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MULTIPLICITY RESULTS FOR A NONLINEAR ROBIN PROBLEM WITH VARIABLE EXPONENT

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Abstract. The nonlinear weighted Robin problem

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)+b(x)|u|^{q(x)-2}u(x) & \\ \qquad\qquad -\lambda c(x)|u|^{r(x)-2}u(x)=f(x,u); & \text{in} \quad \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu}+\beta(x)|\nabla u|^{p(x)-2}u=0, & \text{on} \quad \partial\Omega \end{cases}$$

is studied in the present paper. We are concerned with maximum or minimum growth of the corresponding energy functional by various conditions on p, q, r. We also obtain qualitative properties about the behavior of energy functional and, by applying some variational methods, several existence results for the sequence of weak solutions are deduced. Finally, we study our problem by modeling as a nonlinear eigenvalue problem.

1. Introduction and main results

The variational approach is one of the main ways to study the nonlinear problems. Establishing the behavior of the energy functional corresponding to a certain partial differential equation is a key principle in variational methods and by accepting the weak solution as an acceptable solution, we need to seek the solutions in some appropriate function space.

The Sobolev space $W^{m,p}(\Omega)$, where p is constant, is suitable for studying of many problems in physics and mechanics. Whereas, by introducing problems with $p(x)$-growth conditions that arise by studying some materials with inhomogeneities such as electrorheological fluids, the classical Sobolev space does not work and so the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ are defined, where $p(\cdot)$ is an appropriate function.

Despite the sufficient reasons for developing the Lebesgue and so the Sobolev spaces, the variable exponent Lebesgue and Sobolev spaces can be seen as a mathematical generalization of the classical space which are with constant exponent.

Hence the considerable attention of mathematicians be involved in problems with $p(x)$-growth conditions, for examples see [2, 3, 5, 8, 11, 12, 14–16, 18, 23]. Moreover let us stress [17, 20] and also let us to point that Zhikov in [24] obtained conditions to satisfy Meyers type estimates, for variational problems related to an integrand that have variable exponent p(x). Let us mention the study made by Coscia and Mingione [6] where Hölder continuous functions $p(x)$ are considered to prove Hölder continuity for local minimizers of some kind of functional. Also, among the many

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related papers that there are in the literature, in a noncomprehensive way, we recall the study made by Acerbi and Mingione in [1].

In this article, we examine the nonlinear weighted problem

$$
\text{(P)} \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{q(x)-2}u(x) \\ \qquad\qquad -\lambda c(x)|u|^{r(x)-2}u(x) = f(x,u); & \text{in} \quad \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|\nabla u|^{p(x)-2}u = 0, & \text{on} \quad \partial\Omega; \end{cases}
$$

with Robin boundary value condition. We study maximum or minimum growth of the corresponding energy functional with respect to some functional of $\|u\|$. So by some conditions on $p(x), q(x), r(x)$ we conclude various results about the behavior of energy functional and hence by applying some variational methods such as mountain pass lemma, Ekeland's variational principle, or fountain theorem, several existence results for the sequence of weak solutions are obtained. Finally we study problem (P), by modellins as a nonlinear eigenvalue problem and using the Ljusternik-Schnirelmann principle.

2. Preliminary and auxiliary results

In this paper we suppose

(Ω) Ω is open, bounded subset of $\mathbb{R}^N$ with smooth boundary.

(p) $p \in C(\overline{\Omega})$, $1 < p^- := \mathrm{essinf}_{x\in\bar{\Omega}} p(x) \leq p^+ := \mathrm{esssup}_{x\in\bar{\Omega}} p(x) < \infty$.

Let

$$
E := \mathbf{L}^{p(x)}(\Omega) = \Big\{u: \ u:\Omega \longrightarrow \mathbb{R} \ \text{ is measurable, } \int_\Omega |u|^{p(x)}dx < \infty\Big\}
$$

and $X := W^{1,p(x)}(\Omega) = \{u \in E; \ |\nabla u| \in E\}$.

We refer to [9, 10] for basic information about variable exponent Lebesgue and Sobolev spaces, nevertheless we recall some basic properties.

(i) The space E is a separable, uniform convex Banach space with the norm

$$
|u|_E = \inf\Big\{\sigma > 0: \ \int_\Omega |\frac{u}{\sigma}|^{p(x)}dx \leq 1\Big\}
$$

and its conjugate space is $E' := \mathbf{L}^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. Moreover for any $u \in E$ and $v \in E'$ we have,

$$
\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_E |u|_{E'};
$$

which we named it by generalized Holder inequality.

(ii) If $p_1, p_2 \in C(\overline{\Omega})$ and $1 < p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $\mathbf{L}^{p_2(x)}(\Omega) \hookrightarrow \mathbf{L}^{p_1(x)}(\Omega)$ which is continuous embedding.

(iii)

$$
\min(|u|_E^{p^+}, |u|_E^{p^-}) \leq \int_\Omega |u|^{p(x)}dx \leq \max(|u|_E^{p^+}, |u|_E^{p^-}).
$$

(iv) X is a separable, reflexive Banach space with the norm $\|u\|_X = |u|_E + |\nabla u|_E$.

(v) There is a compact embedding $X \hookrightarrow \mathbf{L}^{q(.)}(\Omega)$; where $q \in \mathbf{C}(\overline{\Omega})$, $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}; & p(x) < N \\ \infty; & p(x) \geq N. \end{cases}$$

(vi) There is a constant $C > 0$, such that

$$|u|_E \leq C|\nabla u|_E \quad \forall u \in E.$$

Thus we can use $|\nabla u|_E$ as an equivalent norm for $\|u\|_X$.

(vii) If Ω possesses the smooth boundary $\partial\Omega$ and $p \in C(\overline{\Omega})$ then there exists a compact embedding $X \hookrightarrow \mathbf{L}^{q(.)}(\partial\Omega)$; where $q \in C(\partial\Omega)$, $1 \leq q(x) < p^{\partial}(x)$ for any $x \in \partial\Omega$ and

$$p^{\partial}(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}; & p(x) < N \\ \infty; & p(x) \geq N. \end{cases}$$

By considering problem (P), for any $v \in X$ we have

$$\begin{aligned}\int_\Omega -\mathrm{div}\,(a(x)|\nabla u|^{p(x)-2}\nabla u)v dx &= -\sum_{i=1}^{N}\int_\Omega (a(x)|\nabla u|^{p(x)-2}u_{x_i})_{x_i} v dx \\ &= \int_{\partial\Omega} a(x)|\nabla u|^{p(x)-2} v\frac{\partial u}{\partial \nu} d\sigma \\ &\quad + \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v dx.\end{aligned}$$

Thus, by letting $F(x,z) = \int_0^z f(x,t)dt$, the energy functional corresponding to problem (P) is defined by

$$\begin{aligned}E_\lambda(u) = &\int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)a(x)}{p(x)}|u|^{p(x)}d\sigma + \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx \\ &- \lambda\int_\Omega \frac{c(x)}{r(x)}|u|^{r(x)}dx - \int_\Omega F(x,u)dx;\end{aligned}$$

which it is known that the weak solutions of (P) are correspond to the critical points of E_λ.

We study problem (P) under the following general conditions.

(qr) $q, r \in C(\Omega)$, $1 < q^-, r^-$ and $q(x), r(x) < p^*(x)$.

(abc) $0 \leq a, b, c \in L^\infty(\Omega)$; $a^-, c^- \neq 0$.

(β) $0 < \beta \in L^\infty(\partial\Omega)$.

$(f1)$ $f \in C^1(\Omega \times \mathbb{R})$ and there exist $\theta, \gamma \in L^\infty(\Omega)$ with $p^+ < \theta^-$ and $\theta(x), \gamma(x) < p^*(x)$ where for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$F(x,z) \leq \varepsilon|z|^{\theta(x)} + C(\varepsilon)|z|^{\gamma(x)}. \tag{2.1}$$

$(f2)$ There exist $\mu > p^+, q^+, r^-$ such that

$$0 < \mu F(x,z) \leq z f(x,z); \quad |z| \neq 0, x \in \Omega. \tag{2.2}$$

$(f3)$ $f(x,t) = -f(x,-t)$; for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

The main results are given by the following theorems.

Theorem 2.1. *Suppose that* $r^- > \max(p^+, q^+)$ *and* $\gamma^- > p^+$. *Then for any* $\lambda > 0$, *problem* (P) *has unbounded sequence of weak solutions.*

Theorem 2.2. *Suppose that* $r^+ < \min(p^-, q^-)$. *Then for any* $\lambda > 0$, *problem* (P) *has a sequence of weak solutions* $\{u_n\}$ *such that* $E_\lambda(u_n) < 0$, $u_n \not\equiv 0$ *and* $\lim_{n\to\infty} u_n = 0$.

Theorem 2.3. *Suppose that* $\max(r^+, \gamma^+) < \min(p^-, q^-)$. *Then there exists* $\lambda^* > 0$ *such that problem* (P) *has nontrivial weak solution for any* $\lambda \in (0, \lambda^*)$; *i.e.,* $\lambda \in (0, \lambda^*)$ *is an eigenvalue for the problem* (P).

Theorem 2.4. *Suppose that* $r^- > \max(p^+, q^+)$ *and* $\max(r^+, \gamma^+) > \theta^+$. *Then for any* $\lambda > 0$, *problem* (P) *has a sequence of weak solutions* $\{u_n\}$ *such that* $E_\lambda(u_n) \to 0$ *as* $n \to +\infty$.

Theorem 2.5. *Suppose that* $r^+ < p^-$, $r^+ < \max(p^-, q^-)$ *and* $p^+ > \gamma^-$. *Then for any* $\lambda > 0$, *problem* (P) *has a sequence of weak solutions* $\{u_n\}$ *such that* $E_\lambda(u_n) < 0$ *and* $E_\lambda(u_n) \to 0$ *as* $n \to \infty$.

Let us first prove some propositions which investigate the behavior of the energy functional E_λ under suitable conditions on the functions $p(x), q(x), r(x)$.

Proposition 2.6. *If* $\min(\gamma^-, r^-) > p^+$ *then for any* $\lambda > 0$ *there exist some* $r, \zeta > 0$ *such that* $E_\lambda(u) \geq \zeta > 0$ *for all* $u \in X$ *with* $\|u\| = r$.

Proof. Since from (f_1) and (qr), we have that $\theta(x), \gamma(x), r(x) < p^*(x)$, there exist positive constants C_θ, C_γ and C_r such that $|u|_{\mathbf{L}^{\theta(\cdot)}(\Omega)} \leq C_\theta \|u\|$, $|u|_{\mathbf{L}^{\gamma(\cdot)}(\Omega)} \leq C_\gamma \|u\|$ and $|u|_{\mathbf{L}^{r(\cdot)}(\Omega)} \leq C_r \|u\|$; for all $u \in X$. Let us assume that $\|u\| < 1$; then by (f_1) for every $\varepsilon > 0$ we have,

$$
\begin{aligned}
E_\lambda(u) &\geq \int_\Omega \frac{a(x)}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_\Omega \frac{c(x)}{r(x)} |u|^{r(x)} dx - \int_\Omega F(x,u) dx \\
&\geq \frac{a^-}{p^+} \int_\Omega |\nabla u|^{p(x)} dx - \frac{\lambda c^+}{r^-} \int_\Omega |u|^{r(x)} dx \\
&\quad - \varepsilon \int_\Omega |u|^{\theta(x)} dx - C(\varepsilon) \int_\Omega |u|^{\gamma(x)} dx \\
&\geq \frac{a^-}{p^+} \|u\|^{p^+} - \frac{\lambda c^+}{r^-} \max(C_r^{r^+} \|u\|^{r^+}, C_r^{r^-} \|u\|^{r^-}) \\
&\quad - \varepsilon \max(C_\theta^{\theta^+} \|u\|^{\theta^+}, C_\theta^{\theta^-} \|u\|^{\theta^-}) \\
&\quad - C(\varepsilon) \max(C_\gamma^{\gamma^+} \|u\|^{\gamma^+}, C_\gamma^{\gamma^-} \|u\|^{\gamma^-}) \\
&\geq \frac{a^-}{p^+} \|u\|^{p^+} - \frac{\lambda c^+}{r^-} \hat{C}_r \|u\|^{r^-} - \varepsilon \hat{C}_\theta \|u\|^{p^+} - C(\varepsilon) \hat{C}_\gamma \|u\|^{\gamma^-}.
\end{aligned}
\tag{2.3}
$$

In which $C(\varepsilon)$ is introduced in (f_2) and the last inequality derives from $p^+ < \theta^-$ and $\hat{C}_\theta := \max(C_\theta^{\theta^+}, C_\theta^{\theta^-})$, $\hat{C}_\gamma := \max(C_\gamma^{\gamma^+}, C_\gamma^{\gamma^-})$ and $\hat{C}_r := \max(C_r^{r^+}, C_r^{r^-})$.

Therefore, if $r^- > \gamma^- > p^+$ we have

$$E_\lambda(u) \geq \|u\|^{p^+}\Big(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta - \Big(\frac{\lambda c^+\hat{C}_r}{r^-} + C(\varepsilon)\hat{C}_\gamma\Big)\|u\|^{\gamma^- - p^+}\Big),$$

and in the case $\gamma^- > r^- > p^+$,

$$E_\lambda(u) \geq \|u\|^{p^+}\Big(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta - \Big(\frac{\lambda c^+\hat{C}_r}{r^-} + C(\varepsilon)\hat{C}_\gamma\Big)\|u\|^{r^- - p^+}\Big).$$

Hence in the both cases for some fixed $\varepsilon \in (0, \frac{a^-}{p^+\tilde{C}_\theta})$ there exists $\zeta > 0$ such that for all $u \in X$ with $\|u\| = r$ small enough, $E_\lambda(u) \geq \zeta > 0$. □

Proposition 2.7. *Let X_1 be a finite dimensional subspace of X. Then the set $S := \{u \in X_1, E_\lambda(u) \geq 0\}$ is bounded, provided that $r^- > \max(p^+, q^+)$.*

Proof. By compact embedding $X \hookrightarrow L^{p(\cdot)}(\partial\Omega)$, there exist a positive constant C_∂ such that $|u|_{\mathbf{L}^{p(\cdot)}(\partial\Omega)} \leq C_\partial\|u\|$. So for $\|u\| > 1$ we have,

$$\begin{aligned}
E_\lambda(u) &\leq \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega}\frac{\beta(x)a(x)}{p(x)}|u|^{p(x)}d\sigma \\
&\quad + \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx - \lambda\int_\Omega\frac{c(x)}{r(x)}|u|^{r(x)}dx \\
&\leq \frac{a^+}{p^-}\int_\Omega|\nabla u|^{p(x)}dx + \frac{\beta^+a^+}{p^-}\int_{\partial\Omega}|u|^{p(x)}d\sigma \\
&\quad + \frac{b^+}{q^-}\int_\Omega|u|^{q(x)}dx - \frac{\lambda c^-}{r^+}\int_\Omega|u|^{r(x)}dx \\
&\leq \Big(\frac{a^+}{p^-} + \frac{\beta^+a^+}{p^-}\hat{C}_\partial\Big)\|u\|^{p^+} + \frac{b^+}{q^-}\hat{C}_q\|u\|^{q^+} - \frac{\lambda c^-}{r^+}\min(|u|^{r^+}_{L^{r(.)}}, |u|^{r^-}_{L^{r(.)}});
\end{aligned}$$

where by similar notation as in previous proposition let $\hat{C}_\delta := \max(C_\partial^{p^+}, C_\partial^{p^-})$. If $u \in X_1$, by equivalency of any two norms on finite dimensional space we have

$$E_\lambda(u) \leq \Big(\frac{a^+}{p^-} + \frac{\beta^+}{p^-}\hat{C}_\partial\Big)\|u\|^{p^+} + \frac{b^+}{q^-}\hat{C}_\partial\|u\|^{q^+} - \kappa\frac{\lambda c^-}{r^+}\|u\|^{r^-};$$

for some positive constant κ. Thus, if $u \in S$, since $r^- > \max(p^+, q^+)$, we obtain that the set S would be bounded. □

Proposition 2.8. *Suppose $p^- > r^+$ or $r^- > \max(p^+, q^+)$ then for any $\lambda > 0$, E_λ satisfies the Palais-Smale condition, that is, if $\{u_n\}$ is a sequence in X with $|E_\lambda(u_n)| < M$, $\nabla E_\lambda(u_n) \to 0$ as $n \to \infty$, where M is positive constant; then $\{u_n\}$ contains a convergent subsequence in X.*

Proof. Firstly, we claim that $\{u_n\}_{n=1}^{\infty}$ is bounded in X. By using $(f2)$, for $\|u_n\| > 1$ we have

$$\begin{aligned} M > E_\lambda(u_n) &\geq \int_\Omega \frac{a(x)}{p(x)}|\nabla u_n|^{p(x)}dx \\ &\quad + \int_{\partial\Omega} \frac{\beta(x)a(x)}{p(x)}|u_n|^{p(x)}d\sigma + \int_\Omega \frac{b(x)}{q(x)}|u_n|^{q(x)}dx \\ &\quad - \lambda\int_\Omega \frac{c(x)}{r(x)}|u_n|^{r(x)}dx - \frac{1}{\mu}\int_\Omega f(x,u_n)u_n dx \\ &= \int_\Omega \Big(\frac{1}{p(x)} - \frac{1}{\mu}\Big)a(x)|\nabla u_n|^{p(x)}dx \\ &\quad + \int_{\partial\Omega} \Big(\frac{1}{p(x)} - \frac{1}{\mu}\Big)\beta(x)a(x)|u_n|^{p(x)}d\sigma \\ &\quad + \int_\Omega \Big(\frac{1}{q(x)} - \frac{1}{\mu}\Big)b(x)|u_n|^{q(x)}dx \\ &\quad - \lambda\int_\Omega \Big(\frac{1}{r(x)} + \frac{1}{\mu}\Big)c(x)|u_n|^{r(x)}dx + \frac{1}{\mu}\langle\nabla E_\lambda(u_n), u_n\rangle \\ &\geq a^-\Big(\frac{1}{p^+} - \frac{1}{\mu}\Big)\|u_n\|^{p^-} - \lambda\Big(\frac{1}{r^-} + \frac{1}{\mu}\Big)c^+\hat{C}_r\|u_n\|^{r^+} \\ &\quad - \frac{1}{\mu}\|\nabla E(u_n)\|\|u_n\|. \end{aligned} \tag{2.4}$$

Thus when $p^- > r^+$ we obtain $\{\|u_n\|\}_{n=1}^{\infty}$ is necessary bounded.

In the case $r^- > \max(p^+, q^+)$, since $\nabla E_\lambda(u_n) \to 0$ as $n \to \infty$, there exist $N_1 > 0$ such that for any $n > N_1$ we have $\|\nabla E_\lambda(u_n)\| \leq 1$ and so

$$| < \nabla E_\lambda(u_n), v > | \leq \|v\|, \quad \forall v \in X, \quad n > N_1.$$

By letting $v := u_n$ we get

$$\begin{aligned} &- \|u_n\| - \int_\Omega a(x)|\nabla u_n|^{p(x)}dx - \int_{\partial\Omega} \beta(x)a(x)|u_n|^{p(x)}d\sigma \\ &\qquad - \int_\Omega b(x)|u_n|^{q(x)}dx + \int_\Omega f(x,u_n)u_n dx \leq -\lambda\int_\Omega c(x)|u_n|^{r(x)}dx; \end{aligned}$$

for any $n > N_1$. Hence by (2.4)

$$\begin{aligned} M > E_\lambda(u_n) &\geq a^-\Big(\frac{1}{p^+} - \frac{1}{r^-}\Big)\int_\Omega |\nabla u_n|^{p(x)}dx + \beta^- a^-\Big(\frac{1}{p^+} - \frac{1}{r^-}\Big)\int_{\partial\Omega} |u_n|^{p(x)}d\sigma \\ &\quad + b^-\Big(\frac{1}{q^+} - \frac{1}{r^-}\Big)\int_\Omega |u_n|^{q(x)}dx + \Big(\frac{1}{r^-} - \frac{1}{\mu}\Big)\int_\Omega f(x,u_n)u_n dx - \|u_n\| \\ &\geq a^-\Big(\frac{1}{p^+} - \frac{1}{r^-}\Big)\|u_n\|^{p^-} - \|u_n\|; \end{aligned}$$

and so $\{\|u_n\|\}_{n=1}^{\infty}$ is bounded.

Hence for two cases $p^- > r^+$ and $r^- > \max(p^+, q^+)$ we can say, up to a subsequence, $\{u_n\}_{n=1}^{\infty}$ converges weakly in X to some $u \in X$. Since $\nabla E(u_n) \longrightarrow 0$, there exists a strictly decreasing sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty}\varepsilon_n = 0$ and

$$|\langle E'(u_n), v\rangle| \leq \varepsilon_n \|v\|; \qquad \forall v \in X, n \in \mathbb{N}. \tag{2.5}$$

Using $v := u_n - u$ in (2.5), we obtain

$$\begin{aligned}
\Big| \int_\Omega & a(x)|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\
&+ \int_{\partial\Omega} \beta(x)a(x)|u_n|^{p(x)-2}u_n(u_n-u)d\sigma \\
&\quad + \int_\Omega b(x)|u_n|^{q(x)-2}u_n(u_n-u)dx - \lambda\int_\Omega c(x)|u_n|^{r(x)-2}u_n(u_n-u)dx \\
&\qquad - \int_\Omega f(x,u_n)(u_n-u)dx\Big| \leq \varepsilon_n\|u_n - u\|.
\end{aligned}$$

On the other hand, by compact embeddings $X \hookrightarrow \mathbf{L}^{p(\cdot)}(\partial\Omega)$, $X \hookrightarrow \mathbf{L}^{q(\cdot)}(\Omega)$ and $X \hookrightarrow \mathbf{L}^{r(\cdot)}(\Omega)$ we can deduce $u_n \longrightarrow u$ strongly in $\mathbf{L}^p(\partial\Omega), \mathbf{L}^q(\Omega)$ and $\mathbf{L}^r(\Omega)$. Hence

$$\begin{aligned}
\lim_{n\to\infty} & \int_{\partial\Omega} \beta(x)a(x)|u_n|^{p(x)-2}u_n(u_n-u)d\sigma \\
&= \lim_{n\to\infty}\int_\Omega b(x)|u_n|^{q(x)-2}u_n(u_n-u)dx \\
&= \lim_{n\to\infty}\int_\Omega c(x)|u_n|^{r(x)-2}u_n(u_n-u)dx = 0.
\end{aligned}$$

Moreover by the hypotheses on f we have $\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)dx = 0$. Hence

$$\limsup_{n\to\infty}\Big|\int_\Omega a(x)|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx\Big| = 0.$$

Since $\langle \nabla E_\lambda(u), u_n - u\rangle \to 0$ as $n\to\infty$ we have

$$\limsup_{n\to\infty}\Big|\int_\Omega a(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)dx\Big| = 0. \tag{2.6}$$

Set $I_n(x) := \int_\Omega a(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)$. We recall two known inequalities

$$\begin{cases} (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta) \geq \frac{1}{2^p}|\xi-\eta|^p; & p \geq 2 \\ (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi-\eta)(|\xi|+|\eta|)^{2-p} \geq (p-1)|\xi-\eta|^p; & 1 < p < 2. \end{cases} \tag{2.7}$$

By applying two inequalities of (2.7) in (2.6), we get

$$\begin{aligned}
I_n(x) \geq & \Big(\frac{a(x)}{2^{p(x)}}|\nabla u_n - \nabla u|^{p(x)}\Big)\chi_{\mathbf{\Omega}^+}(x) \\
&+ a(x)(p(x)-1)\frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}}\chi_{\mathbf{\Omega}^-}(x);
\end{aligned} \tag{2.8}$$

where $\Omega^+ = \{x \in \Omega; p(x) \geq 2\}$ and $\Omega^- = \Omega\backslash\Omega^+$. Then we have

$$\int_{\Omega^+}|\nabla u_n - \nabla u|^{p(x)}dx \leq \kappa\int_\Omega I_n(x)dx; \tag{2.9}$$

for a some positive constant κ and from the last term in (2.8), we get

$$\int_{\Omega^-} |\nabla u_n - \nabla u|^{p(x)} dx \le \kappa \int_{\Omega^-} I_n(x)^{\frac{p(x)}{2}} (|\nabla u_n| + |\nabla u|)^{(2-p(x))\frac{p(x)}{2}} dx := \kappa J_n(x).$$

Since $\lim_{n\to\infty} I_n(x) = 0$, we can consider $0 \le \int_\Omega I_n(x)dx < 1$, and by Young inequality, we have

$$\begin{aligned} J_n(x) &= \int_{\Omega^-} I_n(x)^{\frac{p(x)}{2}} \Big(\int_\Omega I_n(y)dy\Big)^{-\frac{p(x)}{2}} (|\nabla u_n| + |\nabla u|)^{(2-p(x))\frac{p(x)}{2}} \Big(\int_\Omega I_n(y)dy\Big)^{\frac{p(x)}{2}} dx \\ &\le \Big(\int_\Omega I_n(y)dy\Big)^{\frac{1}{2}} \int_{\Omega^-} \frac{p(x)}{2} (I_n(x)\Big(\int_\Omega I_n(y)dy)^{-1}\Big) \\ &\quad + \frac{2-p(x)}{2} (|\nabla u_n| + |\nabla u|)^{p(x)} dx \\ &\le \Big(\int_\Omega I_n(y)dy\Big)^{\frac{1}{2}} (1 + \int_\Omega (|\nabla u_n| + |\nabla u|)^{p(x)} dx). \end{aligned}$$

Hence by using (2.9), we obtain

$$\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \le \kappa (\int_\Omega I_n(y)dy)^{\frac{1}{2}} (1 + \int_\Omega (|\nabla u_n| + |\nabla u|)^{p(x)} dx).$$

Thus $\int_\Omega |\nabla u_n - \nabla u|^{p(x)} dx \to 0$ as $n \to 0$, and so $u_n \to u$ in X. □

For each n, denote by Γ_n, the family of closed symmetric subset Y of X such that $0 \notin Y$ and $\gamma(Y) \ge n$, where $\gamma(Y)$ is the genus of Y, i.e.,

$$\gamma(A) := \inf\{n \in \mathbb{N}; \exists h : A \longrightarrow \mathbb{R}^n \setminus \{0\} \textit{ such that } h \textit{ is continuous and odd}\}.$$

Now, consider the following proposition.

Proposition 2.9. *Suppose $r^+ < \min\{p^-, q^-\}$ and fix $\lambda > 0$, then for each $n \in \mathbb{N}$ there exists $Y_n \in \Gamma_n$ such that $\sup_{u\in Y_n} E_\lambda(u) < 0$.*

Proof. Let $u_1, \dots, u_n \in C_c^\infty(\Omega)$ such that $spt(u_i) \cap spt(u_j) = \emptyset$ and $|spt(u_i)| > 0$ for $i, j \in \{1, \dots, n\}$ and $i \ne j$. Take F_n the subspace generated by $\{u_1, \dots, u_n\}$. Denote $S := \{u \in X, \|u\|\} = 1$ and for any $t \in (0, 1]$ let $Y_n(t) := \{tu; u \in S \cap F_n\}$, then obviously $\gamma(Y_n(t)) = n$, for all t and we have

$$\begin{aligned} \sup_{u\in Y_n(t)} E_\lambda(u) &= \sup_{u\in S\cap F_n} E_\lambda(tu) \\ &\le \sup_{u\in S\cap F_n} \int_\Omega \frac{a(x)}{p(x)} t^{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} t^{p(x)} |u|^{p(x)} d\sigma \\ &\quad + \int_\Omega \frac{b(x)}{q(x)} t^{q(x)} |u|^{p(x)} dx - \lambda \int_\Omega \frac{c(x)}{r(x)} t^{r(x)} |u|^{r(x)} dx \\ &\le \sup_{u\in S\cap F_n} \frac{t^{p^-} a^+}{p^-} \int_\Omega |\nabla u|^{p(x)} dx + \frac{t^{p^-} \beta^+ a^+}{p^-} \int_{\partial\Omega} |u|^{p(x)} d\sigma \\ &\quad + \frac{t^{q^-} b^+}{q^-} \int_\Omega |u|^{q(x)} dx - \frac{\lambda t^{r^+} c^-}{r^+} \int_\Omega |u|^{r(x)} dx \\ &\le \sup_{u\in S\cap F_n} \frac{t^{p^-} a^+}{p^-} + \frac{t^{p^-} \beta^+ a^+}{p^-} \hat{C}_\partial \end{aligned}$$

$$+\frac{t^{q^-}b^+}{q^-}\hat{C}_q - \frac{\lambda t^{r^+}c^-}{r^+}\int_\Omega |u|^{r(x)}dx.$$

Thus since $u \in F_n$, $dim(F_n) < \infty$ and $r^+ < \min\{p^-, q^-\}$ we can conclude the desired result for $t_n \in (0,1]$ which is small enough. □

Proposition 2.10. *If* $\max(r^+, \gamma^+) < \min(p^-, q^-)$*, for any* $\varepsilon \in (0, \frac{a^-}{p^+\hat{c}_\theta})$ *there exist* $R, \zeta, \lambda^* > 0$ *such that* $E_\lambda(u) \geq \zeta > 0$*, for all* $u \in X$ *with* $\|u\| = R$ *and* $\lambda \in (0, \lambda^*)$.

Proof. Without loss of generality suppose that $\max(r^+, \gamma^+) = r^+$. Hence by use of (2.3), for $\|u\| > 1$ we have,

$$\begin{aligned} E_\lambda(u) &\geq \Big(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta\Big)\|u\|^{p^-} - \Big(\frac{\lambda c^+}{r^-}\hat{C}_r + C(\varepsilon)\hat{C}_\gamma\Big)\|u\|^{r^+} \\ &= \|u\|^{r^+}\Big(\Big(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta\Big)\|u\|^{p^- - r^+} - \Big(\frac{\lambda c^+}{r^-}\hat{C}_r + C(\varepsilon)\hat{C}_\gamma\Big)\Big). \end{aligned} \tag{2.10}$$

Let $R > \max\big(1, \big(\frac{C(\varepsilon)\hat{C}_\gamma}{\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta}\big)^{\frac{1}{p^- - r^+}}\big)$, then for fixed $\lambda \in \big(0, \frac{((\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta)R^{p^- - r^+} - C(\varepsilon)\hat{C}_\gamma)r^-}{c^+\hat{C}_r}\big)$ there exists $\zeta > 0$ such that $E_\lambda(u) \geq \zeta > 0$. □

Remark 2.11. If $r^- < \min(p^-, q^-)$ and $r^- < \gamma^-$ then for any $\varepsilon \in (0, \frac{a^-}{p^+\hat{C}_\theta})$ with $\frac{C(\varepsilon)\hat{C}_\gamma}{\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta} < 1$ there exist $R, \zeta, \lambda^* > 0$ such that $E_\lambda(u) \geq \zeta > 0$, for all $u \in X$ with $\|u\| = R$ and $\lambda \in (0, \lambda^*)$.

Proof. In this case by using (2.3) for $\|u\| < 1$ we have

$$E_\lambda(u) \geq \|u\|^{r^-}\Big(\Big(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta\Big)\|u\|^{p^+ - r^-} - \Big(\frac{\lambda c^+}{r^-}\hat{C}_r + C(\varepsilon)\hat{C}_\gamma\Big)\Big).$$

Let $\big(\frac{C(\varepsilon)\hat{C}_\gamma}{\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta}\big)^{\frac{1}{p^+ - r^-}} < R < 1$, then for fixed $\lambda \in \big(0, \frac{(\frac{a^-}{p^+} - \varepsilon\hat{C}_\theta)R^{p^+ - r^-} - C(\varepsilon)\hat{C}_\gamma)r^-}{c^+\hat{C}_r}\big)$ there exists $\zeta > 0$ such that $E_\lambda(u) \geq \zeta > 0$. □

Proposition 2.12. *Suppose* $r^- < \min(p^-, q^-)$ *and fix* $\lambda > 0$ *then there exists* $u \in X$*,* $u \geq 0$*,* $u \not\equiv 0$ *such that* $E_\lambda(tu) < 0$ *for* t *small enough.*

Proof. Since $r^- < \min(p^-, q^-)$, there exists $\varepsilon_0 > 0$ such that

$$r^- + \varepsilon_0 < \min(p^-, q^-); \tag{2.11}$$

and since $r \in C(\Omega)$, there exists an open set $\Omega_0 \subset \Omega$ such that $|r(x) - r^-| < \varepsilon_0$ for all $x \in \Omega_0$. Take $v \in C_c^\infty(\Omega)$ which $\overline{\Omega_0} \subset spt(v)$, $v(x) = 1$ for $x \in \overline{\Omega_0}$ and $0 \leq v(x) \leq 1$ in Ω. Without loss of generality we may assume $\|v\| = 1$. Note that, by this construction of v, we have $\int_{\overline{\Omega_0}} |v|^{r(x)}dx = |\overline{\Omega_0}|$, then for all $t \in (0,1)$ we obtain

$$\begin{aligned} E_\lambda(tv) \leq{}& \frac{t^{p^-}a^+}{p^-}\int_\Omega |\nabla v|^{p(x)}dx + \frac{t^{p^-}\beta^+a^+}{p^-}\int_{\partial\Omega}|v|^{p(x)}d\sigma \\ &+ \frac{t^{q^-}b^+}{q^-}\int_\Omega |v|^{q(x)}dx - \lambda\frac{c^-}{r^+}\int_{\overline{\Omega_0}} t^{r(x)}|v|^{r(x)}dx \end{aligned}$$

$$\leq \frac{t^{p^-}a^+}{p^-} + \frac{t^{p^-}\beta^+ a^+}{p^-}\hat{C}_\partial + \frac{t^{q^-}b^+}{q^-}\hat{C}_q - \lambda \frac{t^{r^-+\varepsilon_0}c^-}{r^+}|\overline{\Omega_0}|.$$

Hence by (2.11), we conclude the proposition. □

Since X is separable, there exist two sequences $\{e_n\} \subset X$ and $\{e_n^*\} \subset X^*$ such that

$$\langle e_n, e_m^* \rangle = \begin{cases} 1, & n = m; \\ 0, & n \neq m, \end{cases}$$

$X = \overline{span}\{e_n;\ n = 1, 2, \dots\}$ and $X^* = \overline{span}^{W^*}\{e_n^*;\ n = 1, 2, \dots\}$. For any $k \in \mathbb{N}$ denote, $X_k := span\{e_k\}$, $Y_k := \oplus_{i=1}^k X_i$ and $Z_k := \oplus_{i=k}^\infty X_i$. Now consider the following proposition.

Proposition 2.13. *Fix $\lambda > 0$, then for any $k \in \mathbb{N}$ there exist $R_k > r_k > 0$ such that*

(i) $\inf\{E_\lambda(u); u \in Z_k, \|u\| = r_k\} \longrightarrow +\infty$ *as* $k \to \infty$ *provided that* $\max(r^+, \gamma^+) > \theta^+$.

(ii) $\max\{E_\lambda(u); u \in Y_k \|u\| = R_k\} \leq 0$ *provided that* $\max(r^-, \mu) > \max(p^+, q^+)$.

Proof. For $\|u\| \geq 1$ we obtain

$$E_\lambda(u) \geq \frac{a^-}{p^+}\|u\|^{p^-} - \lambda \frac{c^+}{r^-}\int_\Omega |u|^{r(x)}dx - \varepsilon \int_\Omega |u|^{\theta(x)}dx - C(\varepsilon)\int_\Omega |u|^{\gamma(x)}dx. \tag{2.12}$$

Let

$$\sigma_{r,k} := \sup\{|u|_{L^{r(x)}(\Omega)}; u \in Z_k, \|u\| \leq 1\};$$
$$\sigma_{\theta,k} := \sup\{|u|_{L^{\theta(x)}(\Omega)}; u \in Z_k, \|u\| \leq 1\};$$

and

$$\sigma_{\gamma,k} := \sup\{|u|_{L^{\gamma(x)}(\Omega)}; u \in Z_k, \|u\| \leq 1\}.$$

Hence for any $u \in Z_k$; $|u|_{L^{r(x)}(\Omega)} \leq \sigma_{r,k}\|u\|$, $|u|_{L^{\theta(x)}(\Omega)} \leq \sigma_{\theta,k}\|u\|$ and $|u|_{L^{\gamma(x)}(\Omega)} \leq \sigma_{\gamma,k}\|u\|$.

First we show that $\sigma_{r,k} \longrightarrow 0$ as $k \longrightarrow \infty$. We have $Z_{k+1} \subset Z_k$ so $\{\sigma_{r,k}\}_{k=1}^\infty$ is positive decreasing real sequence and hence for some nonnegative constant σ_r; $\sigma_{r,k} \to \sigma_r$ as $k \to \infty$. For any k, we can choose $u_k \in Z_k$ such that $|u_k|_{L^{r(x)}(\Omega)} \to \sigma_r$ as $k \to \infty$ and so there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$ such that $u_k \rightharpoonup u$. Moreover

$$\langle e_j^*, u \rangle = \lim_{k\to\infty} \langle e_j^*, u_k \rangle = 0; \quad j = 1, 2, \dots;$$

which implies $u = 0$ and so $u_k \rightharpoonup 0$ in X and hence $u_k \to 0$ in E. So we get $\sigma_r = 0$. By similar argument we get $\sigma_{\theta,k}, \sigma_{\gamma,k} \to 0$ as $k \to \infty$.

Now we can deduce for $u \in Z_k$,

$$\begin{aligned} E_\lambda(u) &\geq \frac{a^-}{p^+}\|u\|^{p^-} - \lambda \frac{c^+}{r^-}\sigma_{r,k}\|u\|^{r^+} - \varepsilon\sigma_{\theta,k}\|u\|^{\theta^+} - C(\varepsilon)\sigma_{\gamma,k}\|u\|^{\gamma^+} \\ &\geq \frac{a^-}{p^+}\|u\|^{p^-} - \Big(\lambda \frac{c^+}{r^-}\sigma_{r,k} - \varepsilon\sigma_{\theta,k} - C(\varepsilon)\sigma_{\gamma,k}\Big)\|u\|^m \end{aligned}$$

where $m = max(r^+, \gamma^+)$.

Let $\|u\| = r_k := \Big(\frac{\frac{a^-}{2p^+}}{\lambda\frac{c^+}{r^-}\sigma_{r,k} - \varepsilon\sigma_{\theta,k} - C(\varepsilon)\sigma_{\gamma,k}}\Big)^{\frac{1}{m-p^+}}$ we obtain

$$E_\lambda(u) \geq \Big(\frac{\frac{a^-}{2p^+}}{\lambda\frac{c^+}{r^-}\sigma_{r,k} - \varepsilon\sigma_{\theta,k} - C(\varepsilon)\sigma_{\gamma,k}}\Big)^{\frac{p^+}{m-p^+}}\Big(\frac{a^-}{2p^+}\Big) \longrightarrow +\infty;$$

as $k \to \infty$ since $\sigma_{r,k}, \sigma_{\theta,k}, \sigma_{\gamma,k} \to 0$. So the proof of (i) is completed.

For (ii), denote $\mathcal{F}(t) = \frac{F(x,tz)}{t^\mu}$; for every $t > 0$. Then using $(f3)$ we get,

$$\mathcal{F}'(t) = \frac{1}{t^{\mu+1}}[tzf(x,tz) - \mu F(x,tz)] \geq 0, \qquad \forall t > 0.$$

Thus, we deduce for any $t \geq 1$, $\mathcal{F}(t) \geq \mathcal{F}(1)$; i.e., $F(x,tz) \geq t^\mu F(x,z)$, for $z \neq 0$. Now let $v \in Y_k$ with $\|v\| = 1$ and $t > 1$ then we have

$$E_\lambda(tv) \leq t^{p^+}\Big(\frac{a^+}{p^-} + \hat{C}_\partial\frac{\beta^+ a^+}{p^-}\Big) + t^{q^+}\hat{C}_q\frac{b^+}{q^-} - \lambda t^{r^-}\frac{c^-}{r^+}\int_\Omega |v|^{r(x)}dx - t^\mu\int_\Omega |v|^\mu dx;$$

which implies for sufficiently large $t > 1$, such as t_k, $E_\lambda(t_k v) < 0$ since $\max(r^-, \mu) > \max(p^+, q^+)$. Note that in this proposition, the condition $\mu > p^+, q^+, r^-$ is not necessary. Thus the proof of assertion (ii) is completed by letting $R_k := t_k$. □

Proposition 2.14. *Fix $\lambda > 0$ then for any $k \in \mathbb{N}$, there exist $R_k > r_k > 0$ such that*

(i) $\inf\{E_\lambda(u); u \in Z_k, \|u\| = R_k\} \geq 0$ *provided that* $p^+ > \max(r^-, \gamma^-)$.
(ii) $\max\{E_\lambda(u); u \in Y_k, \|u\| = r_k\} < 0$ *provided that* $r^+ < \max(p^-, q^-)$.
(iii) $\inf\{E_\lambda(u); u \in Z_k, \|u\| \leq R_k\} \to 0$ *as* $k \to \infty$ *provided that* $r^+ < \max(p^-, q^-)$.

Proof. Let $v \in Z_k$ with $\|v\| = 1$ and let $t \in (0,1)$ then by using the same notation as in Proposition 2.13 and considering (2.12) we can deduce

$$\begin{aligned} E_\lambda(tv) &\geq \frac{a^-}{p^+}t^{p^+} - \lambda\frac{c^+}{r^-}\sigma_{r,k}t^{r^-} - \varepsilon\sigma_{\theta,k}t^{\theta^-} - C(\varepsilon)\sigma_{\gamma,k}t^{\gamma^-} \\ &\geq \frac{a^-}{p^+}t^{p^+} - (\lambda\frac{c^+}{r^-}\sigma_{r,k} - \varepsilon\sigma_{\theta,k} - C(\varepsilon)\sigma_{\gamma,k})t^m \end{aligned}$$

where $m = max(r^-, \gamma^-)$.

Taking

$$R_k = \mathcal{B}\Big(\frac{\frac{a^-}{p^+}}{\lambda\frac{c^+}{r^-}\sigma_{r,k} - \varepsilon\sigma_{\theta,k} - C(\varepsilon)\sigma_{\gamma,k}}\Big)^{(\frac{1}{p^+-m})},$$

then for sufficiently large k, $R_k < 1$. Moreover for $u = R_k v$ we have $E_\lambda(u) \geq 0$.

For (ii), let $v \in Y_k$ with $\|v\| = 1$ and let $t \in (0,1)$. Then

$$E_\lambda(tv) \leq \Big(\frac{a^+}{p^-} + \frac{\beta^+}{p^-}\hat{C}_\partial\Big)t^{p^-} + \frac{b^+}{q^-}t^{q^-}\hat{C}_q - \lambda\frac{c^-}{r^+}t^{r^+}\int_\Omega |v|^{r(x)}dx.$$

Which $r^+ < \max(p^-, q^-)$ implies that $E_\lambda(tv) < 0$, for small enough t. So there exists $r_k \in (0, R_k)$ such that for $t = r_k$, and $u = r_k v$, $E_\lambda(u) < 0$.

Now let $v \in Z_k$ with $\|v\| = 1$ and $t \in [0, R_k]$ then by (2.12) we obtain

$$E_\lambda(tv) \geq -\lambda\frac{c^+}{r^-}\sigma_{r,k}t^{r^-} - \varepsilon\sigma_{\theta,k}t^{\theta^-} - C(\varepsilon)\sigma_{\gamma,k}t^{\gamma^-}.$$

Since $Y_k \cap Z_k \neq \emptyset$ and $r_k < R_k$ by using assertion (ii) we get

$$\inf\{E_\lambda(u); u \in Z_k, \|u\| \leq R_k\} \leq \max\{E_\lambda(u); u \in Y_k, \|u\| = r_k\} < 0.$$

Hence $0 > E_\lambda(tv) \geq -\lambda \frac{c^+}{r^-} \sigma_{r,k} t^{r^-} - \varepsilon \sigma_{\theta,k} t^{\theta^-} - C(\varepsilon)\sigma_{\gamma,k} t^{\gamma^-}$; which shows $E_\lambda(u) \to 0$ by attending $k \to \infty$ and $u \in Z_k$ with $\|u\| \leq R_k$, thus the proof is completed. □

3. Proof of the main results

Now we are ready to conclude the main results that are listed at the end of first section.

Proof of Theorem 2.1:

Proof. By considering Proposition 2.6, Proposition 2.7, Proposition 2.8 and applying $\mathbb{Z}_2$-symmetric version of mountain pass theorem [13], we conclude the result. □

Proof of Theorem 2.2:

Proof. By considering Proposition 2.8 and Proposition 2.9 with applying the symmetric version of the mountain pass theorem [4], we conclude the assertion. □

Proof of Theorem 2.3:

Proof. By Proposition 2.10, we obtain $\inf_{\partial B_R(0)} E_\lambda(u) > 0$, on the other hand by Proposition 2.12, there exists $v \in X$ such that $E_\lambda(tv) < 0$ for $t > 0$ small enough. Moreover for $u \in B_R(0)$, by (2.10) we have

$$E_\lambda(u) \geq \begin{cases} (\frac{a^-}{p^+} - \varepsilon \hat{C}_\theta)\|u\|^{p^-} - (\frac{\lambda c^+}{r^-}\hat{C}_r + C(\varepsilon)\hat{C}_\gamma)\|u\|^{r^+}; & \|u\| \geq 1, \\ (\frac{a^-}{p^+} - \varepsilon \hat{C}_\theta)\|u\|^{p^+} - (\frac{\lambda c^+}{r^-}\hat{C}_r + C(\varepsilon)\hat{C}_\gamma)\|u\|^{r^-}; & \|u\| < 1. \end{cases}$$

Thus $-\infty < c_\lambda := \inf_{\overline{B_R(0)}} E_\lambda(u) < 0$. Hence the functional $E_\lambda : \overline{B_R(0)} \to \mathbb{R}$ is bounded below and continuously differentiable on $\overline{B_R(0)}$. Fix ϵ such that

$$0 < \epsilon < \inf_{\partial B_R(0)} E_\lambda(u) - \inf_{\overline{B_R(0)}} E_\lambda(u). \tag{3.1}$$

Then by Ekeland's variational principle [7], there exists $u_\epsilon \in \overline{B_R(0)}$ such that $E_\lambda(u_\epsilon) - \epsilon \leq c_\lambda$ and

$$0 < E_\lambda(u) - E_\lambda(u_\epsilon) + \epsilon\|u - u_\epsilon\|; \quad u \neq u_\epsilon. \tag{3.2}$$

So by using (3.1), we deduce $u_\epsilon \in B_R(0)$. Now, define $I_\lambda : \overline{B_R(0)} \to \mathbb{R}$ with $I_\lambda(u) = E_\lambda(u) + \epsilon\|u - u_\epsilon\|$. By (3.2), it is obvious that u_ϵ is minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} = \frac{E_\lambda(u_\epsilon + tv) - E_\lambda(u_\epsilon)}{t} + \epsilon\|v\| \geq 0;$$

for small $t > 0$ and any $v \in B_1(0)$. Thus by letting $t \to 0$ we obtain $\|E'_\lambda(u_\epsilon)\| \leq \epsilon$. So the hypothesis of Palais-Smale condition is obtained, that is there exist $\{u_n\} \subset B_R(0)$ such that $E_\lambda(u_n) \longrightarrow c_\lambda$ and $E'_\lambda(u_n) \longrightarrow 0$. Now by applying Proposition 2.8 we have $\{u_n\}$, up to a subsequence converges strongly to some $u_0 \in X$ and since $E_\lambda \in C^1(X, \mathbb{R})$, we get $E_\lambda(u_0) = c_\lambda < 0$ and $E'_\lambda(u_0) = 0$. Hence u_0 is nontrivial weak solution for (P) and since $E_\lambda(|u_0|) = E_\lambda(u_0)$ the problem (P) has a nonnegative one. $\square$

Remark 3.1. In the case $r^- < \min(p^-, q^-)$ and $r^- < \gamma^-$, the assertion of Theorem 2.3 is hold provided that the condition of Remark 2.11 is satisfied .

Proof. By Remark 2.11, we have for some $R \in (0,1)$, $\inf_{\partial B_R(0)} > 0$. Note that for any $u \in B_R(0)$, $\|u\| < 1$ and so by using the same argument as in the proof of Theorem 2.3, we conclude the result. $\square$

Proof of Theorem 2.4:

Proof. By considering Proposition 2.8, Proposition 2.13 and applying Fountain theorem [19], we get the result. $\square$

Proof of Theorem 2.5:

Proof. By considering Proposition 2.8, Proposition 2.14 and applying Dual Fountain theorem [19], we obtain the result. $\square$

4. Modelling as an eigenvalue problem

In this section we study the problem (P) as an eigenvalue problem by Ljusternik-Schnirelmann principle argument ([21,22]), which is one of the more used to find the eigenpair sequences in nonlinear problems, and so we claim the following theorem:

Theorem 4.1. *Consider problem (P) with condition:*

(f_2')
$$0 \leq f(x,z)z \leq F(x,z);$$

instead of (f_2) and not necessarily $p^+ < \theta^-$ in (f_1). Then for any $s > 0$ there exists nondecreasing sequence of nonnegative eigenvalue $\{\lambda_{n,s}\}$ of (R) such that $\lambda_{n,s} = \frac{1}{\mu_{n,s}} \to \infty$ as $n \to \infty$, where each $\mu_{n,s}$ is an eigenvalue of the corresponding equation

$$\begin{cases} \varphi'(u) = \mu \phi'(u) \\ \phi(u) = s, \quad \mu \in \mathbb{R}; \end{cases}$$

which $\varphi(u) = \int_\Omega \frac{c(x)}{r(x)} |u|^{r(x)} dx$ and

$$\begin{aligned} \phi(u) = & \int_\Omega \frac{a(x)}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x)a(x)}{p(x)} |u|^{p(x)} d\sigma \\ & + \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx - \int_\Omega F(x,u) dx. \end{aligned}$$

For this purpose, first we consider the following proposition.

Proposition 4.2.

(i) *φ' is strongly continuous, i.e.; $u_n \rightharpoonup u$ in X implies $\varphi'(u_n) \to \varphi'(u)$.*

(ii) *ϕ' is continuous, bounded and satisfies (S_0) condition, i.e.; as $n \to \infty$,*

$$u_n \rightharpoonup u, \quad \phi'(u_n) \rightharpoonup v, \quad \langle \phi'(u_n), u_n \rangle \to \langle v, u \rangle \quad \text{implies} \quad u_n \to u.$$

(iii) *for any* $s > 0$, *the set* $\{u \in X; \phi(u) = s\}$ *is bounded provided that* $p^- > \max(\theta^+, \gamma^+)$.

Proof. (i): Let $u_n \rightharpoonup u$ in X hence $u_n \to u$ in $L^{r(\cdot)}(\Omega)$ and so for any $v \in X$,

$$\begin{aligned} |\langle \varphi'(u_n) - \varphi'(u), v\rangle| &= \Big| \int_\Omega c(x)(|u_n|^{r(x)-1}u_n - |u|^{r(x)-1}u)v dx \Big| \\ &\leq c^+ C_r \| |u_n|^{r(x)-1}u_n - |u|^{r(x)-1}u\|_{L^{r'(x)}(\Omega)} \|v\|. \end{aligned} \tag{4.1}$$

Note that $r'(x) = \frac{r(x)}{r(x)-1}$, and since $u_n \to u$ in $L^{r(\cdot)}(\Omega)$, $|u_n|^{r(x)-1}u_n \to |u|^{r(x)-1}u$ in $L^{r'(x)}(\Omega)$. Thus the last term in (4.1) tends to zero as $n \to \infty$.

(ii):

$$\begin{aligned} \|\phi'\|_{X*} &= \sup\{| < \phi'(u), v > |; \|v\| \leq 1\} \\ &= \sup \Big| \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)a(x)|u|^{p(x)-2}uv d\sigma \\ &\quad + \int_\Omega b(x)|u|^{q(x)-2}uv dx - \int_\Omega f(x,u)v dx \Big| \\ &\leq \kappa [a^+ ||\nabla u|^{p(x)-1}|_{L^{p'(x)}(\Omega)} \|v\| + \beta^+ a^+ ||u|^{p(x)-1}|_{L^{p'(x)}(\partial\Omega)} |v|_{L^{p(x)}(\partial\Omega)} \\ &\quad + b^+ ||u|^{q(x)-1}|_{L^{q'(x)}(\Omega)} |v|_{L^{q(x)}(\Omega)}] \\ &< \infty; \end{aligned}$$

since $X \hookrightarrow L^{p(x)}(\Omega), L^{p(x)}(\partial\Omega), L^{q(x)}(\Omega)$.

(iii): For $\|u\| > 1$ and $\phi(u) = s$ by using (2.1) we have,

$$s = \phi(u) \geq \frac{a^-}{p^+}\|u\|^{p^-} - \varepsilon \hat{C_\theta}\|u\|^{\theta^+} - C(\varepsilon)\hat{C_\gamma}\|u\|^{\gamma^+};$$

And since $p^- > \max(\theta^+, \gamma^+)$ we obtain that the set $\{u \in X; \phi(u) = s\}$ would be bounded. □

Proof of Theorem 4.1:

Proof. It is obvious that φ, ϕ are even functionals and $\varphi, \phi \in C^1(\Omega)$ with $\varphi(0) = \phi(0) = 0$. Moreover for $t > 1$ and $u \neq 0$, by using (2.1) we have

$$\begin{aligned} \phi(tu) \geq \Big[\int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)a(x)}{p(x)}|u|^{p(x)}d\sigma \Big] t^{p^-} + t^{q^-} \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx \\ - t^{\theta^+}\varepsilon \int_\Omega |u|^{\theta(x)}dx - t^{\gamma^+}C(\varepsilon)\int_\Omega |u|^{\gamma(x)}dx \longrightarrow \infty; \end{aligned}$$

as $n \to \infty$, since $\max(p^-, q^-) > \max(\theta^+, \gamma^+)$. Also it is easy to see that, for $u \in X$ with $\phi(u) = s$ we have

$$\langle \phi'(u), u\rangle \geq s + \int_\Omega (F(x,u) - f(x,u)u)dx;$$

and hence $\inf\{\langle \phi'(u), u\rangle; \phi(u) = s\} > 0$.

Now, by considering Proposition 4.2 and applying the Ljusternik-Schnirelmann principle, [21, 22], we get the result.

□

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