilon^2+|x|^2)^{(2N-\alpha)/2}(\varepsilon^2+|y|^2)^{(2N-\alpha)/2}|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\ &\leq C_1\varepsilon^{2N-\alpha}\left(\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{(\varepsilon^2+|x|^2)^N}\mathrm{d}x\right)^{\frac{2N-\alpha}{2N}}\left(\int_{B_\delta}\frac{1}{(\varepsilon^2+|y|^2)^N}dy\right)^{\frac{2N-\alpha}{2N}}\\ &\leq C_1\varepsilon^{2N-\alpha}\left(\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{|x|^{2N}}\mathrm{d}x\right)^{\frac{2N-\alpha}{2N}}\left(\int_0^\delta\frac{r^{N-1}}{(\varepsilon^2+r^2)^N}dr\right)^{\frac{2N-\alpha}{2N}}\end{aligned}$$

$$
\begin{aligned}
&= O(\varepsilon^{\frac{2N-\alpha}{2}})\left(\int_0^{\delta/\varepsilon}\frac{\rho^{N-1}}{(1+\rho^2)^N}d\rho\right)^{\frac{2N-\alpha}{2N}}\\
&\le O(\varepsilon^{\frac{2N-\alpha}{2}})\left(\int_0^{+\infty}\frac{\rho^{N-1}}{(1+\rho^2)^N}d\rho\right)^{\frac{2N-\alpha}{2N}}\\
&= O(\varepsilon^{\frac{2N-\alpha}{2}}),
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
E &= \varepsilon^{2N-\alpha}\int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{(\varepsilon^2+|x|^2)^{(2N-\alpha)/2}(\varepsilon^2+|y|^2)^{(2N-\alpha)/2}|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\\
&\le C_1\varepsilon^{2N-\alpha}\left(\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{|x|^{2N}}\mathrm{d}x\right)^{\frac{2N-\alpha}{2N}}\left(\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{|y|^{2N}}dy\right)^{\frac{2N-\alpha}{2N}}\\
&= O(\varepsilon^{2N-\alpha}).
\end{aligned}
\tag{4.13}
$$

It follows from (4.11), (4.12) and (4.13), we have that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}\\
&\ge\left[\frac{1}{\widetilde{C}_{N,\alpha,s}^{22^*_{\alpha,s}}}S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}}-O(\varepsilon^{\frac{2N-\alpha}{2}})-O(\varepsilon^{2N-\alpha})\right]^{\frac{1}{2^*_{\alpha,s}}}\\
&=\left[\frac{1}{\widetilde{C}_{N,\alpha,s}^{22^*_{\alpha,s}}}S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}}-O(\varepsilon^{\frac{2N-\alpha}{2}})\right]^{\frac{1}{2^*_{\alpha,s}}}
\end{aligned}
\tag{4.14}
$$

Since $\widetilde{v}_\varepsilon\in C_c^\infty(\mathbb{R}^N)$, and $\widetilde{v}_\varepsilon\in S_{r,a}$, from Lemma 4.2, we see that

$$
m_{r,a,\mu}=\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}}I_\mu\le I_\mu(t_{\widetilde{v}_\varepsilon}\star\widetilde{v}_\varepsilon)=\max_{t\in\mathbb{R}}I_\mu(t\star\widetilde{v}_\varepsilon).
$$

We next give a upper estimation for

$$
I_\mu(t_{\widetilde{v}_\varepsilon}\star\widetilde{v}_\varepsilon)=\max_{t\in\mathbb{R}}I_\mu(t\star\widetilde{v}_\varepsilon).
$$

Step 1) Consider the case $\mu=0$ and estimate

$$
\max_{t\in\mathbb{R}}\Psi^0_{\widetilde{v}_\varepsilon}(t)=I_0(t\star\widetilde{v}_\varepsilon).
$$

In view of

$$
\Psi^0_{\widetilde{v}_\varepsilon}(t)=\frac{e^{2st}}{2}\|\widetilde{v}_\varepsilon\|^2-\frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x,
$$

it is easy to see that for each $\widetilde{v}_\varepsilon \in S_a$ the function $\Psi^0_{\widetilde{v}_\varepsilon}(t)$ has a unique critical point $t_{\varepsilon,0}$, which is a strict maximum point and is given by

$$e^{st_{\varepsilon,0}} = \left(\frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N} (I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}} \mathrm{d}x} \right)^{\frac{1}{22^*_{\alpha,s}-2}}. \tag{4.15}$$

Using the fact that

$$\sup_{t\geq 0} \left(\frac{t^2}{2} a - \frac{t^{22^*_{\alpha,s}}}{22^*_{\alpha,s}} b \right) = \frac{N+2s-\alpha}{2(2N-\alpha)} \left(\frac{a}{b^{1/2^*_{\alpha,s}}} \right)^{\frac{2N-\alpha}{N+2s-\alpha}}, \quad \text{for any fixed } \; a, b > 0,$$

we can deduce by (4.6), (4.14), that

$$\begin{aligned} \Psi^0_{\widetilde{v}_\varepsilon}(t_{\varepsilon,0}) &= \frac{N+2s-\alpha}{2(2N-\alpha)} \left(\frac{\|\widetilde{v}_\varepsilon\|^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}} \mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}} \right)^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} \left(\frac{\|u_\varepsilon\|^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}} \mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}} \right)^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &\leq \frac{N+2s-\alpha}{2(2N-\alpha)} \left(\frac{S^{\frac{N}{2s}} + O(\varepsilon^{N-2s})}{\left[\frac{1}{\widetilde{C}^{22^*_{\alpha,s}}_{N,\alpha,s}} S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}} - O(\varepsilon^{\frac{2N-\alpha}{2}}) \right]^{\frac{1}{2^*_{\alpha,s}}}} \right)^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} \frac{S^{\frac{N}{2s} \frac{2N-\alpha}{N+2s-\alpha}} \widetilde{C}_{N,\alpha,s}^{\frac{2(2N-\alpha)}{N+2s-\alpha}} \left(1 + O(\varepsilon^{N-2s})\right)}{S_{h,l}^{\frac{(2N-\alpha)(N-2s)}{(N-\alpha+2s)^2}} \left(1 - O(\varepsilon^{\frac{2N-\alpha}{2}})\right)} \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} \frac{S_{h,l}^{\frac{(2N-\alpha)^2}{(N-\alpha+2s)^2}} \left(1 + O(\varepsilon^{N-2s})\right)}{S_{h,l}^{\frac{(2N-\alpha)(N-2s)}{(N-\alpha+2s)^2}} \left(1 - O(\varepsilon^{\frac{2N-\alpha}{2}})\right)} \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + O(\varepsilon^{N-2s}). \end{aligned} \tag{4.16}$$

Step 2) Let $t_{\varepsilon,\mu}$ be the unique maximum point of the function

$$\begin{aligned} \Psi^\mu_{\widetilde{v}_\varepsilon}(t) = I_\mu(t \star \widetilde{v}_\varepsilon) &= \frac{e^{2st}}{2} \|\widetilde{v}_\varepsilon\|^2 \\ &\quad - \frac{\mu e^{2st}}{\bar{p}} \int_{\mathbb{R}^N} |\widetilde{v}_\varepsilon|^{\bar{p}} \mathrm{d}x - \frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}} \mathrm{d}x, \end{aligned}$$

and we estimate on $t_{\varepsilon,\mu}$. Since $P_\mu(t_{\varepsilon,\mu}\star\widetilde{v}_\varepsilon) = 0$, we have that

$$\begin{aligned} e^{(22^*_{\alpha,s}-2)st_{\varepsilon,\mu}} &= \frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha \times |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}} - \frac{2\mu}{\bar{p}} \frac{\|\widetilde{v}_\varepsilon\|^{\bar{p}}_{\bar{p}}}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}} \\ &\geq \left(1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s} a^{\frac{4s}{N}}\right) \frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}}, \end{aligned}$$

Step 3 Now we estimate on $\sup_{t\in\mathbb{R}} \Psi^\mu_{v_\varepsilon}$. By steps 1 and 2, we see that

$$\begin{aligned} &\Psi^\mu_{\widetilde{v}_\varepsilon}(t_{\varepsilon,\mu}) \\ &= \Psi^0_{\widetilde{v}_\varepsilon}(t_{\varepsilon,\mu}) - \frac{\mu}{\bar{p}} e^{2st_{\varepsilon,\mu}} \|\widetilde{v}_\varepsilon\|^{\bar{p}}_{\bar{p}} \\ &\leq \sup_{\mathbb{R}} \Psi^0_{\widetilde{v}_\varepsilon} - \frac{\mu}{\bar{p}} \left(1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s} a^{\frac{4s}{N}}\right)^{\frac{2}{22^*_{\alpha,s}-2}} \left(\frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}}\right)^{\frac{2}{22^*_{\alpha,s}-2}} \|\widetilde{v}_\varepsilon\|^{\bar{p}}_{\bar{p}} \\ &\leq \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + O(\varepsilon^{N-2s}) \\ &\quad - \frac{\mu}{\bar{p}} \left(1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s} a^{\frac{4s}{N}}\right)^{\frac{2}{22^*_{\alpha,s}-2}} \left(\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}}\right)^{\frac{2}{22^*_{\alpha,s}-2}} \frac{a^{\frac{4s}{N}} \|u_\varepsilon\|^{\bar{p}}_{\bar{p}}}{\|u_\varepsilon\|_2^{\frac{4s}{N}}} \\ &\leq \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + O(\varepsilon^{N-2s}) - C \frac{\|u_\varepsilon\|^{\bar{p}}_{\bar{p}}}{\|u_\varepsilon\|_2^{\frac{4s}{N}}} \end{aligned} \tag{4.17}$$

From (4.7)-(4.9), we have the following estimate:

$$\frac{\|u_\varepsilon\|^{\bar{p}}_{\bar{p}}}{\|u_\varepsilon\|_2^{\frac{4s}{N}}} = \begin{cases} C\varepsilon^{N - \frac{N-2s}{2}\bar{p} - \frac{4s^2}{N}} = C, & \text{if } N > 4s; \\ C\varepsilon^{4s - s\bar{p} - s} |\ln \varepsilon|^{-\frac{1}{2}} = C|\ln \varepsilon|^{-\frac{1}{2}}, & \text{if } N = 4s; \\ C\varepsilon^{N - \frac{N-2s}{2}\bar{p} - \frac{2s(N-2s)}{N}} = C\varepsilon^{\frac{2s(4s-N)}{N}}, & \text{if } \frac{\bar{p}}{\bar{p}-1}2s < N < 4s \\ C\varepsilon^{\frac{N}{2} - \frac{N-2s}{2}\frac{4s}{N}} |\ln \varepsilon|, & \text{if } \frac{\bar{p}}{\bar{p}-1}2s = N. \end{cases} \tag{4.18}$$

Therefore, we deduce from (4.17)-(4.18) that

$$m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu \leq \max_{t\in\mathbb{R}} \Psi^\mu_{\widetilde{v}_\varepsilon}(t) < \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},$$

and the proof is completed. □

5 L^2-Supercritical Perturbation

In this section, we deal with the case $N > 2s$ and $\bar{p} < q < 2^*_s$. We recall the following decomposition of the Pohozaev manifold:

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^{+} \cup \mathcal{N}_{a,\mu}^{0} \cup \mathcal{N}_{a,\mu}^{-}.$$

Lemma 5.1 *$\mathcal{N}_{a,\mu}^{0} = \emptyset$ and $\mathcal{N}_{a,\mu}$ is a smooth manifold of dimension 1 in S_a.*

Proof Suppose by contradiction that, there exists some $u \in \mathcal{N}_{a,\mu}^{0}$, then

$$\|u\|^2 = \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*} \mathrm{d}x, \tag{5.1}$$

and

$$2s^2\|u\|^2 = \mu q\gamma_{q,s}^2 s^2 \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + 22_{\alpha,s}^* s^2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*} \mathrm{d}x, \tag{5.2}$$

from which, we get

$$(2 - q\gamma_{q,s})\mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x = (22_{\alpha,s}^* - 2) \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*} \mathrm{d}x.$$

Since $2 - q\gamma_{q,s} < 0$, $22_{\alpha,s}^* - 2 > 0$, we see that $u = 0$, which is not possible since $u \in S_a$. The remainder parts of the proof is similar to that of Lemma 3.2, and so we omit the details here. □

Lemma 5.2 *Let $u \in S_a$, then there exists a unique $t_u \in \mathbb{R}$ such that $t_u \times u \in \mathcal{N}_{a,\mu}$, where t_u is the unique critical point of the function of Ψ_u^μ and is a strict maximum point at positive level. Moreover,*

(i) *$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^{-}$;*
(ii) *$\Psi_u^\mu(t)$ is strict decreasing and concave on $(t_u, +\infty)$ and $t_u < 0$ implies that $P_\mu(u) < 0$;*
(iii) *The map $u \in S_a : t_n \in \mathbb{R}$ is of C^1.*
(iv) *If $P_\mu(u) < 0$, then $t_u < 0$.*

Proof Since

$$\begin{aligned}\Psi_u^\mu(t) = I_\mu(t\star u) = \frac{e^{2st}}{2}\|u\|^2 - \frac{\mu e^{q\gamma_{q,s}st}}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\ -\frac{e^{22_{\alpha,s}^* st}}{22_{\alpha,s}^*} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*} \mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}(\Psi_u^\mu)'(t) = se^{2st}\|u\|^2 - \mu\gamma_{q,s} s e^{q\gamma_{q,s}st} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\ -se^{22_{\alpha,s}^* st} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*} \mathrm{d}x,\end{aligned}$$

it is easy to see that $(\Psi_u^\mu)'(t)=0$ if and only if

$$\|u\|^2=\mu\gamma_{q,s}e^{(q\gamma_{q,s}-2)st}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\triangleq f(t).$$

It is easy to see that $f(t)$ is positive, continuous and increasing and $f(t)\to 0^+$ as $t\to-\infty$ and $f(t)\to+\infty$ as $t\to+\infty$. Therefore, there exists a unique point t_u such that $t_u\star u\in\mathcal{N}_{a,\mu}$, where t_u is the unique critical point of $\Psi_u^\mu(t)$ and is a strict maximum point at positive level. Since t_u is the maximum point, we have that $(\Psi_u^\mu)''(t_u)\le 0$. By $\mathcal{N}^0_{a,\mu}=\emptyset$, we have $(\Psi_u^\mu)''(t_u)\ne 0$, this implies that $t_u\star u\in\mathcal{N}^-_{a,\mu}$, and $\mathcal{N}_{a,\mu}=\mathcal{N}^-_{a,\mu}$ since $\Psi_u^\mu(t)$ has exactly one maximum point. To see that the map $u\in S_a:t_n\in\mathbb{R}$ is of C^1, we can apply the implicit function theorem as in Lemma 3.3. Finally, since $(\Psi_u^\mu)'(t)<0$ if and only if $t>t_u$, so $P_\mu(u)=(\Psi_u^\mu)'(0)<0$ if and only if $t_u<0$. □

Lemma 5.3 $m_{a,\mu}=\inf_{\mathcal{N}_{a,\mu}}I_\mu>0$.

Proof Let $u\in\mathcal{N}_{a,\mu}$, then by the fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) and Sobolev inequality (1.12), we have

$$\begin{aligned}\|u\|^2&=\mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\\&\le\mu\gamma_{q,s}C_{N,q,s}a^{(1-\gamma_{q,s})q}\|u\|^{q\gamma_{q,s}}+S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}}.\end{aligned}$$

Hence, by above inequality and $u\in S_a$, we have

$$\mu\gamma_{q,s}C_{N,q,s}a^{(1-\gamma_{q,s})q}\|u\|^{q\gamma_{q,s}-2}+S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}-2}\ge 1,\quad\forall u\in\mathcal{N}_{a,\mu},$$

this implies that $\inf_{u\in\mathcal{N}_{a,\mu}}\|u\|>0$ and so

$$\inf_{u\in\mathcal{N}_{a,\mu}}\left[\mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right]>0.$$

Then, by $P_\mu(u)=0$ and the last inequality, we get

$$\begin{aligned}&\inf_{u\in\mathcal{N}_{a,\mu}}I_\mu(u)\\&=\inf_{u\in\mathcal{N}_{a,\mu}}\left[\frac{1}{2}\|u\|^2-\frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+\frac{1}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right]\\&=\inf_{u\in\mathcal{N}_{a,\mu}}\left[\frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2}-1\right)\int_{\mathbb{R}^N}|u|^q\mathrm{d}x+\frac{N+2s-\alpha}{2(2N-\alpha)}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right]>0.\end{aligned}$$

This completes the proof. □

Lemma 5.4 *For $k>0$ sufficiently small, we have $0<\sup_{\overline{A_k}} I_\mu < m_{a,\mu}$. Moreover,*

$$u\in\overline{A_k}\Rightarrow I_\mu(u)>0,\, P_\mu(u)>0,$$

where $A_k=\{u\in S_a:\ \|u\|^2<k\}$.

Proof By fractional Gagliardo-Nirenberg-Sobolev inequality (2.1) and Sobolev inequality (1.12), we have

$$I_\mu(u)\ge\frac{1}{2}\|u\|^2-\frac{\mu}{q}C_{N,q,s}a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}}-\frac{1}{22^*_{\alpha,s}}S_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}}>0,$$

and

$$\begin{aligned}P_\mu(u)&=s\|u\|^2-s\mu\gamma_{q,s}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x-s\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\\&\ge s\|u\|^2-s\mu\gamma_{q,s}C_{N,q,s}a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}}-sS_{h,l}^{-2^*_{\alpha,s}}\|u\|^{22^*_{\alpha,s}}>0,\end{aligned}$$

provided that $u\in\overline{A_k}$ for k small enough. By Lemma 5.3 we see that $m_{a,\mu}>0$, thus if necessary replacing k with smaller quantity, we also have

$$I_\mu(u)\le\frac{1}{2}\|u\|^2<m_{a,\mu}.$$

This completes the proof. □

In order to apply Proposition 2.2 and recover compactness, we need to estimate from above for the value $m_{r,a,\mu}=\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu$, where $S_{r,a}$ is the subset of the radial functions in S_a.

Lemma 5.5 *If one of the following conditions holds:*

(i) $N>4s$ *and* $\mu a^{q(1-\gamma_{q,s})}<\frac{K_3}{\gamma_{q,s}}$;
(ii) $N=4s$ *or* $\frac{q}{q-2}2s<N<4s$ *or* $2s<N\le\frac{q}{q-2}2s$.

Then we have $m_{r,a,\mu}<\frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}$.

Proof From the definition of $\widetilde{u}_\varepsilon$ and $\widetilde{v}_\varepsilon$ in Lemma 4.5, we know that $\widetilde{u}_\varepsilon\in C_0^\infty(\mathbb{R}^N,[0,1])$ and $\widetilde{v}_\varepsilon\in S_{r,a}$. By Lemma 4.2, we have

$$m_{r,a,\mu}=\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu\le I_\mu(t_{\widetilde{v}_\varepsilon,\mu}\star\widetilde{v}_\varepsilon)=\max_{t\in\mathbb{R}} I_\mu(t\star\widetilde{v}_\varepsilon).$$

By a similar argument as in the step 1 of Lemma 4.5, we have

$$\Psi^0_{\widetilde{v}_\varepsilon}(t_{\widetilde{v}_{\varepsilon,0}})\le\frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}+O(\varepsilon^{N-2s}).$$

Step 1) Let $t_{\widetilde{v}_\varepsilon}$ be the maximum point of

$$\begin{aligned}\Psi^{\mu}_{\widetilde{v}_\varepsilon}(t) &= I_\mu(t\star\widetilde{v}_\varepsilon)\\ &= \frac{e^{2st}}{2}\|\widetilde{v}_\varepsilon\|^2 - \frac{\mu}{q}e^{q\gamma_{q,s}st}\int_{\mathbb{R}^N}|v_\varepsilon|^q\,\mathrm{d}x - \frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha * |v_\varepsilon|^{2^*_{\alpha,s}})|v_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x\end{aligned}$$

and we estimate on $t_{\widetilde{v}_\varepsilon}$. By $(\Psi^{\mu}_{\widetilde{v}_\varepsilon})'(t_{\widetilde{v}_\varepsilon}) = P_\mu(t_{\widetilde{v}_\varepsilon}\star v_\varepsilon) = 0$, we get

$$\begin{aligned}&e^{22^*_{\alpha,s}st_{\widetilde{v}_\varepsilon}}\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x\\ &\quad= e^{2st_{\widetilde{v}_\varepsilon}}\|\widetilde{v}_\varepsilon\|^2 - \mu\gamma_{q,s}e^{q\gamma_{q,s}st}\int_{\mathbb{R}^N}|\widetilde{v}_\varepsilon|^q\,\mathrm{d}x \le e^{2st_{\widetilde{v}_\varepsilon}}\|\widetilde{v}_\varepsilon\|^2,\end{aligned}$$

which implies that

$$e^{st_{\widetilde{v}_\varepsilon}} \le \left(\frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\right)^{\frac{1}{22^*_{\alpha,s}-2}}. \tag{5.3}$$

From (5.3), $q\gamma_{q,s} > 2$ and $\widetilde{v}_\varepsilon = au_\varepsilon/\|u_\varepsilon\|_2$, we infer that

$$\begin{aligned}&e^{(22^*_{\alpha,s}-2)st_{\widetilde{v}_\varepsilon}}\\ &= \frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x} - \mu\gamma_{q,s}e^{(q\gamma_{q,s}-2)st_{\widetilde{v}_\varepsilon}}\frac{\int_{\mathbb{R}^N}|\widetilde{v}_\varepsilon|^q\,\mathrm{d}x}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{*}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\\ &\ge \frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x} - \mu\gamma_{q,s}\left(\frac{\|\widetilde{v}_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\\ &\quad\times\frac{\int_{\mathbb{R}^N}|\widetilde{v}_\varepsilon|^q\,\mathrm{d}x}{\int_{\mathbb{R}^N}(I_\alpha * |\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}})|\widetilde{v}_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\\ &= \frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\\ &- \mu\gamma_{q,s}\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-q}}{a^{22^*_{\alpha,s}-q}}\frac{\|u_\varepsilon\|_q^q}{\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\\ &\quad\times\left(\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\\ &= \frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\frac{\left(\|u_\varepsilon\|^2\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}}{\int_{\mathbb{R}^N}(I_\alpha * |u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\,\mathrm{d}x}\end{aligned}$$

$$\times\left[\left(\|u_\varepsilon\|^2\right)^{\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}-\frac{\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\right]. \tag{5.4}$$

By the estimates in (4.6), (4.10–4.14), we can infer that there exist constants $C_1, C_2, C_3 > 0$ depending only on N, q, s such that

$$\begin{aligned}&\left(\|u_\varepsilon\|^2\right)^{\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}\geq C_1 \ \text{ and } \ C_2\\ &\leq\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\leq\frac{1}{C_2}\end{aligned} \tag{5.5}$$

and

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\leq\begin{cases}C_3\varepsilon^{N-\frac{N-2s}{2}q-sq(1-\gamma_{q,s})}=C_3, & \text{if } N>4s;\\ C_3\varepsilon^{N-\frac{N-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}}, & \text{if } N=4s;\\ C_3\varepsilon^{N-\frac{N-2s}{2}q-\frac{(N-2s)q(1-\gamma_{q,s})}{2}}, & \text{if } \frac{q}{q-1}2s<N<4s;\\ C_3\varepsilon^{\frac{N}{2}-\frac{N-2s}{2}q(1-\gamma_{q,s})}|\ln\varepsilon|, & \text{if } N=\frac{q}{q-1}2s;\\ C_3\varepsilon^{\frac{N-2s}{2}q-\frac{N-2s}{2}q(1-\gamma_{q,s})}, & \text{if } 2s<N<\frac{q}{q-1}2s.\end{cases} \tag{5.6}$$

Next, we show that

$$e^{(22^*_{\alpha,s}-2)st_{\tilde{v}_\varepsilon}}\geq C\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}, \tag{5.7}$$

under suitable conditions.

Case 1: $N > 4s$. In this case, it holds that

$$\varepsilon^{N-\frac{N-2s}{2}q-sq(1-\gamma_{q,s})}=\varepsilon^0=1, \tag{5.8}$$

and from (5.4–5.6) we have

$$e^{(22^*_{\alpha,s}-2)st_{\tilde{v}_\varepsilon}}\geq\frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\left[C_1-\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}\right],$$

and we see that inequality (5.7) holds only when $\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\leq C_1C_2/C_3$. Thus, we have to give a more precise estimate, let us come back to (5.4) and observe that by well-known interpolation inequality, we have

$$\begin{aligned}
&\frac{\|u_\varepsilon\|_q^q}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\\
&\le\frac{\|u_\varepsilon\|_2^{\frac{2(2^*_s-q)}{2^*_s-2}}\|u_\varepsilon\|_{2^*_s}^{\frac{2^*_s(q-2)}{2^*_s-2}}}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\\
&=\frac{\|u_\varepsilon\|_{2^*_s}^{\frac{2^*_s(q-2)}{2^*_s-2}}}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}}.
\end{aligned}\tag{5.9}$$

Therefore, by (5.4) and (5.9) we have

$$\begin{aligned}
e^{(22^*_{\alpha,s}-2)st_{\widetilde{v}_\varepsilon}}\ge{}&\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\frac{\left(\|u_\varepsilon\|^2\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}}{\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x}\\
&\times\left[\left(\|u_\varepsilon\|^2\right)^{\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}-\frac{\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\|u_\varepsilon\|_{2^*_s}^{\frac{2^*_s(q-2)}{2^*_s-2}}}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}}\right].
\end{aligned}\tag{5.10}$$

From the estimations (4.10),(4.6) and (4.14), we see that the right hand side of (5.10) is positive provided that

$$\begin{aligned}
\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}&<\frac{\left(\|u_\varepsilon\|^2\right)^{\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}}{\|u_\varepsilon\|_{2^*_s}^{\frac{2^*_s(q-2)}{2^*_s-2}}}\left(\int_{\mathbb{R}^N}(I_\alpha*|u_\varepsilon|^{2^*_{\alpha,s}})|u_\varepsilon|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\\
&\le\frac{\left(S^{\frac{N}{2s}}+O(\varepsilon^{N-2s})\right)^{\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}}}{\left(S^{\frac{N}{2s}}+O(\varepsilon^N)\right)^{\frac{q-2}{2^*_s-2}}}\left(\mathcal{C}_{N,\alpha,s}S^{\frac{2N-\alpha}{2s}}+O(\varepsilon^N)\right)^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\\
&=S^{\frac{N}{2s}\left(\frac{22^*_{\alpha,s}-q\gamma_{q,s}}{22^*_{\alpha,s}-2}-\frac{q-2}{2^*_s-2}\right)+\frac{(2N-\alpha)(q\gamma_{q,s}-2)}{2s(22^*_{\alpha,s}-2)}}\mathcal{C}_{N,\alpha,s}^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\left(1+O(\varepsilon^{N-2s})\right)\\
&=S^{\frac{(N-\alpha)(2^*_s-2)+2s(2^*_s-q\gamma_{q,s})}{2s(22^*_{\alpha,s}-2)}}\mathcal{C}_{N,\alpha,s}^{\frac{q\gamma_{q,s}-2}{22^*_{\alpha,s}-2}}\left(1+O(\varepsilon^{N-2s})\right)=K_3+O(\varepsilon^{N-2s}),
\end{aligned}$$

where K_3 is given in (1.21). Therefore, if $N>4s$ and $\mu a^{q(1-\gamma_{q,s})}<\frac{K_3}{\gamma_{q,s}}$, we have

$$e^{(22^*_{\alpha,s}-2)st_{\widetilde{v}_\varepsilon}}\ge\frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}.$$

Case 2: $N = 4s$. In this case we have $3 < q < 4$, and

$$\varepsilon^{N-\frac{N-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = |\ln\varepsilon|^{q-4} \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{5.11}$$

Consequently,

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C_3\varepsilon^{N-\frac{N-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = o_\varepsilon(1).$$

Therefore, we get

$$e^{(22^*_{\alpha,s}-2)st_{\tilde{v}_\varepsilon}} \ge C\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\left[C_1 - \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}o_\varepsilon(1)\right] \ge \frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}.$$

Case 3: $\frac{q}{q-1}2s < N < 4s$. By the definition of $\gamma_{q,s}$ and a direct computation we get

$$\begin{aligned} &N - \frac{N-2s}{2}q - \frac{(N-2s)q(1-\gamma_{q,s})}{2} \\ &= (N-2s)\left[\frac{N}{N-2s} - q - \frac{(q-2)N}{4s}\right] = \frac{N-4s}{4s}\left[q - \frac{2N}{N-2s}\right](N-2s) > 0. \end{aligned}$$

Thus,

$$\varepsilon^{N-\frac{N-2s}{2}q-\frac{(N-2s)q(1-\gamma_{q,s})}{2}} \to 0 \ \text{ as } \ \varepsilon \to 0,$$

and so

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \le C\varepsilon^{N-\frac{N-2s}{2}q-\frac{(N-2s)q(1-\gamma_{q,s})}{2}} = o_\varepsilon(1).$$

Therefore, we get

$$e^{(22^*_{\alpha,s}-2)st_{\tilde{v}_\varepsilon}} \ge C\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\left[C_1 - \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}o_\varepsilon(1)\right] \ge \frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}.$$

Case 4: $\frac{q}{q-1}2s = N$. In this case, we may rewrite

$$\frac{N}{2} - \frac{N-2s}{2}q(1-\gamma_{q,s}) = \frac{N}{2} - \frac{N-2s}{2}(q-2) - (N-2s) + \frac{N(N-2s)}{4s}(q-2).$$

Set $h(t)=\frac{N}{2}-\frac{N-2s}{2}t-(N-2s)+\frac{N(N-2s)}{4s}t$, we see that $h(t)$ is increasing about $t\in\mathbb{R}$. Thus,

$$\frac{N}{2}-\frac{N-2s}{2}q(1-\gamma_{q,s})=h(q-2)>h(0)=2s-\frac{N}{2}>0.$$

Using the L'Hospital's principle, we have

$$\varepsilon^{\frac{N}{2}-\frac{N-2s}{2}q(1-\gamma_{q,s})}|\ln\varepsilon|\to 0 \ \text{ as } \ \varepsilon\to 0.$$

Therefore, we get

$$e^{(22^*_{\alpha,s}-2)st_{\widetilde{v}_\varepsilon}}\geq C\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\left[C_1-\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}o_\varepsilon(1)\right]\geq\frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}.$$

Case 5: $2s<N<\frac{q}{q-1}2s$. In this case we see that

$$\frac{N-2s}{2}q-\frac{N-2s}{2}q(1-\gamma_{q,s})=\frac{N-2s}{2}q\gamma_{q,s}>0,$$

and so

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\leq C\varepsilon^{\frac{N-2s}{2}q-\frac{N-2s}{2}q(1-\gamma_{q,s})}=o_\varepsilon(1).$$

Thus, we have

$$e^{(22^*_{\alpha,s}-2)st_{\widetilde{v}_\varepsilon}}\geq C\frac{\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}\left[C_1-\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_3}{C_2}o_\varepsilon(1)\right]\geq\frac{C\|u_\varepsilon\|_2^{22^*_{\alpha,s}-2}}{a^{22^*_{\alpha,s}-2}}.$$

Step 2 We estimate for $\max_{t\in\mathbb{R}}\Psi^\mu_{\widetilde{v}_\varepsilon}(t)$. Note that

$$\begin{aligned}\max_{t\in\mathbb{R}}\Psi^\mu_{\widetilde{v}_\varepsilon}(t)&=\Psi^\mu_{\widetilde{v}_\varepsilon}(t_{\widetilde{v}_\varepsilon})=\Psi^0_{\widetilde{v}_\varepsilon}(t_{\widetilde{v}_\varepsilon})-\mu\frac{e^{q\gamma_{q,s}st_{\widetilde{v}_\varepsilon}}}{q}\int_{\mathbb{R}^N}|\widetilde{v}_\varepsilon|^q\mathrm{d}x\\&\leq\sup_{\mathbb{R}}\Psi^0_{v_\varepsilon}-\frac{C\mu}{q}\frac{\|u_\varepsilon\|_2^{q\gamma_{q,s}}}{a^{q\gamma_{q,s}}}\frac{a^q}{\|u_\varepsilon\|_2^q}\int_{\mathbb{R}^N}|u_\varepsilon|^q\mathrm{d}x\\&=\sup_{\mathbb{R}}\Psi^0_{v_\varepsilon}-\frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q\mathrm{d}x}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\\&\leq\frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}+O(\varepsilon^{N-2s})-\frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\int_{\mathbb{R}^N}|u_\varepsilon|^q\mathrm{d}x}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}.\end{aligned}\tag{5.12}$$

Using estimation (5.6), we can derive that

$$m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu \leq \max_{t\in\mathbb{R}} \Psi^{\mu}_{\widetilde{v}_\varepsilon}(t) < \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}, \tag{5.13}$$

for $\varepsilon > 0$ small enough, which is the desired result. □

6 Proof of Theorem 1.1

Let $\{v_n\}$ be a minimizing sequence for $\inf_{A_{R_0}} I_\mu$, and we may assume that $\{v_n\} \subset S_{r,a}$ is radially decreasing for every $n \in \mathbb{N}$ (if this is not the case, we can replace v_n with $|v_n|^*$, the Schwarz rearrangement of $|v_n|$, and we obtain another function in A_{R_0} with $I_\mu(|v_n|^*) \leq I_\mu(|v_n|)$. By Lemma 3.3, for every n we can take $\alpha_{v_\varepsilon}\star v_n \in \mathcal{N}^+_{a,\mu}$ such that $\|\alpha_{v_n}\star v_n\| \leq R_0$ and

$$I_\mu(\alpha_{v_n}\star v_n) = \min\{I_\mu(t\star v_n) : \ t \in \mathbb{R} \ \text{ and } \ \|t\star v_n\| \leq R_0\} \leq I_\mu(v_n).$$

Therefore, we can get a new minimizing sequence $\{w_n = \alpha_{v_n}\star v_n\}$ with $w_n \in S_{r,a} \cap \mathcal{N}^+_{a,\mu}$ radially decreasing for each n. By Lemma 3.4, we have $\|w_n\| \leq R_0 - r$ for every n and hence by Ekeland's variational principle in a standard way, we know that the existence of a new minimizing sequence $\{u_n\} \subset A_{R_0}$ for $m_{a,\mu}$ with $\|w_n - u_n\| \to 0$ as $n \to +\infty$, which is also a Palais-Smale sequence for I_μ on S_a. Combining the boundedness of $\{w_n\}$, $\|w_n - u_n\| \to 0$, Brezis-Lieb lemma and Sobolev embedding theorem, we infer to

$$\begin{aligned}
&\|u_n\|^2 = \|u_n - w_n\|^2 + \|w_n\|^2 + o_n(1) = \|w_n\|^2 + o_n(1),\\
&\int_{\mathbb{R}^N} |u_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} |u_n - w_n|^p \mathrm{d}x + \int_{\mathbb{R}^N} |w_n|^p \mathrm{d}x + o_n(1) = \int_{\mathbb{R}^N} |w_n|^p \mathrm{d}x + o_n(1),
\end{aligned}$$

for $\forall p \in [2, 2^*_s]$. Now, we set

$$|u_n|^{2^*_{\alpha,s}} = |w_n|^{2^*_{\alpha,s}} + 2^*_{\alpha,s}|w_n + \theta_n(u_n - w_n)|^{2^*_{\alpha,s}-1}(u_n - w_n), \ \ x \in \mathbb{R}^N,$$

where $\theta_n = \theta_n(x) \in [0, 1]$. Thus, by the fact that $\|u_n - w_n\| \to 0$ as $n \to \infty$, and the Hardy-Littlewood-Sobolev inequality (1.12), we can deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}} \mathrm{d}x = \int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}})|w_n|^{2^*_{\alpha,s}} \mathrm{d}x\\
&\quad + 22^*_{\alpha,s} \int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}})[|w_n + \theta_n(u_n - w_n)|^{2^*_{\alpha,s}-1}(u_n - w_n)]\mathrm{d}x\\
&\quad + (2^*_{\alpha,s})^2 \int_{\mathbb{R}^N} (I_\alpha * [|w_n + \theta_n(u_n - w_n)|^{2^*_{\alpha,s}-1}(u_n - w_n)])\\
&\qquad\qquad \times [|w_n + \theta_n(u_n - w_n)|^{2^*_{\alpha,s}-1}(u_n - w_n)]\mathrm{d}x\\
&= \int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}})|w_n|^{2^*_{\alpha,s}} \mathrm{d}x + o_n(1).
\end{aligned}$$

Thus,

$$P_\mu(u_n) = P_\mu(w_n) + o_n(1) \to 0 \text{ as } n \to +\infty.$$

Hence, one of the alternative in Proposition 2.2 occurs. We show that the second alternative in Proposition 2.2 holds. Suppose by contradiction that, there exists a sequence $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$ but not strongly, where $u \not\equiv 0$ is a solution of (1.8) for some $\lambda < 0$, and

$$I_\mu(u) \le m_{a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Since u being a solution of (1.8), by the Pohozaev identity we have $P_\mu(u) = 0$, that is

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x.$$

Thus, by the Gagliardo-Nirenberg inequality

$$\begin{aligned} m_{a,\mu} &\ge I_\mu(u) + \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + \frac{N+2s-\alpha}{2(2N-\alpha)} \|u\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right) \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \\ &\ge \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + \frac{N+2s-\alpha}{2(2N-\alpha)} \|u\|^2 \\ &\quad - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right) C_{N,q,s} a^{q(1-\gamma_{q,s})} \|u\|^{q\gamma_{q,s}}. \end{aligned}$$

Next, we show that the right side of the above inequality is positive, which leads to a contradiction with $m_{a,\mu} < 0$. For this aim, Let

$$g(t) = \frac{N+2s-\alpha}{2(2N-\alpha)} t^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{22^*_{\alpha,s}}\right) C_{N,q,s} a^{q(1-\gamma_{q,s})} t^{q\gamma_{q,s}}, \quad \forall t \ge 0.$$

Since $q\gamma_{q,s} < 2$, the function $g(t)$ has has a global minimum at negative level when $t = t_{\min} > 0$ such that

$$\begin{aligned} g(t_{\min}) &= \min_{t\ge 0} g(t) \\ &= -\frac{1}{2}\left[\mu a^{q(1-\gamma_{q,s})}\right]^{\frac{2}{2-q\gamma_{q,s}}} \left(\frac{C}{q} \frac{22^*_{\alpha,s} - q\gamma_{q,s}}{22^*_{\alpha,s}}\right)^{\frac{2}{2-q\gamma_{q,s}}} \left[\frac{2N-\alpha}{N+2s-\alpha}\right]^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}} \\ &< 0. \end{aligned}$$

By the assumption (1.22), one has

$$\mu a^{q(1-\gamma_{q,s})} < \frac{N+2s-\alpha}{2N-\alpha} \frac{22^*_{\alpha,s} q}{(22^*_{\alpha,s} - q\gamma_{q,s})C_{N,q,s}} S_{h,l}^{\frac{(2N-\alpha)(2-q\gamma_{q,s})}{2(N+2s-\alpha)}},$$

which implies that

$$g(t_{\min}) > -\frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Therefore, we obtain

$$m_{a,\mu} \geq \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + g(\|u\|) \geq \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} + g(t_{\min}) > 0,$$

which contradicts to the fact that $m_{a,\mu} < 0$. Hence, $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$, $I_\mu(u) = m_{a,\mu}$ and u solves (1.8) for some $\lambda < 0$. In order to show that any ground state is a local minimizer for I_μ on A_{R_0}, we use the fact that $I_\mu(u) = m_{a,\mu} < 0$, and then $u \in \mathcal{N}_{a,\mu}$, so by Lemma 3.3 we have that $u \in \mathcal{N}^+_{a,\mu} \subset A_{R_0}$ and

$$I_\mu(u) = m_{a,\mu} = \inf_{A_{R_0}} I_\mu \quad \text{and} \quad \|u\| < R_0.$$

We next prove that the ground state solution is positive. Put $u^+ = \max\{u, 0\}$ the positive part of u. We note that all the calculations above can be repeated word by word, replacing J with the functional

$$\begin{aligned} I^+_\mu(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \\ &\quad - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u^+|^{2^*_{\alpha,s}})|u^+|^{2^*_{\alpha,s}} \mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^N} |u^+|^q \mathrm{d}x \end{aligned} \tag{6.1}$$

Using $u^- = \min\{u, 0\}$ as a test function in (6.1), in view of $(I^+_\mu)'(u)u^- = 0$, and $(a-b)(a^- - b^-) \geq |a^- - b^-|^2$, we conclude that

$$\|u^-\|^2 \leq \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))((u^-(x)-u^-(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y = 0.$$

Thus, $u^- = 0$ and $u \geq 0$ is a solution of (1.8). By some arguments from [14, 37], we can obtain that $u \in L^\infty(\mathbb{R}^3) \cap C^{0,\alpha}(\mathbb{R}^3)$ for some $\alpha \in (0, 1)$. Next we only need to prove that the solution u is positive. Otherwise, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^3$, then $(-\Delta)^s u(x_0) = 0$ and by the definition of $(-\Delta)^s$, we have [19]:

$$(-\Delta)^s u(x_0) = -\frac{C_s}{2}\int_{\mathbb{R}^3} \frac{u(x_0+y) + u(x_0-y) - 2u(x_0)}{|y|^{3+2s}} \mathrm{d}y.$$

Hence, $\int_{\mathbb{R}^3} \frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}} \mathrm{d}y = 0$, which means that $u \equiv 0$, a contradiction. Thus, $u(x) > 0$ in $\mathbb{R}^3$. This completes the proof. □

7 Proof of Theorems 1.2 and 1.3

We first collect some preliminary results, which are useful in proving Theorems 1.2, 1.3. These materials can be found in [25].

Definition 7.1 ([25]) Let B be a closed subset of X. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with boundary B if

(i) each set in $\mathcal{F}$ contains B.
(ii) for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (0 \times X) \cup ([0,1] \times B)$, we have $\eta(1 \times A) \in \mathcal{F}$.

Proposition 7.1 ([25]) *Let ψ be a C^1 function on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy-stable family $\mathcal{F}$ of compact subset of X with a closed boundary B. Set $c = c(\psi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} \psi(x)$ and suppose that*

$$\sup_{x \in B} \psi(x) < c.$$

Then, for any sequence of sets $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that $\lim_n \sup_{A_n} \psi = c$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that

$$\lim_{n \to +\infty} \psi(x_n) = c, \quad \lim_{n \to +\infty} \|d\psi(x_n)\| = 0 \ \textit{and} \ \lim_{n \to +\infty} \textit{dist}(x_n, A_n) = 0.$$

Moreover, if $d\psi$ is uniformly continuous, then x_n can be chosen to be in A_n for each n.

Now we are ready for the Proof of Theorems 1.2,1.3.
Case 1. L^2-critical perturbation for $q = \bar{p}$. Let $k > 0$ be defined by Lemma 4.4, we use the ideas introduced in [31] and use the functional $\widetilde{I}_\mu : \mathbb{R} \times H^s(\mathbb{R}^N) \to \mathbb{R}$ as

$$\begin{aligned} \widetilde{I}_\mu(t,u) :&= I_\mu(t \times u) = \left[\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{p}} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x\right] e^{2st} \\ &\quad - \frac{e^{22^*_{\alpha,s}}}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x. \end{aligned} \tag{7.1}$$

Clearly, $\widetilde{I}_\mu$ is of class of C^1, and $\widetilde{I}_\mu$ is invariant under rotations applied to u, a Palais-Smale sequence for $\widetilde{I}_\mu|_{\mathbb{R} \times S_{r,a}}$ is a palais-Smale sequence $\widetilde{I}_\mu|_{\mathbb{R} \times S_a}$. Set I_μ^c be the closed sublevel set $\{u \in S_a : I_\mu(u) \le c\}$, we give the minimax class

$$\Gamma := \{\gamma = (\alpha, \beta) \in C([0,1], \mathbb{R} \times S_{r,a}) | \gamma(0) \in (0, \overline{A_k}), \gamma(1) \in (0, I_\mu^0)\} \tag{7.2}$$

with associated minimax level

$$\sigma(a,\mu) := \inf_{\gamma\in\Gamma} \max_{(t,u)\in\gamma([0,1])} \widetilde{I}_\mu(t,u).$$

Since $\|t\star u\|^2 \to 0^+$ as $t \to -\infty$ and $I_\mu(t\star u) \to -\infty$ as $t \to +\infty$. Let $u \in S_{r,a}$. There exist $t_0 \ll -1$ and $t_1 \gg 1$ such that

$$\gamma_u : \tau \in [0,1] \to (0, ((1-\tau)t_0 + \tau t_1) \times u) \in \mathbb{R} \times S_{r,a} \tag{7.3}$$

is a path in Γ. Then $\sigma(a,\mu)$ is a real number. Now, for any $\gamma = (\alpha,\beta) \in \Gamma$, we introduce the function

$$T_\gamma : t \in [0,1] \to P_\mu(\alpha(t)\star\beta(t)) \in \mathbb{R}.$$

By Lemmas 4.3, 4.4 we see that $T_\gamma(0) = P_\mu(\beta(0)) > 0$. By virtue of $\Psi^\mu_{\beta(1)}(t) > 0$ for each $t \in (-\infty, t_{\beta(1)})$ and $\Psi^\mu_{\beta(1)}(0) = I_\mu(\beta(1)) \le 0$, we have $t_{\beta(1)} < 0$. Therefore, by Lemma 4.2, we have $T_\gamma(1) = P_\mu(\beta(1)) < 0$. Moreover, the map $\tau \mapsto \alpha(\tau)\star\beta(\tau)$ is continuous from $[0,1]$ to $H^s(\mathbb{R}^N)$, so we infer that there exists $\tau_\gamma \in (0,1)$ such that $T_\gamma(\tau_\gamma) = 0$, consequently, $\alpha(\tau_\gamma)\star\beta(\tau_\gamma) \in \mathcal{N}_{a,\mu}$, which implies that

$$\max_{\gamma([0,1])} \widetilde{I}_\mu \ge \widetilde{I}_\mu(\gamma(\tau_\gamma)) = I_\mu(\alpha(\tau_\gamma) * \beta(\tau_\gamma)) \ge \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} = m_{r,a,\mu}.$$

Therefore, $\sigma(a,\mu) \ge m_{r,a,\mu}$. On the other hand, if $u \in \mathcal{N}_{a,\mu} \cap S_{r,a}$, then γ_u defined in (7.3) is a path in Γ with

$$I_\mu(u) = \max_{\gamma_u([0,1])} \widetilde{I}_\mu \ge \sigma(a,\mu),$$

which implies that

$$m_{r,a,\mu} \ge \sigma(a,\mu).$$

Combining this with Lemmas 4.3, 4.4, we obtain

$$\sigma(a,\mu) = m_{r,a,\mu} > \sup_{(\overline{A_k}\cup I^c_\mu)\cap S_{r,a}} I_\mu = \sup_{(0,\overline{A_k}\cup(0,I^c_\mu))\cap(\mathbb{R}\times S_{r,a})} \widetilde{I}_\mu.$$

Applying Proposition 7.1 we see that $\{\gamma([0,1]) : \gamma \in \Gamma\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_{r,a}$ with closed boundary $(0,\overline{A_k}) \cup (0, I^c_\mu)$ and the superlevel set $\{\widetilde{I}_\mu \ge \sigma(a,\mu)\}$ is a dual set for Γ. By Proposition 7.1, we can take any minimizing sequence $\{\gamma_n = (\alpha_n,\beta_n)\} \subset \Gamma_n$ for $\sigma(a,\mu)$ with the property that $\alpha_n = 0$ and $\beta_n(\tau) \ge 0$ a.e. in $\mathbb{R}^N$ for every $\tau \in [0,1]$, there exists a Pais-Smale sequence $\{(t_n,w_n)\} \subset \mathbb{R} \times S_{r,a}$ for $\widetilde{I}_\mu|_{\mathbb{R}\times S_{r,a}}$ at level $\sigma(a,\mu)$, such that

$$\partial_t \widetilde{I}_\mu(t_n,w_n) \to 0 \ \text{ and } \ \|\partial_u \widetilde{I}_\mu(t_n,w_n)\| \to 0 \ \text{ as } \ n \to +\infty, \tag{7.4}$$

with the property that

$$|t_n| + \mathrm{dist}_{H^s}(w_n, \beta_n([0,1])) \to 0 \ \text{ as } \ n \to +\infty. \tag{7.5}$$

By the definition of $\widetilde{I}_\mu(t_n, w_n)$ in (7.1), and (7.4) we infer that $P_\mu(t_n \star w_n) \to 0$, that is

$$\begin{aligned} dI_\mu(t_n \star w_n)[t_n \star \phi] &= o_n(1)\|\phi\| = o_n(1)\|t_n \star \phi\| \\ &\text{as } \ n \to +\infty \ \text{ for each } \ \phi \in T_{w_n} S_{r,a}. \end{aligned} \tag{7.6}$$

Let $u_n = t_n \star w_n$, by (7.6), we see that $\{u_n\}$ is a Palais-Smale sequence for $I_\mu|_{S_{r,a}}$ at level $\sigma(a,\mu) = m_{r,a,\mu}$ and $P_\mu(u_n) \to 0$. Hence, by Lemmas 4.3–4.5, we infer that $m_{r,a,\mu} \in (0, \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}})$, so by Proposition 2.2, one of the alternatives occurs.

Assume (i) of Proposition 2.2 occurs, then up to a subsequence $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$ but not strongly, where $u \not\equiv 0$ is a solution of (1.8) for some $\lambda < 0$, and

$$I_\mu(u) \le m_{r,a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} < 0.$$

Thus, by Pohozaev identity, $P_\mu(u) = 0$ holds, which implies that

$$\|u\|^2 - \frac{2\mu}{\bar{p}} \int_{\mathbb{R}^N} |u|^{\bar{p}} \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x = 0.$$

Therefore,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{P}} \int_{\mathbb{R}^N} |u|^{\bar{p}} \mathrm{d}x - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x \\ &= \frac{N+2s-\alpha}{2(2N-\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x > 0. \end{aligned}$$

This contradicts the fact that

$$I_\mu(u) \le m_{r,a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} < 0.$$

Thus, the alternative (ii) of Proposition 2.2 holds. There exists a subsequence $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$, $I_\mu(u) = m_{r,a,\mu}$ and u solves (1.8) for some $\lambda < 0$. By $\beta_n(\tau) \ge 0$ a.e. in $\mathbb{R}^N$, (7.5) and the convergence implies that $u \ge 0$, and so by Proposition 2.17 in [60], we see that u is positive. Finally, we show that u is a ground state solution. Note that any normalized solution in $\mathcal{N}_{a,\mu}$ satisfies that

$$I_\mu(u) = m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu} \cap S_{r,a}},$$

It is sufficient to show that

$$\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu = \inf_{\mathcal{N}_{a,\mu}} I_\mu = m_{a,\mu}.$$

Suppose by contradiction that, there is a $w \in \mathcal{N}_{a,\mu}\backslash S_{r,a}$ such that $I_\mu(w) < \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu$. Let $v = |w|^*$ be the symmetric decreasing rearrangement of w. Then by the properties of symmetric decreasing rearrangement, we have

$$\|v\|^2 \leq \|w\|^2, \quad I_\mu(v) \leq I_\mu(w) \text{ and } P_\mu(v) \leq 0 = P_\mu(w).$$

If $P_\mu(v) = 0$, then $P_\mu(v) = P_\mu(w) = 0$, a contradiction to the above inequalities. If $P_\mu(v) < 0$, then by Lemma 4.2, we have that $t_v < 0$ satisfying

$$\begin{aligned} I_\mu(w) &\leq I_\mu(t_v \star v) \\ &= e^{22^*_{\alpha,s} s t_v} \frac{N+2s-\alpha}{2(2N-\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |v|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} \mathrm{d}x \\ &= e^{22^*_{\alpha,s} s t_v} I_\mu(w) < I_\mu(w), \end{aligned}$$

which is a contradiction, here we we use the fact that $t_v * v, u \in \mathcal{N}_{a,\mu}$. Thus,

$$m_{a,\mu} = m_{r,a,\mu},$$

and so, u is a ground state solution.

Case 2: L^2-supercritical perturbation for $\bar{p} < q < 2^*_s$. Proceeding exactly as in the case $q = \bar{p}$, we can obtain a Palais-Smale sequence $\{u_n\} \subset S_{r,a}$ for $I_\mu|_{S_a}$ at level $\sigma(a,\mu) = m_{r,a,\mu}$ and $P_\mu(u_n) \to 0$. Therefore, by Lemma 5.5, we have that $m_r(a,\mu) \in (0, \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}})$, so by Proposition 2.2, one of the alternatives occurs.

Assume (i) of Proposition 2.2 occurs, then up to a subsequence $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$ but not strongly, where $u \not\equiv 0$ is a solution of (1.8) for some $\lambda < 0$, and

$$I_\mu(u) \leq m_{r,a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} < 0.$$

Thus, by the Pohozaev identity, $P_\mu(u) = 0$ holds, which implies that

$$\|u\|^2 - \mu\gamma_{q,s} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x = 0.$$

Therefore, by $q\gamma_{q,s} > 2$, we get

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x \\ &= \frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2} - 1\right) \int_{\mathbb{R}^N} |u|^q \mathrm{d}x + \frac{N+2s-\alpha}{2(2N-\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x > 0. \end{aligned}$$

This contradicts the fact that

$$I_\mu(u) \le m_{r,a,\mu} - \frac{N+2s-\alpha}{2(2N-\alpha)} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} < 0.$$

Thus, the alternative (ii) of Proposition 2.2 holds. There exists a subsequence $u_n \to u$ strongly in $H^s(\mathbb{R}^N)$, $I_\mu(u) = m_{a,\mu}$ and u solves (1.8) for some $\lambda < 0$. By $\beta_n(\tau) \ge 0$ a.e. in $\mathbb{R}^N$, (7.5) and the convergence implies that $u \ge 0$, and by Sect. 6, we see that u is positive. The remainder part of the proof is similar to that of Case 1. This completes the proof. □

8 Proof of Theorem 1.4

In the case $\mu = 0$, the functional of (1.8) is given by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x$$

on S_a. The associated Pohozaev identity is

$$\begin{aligned}\mathcal{N}_{a,0} &= \left\{u \in S_a : s\|u\|^2 - s\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x = 0\right\} \\ &= \left\{u \in S_a : (\Psi_u^0)'(0) = 0\right\},\end{aligned}$$

where

$$\Psi_u^0(t) = \frac{e^{2st}}{2}\|u\|^2 - \frac{e^{22^*_{\alpha,s}st}}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x.$$

Recall the decomposition

$$\mathcal{N}_{a,0} = \mathcal{N}_{a,0}^+ \cup \mathcal{N}_{a,0}^0 \cup \mathcal{N}_{a,0}^-.$$

It is easy to see that for each $u \in S_a$, the function $\Psi_u^0(t)$ has a unique critical point $t_{u,0}$, which achieves a strict maximum point and is given by

$$e^{st_{u,0}} = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x}\right)^{\frac{1}{22^*_{\alpha,s}-2}}. \tag{8.1}$$

From the definition of $\mathcal{N}_{a,0}^+$, we see that $\mathcal{N}_{a,0}^+ = \emptyset$. If $u \in \mathcal{N}_{a,0}^0$, then $u \in \mathcal{N}_{a,0}$ and $(\Psi_u^0)''(0) = 0$, that is,

$$2\|u\|^2 = 22^*_{\alpha,s} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x = 22^*_{\alpha,s}\|u\|^2,$$

which implies that $\|u\| = 0$, contradicting to $u \in S_a$. Then $\mathcal{N}_{a,0} = \mathcal{N}_{a,0}^{-}$.

Next, we show that $\mathcal{N}_{a,0}$ is a smooth manifold codimension 1 on S_a. In view of

$$\mathcal{N}_{a,0} = \left\{ u \in S_a : \|u\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x = 0 \right\},$$

$\mathcal{N}_{a,0}$ can be characterized as $P_0(u) = 0$, $G(u) = 0$, where

$$P_0(u) = s\|u\|^2 - s22^*_{\alpha,s} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x \quad \text{and} \quad G(u) = \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2.$$

Since $P_0(u)$ and $G(u)$ are class of C^1, it is sufficient to check that $d(P_0(u), G(u))$: $H^s(\mathbb{R}^N) \to \mathbb{R}^2$ is surjective. If this is not true, $dP_0(u)$ has to be linearly dependent from $dG(u)$, that is, there exist a $\nu \in \mathbb{R}$ such that

$$2s \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \phi \mathrm{d}x - s22^*_{\alpha,s} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2} u\phi \mathrm{d}x = \nu \int_{\mathbb{R}^N} u\phi \mathrm{d}x$$

for each $\phi \in H^s(\mathbb{R}^N)$, which shows that

$$2s(-\Delta)^s u = \nu u + s22^*_{\alpha,s} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2} u \quad x \in \mathbb{R}^N.$$

Using the Pohozaev identity for above equation, we infer that

$$2s\|u\|^2 = 22^*_{\alpha,s} s \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}} \mathrm{d}x,$$

that is $u \in \mathcal{N}_{a,0}^0$, which yields a contradiction. Thus, $u \in \mathcal{N}_{a,0}$ is a natural constraint. Indeed, if $u \in \mathcal{N}_{a,0}$ is a critical point of $I_0|_{\mathcal{N}_{a,0}}$, then u is a critical point of $I_0|_{S_a}$. Thus, for each $u \in S_a$, there exist a unique $t_{u,0} \in \mathbb{R}$ such that $t_{u,0} \star u \in \mathcal{N}_{a,0}$ and $t_{u,0}$ is a strict maximum point of $\Psi_u^0(t)$, if $u \in \mathcal{N}_{a,0}$, we have that $t_{u,0} = 0$ and

$$I_0(u) = \max_{t \in \mathbb{R}} I_0(t \star u) \geq \inf_{u \in S_a} \max_{t \in \mathbb{R}} I_0(t \star u).$$

On the other hand, if $u \in S_a$, then $t_{u,0} \star u \in \mathcal{N}_{a,0}$, and

$$\max_{t \in \mathbb{R}} I_0(t \star u) = I_0(t_{u,0} \star u) \geq \inf_{u \in \mathcal{N}_{a,0}} I_0(u).$$

Therefore,

$$\inf_{u \in \mathcal{N}_{a,0}} I_0(u) = \inf_{u \in S_a} \max_{t \in \mathbb{R}} I_0(t \star u).$$

Now, we have by (8.1),

$$
\begin{aligned}
&\inf_{u\in\mathcal{N}_{a,0}} I_0(u)\\
&= \inf_{u\in S_a}\max_{t\in\mathbb{R}} I_0(t\star u)\\
&= \inf_{u\in\mathcal{N}_{a,0}}\left[\frac{1}{2}\left(\frac{\|u\|^2}{\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x}\right)^{\frac{2}{22^*_{\alpha,s}-2}}\|u\|^2\right.\\
&\quad\left.-\frac{1}{22^*_{\alpha,s}}\left(\frac{\|u\|^2}{\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x}\right)^{\frac{22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right]\\
&= \inf_{u\in\mathcal{N}_{a,0}}\frac{N+2s-\alpha}{2(2N-\alpha)}\left(\frac{\|u\|^2}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}}\right)^{\frac{22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}\\
&= \inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}}\frac{N+2s-\alpha}{2(2N-\alpha)}\left(\frac{\|u\|^2}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.
\end{aligned}
$$

Thus, it follows from the definition of $S_{h,l}$ that

$$
\begin{aligned}
&\inf_{u\in H^s(\mathbb{R}^N)\setminus\{0\}}\frac{N+2s-\alpha}{2(2N-\alpha)}\left(\frac{\|u\|^2}{\left(\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\alpha,s}}}}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}\\
&\quad=\frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}
\end{aligned}
$$

and the infimum is attained if and only if by the extremal functions $\widetilde{U}_{\varepsilon,y}$ defined in (1.13) when $N>4s$ and stay in $L^2(\mathbb{R}^N)$. In the case $2s<N\le 4s$, we show that the infimum of I_0 in $\mathcal{N}_{a,0}$ is not achieved. Suppose by contradiction that there exists a minimizer u, let $v=|u|^*$ the symmetric decreasing rearrangement of u. Then by the properties of symmetric decreasing rearrangement, we infer to

$$
\|v\|^2\le\|u\|^2,\quad I_0(v)\le I_0(u)\ \text{ and }\ P_0(v)\le 0=P_0(u).
$$

If $P_0(v)<0$, then by (8.1), we have $t_{v,0}<0$, and so

$$
\begin{aligned}
I_0(u)\le I_0(t_{v,0}\star v)&=e^{2st_{v,0}}\frac{N+2s-\alpha}{2(2N-\alpha)}\|v\|^2\\
&\le e^{2st_{v,0}}\frac{N+2s-\alpha}{2(2N-\alpha)}\|u\|^2\\
&=e^{2st_{v,0}}I_0(u)<I_0(u).
\end{aligned}
$$

which is a contradiction. Thus $P_0(v) = 0$, and so $v \in \mathcal{N}_{a,0}$. Since $\mathcal{N}_{a,0}$ is a natural constraint, we obtain

$$(-\Delta)^s v = \lambda v + (I_\alpha * |v|^{2^*_{\alpha,s}}) v^{2^*_{\alpha,s}-1}, \quad v \geq 0 \text{ in } \mathbb{R}^N, \tag{8.2}$$

for some $\lambda < 0$. By $P_0(v) = 0$, we see that $\lambda = 0$. Using the maximum principle [15], we have $v > 0$ in $\mathbb{R}^N$. From [29], we know that $v = \theta \widetilde{U}_{\varepsilon,0}$ for some $\theta > 0$, this is not possible, since $\widetilde{U}_{\varepsilon,0} \notin H^s(\mathbb{R}^N)$ for $2s < N \leq 4s$. This completes the proof.

9 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. The following two lemmas are necessary to the proof.

Lemma 9.1 *Let $a > 0$, $\mu \geq 0$, $\bar{p} \leq q < 2^*_s$ and (1.22) holds. Then*

$$\inf_{u \in \mathcal{N}_{a,\mu}} I_\mu(u) = \inf_{u \in S_a} \max_{t \in \mathbb{R}} I_\mu(t * u).$$

Proof Since $\bar{p} \leq q < 2^*_s$ and $\mu \geq 0$, by Lemmas 4.2 and 5.2, we have that $\mathcal{N}_{a,0} = \mathcal{N}^-_{a,0}$. For any fixed $u \in S_a$, there is a unique $t_{u,\mu} \in \mathbb{R}$ such that $t_{u,\mu} \star u \in \mathcal{N}_{a,\mu}$, and $t_{u,\mu}$ is the unique critical point of the functional Ψ^μ_u. So, if $u \in \mathcal{N}_{a,\mu}$ we have that $t_{u,\mu} = 0$ and

$$I_\mu(u) = \max_{t \in \mathbb{R}} I_\mu(t \star u) \geq \inf_{v \in S_a} \max_{t \in \mathbb{R}} I_\mu(t \star v).$$

On the other hand, if $u \in S_a$, then $t_{u,\mu} \star u \in \mathcal{N}_{a,\mu}$ and hence

$$\max_{t \in \mathbb{R}} I_\mu(t \star u) = I_\mu(t_{u,\mu} \star u) \geq \inf_{v \in \mathcal{N}_{a,\mu}} I_\mu(v).$$

This completes the proof. □

Lemma 9.2 *Let $a > 0$, $\bar{p} \leq q < 2^*_s$, $\mu^* \geq 0$ such that (1.22) holds. Then the function $\mu : [0, \mu^*] \to m_{a,\mu} \in \mathbb{R}$ is monotone and non-increasing.*

Proof Let $0 \leq \mu_1 \leq \mu_2 \leq \mu^*$, by Lemma 9.1, we know that

$$\begin{aligned}
m_{a,\mu_2} &= \inf_{u \in S_a} \max_{t \in \mathbb{R}} I_{\mu_2}(t \star u) = \inf_{u \in S_a} I_{\mu_2}(t_{u,\mu_2} \star u) \\
&= \inf_{u \in S_a} \left[I_{\mu_1}(t_{u,\mu_2} \star u) + (\mu_1 - \mu_2) \frac{e^{q\gamma_{q,s} st}}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x \right] \\
&\leq \inf_{u \in S_a} \max_{t \in \mathbb{R}} I_{\mu_1}(t \star u) = m_{a,\mu_1},
\end{aligned}$$

and the conclusion follows. □

Proof of Theorem 1.5 We divide the proof into two cases.

Case 1: $2 < q < \bar{p} < 2_s^*$. Since u_μ is a positive ground state solution of I_μ on $\{u \in S_a : \|u\|^2 < R_0\}$, where R_0 is given in Lemma 3.1, such that $h(R_0) = 0$, and h is defined in (3.2), we can check that $R_0 = R_0(a,\mu) \to 0^+$. Therefore, $\|u_\mu\|^2 < R_0 \to 0$ as $\mu \to 0^+$. For any $u \in S_a$, by (2.3) and (1.11)

$$
\begin{aligned}
0 > m_{a,\mu} = I_\mu(u_\mu) &\geq \frac{1}{2}\|u_\mu\|^2 - \frac{\mu}{q} C_{N,q,s}\|u_\mu\|^{q\gamma_{q,s}} a^{q(1-\gamma_{q,s})} \\
&\quad - \frac{1}{22^*_{\alpha,s}} S_{h,l}^{-2^*_{\alpha,s}} \|u_\mu\|^{22^*_{\alpha,s}} \to 0
\end{aligned}
$$

as $\mu \to 0^+$.

Case 2: $\bar{p} \leq q < 2_s^*$. Let $\mu^* \geq 0$ and (1.22) holds. Firstly, we show that the family of positive radial ground states $\{u_\mu : \mu \in (0,\mu^*)\}$ is a bounded set in $H^s(\mathbb{R}^N)$. If $q = \bar{p} = 2 + \frac{4s}{N}$, then by Lemma 9.2 and $P_\mu(u_\mu) = 0$, we deduce that

$$
\begin{aligned}
m_{a,0} \geq m_{a,\mu} = I_\mu(u_\mu) &= \frac{N+2s-\alpha}{2(2N-\alpha)}\left[\|u\|^2 - \frac{2\mu}{\bar{p}}\int_{\mathbb{R}^N}|u|^{\bar{p}}\mathrm{d}x\right] \\
&\geq \frac{N+2s-\alpha}{2(2N-\alpha)}\left[1 - \frac{2\mu}{\bar{p}} C_{N,\bar{p},s}^{\bar{p}} a^{\frac{4s}{N}}\right]\|u\|^2.
\end{aligned}
$$

If $\bar{p} < q < 2_s^*$, by a similar argument as above we infer to

$$
\begin{aligned}
m_{a,0} &\geq m_{a,\mu} = I_\mu(u_\mu) \\
&= \frac{N+2s-\alpha}{2(2N-\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u_\mu|^{2^*_{\alpha,s}})|u_\mu|^{2^*_{\alpha,s}}\mathrm{d}x + \frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2} - 1\right)\int_{\mathbb{R}^N}|u_\mu|^q\mathrm{d}x.
\end{aligned}
$$

Therefore, $\{u_\mu\}$ is bounded in $L^q(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$. By $P_\mu(u_\mu) = 0$, we also have $\{u_\mu\}$ is bounded in $H^s(\mathbb{R}^N)$. Since

$$
\begin{aligned}
\overline{\lambda_\mu} a^2 &= \|u_\mu\|^2 - \mu\int_{\mathbb{R}^N}|u_\mu|^q\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * |u_\mu|^{2^*_{\alpha,s}})|u_\mu|^{2^*_{\alpha,s}}\mathrm{d}x \\
&= \mu(\gamma_{q,s} - 1)\int_{\mathbb{R}^N}|u_\mu|^q\mathrm{d}x \to 0
\end{aligned}
$$

as $\mu \to 0^+$. Therefore, $u_\mu \rightharpoonup u$ weakly in $H^s(\mathbb{R}^N)$, $D^s(\mathbb{R}^N)$, $L^{2_s^*}(\mathbb{R}^N)$ and $u_\mu \to u$ in $L^q(\mathbb{R}^N)$, $\overline{\lambda_\mu} \to 0$. Let $\|u_\mu\|^2 \to \ell \geq 0$. If $\ell = 0$, then $u_\mu \to 0$ in $D^s(\mathbb{R}^N)$, and so $I_\mu(u_\mu) \to 0$. But, by Lemma 9.2, we have $I_\mu(u_\mu) \geq m_{a,\mu^*} > 0$ for each $\mu \in (0,\mu^*)$, a contradiction. Hence, $\ell > 0$. From $P_\mu(u_\mu) = 0$, we have

$$
\int_{\mathbb{R}^N}(I_\alpha * |u_\mu|^{2^*_{\alpha,s}})|u_\mu|^{2^*_{\alpha,s}}\mathrm{d}x = \|u_\mu\|^2 - \mu\gamma_{q,s}\int_{\mathbb{R}^N}|u_\mu|^q\mathrm{d}x \to \ell, \quad \text{as } \mu \to 0^+.
$$

Recalling the definition of $S_{h,l}$ in (1.13), we have $\ell \geq S_{h,l}^{\frac{N+2s-\alpha}{2N-\alpha}}$. Meanwhile, we see that

$$\begin{aligned}\frac{N+2s-\alpha}{2(2N-\alpha)}\ell &= \lim_{\mu\to 0^+}\left[\frac{N+2s-\alpha}{2(2N-\alpha)}\int_{\mathbb{R}^N}(I_\alpha * |u_\mu|^{2^*_{\alpha,s}})|u_\mu|^{2^*_{\alpha,s}}\mathrm{d}x\right.\\ &\quad\left.+\frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2}-1\right)\int_{\mathbb{R}^N}|u_\mu|^q\mathrm{d}x\right]\\ &= \lim_{\mu\to 0^+} I_\mu(u_\mu) \leq m_{a,0} = \frac{N+2s-\alpha}{2(2N-\alpha)}S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.\end{aligned}$$

Therefore, $\ell = S_{h,l}^{\frac{N+2s-\alpha}{2N-\alpha}}$ and the conclusion follows. □

Acknowledgements The authors would like to thank two anonymous reviewers for the careful reading of the manuscript and for their valuable comments. Xiaoming He is supported by the National Natural Science Foundation of China (11771468, 11971027, 12171497), while Wenming Zou is supported by the National Natural Science Foundation of China (11771234, 11926323). The research of Vicenţiu D. Rădulescu is supported by a grant of the Romanian Ministry of Research, Innovation and Digitization, CNCS/CCCDI-UEFISCDI, project number PCE 137/2021, within PNCDI III.

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