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Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type

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Abstract

We are concerned with the generalized Lane–Emden–Fowler equation $-\Delta u = \lambda f(u) + a(x)g(u)$ in Ω, subject to the Dirichlet boundary condition $u|_{\partial\Omega} = 0$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, a is a nonnegative Hölder function, and f is positive and nondecreasing such that the mapping $f(s)/s$ is nonincreasing in $(0,\infty)$. Here, the singular character of the problem is given by the nonlinearity g which is assumed to be unbounded around the origin. We distinguish two different cases which are related to the sublinear (respectively linear) growth of f at infinity.

Résumé

On étudie l'équation de Lane–Emden–Fowler généralisée $-\Delta u = \lambda f(u) + a(x)g(u)$ dans Ω avec une condition de Dirichlet $u|_{\partial\Omega} = 0$, où $\Omega \subset \mathbb{R}^N$ est un domaine borné régulier, $\lambda \in \mathbb{R}$, a est une fonction de Hölder non-négative et f est positive et croissante telle que l'application $f(s)/s$ soit décroissante sur $(0,\infty)$. Le caractère singulier de ce problème est donné par la nonlinéarité g, qui est non bornée autour de l'origine. Sous des hypothèses différentes concernant f et g, on discute l'existence et l'unicité d'une solution classique positive. On distingue deux cas différents, correspondant aux situations où f a une croissance sous-linéaire ou linéaire à l'infini.

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1. Introduction and statement of the problem

The study of singular semilinear equations has an important place in the literature. From a physical point of view, these equations arise in the context of chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials, as well as in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids. Nonlinear singular elliptic equations are also encountered in glacial advance (see [32]), in transport of coal slurries down conveyor belts (see [4]) and in several other geophysical and industrial contents (see [3] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence). Singular problems have also been considered in the context of integral equations. In this sense we mention the papers [16,19,24,27]. For elliptic operators more general than the Laplacian, this kind of problems were treated in [9,28]. For more details we refer to [7,11,22,25,26,30] and the references therein.

This paper is motivated by our recent work [14] in which we have studied the role of positive parameters λ and μ in the boundary value problem:

$$\begin{cases} -\Delta u + a(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where f is a positive function with sublinear growth, and g is a singular nonlinearity. The aim of this paper is to study the bifurcation problem:

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\lambda \in \mathbb{R}$ is a parameter and $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded domain with smooth boundary $\partial\Omega$. Let $0 < f \in C^{0,\beta}[0,\infty)$ and $0 \leqslant g \in C^{0,\beta}(0,\infty)$ $(0 < \beta < 1)$ fulfill the hypotheses

$(f1)$ f is nondecreasing on $(0,\infty)$ while $f(s)/s$ is nonincreasing for $s > 0$;
$(g1)$ g is nonincreasing on $(0,\infty)$ with $\lim_{s\searrow 0} g(s) = +\infty$;
$(g2)$ there exist $C_0, \eta_0 > 0$ and $\alpha \in (0,1)$ so that $g(s) \leqslant C_0 s^{-\alpha}$, $\forall s \in (0,\eta_0)$.

The assumption $(g2)$ has been used in [14] and it implies the following Keller–Osserman-type growth condition around the origin:

$$\int_0^1 \left(\int_0^t g(s)\,\mathrm{d}s \right)^{-1/2} \mathrm{d}t < +\infty. \tag{2}$$

As proved by Bénilan, Brezis and Crandall in [1], condition (2) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + g(u) = h \quad \text{a.e. in } \mathbb{R}^N.$$

In many papers (see, e.g., [10,20]) the potential $a(x)$ is assumed to depend "almost" radially on x, in the sense that

$$C_1 p\big(|x|\big) \leqslant a(x) \leqslant C_2 p\big(|x|\big),$$

where C_1, C_2 are positive constants and $p(|x|)$ is a positive function satisfying some integrability condition. We do not impose any growth assumption on a, but we suppose throughout this paper that the variable potential $a(x)$ satisfies $a \in C^{0,\beta}(\overline{\Omega})$ and $a > 0$ in Ω.

If $\lambda = 0$ this equation is called the Lane–Emden–Fowler equation and arises in the boundary-layer theory of viscous fluids (see [33]). Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, superdiffusivities equations of this type have been proposed by de Gennes [12] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [5]), as well as to describe the flow over an impermeable plate (see [2,3]). Problems of this type are obtained from evolution equations of the form,

$$u_{tt} = \mathrm{div}(u^{m-1}\nabla u) + h(x,u) \quad \text{in } \Omega \times (0,T)$$

through the implicit discretization in time arising in nonlinear semigroup theory (see [8, 31]). In [13], Fulks and Maybee studied the existence of solutions to singular parabolic equations of the form,

$$u_t - \Delta u = g(x,t,u) \quad \text{in } \Omega \times (0,T)$$

coupled with initial and boundary conditions. Under the hypotheses that g is nonincreasing in u and $g(x,t,r) \to g(x,r)$ as $t \to \infty$, they obtain classical solutions of the corresponding elliptic boundary value problem.

The problem (P_λ) has been widely studied for the special nonlinearities $f(t) = t^p$ and $g(t) = t^{-\gamma}$, where p and γ are positive parameters. In this case, (P_λ) becomes:

$$\begin{cases} -\Delta u = \lambda u^p + a(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

In [29], Shi and Yao studied the problem (3) for $p, \gamma \in (0,1)$ and $\lambda > 0$. For $\lambda = 0$ and $a \equiv 1$, Lazer and McKenna [21] proved that (3) has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, if $\gamma > 1$ then u is *not* in $C^1(\overline{\Omega})$. If $a \equiv 1$ and $\gamma \in (0,1)$, Coclite and Palmieri [6] showed that (3) has at least one solution provided that $\lambda \geqslant 0$ and $p \in (0,1)$. In turns, if $p \geqslant 1$, they proved that there exists $\lambda^* > 0$ such that (3) has a solution for $\lambda \in [0, \lambda^*)$ and no solutions exist if $\lambda > \lambda^*$. A similar problem to (3) when $p = 1$ and $\lambda \leqslant 0$ was studied in [9].

2. The main results

Our purpose is to study the effect of the asymptotically linear perturbation $f(u)$ in (P_λ), as well as to describe the set of values of the positive parameter λ such that problem (P_λ) admits a solution. In this case, we also prove a uniqueness result. Due to the singular character of (P_λ), we cannot expect to find solutions in $C^2(\overline{\Omega})$. However, under the above assumptions we will show that (P_λ) has solutions in the class:

$$\mathcal{E} := \left\{ u \in C^2(\Omega) \cap C^{1,1-\alpha}\left(\overline{\Omega}\right);\ \Delta u \in L^1(\Omega) \right\}.$$

We first observe that, in view of the assumption $(f1)$, there exists

$$m := \lim_{s \to \infty} \frac{f(s)}{s} \in [0, \infty).$$

This number plays a crucial role in our analysis. More precisely, the existence of the solutions to (P_λ) will be separately discussed for $m > 0$ and $m = 0$. We point out that in [14] we studied in detail the problem (1) in the case where $m = 0$ and a is a sign-changing potential. In that case, a significant role in the study of the existence of solutions was played by the decay rate of g combined with the signs of the extremal values of the potential $a(x)$ in $\overline{\Omega}$. Let $a_* = \min_{x \in \overline{\Omega}} a(x)$.

Our first result is:

Theorem 1. *Assume* $(f1)$, $(g1)$, $(g2)$ *and* $m = 0$. *If* $a_* > 0$ (*respectively,* $a_* = 0$)*, then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $\lambda \in \mathbb{R}$ (*respectively,* $\lambda \geqslant 0$) *with the properties*:

(i) u_λ *is strictly increasing with respect to* λ;
(ii) *there exist two positive constant* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \leqslant u_\lambda \leqslant c_2 d(x)$ *in* Ω.

The bifurcation diagram in the "sublinear" case $m = 0$ is depicted in Fig. 1. We now consider the case $m > 0$. The results in this case are different from those presented in Theorem 1. A careful examination of (P_λ) reveals the fact that the singular term $g(u)$ is not significant. Actually, the conclusions are close to those established in [23, Theorem A], where an elliptic problem associated to an asymptotically linear function is studied.

Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω and $\lambda^* = \lambda_1/m$. Our result in this case is the following:

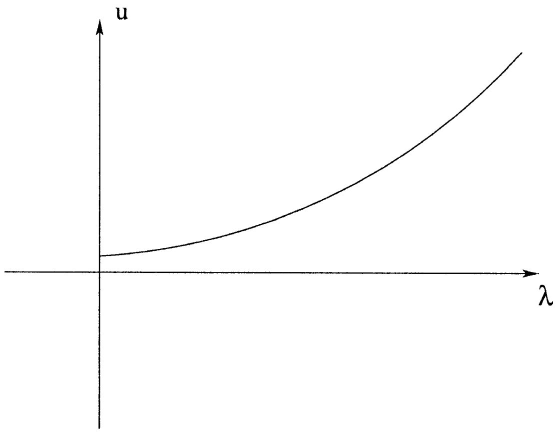

Fig. 1. The “sublinear” case $m = 0$.

Theorem 2. *Assume* $(f1)$, $(g1)$, $(g2)$ *and* $m > 0$. *Then the following hold*:

(i) *If* $\lambda \geqslant \lambda^*$, *then* (P_λ) *has no solutions in* $\mathcal{E}$.
(ii) *If* $a_* > 0$ (*respectively* $a_* = 0$) *then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $-\infty < \lambda < \lambda^*$ (*respectively* $0 < \lambda < \lambda^*$) *with the properties*:
  (ii1) u_λ *is strictly increasing with respect to* λ;
  (ii2) *there exist two positive constants* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \leqslant u_\lambda \leqslant c_2 d(x)$ *in* Ω;
  (ii3) $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, *uniformly on compact subsets of* Ω.

The bifurcation diagram in the “linear” case $m > 0$ is depicted in Fig. 2.

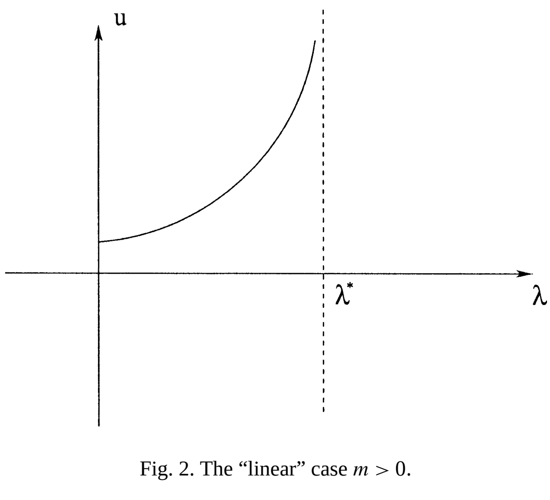

Fig. 2. The “linear” case $m > 0$.

3. Proof of Theorem 1

We first recall some auxiliary results that we need in the proof.

Lemma 3 (see [29]). *Let $F:\overline{\Omega}\times(0,\infty)\to\mathbb{R}$, be a Hölder continuous function with exponent $\beta\in(0,1)$, on each compact subset of $\overline{\Omega}\times(0,\infty)$ which satisfies*:

(F1) $\limsup_{s\to+\infty}(s^{-1}\max_{x\in\overline{\Omega}}F(x,s))<\lambda_1$;
(F2) *for each $t>0$, there exists a constant $D(t)>0$, such that*

$$F(x,r)-F(x,s)\geqslant -D(t)(r-s),\quad \textit{for } x\in\overline{\Omega} \textit{ and } r\geqslant s\geqslant t;$$

(F3) *there exists $\eta_0>0$, and an open subset $\Omega_0\subset\Omega$, such that*

$$\min_{x\in\overline{\Omega}}F(x,s)\geqslant 0\quad \textit{for } s\in(0,\eta_0),$$

and

$$\lim_{s\searrow 0}\frac{F(x,s)}{s}=+\infty\quad \textit{uniformly for } x\in\Omega_0.$$

Then, for any nonnegative function $\varphi_0\in C^{2,\beta}(\partial\Omega)$, the problem,

$$\begin{cases}-\Delta u=F(x,u) & \textit{in } \Omega,\\ u>0 & \textit{in } \Omega,\\ u=\varphi_0 & \textit{on } \partial\Omega,\end{cases}$$

has at least one positive solution $u\in C^{2,\beta}(G)\cap C(\overline{\Omega})$, for any compact set $G\subset\Omega\cup\{x\in\partial\Omega;\ \varphi_0(x)>0\}$.

Lemma 4 (see [29]). *Let $F:\overline{\Omega}\times(0,\infty)\to\mathbb{R}$, be a continuous function such that the mapping $(0,\infty)\ni s\mapsto\frac{F(x,s)}{s}$, is strictly decreasing at each $x\in\Omega$. Assume that there exists $v,w\in C^2(\Omega)\cap C(\overline{\Omega})$ such that*

(a) $\Delta w+F(x,w)\leqslant 0\leqslant\Delta v+F(x,v)$ *in* Ω;
(b) $v,w>0$ *in* Ω *and* $v\leqslant w$ *on* $\partial\Omega$;
(c) $\Delta v\in L^1(\Omega)$.

Then $v\leqslant w$ in Ω.

Now we are ready to give the proof of Theorem 1. This will be divided into four steps.

Step 1. Existence of solutions to problem (P_λ)

For any $\lambda \in \mathbb{R}$, define the function:

$$\Phi_\lambda(x,s) = \lambda f(s) + a(x)g(s), \quad (x,s) \in \overline{\Omega} \times (0,\infty). \tag{4}$$

Taking into account the assumptions of Theorem 1, it follows that Φ_λ verifies the hypotheses of Lemma 3 for $\lambda \in \mathbb{R}$ if $a_* > 0$ and $\lambda \geqslant 0$ if $a_* = 0$. Hence, for λ in the above range, (P_λ) has at least one solution $u_\lambda \in C^{2,\beta}(\Omega) \cap C(\overline{\Omega})$.

Step 2. Uniqueness of solution

Fix $\lambda \in \mathbb{R}$ (respectively, $\lambda \geqslant 0$) if $a_* > 0$ (respectively, $a_* = 0$). Let u_λ be a solution of (P_λ). Denote $\lambda^- = \min\{0,\lambda\}$ and $\lambda^+ = \max\{0,\lambda\}$. We claim that $\Delta u_\lambda \in L^1(\Omega)$. Since $a \in C^{0,\beta}(\overline{\Omega})$, by [15, Theorem 6.14], there exists a unique nonnegative solution $\zeta \in C^{2,\beta}(\overline{\Omega})$ of

$$\begin{cases} -\Delta\zeta = a(x) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle (see, e.g., [15, Theorem 2.2]), $\zeta > 0$ in Ω. Moreover, we are going to prove that

(a) $z(x) := c\zeta(x)$ is a subsolution of (P_λ), for $c > 0$ small enough;
(b) $z(x) \geqslant c_1 d(x)$ in $\overline{\Omega}$, for some positive constant $c_1 > 0$;
(c) $u_\lambda \geqslant z$ in $\overline{\Omega}$.

Therefore, by (b) and (c), $u_\lambda \geqslant c_1 d(x)$ in $\overline{\Omega}$. Using $(g2)$, we obtain $g(u_\lambda) \leqslant C d^{-\alpha}(x)$ in Ω, where $C > 0$ is a constant. So, $g(u_\lambda) \in L^1(\Omega)$. This implies:

$$\Delta u_\lambda \in L^1(\Omega).$$

Proof of (a). Using $(f1)$ and $(g1)$, we have:

$$\begin{aligned} \Delta z(x) + \Phi_\lambda(x,z) &= -ca(x) + \lambda f(c\zeta) + a(x)g(c\zeta) \\ &\geqslant -ca(x) + \lambda^- f\big(c\|\zeta\|_\infty\big) + a(x)g\big(c\|\zeta\|_\infty\big) \\ &\geqslant ca(x)\left[\frac{g(c\|\zeta\|_\infty)}{2c} - 1\right] + f\big(c\|\zeta\|_\infty\big)\left[a_* \frac{g(c\|\zeta\|_\infty)}{2f(c\|\zeta\|_\infty)} + \lambda^-\right] \end{aligned}$$

for each $x \in \Omega$. Since $\lambda < 0$ corresponds to $a_* > 0$, using $\lim_{t \searrow 0} g(t) = +\infty$ and $\lim_{t \to 0} f(t) \in (0,\infty)$, we can find $c > 0$ small such that

$$\Delta z + \Phi_\lambda(x,z) \geqslant 0, \quad \forall x \in \Omega.$$

This concludes (a). $\square$

Proof of (b). Since $\zeta \in C^{2,\beta}(\overline{\Omega})$, $\zeta > 0$ in Ω and $\zeta = 0$ on $\partial\Omega$, by Lemma 3.4 in [15], we have:

$$\frac{\partial\zeta}{\partial\nu}(y) < 0, \quad \forall y \in \partial\Omega.$$

Therefore, there exists a positive constant c_0 such that

$$\frac{\partial\zeta}{\partial\nu}(y) := \lim_{x\in\Omega,\ x\to y} \frac{\zeta(y) - \zeta(x)}{|x-y|} \leqslant -c_0, \quad \forall y \in \partial\Omega.$$

So, for each $y \in \Omega$, there exists $r_y > 0$ such that

$$\frac{\zeta(x)}{|x-y|} \geqslant \frac{c_0}{2}, \quad \forall x \in B_{r_y}(y) \cap \Omega. \tag{5}$$

Using the compactness of $\partial\Omega$, we can find a finite number k of balls $B_{r_{y_i}}(y_i)$ such that $\partial\Omega \subset \bigcup_{i=1}^{k} B_{r_{y_i}}(y_i)$. Moreover, we can assume that for small $d_0 > 0$,

$$\{x \in \Omega\colon\ d(x) < d_0\} \subset \bigcup_{i=1}^{k} B_{r_{y_i}}(y_i).$$

Therefore, by (5) we obtain:

$$\zeta(x) \geqslant \frac{c_0}{2} d(x), \quad \forall x \in \Omega \text{ with } d(x) < d_0.$$

This fact, combined with $\zeta > 0$ in Ω, shows that for some constant $\tilde{c} > 0$

$$\zeta(x) \geqslant \tilde{c} d(x), \quad \forall x \in \Omega.$$

Thus, (b) follows by the definition of z. □

Proof of (c). We distinguish two cases:

Case 1. $\lambda \geqslant 0$. We see that Φ_λ verifies the hypotheses in Lemma 4. Since

$$\begin{gathered}
\Delta u_\lambda + \Phi_\lambda(x, u_\lambda) \leqslant 0 \leqslant \Delta z + \Phi_\lambda(x, z) \quad \text{in } \Omega, \\
u_\lambda,\ z > 0 \quad \text{in } \Omega, \\
u_\lambda = z \quad \text{on } \partial\Omega, \\
\Delta z \in L^1(\Omega),
\end{gathered}$$

by Lemma 4 it follows that $u_\lambda \geqslant z$ in $\overline{\Omega}$.

Now, if u_1 and u_2 are two solutions of (P_λ), we can use Lemma 4 in order to deduce that $u_1 = u_2$.

Case 2. $\lambda < 0$ (corresponding to $a_* > 0$). Let $\varepsilon > 0$ be fixed. We prove that

$$z \leqslant u_\lambda + \varepsilon\big(1 + |x|^2\big)^\tau \quad \text{in } \Omega, \tag{6}$$

where $\tau < 0$ is chosen such that $\tau|x|^2 + 1 > 0$, $\forall x \in \Omega$. This is always possible since $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is bounded.

We argue by contradiction. Suppose that there exists $x_0 \in \Omega$ such that $u_\lambda(x_0) + \varepsilon(1 + |x_0|)^\tau < z(x_0)$. Then $\min_{x\in\overline{\Omega}}\{u_\lambda(x) + \varepsilon(1 + |x|^2)^\tau - z(x)\} < 0$ is achieved at some point $x_1 \in \Omega$. Since $\Phi_\lambda(x, z)$ is nonincreasing in z, we have:

$$\begin{aligned} 0 &\geqslant -\Delta\big[u_\lambda(x) - z(x) + \varepsilon\big(1 + |x|^2\big)^\tau\big]\big|_{x=x_1} \\ &= \Phi_\lambda\big(x_1, u_\lambda(x_1)\big) - \Phi_\lambda\big(x_1, z(x_1)\big) - \varepsilon\Delta\big[\big(1 + |x|^2\big)^\tau\big]\big|_{x=x_1} \\ &\geqslant -\varepsilon\Delta\big[\big(1 + |x|^2\big)^\tau\big]\big|_{x=x_1} = -2\varepsilon\tau\big(1 + |x_1|^2\big)^{\tau-2}\big[(N + 2\tau - 2)|x_1|^2 + N\big] \\ &\geqslant -4\varepsilon\tau\big(1 + |x_1|^2\big)^{\tau-2}\big(\tau|x_1|^2 + 1\big) > 0. \end{aligned}$$

This contradiction proves (6). Passing to the limit $\varepsilon \to 0$, we obtain (c). □

In a similar way we can prove that (P_λ) has a unique solution.

Step 3. Dependence on λ

We fix $\lambda_1 < \lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ if $a_* > 0$ respectively, $\lambda_1, \lambda_2 \in [0, \infty)$ if $a_* = 0$. Let $u_{\lambda_1}, u_{\lambda_2}$ be the corresponding solutions of (P_{λ_1}) and (P_{λ_2}) respectively.

If $\lambda_1 \geqslant 0$, then Φ_{λ_1} verifies the hypotheses in Lemma 4. Furthermore, we have:

$$\begin{gathered} \Delta u_{\lambda_2} + \Phi_{\lambda_1}(x, u_{\lambda_2}) \leqslant 0 \leqslant \Delta u_{\lambda_1} + \Phi_{\lambda_1}(x, u_{\lambda_1}) \quad \text{in } \Omega, \\ u_{\lambda_1}, u_{\lambda_2} > 0 \quad \text{in } \Omega, \\ u_{\lambda_1} = u_{\lambda_2} \quad \text{on } \partial\Omega, \\ \Delta u_{\lambda_1} \in L^1(\Omega). \end{gathered}$$

Again by Lemma 4, we conclude that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$. Moreover, by the maximum principle, $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Let $\lambda_2 \leqslant 0$; we show that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$. Indeed, supposing the contrary, there exists $x_0 \in \Omega$ such that $u_{\lambda_1}(x_0) > u_{\lambda_2}(x_0)$. We conclude now that $\max_{x\in\overline{\Omega}}\{u_{\lambda_1}(x) - u_{\lambda_2}(x)\} > 0$ is achieved at some point in Ω. At that point, say $\bar{x}$, we have:

$$0 \leqslant -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = \Phi_{\lambda_1}\big(\bar{x}, u_{\lambda_1}(\bar{x})\big) - \Phi_{\lambda_2}\big(\bar{x}, u_{\lambda_2}(\bar{x})\big) < 0,$$

which is a contradiction. It follows that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$, and by maximum principle we have $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

If $\lambda_1 < 0 < \lambda_2$, then $u_{\lambda_1} < u_0 < u_{\lambda_2}$ in Ω. This finishes the proof of Step 3.

Step 4. Regularity of the solution

Fix $\lambda \in \mathbb{R}$ and let $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique solution of (P_λ). An important result in our approach is the following estimate:

$$c_1 d(x) \leqslant u_\lambda(x) \leqslant c_2 d(x), \quad \text{for all } x \in \Omega, \tag{7}$$

where c_1, c_2 are positive constants. The first inequality in (7) was established in Step 2. For the second one, we apply an idea found in [17].

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega;\ d(x) \leqslant \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let w be the unique solution of the Dirichlet problem:

$$\begin{cases} -\Delta w = \lambda^+ f(w) + g(w) & \text{in } B_K(0) \setminus \overline{B_1}(0), \\ w > 0 & \text{in } B_K(0) \setminus \overline{B_1}(0), \\ w = 0 & \text{on } \partial(B_K(0) \setminus \overline{B_1}(0)), \end{cases} \tag{8}$$

where $B_r(0)$ is the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, w is radially symmetric. Hence $w(x) = \widetilde{w}(|x|)$ and

$$\begin{cases} \widetilde{w}'' + \frac{N-1}{r}\widetilde{w}' + \lambda^+ f(\widetilde{w}) + g(\widetilde{w}) = 0 & \text{for } r \in (1, K), \\ \widetilde{w} > 0 & \text{in } (1, K), \\ \widetilde{w}(1) = \widetilde{w}(K) = 0. \end{cases} \tag{9}$$

Integrating in (9) we have:

$$\begin{aligned} \widetilde{w}'(t) &= \widetilde{w}'(a) a^{N-1} t^{1-N} - t^{1-N} \int_a^t r^{N-1} \big[\lambda^+ f\big(\widetilde{w}(r)\big) + g\big(\widetilde{w}(r)\big)\big]\,\mathrm{d}r, \\ &= \widetilde{w}'(b) b^{N-1} t^{1-N} + t^{1-N} \int_t^b r^{N-1} \big[\lambda^+ f\big(\widetilde{w}(r)\big) + g\big(\widetilde{w}(r)\big)\big]\,\mathrm{d}r, \end{aligned}$$

where $1 < a < t < b < K$. Since $g(\widetilde{w}) \in L^1(1, K)$, we deduce that both $\widetilde{w}'(1)$ and $\widetilde{w}'(K)$ are finite, so $\widetilde{w} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$w(x) \leqslant C \min\big\{K - |x|, |x| - 1\big\}, \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{10}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y_0, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $v(x) = cw((x - y_0)/\delta)$, $x \in \overline{\Omega}$. We show that v is a supersolution of (P_λ), provided that c is large enough. Indeed, if $c > \max\{1, \delta^2 \|a\|_\infty\}$, then for all $x \in \Omega$ we have:

$$\Delta v+\lambda f(v)+a(x)g(v)\leqslant \frac{c}{\delta^2}\left(\widetilde{w}''(r)+\frac{N-1}{r}\widetilde{w}'(r)\right)\\ +\lambda^+ f\big(c\widetilde{w}(r)\big)+a(x)g\big(c\widetilde{w}(r)\big),$$

where $r=|x-y_0|/\delta\in(1,K)$. Using the assumption $(f1)$ we get $f(c\widetilde{w})\leqslant cf(\widetilde{w})$ in $(1,K)$. The above relations lead us to

$$\begin{aligned}\Delta v+\lambda f(v)+a(x)g(v)&\leqslant \frac{c}{\delta^2}\left(\widetilde{w}''+\frac{N-1}{r}\widetilde{w}'\right)+\lambda^+cf(\widetilde{w})+\|a\|_\infty g(\widetilde{w})\\ &\leqslant \frac{c}{\delta^2}\left(\widetilde{w}''+\frac{N-1}{r}\widetilde{w}'+\lambda^+f(\widetilde{w})+g(\widetilde{w})\right)\\ &=0.\end{aligned}$$

Since $\Delta u_\lambda\in L^1(\Omega)$, with a similar proof as in Step 2 we get $u_\lambda\leqslant v$ in Ω. This combined with (10) yields:

$$u_\lambda(x_0)\leqslant v(x_0)\leqslant C\min\left\{K-\frac{|x_0-y_0|}{\delta},\frac{|x_0-y_0|}{\delta}-1\right\}\leqslant \frac{C}{\delta}d(x_0).$$

Hence $u_\lambda\leqslant \frac{C}{\delta}d(x)$ in Ω_δ and the last inequality in (7) follows.

Let G be the Green's function associated with the Laplace operator in Ω. Then, for all $x\in\Omega$ we have:

$$u_\lambda(x)=-\int_\Omega G(x,y)\big[\lambda f\big(u_\lambda(y)\big)+a(y)g\big(u_\lambda(y)\big)\big]\,\mathrm{d}y,$$

and

$$\nabla u_\lambda(x)=-\int_\Omega G_x(x,y)\big[\lambda f\big(u_\lambda(y)\big)+a(y)g\big(u_\lambda(y)\big)\big]\,\mathrm{d}y.$$

If $x_1,x_2\in\Omega$, using $(g2)$ we obtain:

$$\big|\nabla u_\lambda(x_1)-\nabla u_\lambda(x_2)\big|\leqslant|\lambda|\int_\Omega\big|G_x(x_1,y)-G_x(x_2,y)\big|\cdot f\big(u_\lambda(y)\big)\,\mathrm{d}y\\ +\tilde{c}\int_\Omega\big|G_x(x_1,y)-G_x(x_2,y)\big|\cdot u_\lambda^{-\alpha}(y)\,\mathrm{d}y.$$

Now, taking into account that $u_\lambda\in C(\overline{\Omega})$, by the standard regularity theory (see [15]) we get:

$$\int_\Omega\big|G_x(x_1,y)-G_x(x_2,y)\big|\cdot f\big(u_\lambda(y)\big)\leqslant\tilde{c}_1|x_1-x_2|.$$

On the other hand, with the same proof as in [17, Theorem 1], we deduce:

$$\int_\Omega \left|G_x(x_1, y) - G_x(x_2, y)\right| \cdot u_\lambda^{-\alpha}(y) \leqslant \tilde{c}_2|x_1 - x_2|^{1-\alpha}.$$

The above inequalities imply $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})$. The proof of Theorem 1 is now complete.

4. Proof of Theorem 2

(i) Let φ_1 be the first eigenfunction of the Laplace operator in Ω with Dirichlet boundary condition. Arguing by contradiction, let us suppose that there exists $\lambda \geqslant \lambda^*$ such that (P_λ) has a solution $u_\lambda \in \mathcal{E}$.
Multiplying by φ_1 in (P_λ) and then integrating over Ω we get:

$$-\int_\Omega \varphi_1\, \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1. \tag{11}$$

Since $\lambda \geqslant \lambda_1/m$, in view of the assumption $(f1)$ we get $\lambda f(u_\lambda) \geqslant \lambda_1 u_\lambda$ in Ω. Using this fact in (11) we obtain:

$$-\int_\Omega \varphi_1 \Delta u_\lambda > \lambda_1 \int_\Omega u_\lambda \varphi_1.$$

The regularity of u_λ yields $-\int_\Omega u_\lambda \Delta\varphi_1 > \lambda_1 \int_\Omega u_\lambda\varphi_1$. This is clearly a contradiction since $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω. Hence (P_λ) has no solutions in $\mathcal{E}$ for any $\lambda \geqslant \lambda^*$.

(ii) From now, the proof of the existence, uniqueness and regularity of solution is the same as in Theorem 1.

(ii3) In what follows we shall apply some ideas developed in [23]. Due to the special character of our problem, we will be able to prove that, in certain cases, L^2-boundedness implies H_0^1-boundedness!

Let $u_\lambda \in \mathcal{E}$ be the unique solution of (P_λ) for $0 < \lambda < \lambda^*$. We prove that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. Suppose the contrary. Since $(u_\lambda)_{0<\lambda<\lambda^*}$ is a sequence of nonnegative superharmonic functions in Ω, by Theorem 4.1.9 in [18], there exists a subsequence of $(u_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(u_\lambda)_{\lambda<\lambda^*}$) which is convergent in $L^1_{\mathrm{loc}}(\Omega)$.

We first prove that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We argue by contradiction. Suppose that $(u_\lambda)_{\lambda<\lambda^*}$ is not bounded in $L^2(\Omega)$. Thus, passing eventually at a subsequence we have $u_\lambda = M(\lambda)w_\lambda$, where

$$M(\lambda) = \|u_\lambda\|_{L^2(\Omega)} \to \infty \text{ as } \lambda \nearrow \lambda^* \quad \text{and} \quad w_\lambda \in L^2(\Omega),\ \|w_\lambda\|_{L^2(\Omega)} = 1. \tag{12}$$

Using $(f1)$, $(g2)$ and the monotonicity assumption on g, we deduce the existence of $A, B, C, D > 0$ $(A > m)$ such that

$$f(t) \leqslant At + B, \quad g(t) \leqslant Ct^{-\alpha} + D, \quad \text{for all } t > 0. \tag{13}$$

This implies:

$$\frac{1}{M(\lambda)}\big(\lambda f(u_\lambda) + a(x)g(u_\lambda)\big) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*$$

that is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{14}$$

By Green's first identity, we have:

$$\int_\Omega \nabla w_\lambda \cdot \nabla\varphi \,\mathrm{d}x = -\int_\Omega \varphi\Delta w_\lambda \,\mathrm{d}x = -\int_{\mathrm{Supp}\,\varphi} \varphi\Delta w_\lambda \,\mathrm{d}x \quad \forall\varphi \in C_0^\infty(\Omega). \tag{15}$$

Using (14) we derive that

$$\begin{aligned}\left|\int_{\mathrm{Supp}\,\varphi} \varphi\Delta w_\lambda \,\mathrm{d}x\right| &\leqslant \int_{\mathrm{Supp}\,\varphi} |\varphi||\Delta w_\lambda| \,\mathrm{d}x \\ &\leqslant \|\varphi\|_{L^\infty} \int_{\mathrm{Supp}\,\varphi} |\Delta w_\lambda| \,\mathrm{d}x \to 0 \quad \text{as } \lambda \nearrow \lambda^*.\end{aligned} \tag{16}$$

Combining (15) and (16), we arrive at

$$\int_\Omega \nabla w_\lambda \cdot \nabla\varphi \,\mathrm{d}x \to 0 \text{ as } \lambda \nearrow \lambda^*, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{17}$$

By definition, the sequence $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$.

We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (13) and Hölder's inequality, we have:

$$\begin{aligned}\int_\Omega |\nabla w_\lambda|^2 = -\int_\Omega w_\lambda \Delta w_\lambda = \frac{-1}{M(\lambda)}\int_\Omega w_\lambda \Delta u_\lambda \\ = \frac{1}{M(\lambda)}\int_\Omega \big[\lambda w_\lambda f(u_\lambda) + a(x)g(u_\lambda)w_\lambda\big] \\ \leqslant \frac{\lambda}{M(\lambda)}\int_\Omega w_\lambda(Au_\lambda + B) + \frac{||a||_\infty}{M(\lambda)}\int_\Omega w_\lambda(Cu_\lambda^{-\alpha} + D)\end{aligned}$$

$$= \lambda A \int_\Omega w_\lambda^2 + \frac{\|a\|_\infty C}{M(\lambda)^{1+\alpha}} \int_\Omega w_\lambda^{1-\alpha} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)} \int_\Omega w_\lambda$$
$$\leqslant \lambda^* A + \frac{\|a\|_\infty C}{M(\lambda)^{1+\alpha}} |\Omega|^{(1+\alpha)/2} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)} |\Omega|^{1/2}.$$

From the above estimates, it is easy to see that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim is proved. Then, there exists $w \in H_0^1(\Omega)$ such that (up to a subsequence)

$$w_\lambda \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \nearrow \lambda^* \tag{18}$$

and, because $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$w_\lambda \to w \quad \text{strongly in } L^2(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{19}$$

On the one hand, by (12) and (19), we derive that $\|w\|_{L^2(\Omega)} = 1$. Furthermore, using (17) and (18), we infer that

$$\int_\Omega \nabla w \cdot \nabla \varphi \, \mathrm{d}x = 0, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Since $w \in H_0^1(\Omega)$, using the above relation and the definition of $H_0^1(\Omega)$, we get $w = 0$. This contradiction shows that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As above for w_λ, we can derive that u_λ is bounded in $H_0^1(\Omega)$. So, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases} u_\lambda \rightharpoonup u^* & \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\ u_\lambda \to u^* & \text{strongly in } L^2(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\ u_\lambda \to u^* & \text{a.e. in } \Omega \text{ as } \lambda \nearrow \lambda^*. \end{cases} \tag{20}$$

Now we can proceed to get a contradiction. Multiplying by φ_1 in (P_λ) and integrating over Ω we have:

$$-\int_\Omega \varphi_1 \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\varphi_1 + \int_\Omega a(x) g(u_\lambda)\varphi_1, \quad \text{for all } 0 < \lambda < \lambda^*. \tag{21}$$

On the other hand, by $(f1)$ it follows that $f(u_\lambda) \geqslant m u_\lambda$ in Ω, for all $0 < \lambda < \lambda^*$. Combining this with (21) we obtain:

$$\lambda_1 \int_\Omega u_\lambda \varphi_1 \geqslant \lambda m \int_\Omega u_\lambda \varphi_1 + \int_\Omega a(x) g(u_\lambda)\varphi_1, \quad \text{for all } 0 < \lambda < \lambda^*. \tag{22}$$

Notice that by $(g1)$, (20) and the monotonicity of u_λ with respect to λ we can apply the Lebesgue convergence theorem to find:

$$\int_\Omega a(x)g(u_\lambda)\varphi_1\,\mathrm{d}x \to \int_\Omega a(x)g(u^*)\varphi_1\,\mathrm{d}x \quad \text{as } \lambda \nearrow \lambda_1.$$

Passing to the limit in (22) as $\lambda \nearrow \lambda^*$, and using (20), we get:

$$\lambda_1 \int_\Omega u^*\varphi_1 \geqslant \lambda_1 \int_\Omega u^*\varphi_1 + \int_\Omega a(x)g(u^*)\varphi_1. \tag{23}$$

Hence $\int_\Omega a(x)g(u^*)\varphi_1 = 0$, which is a contradiction. This fact shows that $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. This ends the proof.

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