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Long-time behavior for the Kirchhoff diffusion problem with magnetic fractional Laplace operator

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ABSTRACT

We consider a Kirchhoff-type diffusion problem driven by the magnetic fractional Laplace operator. The main result in this paper establishes that infinite time blow-up cannot occur for the problem. The proof is based on the potential well method, in relationship with energy and Nehari functionals.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n > 2s)$ be a bounded domain with smooth boundary. In this paper we study the following Kirchhoff-type diffusion problem

$$\begin{cases} u_t + M\left(\|u\|^2_{X_{0,A}}\right)(-\Delta)^s_A u = f(|u|)u, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{in } (\mathbb{R}^n \setminus \Omega) \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1}$$

Given $s \in (0,1)$ and $A \in L^\infty_{loc}(\mathbb{R}^n)$, we define the magnetic fractional Laplace operator $(-\Delta)^s_A$ defined by

$$(-\Delta)^s_A u(x,t) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x,t) - e^{i(x-y)\cdot A(\frac{x+y}{2})} u(y,t)}{|x-y|^{n+2s}}\, dy, \tag{2}$$

for $x \in \mathbb{R}^n$ and $u \in C^\infty_0(\mathbb{R}^n, \mathbb{C})$. This differential operator is weighted by a Kirchhoff-type function $M : [0,\infty) \to [0,\infty)$ (see [1]), satisfying $(M1)$-$(M2)$ below. When $A \equiv 0$ in (2), then we have the usual fractional Laplacian differential operator denoted by $(-\Delta)^s$. Such differential operator was studied in the context of problems in quantum mechanics and of the motion of chains or arrays of particles connected by elastic springs, as well as in the context of problems of unusual diffusion processes in turbulent fluids and

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of mass transport in fractured media. We refer to [2] (Lévy processes), [3] (nonlocal diffusions, drifts and games), [4–9] for other classes of nonlocal operators. In all the aforementioned works, the authors deal with Schrödinger operators with magnetic fields. For instance, [7] establish the existence of nontrivial solutions to a parametric fractional Schrödinger equation in the case of critical or supercritical nonlinearity. Next, the operator $(-\Delta)^s_A$ (see (2)) was introduced in [10], as a fractional counterpart of the magnetic Laplacian $(\nabla - iA)^2$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a L^∞_{loc}-vector potential. Zuo & Lopes [11] established the existence of weak solutions to problem (1). The strategy is based on the potential well method, hence they obtain global in time solutions and blow-up in finite time solutions. Here, we show that the global in time solutions to (1) cannot blow-up in infinite time. Recently, the asymptotic behavior of solutions was investigated by Qi et al.[12] in dealing with fractional p-Kirchhoff type equations. We also mention [13] (non-local parabolic equation in a bounded convex domain), [10] (for the equation $(-\Delta)^s_A u + u = |u|^{p-2}u$ posed in $\mathbb{R}^3$), [14] (fractional Choquard equation), and [15,16] (system of Kirchhoff type equations). Finally, we point out that the authors [17] study pseudo-parabolic equation involving fractional derivative. They focus on the formula of mild solution, given in the form of Fourier series by some operators. Hence, they distinguish two cases. In the linear case, the authors obtain the continuity of mild solution with respect to the fractional order. For the nonlinear case, they establish the existence and uniqueness of a global solution. The approach is based on fixed-point arguments and Sobolev embeddings.

2. Mathematical background and hypotheses

The right framework for the analysis of Eq. (1) is the function space $X_{0,A}$ (hence $H^s_A(\Omega)$) defined as follows. For an open and bounded set $\Omega \subset \mathbb{R}^n$ ($n > 2s$), let $|\Omega|$ be the measure of the set Ω. By $L^p(\Omega, \mathbb{C})$ we mean the Lebesgue space of complex valued functions with norm $\|\cdot\|_{L^p(\Omega)}$ and inner product $\langle \cdot, \cdot \rangle$. For $p = 2$, $s \in (0,1)$ and $A \in L^\infty_{loc}(\mathbb{R}^n)$, we consider the magnetic Gagliardo semi-norm defined by

$$[u]^2_{H^s_A(\Omega)} := \int\int_{\Omega\times\Omega} \frac{|u(x,t) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y,t)|^2}{|x-y|^{n+2s}}\, dx\, dy.$$

Hence we consider the space $H^s_A(\Omega)$ of functions $u \in L^2(\Omega, \mathbb{C})$ with $[u]_{H^s_A(\Omega)} < \infty$ and furnished with the norm $\|u\|_{H^s_A(\Omega)} := (\|u\|^2_{L^2(\Omega)} + [u]^2_{H^s_A(\Omega)})^{\frac{1}{2}}$. Referring to [18,19], we define $X_{0,A} := \{u \in H^s_A(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$, with the real scalar product (see [10]) given as

$$\langle u, v\rangle_{X_{0,A}} := \mathcal{R} \int\int_{\mathbb{R}^{2n}} \frac{\left(u(x,t) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y,t)\right) \overline{\left(v(x,t) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} v(y,t)\right)}}{|x-y|^{n+2s}}\, dx\, dy,$$

where, for every $z \in \mathbb{C}$, by $\mathcal{R}z$ we mean the real part of z and by $\bar{z}$ its complex conjugate. This scalar product induces the following norm

$$\|u\|_{X_{0,A}} := \left(\int\int_{\mathbb{R}^{2n}} \frac{|u(x,t) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y,t)|^2}{|x-y|^{n+2s}}\, dx\, dy\right)^{\frac{1}{2}}.$$

We will use $\overset{w}{\longrightarrow}$ and $\to$ to denote weak and strong convergences, respectively. Our hypotheses on problem (1) are the following:

(F) $f \in C^1([0,\infty))$, and we can find $C > 0$ and $\gamma \geq p$, for $p \in (2, 2^*_s)$ with $2^*_s = \frac{2n}{n-2s}$, such that

$$\begin{aligned} &|F(u)| \leq Cu^p,\ f(u)u^2 \leq pCu^p,\ \text{ for } u \geq 0,\\ &0 < \gamma F(u) \leq f(u)u^2,\ u^2(uf'(u) - (p-2)f(u)) \geq 0 \text{ for } u > 0, \end{aligned} \tag{3}$$

where $F(u) := \int_0^u f(\tau)\tau\, d\tau$.

Further, the Kirchhoff function $M : [0,\infty) \to [0,\infty)$ is as follows:

(M1) it is a continuous function and there exist constants $m_0 > 0$ and $\theta \in (1, \frac{2^*_s}{2})$ such that $M(u) \geq m_0 u^{\theta-1}$ for all $u \in [0,\infty)$;
(M2) there exists a constant $\mu \in (1, \frac{2^*_s}{2})$ such that $\mu \mathscr{M}(u) \geq M(u)u$ for all $u \in [0,\infty)$, where $\mathscr{M}(u) := \int_0^u M(\tau)\, d\tau$.

Here, θ and μ satisfy the condition: (P) $2\max\{\theta, \mu\} < p < 2^*_s$.
We consider the C^1-functional related to problem (1) and defined by

$$J(u) := \frac{1}{2}\mathscr{M}(\|u\|^2_{X_{0,A}}) - \int_\Omega F(|u|)\, dx. \tag{4}$$

We have $\langle J'(u), \phi\rangle = M(\|u\|^2_{X_{0,A}})\langle u, \phi\rangle_{X_{0,A}} - \mathcal{R}\int_\Omega f(|u|)u\overline{\phi}\, dx$, for any $\phi \in X_{0,A}$, and we introduce the Nehari functional for (1) given by $I(u) := \langle J'(u), u\rangle$. Hence, we can consider the non-negative value d (i.e., mountain pass level) given as $d = \inf_{u\in X_{0,A}\setminus\{0\} : I(u)=0} J(u)$. According to Sattinger [20] and Payne & Sattinger [21] we know that if the initial energy $J(u_0)$ is less than the mountain pass level d, then the solution to problem (1) exists globally if it begins in the stable set $\mathcal{W} = \{u \in X_{0,A} : I(u) > 0, J(u) < d\} \cup \{0\}$, and fails to exist globally if it starts from the unstable set $\mathcal{U} = \{u \in X_{0,A} : I(u) < 0, J(u) < d\}$. The functionals $I(\cdot)$ and $J(\cdot)$ are energy functionals of the stationary problem

$$\begin{cases} M(\|u\|^2_{X_{0,A}})(-\Delta)^s_A u = f(|u|)u, & \text{in}\Omega, \\ u(x) = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5}$$

and we recall that $u \in X_{0,A}(\Omega)$ is a solution to (5) (i.e., stationary solution to problem (1)) if $\langle J'(u), \phi\rangle = 0$, namely $M(\|u\|^2_{X_{0,A}})\langle u, \phi\rangle_{X_{0,A}} - \mathcal{R}\int_\Omega f(|u|)u\overline{\phi}\,dx = 0$, for all $\phi \in X_{0,A}(\Omega)$, see also [11, Definition 1]. So, $u \in X_{0,A}(\Omega)$ solves (1) on $(0,T)$ for $T > 0$ if $u \in L^\infty(0,\infty; L^2(\Omega)) \cap L^\infty(0,\infty; X_{0,A}) \cap C(0,\infty; L^2(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$ and

$$\begin{aligned} &\mathcal{R}\int_0^T \int_\Omega u_t\overline{\phi}\,dx\,dt + \int_0^T M(\|u\|^2_{X_{0,A}})\langle u, \phi\rangle_{X_{0,A}}\,dt \\ &- \mathcal{R}\int_0^T \int_\Omega f(|u|)u\overline{\phi}\,dx\,dt = 0 \text{ for all } \phi \in X_{0,A}(\Omega). \end{aligned}$$

Remark 2.1. We note that [11] established the existence of a weak solution $u \in X_{0,A}(\Omega)$ to problem (1), based on hypotheses (F) and (M1) (see [11, Theorem 2.1] for the precise requirements). Imposing also (M2), they obtained that weak solutions blow-up in a finite time, provided that the initial energy is negative (see [11, Theorem 2.2]). Further, problem (1) has a weak solution for all $T > 0$ (namely, a global solution) such that $u \in L^\infty(0,\infty; L^2(\Omega)) \cap L^\infty(0,\infty; X_{0,A}) \cap C(0,\infty; L^2(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$, provided that $u_0 \in \mathcal{W}$ (see [11, Theorem 2.3]).

3. Main result

Let $u = u(t)$ be solution of (1) with initial data $u_0 \in X_{0,A}$, then by $T = T(u_0)$ we denote the maximal existence time of $u = u(t)$ given as

(i) $T = \infty$ if $u(t) \in X_{0,A}$ for $t \in [0,\infty)$;
(ii) $T = t_{\max} (> 0)$ if $u(t) \in X_{0,A}$ for $t \in [0, t_{\max})$, $u(t_{\max}) \notin X_{0,A}$.

Theorem 3.1. *Assume that hypotheses (F), (M1), (M2) and (P) hold and $p \in (2, 2^*_s)$ $(n > 2s)$. Let $u = u(t)$ be a solution of (1) with $u_0 \in X_{0,A}$ such that $J(u_0) \le d$ and $I(u_0) > 0$. If the maximal existence time is $T = \infty$, then there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to \infty$ as $k \to \infty$, such that $u(t_k)$ converges to a stationary solution $v \in X_{0,A}(\Omega)$ of problem (1), that is $u(t_k) \to v$ as $k \to \infty$.*

Proof. We show global in time solutions to (1) cannot blow-up in infinite time, by arguing in three steps.

Step 1: Existence of an increasing sequence $\{t_k\}$.

Let $u = u(t)$ be a solution of problem (1) with $u_0 \in X_{0,A}$ and maximal existence time $T = \infty$. Now, [11, Theorem 2.3] gives us $u \in L^\infty(0,\infty; L^2(\Omega)) \cap L^\infty(0,\infty; X_{0,A}) \cap C(0,\infty; L^2(\Omega))$, $u_t \in L^2(0,\infty; L^2(\Omega))$. Multiplying Eq. (1) by $\phi \in X_{0,A}$ and integrating over Ω, we get

$$(u'(t), \phi) = -M(\|u\|^2_{X_{0,A}})\langle u(t), \phi\rangle_{X_{0,A}} + \mathcal{R}\int_\Omega f(|u(t)|)u(t)\overline{\phi}\,dx. \tag{6}$$

By [11, Lemma 3.3] $J(u(t))$ is non-increasing with respect to t, hence

$$0 \le J(u(t)) \le J(u_0), \tag{7}$$

where we use a contradiction argument to conclude the first inequality in (7). So, we assume that there exists a time t_0 such that $J(u(t_0)) < 0$, then by (4) we deduce that $0 > J(u(t_0)) = \frac{1}{2}\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) - \int_\Omega F(|u(t_0)|)\,dx$. It follows that

$$\frac{1}{2}\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) < \int_\Omega F(|u(t_0)|)\,dx. \tag{8}$$

Combining the Nehari functional $I(\cdot)$, together with assumptions (3), (M2), (P) and inequality (8), we conclude that

$$\begin{aligned} I(u(t_0)) = &\ M(\|u(t_0)\|^2_{X_{0,A}})\|u(t_0)\|^2_{X_{0,A}} - \mathcal{R}\int_\Omega f(|u(t_0)|)|u(t_0)|^2\,dx \\ \le &\ \mu\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) - \gamma\int_\Omega F(|u(t_0)|)\,dx < \mu\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) - \frac{\gamma}{2}\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) \\ \le &\ (\mu - \frac{p}{2})\mathscr{M}(\|u(t_0)\|^2_{X_{0,A}}) < 0 \quad \text{(recall that } \gamma \ge p). \end{aligned}$$

So, at time $t = t_0$ we have $J(u(t_0)) < 0$ and $I(u(t_0)) < 0$, hence $u(t_0)$ is in the unstable set $\mathcal{U} = \{u \in X_{0,A} : I(u) < 0, J(u) < d(\text{d}> 0)\}$. By [11, Theorem 2.4], $u(t)$ blows-up in a finite time, which contradicts $T = \infty$. Since $J(u(t))$ is non-increasing with respect to t, by (7) we can find c with $0 \le c \le J(u_0)$ and such that $J(u(t)) \to c$ as $t \to \infty$. Passing to the limit as $t \to \infty$ in $\int_0^t \|u'(s)\|^2_{L^2(\Omega)}\,ds + J(u) = J(u_0)$, then we get $c = J(u_0) - \int_0^\infty \|u'(s)\|^2_{L^2(\Omega)}\, ds$. It follows that we can find an increasing sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to \infty$ as $k \to \infty$ satisfying

$$\lim_{k\to\infty} \|u'(t_k)\|_{L^2(\Omega)} = 0. \tag{9}$$

Step 2: Convergence of $\{u(t_k)\}$ to a function $v \in X_{0,A}$.

Eq. (6) leads to the following

$$\langle J'(u(t)), \phi\rangle = M(\|u(t)\|^2_{X_{0,A}})\langle u(t), \phi\rangle_{X_{0,A}} - \mathcal{R}\int_\Omega f(|u(t)|)u(t)\overline{\phi}\, dx = (-u'(t), \phi) \text{ for all } \phi \in X_{0,A}.$$

Combining the Schwartz inequality, the definition of the first eigenvalue λ_1 of $(-\Delta)^s_A$ (see [19, Proposition 3.3]) and the limit in (9), we conclude

$$\begin{aligned}
&\|J'(u(t_k))\|_{X'_{0,A}} = \sup_{\substack{\phi\in X_{0,A}\\ \|\phi\|_{X_{0,A}}=1}} \langle J'(u(t_k)), \phi\rangle = \sup_{\substack{\phi\in X_{0,A}\\ \|\phi\|_{X_{0,A}}=1}} (-u'(t_k), \phi)\\
\leq &\sup_{\substack{\phi\in X_{0,A}\\ \|\phi\|_{X_{0,A}}=1}} \|u'(t_k)\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \leq \|u'(t_k)\|_{L^2(\Omega)} \sup_{\phi\in X_{0,A}\setminus\{0\}} \left(\frac{\|\phi\|_{L^2(\Omega)}}{\|\phi\|_{X_{0,A}}}\right)\\
\leq &\|u'(t_k)\|_{L^2(\Omega)} \left(\inf_{\phi\in X_{0,A}\setminus\{0\}} \left(\frac{\|\phi\|_{X_{0,A}}}{\|\phi\|_{L^2(\Omega)}}\right)\right)^{-1} \leq \frac{1}{\sqrt{\lambda_1}}\|u'(t_k)\|_{L^2(\Omega)} \to 0
\end{aligned} \tag{10}$$

as $k \to \infty$. As usual, by $X'_{0,A}$ we denote the dual space of $X_{0,A}$, hence we can find $c_1 > 0$, independent of the index k, such that

$$\|J'(u(t_k))\|_{X'_{0,A}} \leq c_1,\ \ k = 1, 2, \ldots. \tag{11}$$

For the Nehari energy functional $I(\cdot)$, the bound from above in (11) and the Young inequality lead to

$$|I(u(t_k))| \leq |\langle J'(u(t_k)), u(t_k)\rangle| \leq \|J'(u(t_k))\|_{X'_{0,A}}\|u(t_k)\|_{X_{0,A}} \leq c_1\|u(t_k)\|_{X_{0,A}} \leq c_2 + \frac{(p-2\mu)m_0}{4p\mu}\|u(t_k)\|^{2\theta}_{X_{0,A}}.$$

Using $J(\cdot)$ and $I(\cdot)$, together with (12) and (M1), we obtain

$$\begin{aligned}
&J(u(t_k)) = \frac{1}{2}\mathscr{M}\left(\|u(t_k)\|^2_{X_{0,A}}\right) - \int_\Omega F(|u(t_k)|)\, dx\\
\geq &\frac{1}{2\mu}M(\|u(t_k)\|^2_{X_{0,A}})\|u(t_k)\|^2_{X_{0,A}} - \frac{1}{\gamma}\int_\Omega f(|u(t_k)|)|u(t_k)|^2\, dx\\
= &\frac{1}{2\mu}M(\|u(t_k)\|^2_{X_{0,A}})\|u(t_k)\|^2_{X_{0,A}} + \frac{1}{p}I(u(t_k)) - \frac{1}{p}M(\|u(t_k)\|^2_{X_{0,A}})\|u(t_k)\|^2_{X_{0,A}}\\
\geq &\frac{(p-2\mu)}{2p\mu}M(\|u(t_k)\|^2_{X_{0,A}})\|u(t_k)\|^2_{X_{0,A}} - c_2 - \frac{(p-2\mu)}{4p\mu}m_0\|u(t_k)\|^{2\theta}_{X_{0,A}}\\
\geq &\frac{(p-2\mu)}{2p\mu}m_0\|u(t_k)\|^{2\theta}_{X_{0,A}} - \frac{(p-2\mu)}{4p\mu}m_0\|u(t_k)\|^{2\theta}_{X_{0,A}} - c_2.
\end{aligned}$$

This inequality, taking into account the bound from above in (7), gives us $J(u_0) + c_2 \geq \frac{(p-2\mu)}{4p\mu}m_0\|u(t_k)\|^{2\theta}_{X_{0,A}}$, which implies that

$$\|u(t_k)\|_{X_{0,A}} \leq \left[\frac{(J(u_0)+c_2)4p\mu}{(p-2\mu)m_0}\right]^{\frac{1}{2\theta}}, \quad k = 1, 2, \ldots. \tag{12}$$

From (12) and the fact that the embedding $X_{0,A} \hookrightarrow L^q(\Omega, \mathbb{C})$ is compact for all $q \in [1, 2^*_s)$ (see [18, Lemma 2.2]), then there exist an increasing subsequence, still denoted by $\{t_k\}_{k=1}^\infty$, and a function $v \in X_{0,A}$ such that $u_k := u(t_k)$ satisfies the following convergences

$$u_k \overset{w}{\longrightarrow} v \text{ in } X_{0,A} \text{ as } k \to \infty, \tag{13}$$

$$u_k \to v \text{ in } L^q(\Omega, \mathbb{C}) \text{ for all } q \in [1, 2^*_s) \text{ as } k \to \infty. \tag{14}$$

By (14), there exist a subsequence, still denoted by $\{u_k\}_{k=1}^\infty$, and a function $w \in L^q(\Omega, \mathbb{C})$ for all $q \in [1, 2^*_s)$, such that

$$u_k(x) \to v(x) \text{ a.e. in } \Omega \text{ as } k \to \infty, \tag{15}$$

$$\text{for all } k, |u_k(x)| \leq w(x) \text{ a.e. in } \Omega. \tag{16}$$

Step 3: v is a solution to the stationary problem (5).

Let us prove that for all $\phi \in X_{0,A}$ we have

$$\mathcal{R}\int_\Omega f(|u_k|)u_k\overline{\phi}\, dx \to \mathcal{R}\int_\Omega f(|v|)v\overline{\phi}\, dx \quad \text{as } k \to \infty. \tag{17}$$

By (15) and $f \in C^1([0,\infty))$, we get $f(|u_k|)u_k\overline{\phi} \to f(|v|)v\overline{\phi}$, for a.e. $x \in \Omega$ as k to ∞. Using (3) and (16) we get $|f(|u_k|)u_k\overline{\phi}| \leq pC|w|^{p-2}|w||\phi| = pC|w|^{p-1}|\phi|$, for a.e. $x \in \Omega$. The Hölder inequality implies

$$\int_\Omega pC|w|^{p-1}|\phi|\ dx \leq c\|\phi\|_{L^p}\left(\int_\Omega |w|^{(p-1)\frac{p}{p-1}}\ dx\right)^{\frac{p-1}{p}} = c\|\phi\|_{L^p}\|w\|_{L^p}^{p-1} \leq c,$$

thanks to $w \in L^q(\Omega,\mathbb{C})$ for all $q \in [1,2_s^*)$ and $\phi \in X_{0,A} \hookrightarrow L^p(\Omega,\mathbb{C})$ for $p \in (2,2_s^*)$. We conclude $|w|^{p-1}|\phi| \in L^1(\Omega)$. The Lebesgue dominated convergence theorem gives us (17). We show that

$$\langle J'(u_k),\phi\rangle = M(\|u_k\|_{X_{0,A}}^2)\langle u_k,\phi\rangle - \mathcal{R}\int_\Omega f(|u_k|)u_k\overline{\phi}\ dx, \tag{18}$$

converges to $0 = M(\|v\|_{X_{0,A}}^2)\langle v,\phi\rangle - \mathcal{R}\int_\Omega f(|v|)v\overline{\phi}\ dx$ as $k\to\infty$ and $u_k \to v$ in $X_{0,A}$. We show that $M(\|u_k\|_{X_{0,A}}^2) \to M(\|v\|_{X_{0,A}}^2)$ and $u_k \to v$ strongly in $X_{0,A}$ (see [22]). Since M is continuous, (12) implies $M(\|u_k\|_{X_{0,A}}^2) \leq c$ for all $k\in\mathbb{N}$, some $c>0$, hence $\{M(\|u_k\|_{X_{0,A}}^2)\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Now, there is a subsequence, still say $\{M(\|u_k\|_{X_{0,A}}^2)\}_{k\in\mathbb{N}}$, converging to $\overline{M}$, so $\lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\langle v,\phi\rangle_{X_{0,A}} = \overline{M}\langle v,\phi\rangle_{X_{0,A}}$ and

$$\lim_{k\to\infty}\int\int_{\mathbb{R}^{2n}}\left[M(\|u_k\|_{X_{0,A}}^2) - \overline{M}\right]^2 \frac{\left|v(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(y)\right|^2}{|x-y|^{n+2s}}\ dx\ dy = 0,$$

that is, $M(\|u_k\|_{X_{0,A}}^2)v \to \overline{M}v$ in $X_{0,A}$. This together with (13) imply $\lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\langle u_k,v\rangle_{X_{0,A}} = \overline{M}\langle v,v\rangle_{X_{0,A}}$. By (10), (13) and (17), we pass to the limit as $k\to\infty$ in (18) to get $0 = \overline{M}\langle v,\phi\rangle_{X_{0,A}} - \mathcal{R}\int_\Omega f(|v|)v\overline{\phi}\ dx$ for all $\phi \in X_{0,A}$. For $\phi = v$, we get $\overline{M}\langle v,v\rangle_{X_{0,A}} = \int_\Omega f(|v|)|v|^2\ dx$. Similar to (17), we deduce $\lim_{k\to\infty}\int_\Omega f(|u_k|)|u_k|^2\ dx = \int_\Omega f(|v|)|v|^2\ dx$. So, (13), (10) lead to $|\langle J'(u_k),u_k\rangle| \leq \|J'(u_k)\|_{X'_{0,A}}\|u_k\|_{X_{0,A}}$
$\leq \|J'(u_k)\|_{X'_{0,A}}\left[\frac{(J(u_0)+c_2)4p\mu}{(p-2\mu)m_0}\right]^{\frac{1}{2\theta}} \to 0$ as $k\to\infty$. We first deduce that

$$\begin{aligned}&\lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\langle u_k,u_k\rangle_{X_{0,A}} = \lim_{k\to\infty}(\langle J'(u_k),u_k\rangle + \int_\Omega f(|u_k|)|u_k|^2\ dx)\\ &= \int_\Omega f(|v|)|v|^2\ dx = \lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\langle u_k,v\rangle_{X_{0,A}},\end{aligned}$$

then $\lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)(\langle u_k,u_k\rangle_{X_{0,A}} - \langle u_k,v\rangle_{X_{0,A}}) = 0$, and so

$$\begin{aligned}0 &= \lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\langle u_k,u_k - v\rangle_{X_{0,A}}\\ &= \lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\left[\|u_k - v\|_{X_{0,A}}^2 + \langle v,u_k - v\rangle_{X_{0,A}}\right]\\ &= \lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2)\|u_k - v\|_{X_{0,A}}.\end{aligned}$$

Since $M(\sigma) \geq m_0\sigma^{\theta-1}$ for all $\sigma \geq 0$, then $\lim_{k\to\infty}\|u_k - v\|_{X_{0,A}} = 0$. So, $u_k(x) \to v(x)$ in $X_{0,A}$, which implies that $\|u_k\|_{X_{0,A}}^2 \to \|v\|_{X_{0,A}}^2$ as $k\to\infty$. Using the continuity of M, we get $\lim_{k\to\infty} M(\|u_k\|_{X_{0,A}}^2) = M(\|v\|_{X_{0,A}}^2)$, which allows us to conclude that $\overline{M} = M(\|v\|_{X_{0,A}}^2)$. □

Data availability

No data was used for the research described in the article.

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