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Non-autonomous (p, q)-equations with unbalanced growth

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Abstract

We consider a nonlinear elliptic Dirichlet equation driven by a double phase operator and a Carathéodory $(p-1)$-linear reaction. First, we conduct a detailed spectral analysis of the double phase operator. Next, we use the results of this analysis to prove existence and multiplicity properties for problems in which there is resonance asymptotically as $x \to \pm\infty$.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this paper we study the following double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \;\; 1 < q < p < N. \end{cases} \tag{1.1}$$

Given $r \in (1, \infty)$ and $a \in L^\infty(\Omega) \setminus \{0\}$, $a(z) \geq 0$ for a.a. $z \in \Omega$, we denote by Δ_r^a the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}\left(a(z)|Du|^{r-2}Du\right).$$

The interest in the study of this type of problem is twofold. On the one hand, there are physical motivations, since the double phase operator has been applied to describe steady-state solutions of reaction-diffusion problems in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^a u(z) + \Delta_q u + g(x, u).$$

In this framework, the function u generally stands for a concentration, the term $\Delta_p^a u(z) + \Delta_q u$ corresponds to the diffusion with coefficient $a(z)|Du|^{p-2} + |Du|^{q-2}$, while $g(x, u)$ represents the reaction term related to source and loss processes; see Cherfils & Il'yasov [5] and Singer [26]. On the other hand, such operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point, where the modulating coefficient $a(z)$ dictates the geometry of a composite made by two different materials.

If $a \equiv 1$, then we have the standard r-Laplace differential operator denoted by Δ_r. In problem (1.1), we have the sum of two such operators and so the left-hand side of problem (1.1) is not homogeneous. In fact, the differential operator $u \mapsto -\Delta_p^a u - \Delta_q u$ driving problem (1.1) is related to the so-called "double-phase" integral functional given by

$$u \mapsto \int_\Omega \left(a(z)|Du|^p + |Du|^q\right) dz.$$

The integrand of this functional is the function

$$\xi(z, t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \text{ all } t \geq 0.$$

A feature of this paper is that we do not assume that the weight function $a(\cdot)$ is bounded away from zero, that is, we do not require that $\operatorname*{essinf}_\Omega a > 0$. This implies that the integrand $\xi(z, \cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \xi(z, t) \leq C_0 \left(t^p + t^q\right) \text{ for a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ for some } C_0 > 0.$$

Such functionals were first investigated by Marcellini [18] and Zhikov [27], in the context of problems of the calculus of variations and of nonlinear elasticity for strongly anisotropic materials. For such problems, there is no global (that is, up to the boundary) regularity theory. There are only interior regularity results primarily due to Marcellini and coworkers and to Mingione and coworkers. We mention the papers of Marcellini [19] and Baroni, Colombo & Mingione [3] and the references therein. There is also the work of Ragusa & Tachikawa [25] on anisotropic double phase problems. An informative survey of the recent developments on the subject can be found in Mingione & Rădulescu [20]. The lack of global regularity theory eliminates from consideration many of the tools used in the study of balanced (p, q)-equations.

The double-phase problem (1.1) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [4] that appears in electromagnetism:

$$-\text{div}\left(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2}x^2 + \frac{5!!}{3! \cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots \quad \text{for } |x| < 1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we obtain problem (P_λ) for $p = 4$ and $q = 2$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \text{div}\left(\frac{|\nabla u|^2}{(1 - |\nabla u|^4)^{3/4}}\,\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots.$$

This shows that the fourth-order relativistic operator can be approximated by the following autonomous double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

In the reaction (right-hand side of problem (1.1)), we have a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(x,z)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z,x)$ is continuous). We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth as $x \mapsto \pm\infty$. Recently there have been some existence and multiplicity results for double phase equations with $(p-1)$-superlinear reaction. We mention the works of Gasinski & Papageorgiou [10], Gasinski & Winkert [11], Ge, Lv & Lu [12], Liu & Dai [16], and Papageorgiou, Vetro & Vetro [24]. For equations with $(p-1)$-linear reaction $f(z,\cdot)$, there is only the very recent work of Papageorgiou, Rădulescu & Zhang [22], where it is stressed that the study of such problems is based on the spectral analysis of the Dirichlet $-\Delta_p^a$ operator. In the present paper, we conduct a detailed such spectral analysis, extending the work initiated by Papageorgiou, Rădulescu & Zhang [22]. Then we prove existence and multiplicity results for resonant double phase equations.

The features of the present paper are the following:

(i) The source term of problem (1.1) is driven by a differential operator with a power-type nonhomogeneous term.

(ii) The corresponding energy functional is a non-autonomous variational integral that satisfies nonstandard growth conditions of (p,q)-type, following the terminology introduced in the basic papers of Marcellini [17–19].

(iii) The potential that describes the differential operator satisfies general regularity assumptions and it belongs to the p-Muckenhoupt class. Accordingly, the thorough spectral and the qualitative analysis contained in this paper are developed in Musielak–Orlicz–Sobolev spaces.

(iv) The paper covers both the coercive resonant case and the noncoercive (asymptotic resonance or nonresonance) case.

2 Mathematical background

The unbalanced growth integrand $\xi(z,\cdot)$ implies that the right functional framework for problem (1.1) is provided by the so called Musielak–Orlicz–Sobolev spaces (or generalized Orlicz–Sobolev spaces). A comprehensive account of the abstract theory of these spaces can be found in the recent book of Harjulehto & Hästo [13].

In what follows. we denote

$$C^{0,1}(\overline{\Omega}) = \{a : \overline{\Omega} \to \mathbb{R} \text{ Lipschitz continuous}\}.$$

We impose the following conditions on the weight function $a(\cdot)$ and the exponents p, q.

H_0: $a \in C^{0,1}(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 1 The last condition relating p, q, N is standard in Dirichlet double phase problems and it implies that $p < q^* = \frac{Nq}{N-q}$. This then leads to useful compact embeddings of some relevant function spaces. Moreover, this condition together with the Lipschitz continuity of $a(\cdot)$, implies that the Poincaré inequality is valid for the Musielak–Orlicz–Sobolev space corresponding to the function $\xi(z, t)$.

Let $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. We identify two such functions which differ only on a Lebesgue null subset of Ω. The Musielak–Orlicz space $L^{\xi}(\Omega)$ is defined by

$$L^{\xi}(\Omega) = \{u \in M(\Omega) : \ \rho_{\xi}(u) < \infty\},$$

where $\rho_{\xi}(\cdot)$ is the modular function defined by

$$\rho_{\xi}(u) = \int_{\Omega} \xi(z, |u|)dz = \int_{\Omega} \left[a(z)|u|^p + |u|^q\right] dz.$$

This space is equipped with the so called "Luxemburg norm" given by

$$\|u\|_{\xi} = \inf \left\{\lambda > 0 : \ \rho_{\xi}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

With this norm $L^{\xi}(\Omega)$ becomes a Banach space which is separable and uniformly convex (thus, reflexive). Using $L^{\xi}(\Omega)$ we can introduce the corresponding Musielak–Orlicz–Sobolev space by

$$W^{1,\xi}(\Omega) = \left\{u \in L^{\xi}(\Omega) : \ |Du| \in L^{\xi}(\Omega)\right\}.$$

Here, Du denotes the weak gradient of u. This space is equipped with the norm

$$\|u\|_{1,\xi} = \|u\|_{\xi} + \|Du\|_{\xi},$$

with $\|Du\|_{\xi} = \||Du|\|_{\xi}$. Evidently, $W^{1,\xi}(\Omega)$ is a Banach space which is separable and uniformly convex (thus, reflexive). Also, we define

$$W_0^{1,\xi}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_{1,\xi}}.$$

This is also a Banach space which is separable and uniformly convex. Moreover, as we already mentioned on account of hypotheses H_0, the Poincaré inequality is valid on $W_0^{1,\xi}(\Omega)$. So, on $W_0^{1,\xi}(\Omega)$ we can consider the norm

$$\|u\| = \|Du\|_{\xi} \text{ for all } u \in W_0^{1,\xi}(\Omega).$$

We have the following useful embeddings. For details we refer to [13, Chapter 6].

Proposition 1 *If hypotheses H_0 hold, then*

(a) $L^{\xi}(\Omega) \hookrightarrow L^{\mu}(\Omega)$ *and* $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\mu}(\Omega)$ *continuously for all* $\mu \in [1, q]$*;*
(b) $W_0^{1,\xi}(\Omega) \hookrightarrow L^{\mu}(\Omega)$ *continuously for all* $\mu \in [1, q^*]$ *and compactly for all* $[1, q^*)$.
(c) $L^{p}(\Omega) \hookrightarrow L^{\xi}(\Omega)$ *continuously.*

The next proposition establishes a close relation between the norm and the modular function; see [13, Section 3.2].

Proposition 2 *If hypotheses* H_0 *hold and* $u \in W_0^{1,\xi}(\Omega)$*, then*

(a) $\|u\| = \lambda \Leftrightarrow \rho_\xi\left(\frac{|Du|}{\lambda}\right) = 1$*;*
(b) $\|u\| < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_\xi(Du) < 1$ *(resp.* $= 1, > 1$*);*
(c) $\|u\| \le 1 \Rightarrow \|u\|^p \le \rho_\xi(Du) \le \|u\|^q$*;*
(d) $\|u\| > 1 \Rightarrow \|u\|^q \le \rho_\xi(Du) \le \|u\|^p$*;*
(e) $\|u\| \to 0$ *(resp.* $\to +\infty$*)* $\Leftrightarrow \rho_\xi(Du) \to 0$ *(resp.* $\to +\infty$*).*

We introduce the operators $A_p^a, A_q : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ defined by

$$\langle A_p^a(u), h\rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz$$
$$\langle A_q(u), h\rangle = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,\xi}(\Omega).$$

We set $V = A_p^a + A_q$. For this operator, we have the following result (see Liu & Dai [16]).

Proposition 3 *If hypotheses* H_0 *hold, then* $V : W_0^{1,\xi}(\Omega) \to W_0^{1,\xi}(\Omega)^*$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type* $(S)_+$*, that is,*

$$\text{“}u_n \xrightarrow{w} u \text{ in } W_0^{1,\xi}(\Omega) \text{ and } \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$$
$$\text{imply that}$$
$$u_n \to u \text{ in } W_0^{1,\xi}(\Omega)\text{”}.$$

As we already mentioned in the Introduction, our analysis of problem (1.1) depends on the spectral properties of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$ with $\xi_0(z,t) = a(z)t^p$, $z \in \overline{\Omega}$, $t \ge 0$. In the next section we conduct a comprehensive study of these spectral properties.

3 Spectral analysis of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$

Let $m \in L^\infty(\Omega)$, $m(z) > 0$ for a.a. $z \in \Omega$ and consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p^a u(z) = \hat{\lambda} m(z)a(z)|u(z)|^{p-2}u(z) \text{ in } \Omega \\ u|_{\partial\Omega} = 0,\ 1 < p < N. \end{cases} \tag{3.1}$$

Recall that $\xi_0(z,t) = a(z)t^p$ and consider the corresponding spaces $L^{\xi_0}(\Omega)$ and $W_0^{1,\xi_0}(\Omega)$. These are Banach spaces which are separable and reflexive (see Harjulehto & Hästo [13]). Let $\tilde{A}_p$ denote the p-Muckenhoupt class (see Cruz Uribe & Fiorenza [7, p.142]). We strengthen hypotheses H_0 as follows:

H_0': $a \in C^{0,1}(\overline{\Omega}) \cap \tilde{A}_p$, $a(z) > 0$, for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

By an eigenvalue of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega), m\right)$ we mean a real number $\hat{\lambda}$ such that problem (3.1) has a nontrivial weak solution (eigenfunction). So, there exists $\hat{u} \in W_0^{1,\xi_0}(\Omega) \setminus \{0\}$ such that

$$\langle A_p^a(\hat{u}), h\rangle = \int_\Omega \hat{\lambda} m(z)a(z)|\hat{u}|^{p-2}\hat{u}h dz \text{ for all } h \in W_0^{1,\xi_0}(\Omega).$$

The starting point of our spectral analysis is the following compact embedding theorem proved in [22].

Proposition 4 *If hypotheses H_0' hold, then $W_0^{1,\xi_0}(\Omega) \hookrightarrow L^{\xi_0}(\Omega)$ compactly.*

Using this compact embedding result, we can show the existence of a smallest (first) eigenvalue for problem (3.1).

Proposition 5 *If hypotheses H_0' hold, then problem (3.1) has a smallest eigenvalue*

$$\hat{\lambda}_1^a(p, m) > 0$$

and every corresponding eigenfunction $\hat{u} \in W_0^{1,\xi_0}(\Omega)$ satisfies

$$\hat{u} \in L^\infty(\Omega), \hat{u}(z) > 0 \text{ or } \hat{u}(z) < 0 \text{ for a.a. } z \in \Omega.$$

Proof Let $\rho_a(Du) = \int_\Omega a(z)|Du|^p dz$ for all $u \in W_0^{1,\xi_0}(\Omega)$ and define

$$\begin{aligned}\hat{\lambda}_1^a(p, m) &= \inf\left\{\frac{\rho_a(Du)}{\int_\Omega m(z)a(z)|u|^p dz} : u \in W_0^{1,\xi_0}(\Omega), u \neq 0\right\}\\ &= \inf\left\{\rho_a(Du) : \int_\Omega m(z)a(z)|u|^p dz = 1\right\}\\ &\quad \text{(by homogeneity).}\end{aligned} \tag{3.2}$$

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega)$ such that

$$\rho_a(Du_n) \downarrow \hat{\lambda}_1^a(p, m) = \hat{\lambda}_1 \text{ and } \int_\Omega m(z)a(z)|u_n|^p dz = 1$$

for all $n \in \mathbb{N}$.

We have that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } u_n \to \hat{u} \text{ in } L^{\xi_0}(\Omega) \text{ (see Proposition 4).} \tag{3.3}$$

Note that $\rho_a(\cdot)$ is continuous, convex, thus it is sequentially weakly semicontinuous. Therefore from (3.3) we have

$$\rho_a(D\hat{u}) \leq \liminf_{n\to\infty} \rho_a(Du_n). \tag{3.4}$$

Also we have

$$\int_\Omega m(z)a(z)|u_n|^p dz \to \int_\Omega m(z)a(z)|u|^p dz = 1 \\ \text{(see (3.2)).} \tag{3.5}$$

From (3.4) and (3.5) we have

$$\begin{aligned} &\rho_a(D\hat{u}) \leq \hat{\lambda}_1 \text{ and } \int_\Omega m(z)a(z)|\hat{u}|^p dz = 1, \\ \Rightarrow\ &\rho_a(D\hat{u}) = \hat{\lambda}_1 \text{ and } \int_\Omega m(z)a(z)|\hat{u}|^p dz = 1 \text{ (see (3.2)),} \\ \Rightarrow\ &\hat{\lambda}_1 = \hat{\lambda}_1^a(p, m) > 0. \end{aligned}$$

The Lagrange multiplier rule (see Papageorgiou, Rădulescu & Repovš [21, p.422]), implies that

$$-\Delta_p^a \hat{u} = \hat{\lambda}_1 m(z)a(z)|\hat{u}|^{p-2}\hat{u} \text{ in } \Omega,\ \hat{u}|_{\partial\Omega} = 0. \tag{3.6}$$

Suppose that $\hat{u}^+ \neq 0$. Acting on (3.6) with $\hat{u} \in W_0^{1,\xi_0}(\Omega)$ we obtain

$$\begin{aligned} &\rho_a(D\hat{u}^+) = \hat{\lambda}_1 \int_\Omega m(z)a(z)(\hat{u}^+)^p dz, \\ \Rightarrow\ &\hat{u}^+ \text{ realizes the infimum in (3.2),} \\ \Rightarrow\ &\hat{u}^+ \text{ is an eigenfunction for } \hat{\lambda}_1 > 0. \end{aligned}$$

Let $(\xi_0)_*(z, t)$ denote the Sobolev conjugate of $\xi_0(z, t)$ (see Adams [1, p.248]). We know that $W_0^{1,\xi_0}(\Omega) \hookrightarrow L^{(\xi_0)_*}(\Omega)$ continuously (see Adams [1, p.249]). Then as in Colasuonno & Squassina [6, Section 3.3] we infer that

$$\hat{u}^+ \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega).$$

Moreover, Proposition 2.4 of Papageorgiou, Vetro & Vetro [24], implies

$$\begin{aligned} &\hat{u}^+(z) > 0 \text{ for a.a. } z \in \Omega, \\ \Rightarrow\ &\hat{u} = \hat{u}^+. \end{aligned}$$

If $\hat{u}^+ \equiv 0$, then $\hat{u} = -\hat{u}^-$ is an eigenfunction and so

$$\hat{u}(z) < 0 \text{ for a.a. } z \in \Omega.$$

The proof is now complete. ◻

We consider the map $m \mapsto \hat{\lambda}_1^a(p, m)$. For this map we have the following monotonicity result.

Proposition 6 *If hypotheses H_0' hold, $m, \hat{m} \in L^\infty(\Omega)$, $0 < m(z) \leq \hat{m}(z)$, for a.a. $z \in \Omega$ and $m \neq \hat{m}$, then $\hat{\lambda}_1^a(p, \hat{m}) < \hat{\lambda}_1^a(p, m)$.*

Proof Let $\hat{u} \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$ be an eigenfunction corresponding to the eigenvalue $\hat{\lambda}_1^a(p, m)$ and $\hat{v} \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$ an eigenfunction corresponding to the eigenvalue $\hat{\lambda}_1^a(p, \hat{m})$. We have

$$\begin{cases} \hat{\lambda}_1^a(p, m) = \dfrac{\rho_a(D\hat{u})}{\int_\Omega m(z)a(z)|\hat{u}|^p dz}, \\ \hat{\lambda}_1^a(p, \hat{m}) = \dfrac{\rho_a(D\hat{v})}{\int_\Omega \hat{m}(z)a(z)|\hat{v}|^p dz}. \end{cases} \tag{3.7}$$

We may assume that

$$\hat{u}(z) > 0 \text{ and } \hat{v}(z) > 0 \text{ for a.a. } z \in \Omega. \tag{3.8}$$

Then from (3.7) and (3.8) and since $a(z) > 0$ for all $z \in \Omega$ (see hypotheses H_0'), we infer that $\hat{\lambda}_1^a(p, \hat{m}) < \hat{\lambda}_1^a(p, m)$. ◻

We show that for fixed sign we have only the eigenfunctions corresponding to $\hat{\lambda}_1$.

Proposition 7 *If hypotheses H_0' hold and $\hat{\lambda} \in (\hat{\lambda}_1, \mu)$ is an eigenvalue of (3.2), then every eigenfunction $\hat{u} \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$ corresponding to $\hat{\lambda}$ is nodal (sign changing) and if $\Omega_+ = \{z \in \Omega : \hat{u}(z) > 0\}$, $\Omega_- = \{z \in \Omega : \hat{u}(z) < 0\}$, then there exists $\hat{C} > 0$ such that*

$$0 < \hat{C} \leq |\Omega_\pm|_N \text{ for all } \hat{\lambda} \in (\hat{\lambda}_1, \mu)$$

(here $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$).

Proof We argue by contradiction. So, suppose that $\hat{u}(\cdot)$ has fixed sign. We may assume that $\hat{u} \geq 0$. As before, by Proposition 2.4 of [24], we have $\hat{u}(z) > 0$ for a.a. $z \in \Omega$. Let $\hat{u}_1 \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$ be a positive eigenfunction corresponding to $\hat{\lambda}_1 = \hat{\lambda}_1^a(p,m) > 0$. We know that $\hat{u}_1(z) > 0$ for a.a. $z \in \Omega$ and $\hat{u} \geq \hat{u}_1$ (see (3.1)). For $\varepsilon > 0$ se set

$$\hat{u}_\varepsilon = \hat{u} + \varepsilon \text{ and } (\hat{u}_1)_\varepsilon = \hat{u}_1 + \varepsilon.$$

We consider the following test functions

$$h_1 = \frac{(\hat{u}_1)_\varepsilon^p - \hat{u}^p}{(\hat{u}_1)_\varepsilon^{p-1}} \in W_0^{1,\xi_0}(\Omega),$$
$$h_2 = \frac{\hat{u}_\varepsilon^p - (\hat{u}_1)_\varepsilon^p}{\hat{u}^{p-1}} \in W_0^{1,\xi_0}(\Omega).$$

The gradients of these functions are

$$Dh_1 = \left[1 + (p-1)\left(\frac{\hat{u}_\varepsilon}{(\hat{u}_1)_\varepsilon}\right)^p\right] D(\hat{u}_1)_\varepsilon - p\left(\frac{\hat{u}_\varepsilon}{(\hat{u}_1)_\varepsilon}\right)^{p-1} D\hat{u}_\varepsilon, \tag{3.9}$$

$$Dh_2 = \left[1 + (p-1)\left(\frac{(\hat{u}_1)_\varepsilon}{\hat{u}\varepsilon}\right)^p\right] D\hat{u}_\varepsilon - p\left(\frac{(\hat{u}_1)_\varepsilon}{\hat{u}_\varepsilon}\right)^{p-1} D(\hat{u}_1)_\varepsilon. \tag{3.10}$$

We have

$$\langle A_p^a(\hat{u}_1), h_1\rangle = \hat{\lambda}_1 \int_\Omega m(z)a(z)\hat{u}_1^{p-1} h_1 dz, \tag{3.11}$$

$$\langle A_p^a(\hat{u}), h_2\rangle = \hat{\lambda} \int_\Omega m(z)a(z)\hat{u}^{p-1} h_2 dz. \tag{3.12}$$

Adding and using (3.9), (3.10), we obtain

$$\begin{aligned}
&\hat{\lambda}_1 \int_\Omega m(z)a(z)\hat{u}_1^{p-1}\left(\frac{(\hat{u}_1)_\varepsilon^p - \hat{u}_\varepsilon^p}{(\hat{u}_1)_\varepsilon^{p-1}}\right) dz + \hat{\lambda}\int_\Omega m(z)a(z)\hat{u}^{p-1}\left(\frac{\hat{u}_\varepsilon^p - (\hat{u}_1)_\varepsilon^p}{\hat{u}_\varepsilon^{p-1}}\right) dz \\
&= \int_\Omega m(z)a(z)\left[\hat{\lambda}_1 \frac{\hat{u}_1^{p-1}}{(\hat{u}_1)^{p-1}} - \hat{\lambda}\frac{\hat{u}^{p-1}}{\hat{u}_\varepsilon^{p-1}}\right]\left((\hat{u}_1)_\varepsilon^p - \hat{u}^p\right) dz \\
&= \int_\Omega \left(\left[1 + (p-1)\left(\frac{\hat{u}_\varepsilon}{(\hat{u}_1)_\varepsilon}\right)^p\right]|D\hat{u_1}|^p + \left[1 + (p-1)\left(\frac{(\hat{u}_1)_\varepsilon}{\hat{u}_\varepsilon}\right)^p\right]|D\hat{u}|^p\right) dz \\
&\quad -p\int_\Omega \left(\left(\frac{\hat{u}_\varepsilon}{(\hat{u}_1)_\varepsilon}\right)^{p-1}|D\hat{u}_1|^{p-2} + \left(\frac{(\hat{u}_1)_\varepsilon}{\hat{u}_\varepsilon}\right)^{p-1}|D\hat{u}|^{p-2}\right)(D\hat{u}_1, D\hat{u})_{\mathbb{R}^N} dz \\
&= \int_\Omega \left((\hat{u}_1)_\varepsilon^p - \hat{u}_\varepsilon^p\right)\left(|D\ln(\hat{u}_1)_\varepsilon|^p - |D\ln\hat{u}_\varepsilon|^p\right) dz \\
&\quad -p\int_\Omega \hat{u}_\varepsilon^p |D\ln(\hat{u}_1)_\varepsilon|^{p-2}\left(D\ln(\hat{u}_1)_\varepsilon, D\ln\hat{u}_\varepsilon - D\ln(\hat{u}_1)_\varepsilon\right)_{\mathbb{R}^N} dz
\end{aligned}$$

$$-p \int_\Omega (\hat{u}_1)_\varepsilon^p |D \ln \hat{u}_\varepsilon|^{p-2} \left(D \ln \hat{u}_\varepsilon, D \ln(\hat{u}_1)_\varepsilon - D \ln \hat{u}_\varepsilon\right)_{\mathbb{R}^N} dz$$

(see Lindqvist [15, Appendix]).

We let $\varepsilon \to 0^+$ and obtain

$$(\hat{\lambda}_1 - \hat{\lambda}) \int_\Omega m(z)a(z) \left(\hat{u}^p - \hat{u}_1^p\right) dz \geq 0,$$
$$\Rightarrow 0 \leq \hat{\lambda}_1 - \hat{\lambda} < 0 \text{ (since } \hat{u} \geq \hat{u}_1),$$

a contradiction. So, we infer that every eigenvalue $\hat{\lambda} > \hat{\lambda}_1$ has nodal eigenfunctions.

Since we can always multiply with $\eta > 1$ big without changing the set Ω_+, without any loss of generality, we may assume that

$$\|\hat{u}^+\| \geq 1 \text{ and } \int_\Omega m(z)a(z)(\hat{u}^+)dz \geq 1.$$

We have

$$\begin{aligned}\rho_a(D\hat{u}^+) &= \hat{\lambda} \int_\Omega m(z)a(z)(\hat{u}^+)^p \chi_{\Omega_+} dz \\ &\leq \hat{\lambda} C_1 \left[\int_\Omega m(z)a(z)(\hat{u}^+)^p dz\right]^{\frac{q}{p}} |\Omega_+|_N^{\frac{p-q}{p}},\end{aligned}$$

for some $C_1 > 0$ (use Hölder's inequality and recall that $\hat{u} \in L^\infty(\Omega)$).

It follows that

$$\begin{aligned}\rho_a(D\hat{u}^+) &\leq \hat{\lambda} C_1 \int_\Omega m(z)a(z)(\hat{u}^+)^p dz |\Omega_+|_N^{\frac{p-q}{p}} \text{ (since} q < p), \\ &\Rightarrow 1 \leq \frac{\hat{\lambda}}{\hat{\lambda}_1} C_1 |\Omega_+|_N^{\frac{p-q}{p}}, \\ &\Rightarrow 0 < \hat{C} \leq |\Omega_+|_N \text{ for all } \hat{\lambda} \in (\hat{\lambda}_1, \mu).\end{aligned}$$

A similar argument holds for Ω_-. □

Proposition 8 *If hypotheses H_0' hold, then $\hat{\lambda}_1 = \hat{\lambda}_1^a(p, m) > 0$ is isolated in the spectrum.*

Proof We argue by contradiction. So, we suppose we can find a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \downarrow \hat{\lambda}_1$. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$ of corresponding eigenfunctions satisfying the normalization condition

$$\int_\Omega m(z)a(z)|u_n|^p dz = 1 \text{ for all } n \in \mathbb{N}.$$

We have

$$\rho_a(Du_n) = \lambda_n \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega) \text{ is bounded.} \tag{3.13}$$

So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } u_n \to \hat{u} \text{ in } L^{\xi_0}(\Omega)$$
$$\text{(see Proposition 4).} \tag{3.14}$$

On account of (3.14) we have

$$\rho_a(D\hat{u}) \leq \liminf_{n\to\infty} \rho_a(Du_n), \quad \int_\Omega m(z)a(z)|\hat{u}|^p dz = 1$$
$$\Rightarrow \rho_a(D\hat{u}) \leq \hat{\lambda}_1 \text{ (see (3.13))}$$
$$\Rightarrow \rho_a(D\hat{u}) = \hat{\lambda}_1 \text{ (see Proposition 2) and so } \hat{u} \neq 0.$$

Therefore $\hat{u}$ is an eigenfunction corresponding to $\hat{\lambda}_1$. Hence $\hat{u} \in W_0^{1,\xi_0}(\Omega)\cap L^\infty(\Omega)$ and for every compact $K \subseteq \Omega$ we have

$$0 < C_K \leq \hat{u}(z) \text{ for a.a. } z \in K$$
$$\text{(see [24, Proposition 2.4]).}$$

From (3.14) and Lemma 3.3.1 of Harjulehto & Hästo [13, p.51] we know that

$$u_n(z) \to \hat{u}(z) \text{ for a.a. } z \in \Omega.$$

By Egorov's theorem, we can find $D_\varepsilon \subseteq \Omega$ compact with $|\Omega \setminus D_\varepsilon|_N \leq \varepsilon$ $(0 < \varepsilon < |\Omega|_N)$ such that

$$u_n \to \hat{u} \text{ uniformly on } D_\varepsilon.$$

Then we can find $n_0 = n_0(\varepsilon) > 0$ such that

$$|u_n(z) - \hat{u}(z)| \leq \frac{C_{D_\varepsilon}}{2} \text{ for all } n \geq n_0, \text{ all } z \in D_\varepsilon,$$
$$\Rightarrow \frac{C_{D_\varepsilon}}{2} \leq u_n(z) \text{ for all } n \geq n_0, \text{ all } z \in D_\varepsilon.$$

Let $\Omega_-^n = \{z \in \Omega : u_n(z) < 0\}$, $n \in \mathbb{N}$. From Proposition 7 we know that

$$0 < \hat{C} \leq |\Omega_-^n|_N \text{ for all } n \in \mathbb{N}.$$

So, if we choose $\varepsilon = \frac{\hat{C}}{2}$, then for $n \geq n_0$ we have

$$
\begin{aligned}
|\Omega|_N &\geq |\Omega_+^n|_N + |\Omega_-^n|_N \\
&\geq |D_\varepsilon|_N + \hat{C} \\
&\geq |\Omega|_N - \frac{\hat{C}}{2} + \hat{C} > |\Omega|_N,
\end{aligned}
$$

a contradiction. Therefore $\hat{\lambda}_1 > 0$ is isolated in the spectrum. ◻

We can also show that $\hat{\lambda}_1 = \hat{\lambda}_1^a(p, m)$ is simple, that is, if u and v are two eigenfunctions corresponding to $\hat{\lambda}_1$, then $u = \mu v$ with $\mu \in \mathbb{R} \setminus \{0\}$. We follow Lindqvist [15], who proved the corresponding result for the Dirichlet p-Laplacian.

Proposition 9 *If hypotheses H_0' hold, then $\hat{\lambda}_1 > 0$ is simple.*

Proof As in the proof of Proposition 7, due to the lack of global regularity theory for eigenfunctions, we need to perturb them. So, let u and v be two eigenfunctions for $\hat{\lambda}_1 > 0$ and $\varepsilon > 0$.

We set

$$u_\varepsilon = u + \varepsilon \text{ and } v_\varepsilon = v + \varepsilon.$$

On account of Proposition 5, we may assume that $u, v \geq 0$. We have

$$\langle A_p^a(u), h \rangle = \hat{\lambda}_1 \int_\Omega m(z) a(z) u^{p-1} h dz, \tag{3.15}$$

$$\langle A_p^a(v), h \rangle = \hat{\lambda}_1 \int_\Omega m(z) a(z) v^{p-1} h dz$$

$$\text{for all } h \in W_0^{1,\xi_0}(\Omega). \tag{3.16}$$

As in the proof of Proposition 7, in (3.15) we use the test function

$$h_1 = \frac{u_\varepsilon^p - v_\varepsilon^p}{u_\varepsilon^{p-1}} \in W_0^{1,\xi_0}(\Omega)$$

and in (3.16) we use the test function

$$h_2 = \frac{v_\varepsilon^p - u_\varepsilon^p}{v_\varepsilon^{p-1}} \in W_0^{1,\xi_0}(\Omega).$$

Reasoning as in the proof of Proposition 7, we obtain:

If $2 \leq p$, then

$$0 \leq C_2 \int_\Omega a(z) \left[\frac{1}{v_\varepsilon^p} - \frac{1}{u_\varepsilon^p} \right] |v_\varepsilon Du - u_\varepsilon Dv| dz$$

$$
\begin{aligned}
&\leq -\hat{\lambda}_1 \int_\Omega m(z)a(z)\left[\left(\frac{u}{u_\varepsilon}\right)^{p-1} - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right](u_\varepsilon^p - v_\varepsilon^p)dz \\
&\quad \text{for some } C_2 > 0.
\end{aligned} \tag{3.17}
$$

If $1 < p < 2$, then

$$
\begin{aligned}
0 &\leq C_3 \int_\Omega a(z)\,[u_\varepsilon + v_\varepsilon]^p\,(u_\varepsilon v_\varepsilon)^p \frac{|u_\varepsilon Dv - v_\varepsilon Du|}{(v_\varepsilon|Du| - u_\varepsilon|Dv|)^{2-p}}dz \\
&\leq -\hat{\lambda}_1 \int_\Omega m(z)a(z)\left[\left(\frac{u}{u_\varepsilon}\right)^{p-1} - \left(\frac{v}{v_\varepsilon}\right)^{p-1}\right](u_\varepsilon^p - v_\varepsilon^p)dz \\
&\quad \text{for some } C_3 > 0.
\end{aligned} \tag{3.18}
$$

We let $\varepsilon \to 0^+$ in both (3.17) and (3.18) and obtain

$$
\begin{aligned}
&uDv = vDu \\
\Rightarrow\ &u = \mu v \text{ for some } \mu > 0.
\end{aligned}
$$

The proof is now complete. □

Proposition 10 *If hypotheses H_0' hold, then the operator $A_p^a : W_0^{1,\xi_0}(\Omega) \to W_0^{1,\xi_0}(\Omega)^*$ is of type $(S)_+$.*

Proof Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega)$ such that

$$
u_n \xrightarrow{w} u \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } \limsup_{n\to\infty}\langle A_p^a(u_n), u_n - u\rangle \leq 0. \tag{3.19}
$$

From the convexity of the modular function $\rho_a(\cdot)$, we have

$$
\begin{aligned}
&\frac{1}{p}\left[\rho_a(Du_n) - \rho_a(Du)\right] \leq \langle A_p^a(u_n), u_n - u\rangle, \\
\Rightarrow\ &\limsup_{n\to\infty} \rho_a(Du_n) \leq \rho_a(Du) \text{ (see (3.19))}.
\end{aligned} \tag{3.20}
$$

On the other hand, $\rho_a(\cdot)$ is continuous and convex, hence it is sequentially weakly lower semicontinuous. So, we have

$$
\begin{aligned}
&\rho_a(Du) \leq \liminf_{n\to\infty} \rho_a(Du_n), \\
\Rightarrow\ &\rho_a(Du_n) \to \rho_a(Du) \text{ (see (3.20))}.
\end{aligned} \tag{3.21}
$$

Since $\xi_0(z,\cdot)$ is uniformly convex, then from (3.21) and Lemma 3.6.5 of Harjulehto & Hästo [13, p.64], it follows that

$$
\rho_a(Du_n - Du) \to 0,
$$

$$\Rightarrow u_n \to u \text{ in } W_0^{1,\xi_0}(\Omega) \text{ (see Proposition 2 for } \xi_0).$$

The proof is now complete. □

In the sequel, for the sake of simplicity, we assume that $m \equiv 1$. Consider the Banach manifold M defined by

$$M = \{u \in W_0^{1,\xi_0}(\Omega) : \|u\|_{\xi_0} = 1\}.$$

Also let τ_p^a denote the spectrum of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$. It is easy to see that $\tau_p^a \subseteq [\hat{\lambda}_1, +\infty)$ is closed and unbounded (Ljusternik-Schnirelmann theory). From Proposition 8, we know that $\hat{\lambda}_1 > 0$ is isolated in τ_p^a. So, the second eigenvalue $\hat{\lambda}_2 = \hat{\lambda}_2^a(p)$ of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$ is defined by

$$\hat{\lambda}_2 = \min\left\{\hat{\lambda} \in \tau_p^a : \hat{\lambda} > \hat{\lambda}_1\right\}.$$

Using the Banach manifold M, we will produce a min-max characterization of $\hat{\lambda}_2$ (for the corresponding result for the Dirichlet p-Laplacian, we refer to Cuesta, de Figueiredo & Gossez [8]). In what follows, we denote by $\hat{u}_1$ the positive, $L^{\xi_0}(\Omega)$-normalized (that is, $\|\hat{u}_1\|_{\xi_0} = 1$) eigenfunction for $\hat{\lambda}_1 > 0$. From Proposition 5, we know that $\hat{u}_1 \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega)$, $\hat{u}_1(z) > 0$ for a.a. $z \in \Omega$.

Proposition 11 *If hypotheses H_0' hold, then*

$$\hat{\lambda}_2 = \inf_{\gamma \in \Gamma} \max_{-1 \leq t \leq 1} \rho_a(D\gamma(t)),$$

where

$$\Gamma = \left\{\gamma \in C([-1, 1], M) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\right\}.$$

Proof We start by showing that $\rho_a\Big|_M$ satisfies the Palais-Smale condition. So, we consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq M$ such that

$$|\rho_a(Du_n)| \leq C_4 \text{ for some } C_4 > 0, \text{ all } n \in \mathbb{N} \tag{3.22}$$
$$|\langle A_p^a(u_n), h\rangle| \leq \varepsilon_n \|h\| \tag{3.23}$$

for all $u \in T_{u_n}M$, with $\varepsilon_n \to 0^+$.

Here, $T_{u_n}M$ denotes the tangent space to the manifold M at u_n. From Ljusternik's theorem (see Gasinski & Papageorgiou [9, p.697]) we have

$$T_{u_n}M = \left\{h \in W_0^{1,\xi_0}(\Omega) : \int_\Omega a(z)|u_n|^{p-2}u_n h dz = 0\right\}, \ n \in \mathbb{N}. \tag{3.24}$$

Let $y \in W_0^{1,\xi_0}(\Omega)$. We have

$$h = y - \left(\int_\Omega a(z)|u_n|^{p-2} u_n y dz \right) u_n \in T_{u_n} M \text{ for all } n \in \mathbb{N}$$
$$\text{(see (3.24) and recall that } \rho_a(u_n) = 1 \Leftrightarrow \|u_n\|_{\xi_0} = 1).$$

We use this particular h as a test function in (3.23). We obtain

$$\begin{aligned} |\langle A_p^a(u_n), y\rangle - \int_\Omega a(z)|u_n|^{p-2} u_n y dz \rho_a(Du_n)| &\leq \varepsilon_n \|h\| \\ &\leq \varepsilon_n C_5 \|y\| \\ \text{for some } C_5 > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.25}$$

From (3.22), we deduce that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } u_n \to u \text{ in } L^{\xi_0}(\Omega) \\ \text{(see Proposition 4).} \tag{3.26}$$

Using Hölder's inequality (see Harjulehto & Hästo [13, p.54]) we have

$$\begin{aligned} &\left| \int_\Omega (u_n - u) a(z)|u_n|^{p-2} u_n dz \right| \\ &\leq C_6 \rho_{\xi_0}(u_n - u) \text{ for some } C_6 > 0, \text{ all } n \in \mathbb{N}, \\ &\Rightarrow \int_\Omega (u_n - u) a(z)|u_n|^{p-2} u_n dz \to 0 \text{ as } n \to \infty \text{ (see (3.26)).} \end{aligned} \tag{3.27}$$

So, if in (3.25) we choose $y = u_n - u \in W_0^{1,\xi_0}(\Omega)$ and pass to the limit as $n \to \infty$, then using (3.26) and (3.27), we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \langle A_p^a(u_n), u_n - u\rangle \leq 0, \\ &\Rightarrow u_n \to u \text{ in } W_0^{1,\xi_0}(\Omega) \text{ (see Proposition 10),} \\ &\Rightarrow \rho_a \Big|_M \text{ satisfies the Palais-Smale condition.} \end{aligned}$$

We know that $\pm\hat{u}_1 \in M$ are minimizers of $\rho_a\Big|_M$ and $\rho_a(\pm D\hat{u}_1) = \hat{\lambda}_1$. Choose $\rho > 0$ small such that

$$\rho_a(\pm D\hat{u}_1) < \inf\left\{\rho_a(Du) : \|u - (\pm\hat{u}_1)\|_{1,\xi_0} = \rho\right\} = m_\rho \\ \text{(see [21, Theorem 5.7.6, p.449]).} \tag{3.28}$$

From (3.28) and since $\rho_a\Big|_M$ satisfies the Palais-Smale condition, we can use the mountain pass theorem for manifolds and have

$$\tilde{\lambda} = \inf_{\gamma\in\Gamma} \max_{-1\le t\le 1} \rho_a(D\gamma(t)) \geq m_\rho > \rho_a(\pm D\hat{u}_1) \tag{3.29}$$

is a critical value of $\rho_a\Big|_M$. Hence $\tilde{\lambda} > 0$ is an eigenvalue of $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$. We show that $\tilde{\lambda} = \hat{\lambda}_2$. It suffices to show that $(\hat{\lambda}_1, \tilde{\lambda}) \cap \tau_p^a = \emptyset$. Arguing by contradiction, suppose that we can find an eigenvalue $\lambda \in (\hat{\lambda}_1, \tilde{\lambda})$ and $u \in M$ is corresponding eigenfunction. From Proposition 7, we know that u is nodal. Therefore $u^+ \neq 0$, $u^- \neq 0$. Consider the following two continuous paths on M

$$\gamma_1(t) = \frac{u^+ - tu^-}{\|u^+ - tu^-\|_{\xi_0}} \text{ for all } t \in [0, 1], \tag{3.30}$$

$$\gamma_2(t) = \frac{-u^- + (1-t)u^+}{\| -u^- + (1-t)u^+\|_{\xi_0}} \text{ for all } t \in [0, 1]. \tag{3.31}$$

We see that the path $\gamma_1(\cdot)$ connects $\frac{u^+}{\|u^+\|_{\xi_0}}$ with u (recall that $u \in M$ and so $\|u\|_{\xi_0} = 1$). On the other hand, the path $\gamma_2(\cdot)$ connects u and $\frac{-u^-}{\|u^-\|_{\xi_0}}$. We concatenate γ_1 and γ_2 and generate a continuous path on M which connects $\frac{u^+}{\|u^+\|_{\xi_0}}$ and $\frac{u^-}{\|u^-\|_{\xi_0}}$. We have

$$\begin{aligned} &\rho_a(Du^+) = \lambda\rho_a(u^+) \text{ and } \rho_a(Du^-) = \lambda\rho_a(u^-), \\ \Rightarrow\ &\rho_a(D\gamma_1(t)) = \rho_a(D\gamma_2(t)) = \lambda \text{ for all } t \in [0, 1] \text{ (see (3.30), (3.31))}. \end{aligned}$$

Consider the following set

$$L = \{u \in W_0^{1,\xi_0}(\Omega) : \ \rho_a(Du) < \lambda\}.$$

This set cannot be path connected or otherwise we violate (3.29). Using the Ekeland variational principle and since $\rho_a\Big|_M$ satisfies the Palais-Smale condition, we see that each path connected component of L has a critical point $\rho_a\Big|_M$. We already know that $\pm\hat{u}_1$ are the only critical points of $\rho_a\Big|_M$ (see Proposition 9). Therefore L has exactly two path connected components with $\hat{u}_1$ in one and $-\hat{u}_1$ in the other. Note that $\frac{u^+}{\|u^+\|_{\xi_0}}$ cannot be a critical point of $\rho_a\Big|_M$. So, we can find a continuous path $S : [-\varepsilon, \varepsilon] \to M$ such that

$$S(0) = \frac{u^+}{\|u^+\|_{\xi_0}}, \ \frac{d}{dt}\left(\rho_a\Big|_M\right)(S(t)) \neq 0 \text{ for all } t \in [-\varepsilon, \varepsilon].$$

Hence starting with this path, we can move from $\frac{u^+}{\|u^+\|_{\xi_0}}$ to $y \in M$ staying in L, except at the starting point. So, $y \in L$ and let U_1 be the path connected component of L which contains y. We may assume that $\hat{u}_1 \in U_1$. We can connect y and $\hat{u}_1$ with a continuous path which stays in U_1. We concatenate this path with $S(\cdot)$ and produce a continuous path $\gamma_+ : [0, 1] \to \overline{U}_1$ such that

$$\gamma_+(0) = \hat{u}_1, \ \gamma_+(1) = \frac{u^+}{\|u^+\|_{\xi_0}}, \gamma_+(t) \in U_1 \subseteq L \text{ for all } t \in [0, 1).$$

Similarly, if U_2 is the other path connected component of L with $-\hat{u}_1 \in U_2$, then we produce a continuous path $\gamma_- : [0, 1] \to \overline{U}_2$ such that

$$\gamma_-(0) = \frac{-u^-}{\|u^-\|_{\xi_0}}, \ \gamma_-(1) = -\hat{u}_1, \ \gamma_-(t) \in U_2 \subseteq L \text{ for all } t \in (0, 1].$$

We concatenate $\gamma_-, \gamma, \gamma_+$ to produce a continuous path $\gamma_* \in \Gamma$ such that

$$\begin{aligned} &\rho_a(D\gamma(t)) \le \lambda \text{ for all } t \in [-1, 1], \\ \Rightarrow\ &\tilde{\lambda} \le \lambda, \ \text{ a contradiction.} \end{aligned}$$

Therefore $\tilde{\lambda} = \hat{\lambda}_2$. □

Let $\hat{\lambda}_1(q) > 0$ be the principal eigenvalue of $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$ and $\hat{u}_1(q)$ be the corresponding L^q-normalized, positive eigenfunction.

We define

$$V = \left\{u \in W_0^{1,q}(\Omega) : \int_\Omega \hat{u}_1(q)^{q-1} u dz = 0\right\}, \ \hat{\lambda}_V = \inf\left\{\frac{\|Du\|_q^q}{\|u\|_q^q} \ u \in V, u \neq 0\right\}. \tag{3.32}$$

Proposition 12 *We have* $\hat{\lambda}_1(q) < \hat{\lambda}_V \le \hat{\lambda}_2(q)$.

Proof Clearly $\hat{\lambda}_1(q) \le \hat{\lambda}_V$. Suppose that $\hat{\lambda}_1(q) = \hat{\lambda}_V$. Then we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq V$ such that

$$\|Du\| \downarrow \hat{\lambda}_V \text{ and } \|Du_n\|_q = 1 \text{ for all } n \in \mathbb{N}.$$

We have that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,q}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,q}(\Omega) \text{ and } u_n \to u \text{ in } L^q(\Omega). \tag{3.33}$$

From (3.33) we have

$$\|Du\|_q^q \le \liminf_{n\to\infty} \|Du_n\|_q^q, \ \|u\|_q = 1. \tag{3.34}$$

Note that

$$\begin{aligned}&\left|\int_\Omega \hat{u}_1(q)^{q-1}(u_n-u)dz\right| \le \|\hat{u}_1(q)\|_q^{q-1}\|u_n-u\|_q\\ &\text{(by Hölder's inequality),}\\ \Rightarrow &\int_\Omega \hat{u}_1(q)^{q-1}(u_n-u)dz \to 0 \text{ as } n\to\infty \text{ (see (3.33)),}\\ \Rightarrow &\int_\Omega \hat{u}_1(q)^{q-1}u dz = 0 \text{ (since } u_n\in V \text{ for all } n\in\mathbb{N}),\\ \Rightarrow\ & u\in V. \end{aligned} \tag{3.35}$$

Then from (3.34) and (3.35) we infer that

$$\|Du\|_q^q = \hat{\lambda}_V = \hat{\lambda}_1.$$

According to Proposition 9, we have $u = \pm\hat{u}_1$, which contradicts (3.35). Therefore $\hat{\lambda}_1 < \hat{\lambda}_V$.

Next we suppose that $\hat{\lambda}_2 < \hat{\lambda}_V$. Hence by Proposition 8, we can find $\gamma \in \Gamma$ such that

$$\|D\gamma(t)\|_q^q < \hat{\lambda}_V \text{ for all } t\in[-1,1]. \tag{3.36}$$

Recall that

$$\gamma(-1) = -\hat{u}_1(q) \text{ and } \gamma(1) = \hat{u}_1(q).$$

Let $\mu_\gamma : [-1,1] \to \mathbb{R}$ be the function defined by

$$\mu_\gamma(t) = \int_\Omega \hat{u}_1(q)^{q-1}\gamma(t)dz \text{ for all } t\in[-1,1].$$

Evidently $\mu_\gamma(\cdot)$ is continuous and

$$\mu_\gamma(-1) = -\|\hat{u}_1(q)\|_q^q = -1$$

and

$$\mu_\gamma(1) = \|\hat{u}_1(q)\|_q^q = 1.$$

So, by Bolzano's theorem, we can find $t_0 \in (-1,1)$ such that $\mu_\gamma(t_0) = 0$. We have

$$\begin{aligned}&\int_\Omega \hat{u}_1(q)^{q-1}\gamma(t_0)dz = 0\\ \Rightarrow\ & \gamma(t_0)\in V.\end{aligned}$$

which contradicts (3.36). Therefore we conclude that $\hat{\lambda}_V \le \hat{\lambda}_2(q)$. □

Remark 2 Is it true that $\hat{\lambda}_V = \hat{\lambda}_2(q)$? The proposition also holds for $\left(-\Delta_q^a, W_0^{1,\xi_0}(\Omega)\right)$.

Using the results of this section, we can prove the following useful estimate.

Lemma 1 *If $\beta \in L^\infty(\Omega)$, $\beta(z) \le \hat{\lambda}_1$ for a.a. $z \in \Omega$ and $\beta \not\equiv \hat{\lambda}_1$, then there exists $C_7 > 0$ such that*

$$C_7 \|u\|_{1,\xi_0}^p \le \rho_a(Du) - \int_\Omega \beta(z)a(z)|u|^p dz$$

for all $u \in W_0^{1,\xi_0}(\Omega)$.

Proof We argue by contradiction. So, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega)$ such that

$$\rho_a(Du_n) - \int_\Omega \beta(z)a(z)|u_n|^p dz < \frac{1}{n}\|u_n\|_{1,\xi_0}^p \text{ for all } n \in \mathbb{N}.$$

The p-homogeneity of the modular function $\rho_a(\cdot)$ implies that we may assume that $\|u_n\|_{1,\xi_0} = 1$ for all $n \in \mathbb{N}$. So, we have

$$\rho_a(Du_n) - \int_\Omega \beta(z)a(z)|u_n|^p dz \le \frac{1}{n}, \quad \|u_n\|_{1,\xi_0} = 1 \tag{3.37}$$

for all $n \in \mathbb{N}$.

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } u_n \to u \text{ in } L^{\xi_0}(\Omega). \tag{3.38}$$

If $u = 0$, then

$$\begin{aligned} &\left|\int_\Omega \beta(z)a(z)|u_n|^p dz\right| \le \|\beta\|_\infty \rho_a(u_n) \to 0 \text{ (see (3.38))} \\ &\Rightarrow \rho_a(Du_n) \to 0 \text{ (see (3.37))}, \\ &\Rightarrow u_n \to 0 \text{ in } W_0^{1,\xi_0}(\Omega) \text{ (see Proposition 2)}. \end{aligned}$$

But this contradicts the fact that $\|u_n\|_{1,\xi_0} = 1$ for all $n \in \mathbb{N}$.
So, $u \ne 0$. In (3.37) we pass to the limit as $n \to \infty$ and use (3.38). We obtain

$$\begin{aligned} &\rho_a(Du) \le \int_\Omega \beta(z)a(z)|u|^p dz \le \hat{\lambda}_1 \rho_a(u), \\ &\Rightarrow \rho_a(Du) = \hat{\lambda}_1 \rho_a(u) \text{ (see (3.2))}, \\ &\Rightarrow u \in W_0^{1,\xi_0}(\Omega) \cap L^\infty(\Omega) \text{ is an eigenfunction of } \hat{\lambda}_1. \end{aligned} \tag{3.39}$$

Replacing u by $|u|$ we can always assume that $u \geq 0$. From Proposition 5 we have that

$$0 < u(z) \text{ for a.a. } z \in \Omega.$$

Then from (3.39), we have

$$\rho_a(Du) < \hat{\lambda}_1 \rho_a(u),$$

which contradicts (3.2). □

4 Coercive resonant problems

In this section we consider problem (1.1) when resonance occurs with respect to the principal eigenvalue $\hat{\lambda}_1 = \hat{\lambda}_1^a(p, 1) > 0$. The resonance occurs from the left of $\hat{\lambda}_1$ making the energy functional of the problem coercive.

The precise conditions on the forcing term $f(z, x)$ are the following:

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)\left[1 + |x|^{r-1}\right]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < q^* = \frac{Nq}{N-q}$;

(ii) if $F(z, x) = \displaystyle\int_0^x f(z, s)ds$, then $\displaystyle\limsup_{x\to\pm\infty} \frac{pF(z, x)}{a(z)|x|^p} \leq \hat{\lambda}_1$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\mu_0 > 0$ such that

$$-\mu_0 \leq f(z, x)x - pF(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iv) there exist $\eta \in L^\infty(\Omega)$ and $\delta_0 > 0$ such that

$$\begin{aligned}
&\hat{\lambda}_1(q) \leq \eta(z) \text{ for a.a. } z \in \Omega, \ \eta \not\equiv \hat{\lambda}_1(q),\\
&\eta(z) \leq \liminf_{x\to 0} \frac{f(z, x)}{|x|^{q-2}x} \text{ uniformly for a.a. } z \in \Omega,\\
&F(z, x) \leq \frac{1}{q}\hat{\lambda}_V |x|^q \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.
\end{aligned}$$

Remark 3 Hypothesis $H_1(ii)$ implies that we can have resonance with respect to $\hat{\lambda}_1 = \hat{\lambda}_1^a(p, 1) > 0$ as $x \to \pm\infty$. As we will see in the process of the proof, this resonance will occur from the left of $\hat{\lambda}_1$ (that is, $\hat{\lambda}_1 a(z)|x|^p - pF(z, x) \geq -\mu_0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$) and this implies that the energy functional of the problem is coercive. Hypothesis $H_1(iv)$ implies that at zero we have nonuniform nonresonance with respect to $\hat{\lambda}_1(q) > 0$, that is, partial interaction with the spectrum of $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$.

Recall that $\xi(z,x) = a(z)|x|^p + |x|^q$ and let $\varphi : W_0^{1,\xi}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,\xi}(\Omega).$$

Evidently, $\varphi \in C^1\left(W_0^{1,\xi}(\Omega)\right)$.

Proposition 13 *If hypotheses H_0', H_1 hold, then the energy functional $\varphi(\cdot)$ is coercive.*

Proof We argue by contradiction. So, suppose that $\varphi(\cdot)$ is not coercive. We can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ such that

$$\|u_n\| \to \infty \text{ and } \varphi(u_n) \le C_8 \text{ for some } C_8 > 0,\ \text{all } n \in \mathbb{N}. \tag{4.1}$$

We know that $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\xi_0}(\Omega)$ continuously. So, we have $u_n \in W_0^{1,\xi_0}(\Omega)$ for all $n \in \mathbb{N}$.

First assume that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi_0}(\Omega) \text{ is bounded.} \tag{4.2}$$

Hypotheses $H_1(i)$, (ii) imply that we can find $C_9 > 0$ such that

$$F(z,x) \le \frac{1}{p}\left[\hat{\lambda}_1 + 1\right]a(z)|x|^p + C_9 \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{4.3}$$

From (4.1) and (4.3) we have

$$\begin{aligned}
&\frac{1}{p}\left[\rho_a(Du_n) - \hat{\lambda}_1\rho_a(u_n)\right] - \frac{1}{p}\rho_a(u_n) + \frac{1}{q}\|Du_n\|_q^q \le C_{10} \\
&\text{for some } C_{10} > 0,\ \text{all } n \in \mathbb{N}, \\
\Rightarrow\ &\|Du_n\|_q \le C_{11} \text{ for some } C_{11} > 0,\ \text{all } n \in \mathbb{N} (\text{see (3.2) and (4.2)}) \\
\Rightarrow\ &\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega) \text{ is bounded, a contradiction (see (4.1)).}
\end{aligned}$$

So, we may assume that

$$\|u_n\|_{1,\xi_0} \to \infty \text{ as } n \to \infty. \tag{4.4}$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,\xi_0}}$ for all $n \in \mathbb{N}$. Then $\|y_n\|_{1,\xi_0} = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } y_n \to y \text{ in } L^{\xi_0}(\Omega) \\ \text{(see Proposition 4).} \tag{4.5}$$

From (4.1) we have

$$\frac{1}{p}\rho_a(Dy_n) + \frac{1}{q\|u_n\|_{1,\xi_0}^{p-q}}\|Dy_n\|_q^q - \int_\Omega \frac{F(z,u_n)}{\|u_n\|_{1,\xi_0}^p}dz \le \frac{C_8}{\|u_n\|_{1,\xi_0}^p} \tag{4.6}$$

for all $n \in \mathbb{N}$.

Claim: $-\mu_0 \leq \hat{\lambda}_1 a(z)|x|^p - pF(z,x)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

We have

$$\begin{aligned}
\frac{d}{dx}\left[\frac{F(z,x)}{|x|^p}\right] &= \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} \\
&= \frac{f(z,x)x - pF(z,x)}{|x|^p x} \\
&\begin{cases} \geq -\frac{\mu_0}{x^{p+1}} \text{ if } x > 0 \\ \leq -\frac{\mu_0}{|x|^p x} \text{ if } x < 0 \end{cases} \\
&\text{(see hypothesis } H_1(iii)) \\
\Rightarrow \frac{F(z,v)}{|v|^p} - \frac{F(z,x)}{|x|^p} &\geq \frac{\mu_0}{p}\left[\frac{1}{|v|^p} - \frac{1}{|x|^p}\right] \\
&\text{for a.a.} z \in \Omega, \text{ all } |v| \geq |x| > 0.
\end{aligned}$$

We let $v \to \pm\infty$. Using hypothesis $H_1(ii)$ we obtain

$$\begin{aligned}
&\frac{\hat{\lambda} a(z)}{p} - \frac{F(z,x)}{|x|^p} \geq -\frac{\mu_0}{p}\frac{1}{|x|^p} \text{ for a.a. } z \in \Omega, \text{ all } x \neq 0, \\
\Rightarrow\ & \hat{\lambda}_1 a(z)|x|^p - pF(z,x) \geq -\mu_0 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.
\end{aligned}$$

This proves the Claim.

We return to (4.6) and use the Claim. We obtain

$$\frac{1}{p}\rho_a(Dy_n) - \frac{\hat{\lambda}_1}{p}\rho_a(y_n) \leq \left[C_8 + \frac{\mu_0}{p}|\Omega|_N\right]\frac{1}{\|u_n\|_{1,\xi_0}^p} \tag{4.7}$$

for all $n \in \mathbb{N}$.

Passing to the limit as $n \to \infty$ and using (4.4) and (4.5) we have

$$\begin{aligned}
&\rho_a(Dy) \leq \hat{\lambda}_1\rho_a(y) \\
\Rightarrow\ &\rho_a(Dy) = \hat{\lambda}_1\rho_a(y) \text{ (see (3.2))} \\
\Rightarrow\ &y = 0 \text{ or } y = \text{ eigenfunction for } \hat{\lambda}_1.
\end{aligned}$$

If $y = 0$, then from (4.7) and (4.5), we see that

$$y_n \to 0 \text{ in } W_0^{1,\xi_0}(\Omega)$$

which contradicts the fact that $\|y_n\|_{1,\xi_0} = 1$ for all $n \in \mathbb{N}$.

If y is an eigenfunction for $\hat{\lambda}_1$, then by Proposition 5 we may assume that $y(z) > 0$ for a.a. $z \in \Omega$. So, $u_n(z) \to +\infty$ for a.a. $z \in \Omega$ as $n \to \infty$. From (4.1) we have

$$\frac{1}{p}\rho_a(Du_n) + \frac{1}{q}\|Du_n\|_q^q - \int_\Omega F(z,u_n)dz \leq C_8 \text{ for all } n \in \mathbb{N},$$

$$\Rightarrow \frac{1}{p}\int_\Omega \left[\hat{\lambda}_1 a(z)|u_n|^p - pF(z,u_n)\right] dz + \frac{1}{q}\|Du_n\|_q^q \le C_8$$
for all $n \in \mathbb{N}$(see (3.2)),
$$\Rightarrow \frac{\hat{\lambda_1(q)}}{q}\|u_n\|_q^q \le C_{12}$$
for some $C_{12} > 0$, all $n \in \mathbb{N}$(see the Claim).

But $\int_\Omega |u_n|^q dz \to +\infty$ (by Fatou's lemma), a contradiction. This proves that the energy functional $\varphi(\cdot)$ is coercive. □

We know that $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ continuously (see Proposition 1). Let $\hat{V} = V \cap W_0^{1,\xi}(\Omega)$. We have

$$W_0^{1,\xi}(\Omega) = \mathbb{R}\hat{u}_1(q) \oplus \hat{V}$$
$$\text{(recall that } \hat{u}_1(q) \in C_0^1(\overline{\Omega})). \tag{4.8}$$

Proposition 14 *If hypotheses H_0', H_1 hold, then $\varphi(\cdot)$ has local linking at* 0 *with respect to* (4.8) *(see [21, p.408]).*

Proof Hypotheses $H_1(i)$, (iv) imply that given $\varepsilon > 0$, we can find $C_{13} = C_{13}(\varepsilon) > 0$ such that

$$F(z,x) \ge \frac{1}{q}\left[\eta(z) - \varepsilon\right]|x|^q - C_{13}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{4.9}$$

Let $u \in \mathbb{R}\hat{u}_1(q)$. Then $u = \tau\hat{u}_1(q)$ for some $\tau \in \mathbb{R}$. We have

$$\begin{aligned}
\varphi(u) &= \varphi(\tau\hat{u}_1(q)) \\
&\le \frac{|\tau|^p}{p}\rho_a\left(D\hat{u}_1(q)\right) + \frac{|\tau|^q}{q}\|D\hat{u}_1(q)\|_q^q \\
&\quad - \frac{|\tau|^q}{q}\int_\Omega \eta(z)\hat{u}_1(q)^q dz + \frac{\varepsilon}{q}|\tau|^q + C_{13}|\tau|^r\|\hat{u}_1(q)\|_r^r \\
&\quad \text{(see (4.9) and recall that } \|\hat{u}_1(q)\|_q = 1) \\
&\le C_{14}\left[|\tau|^p + |\tau|^r\right] + \frac{|\tau|^q}{q}\left[\int_\Omega (\hat{\lambda}_1(q) - \eta(z))\hat{u}_1(q)^q dz - \varepsilon\right] \\
&\quad \text{for some } C_{14} > 0.
\end{aligned}$$

We know that $\hat{u}_1(q)(z) > 0$ for all $z \in \Omega$. Therefore

$$\int_\Omega \left(\eta(z) - \hat{\lambda}_1(q)\right)\hat{u}_1(q)^q dz = C_{15} > 0.$$

Hence choosing $\varepsilon \in (0, C_{15})$, we obtain

$$\varphi(u) = \varphi(\tau\hat{u}_1(q)) \le C_{14}\left[|\tau|^p + |\tau|^q\right] - C_{16}|\tau|^q$$

for some $C_{16} > 0$.

Since $q < p < r$, choosing $|\tau| \in (0, 1)$ small, we infer that

$$\varphi(u) = \varphi(\tau \hat{u}_1(q)) < 0. \tag{4.10}$$

On the other hand, using once again hypotheses $H_1(i), (iv)$, we can find $C_{17} > 0$ such that

$$F(z, x) \leq \frac{1}{q}\hat{\lambda}_V |x|^q + C_{17}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{4.11}$$

Let $v \in \hat{V}$ with $\|v\| \leq 1$. Then

$$\begin{aligned} \varphi(v) \geq &\ \frac{1}{p}\rho_a(Dv) + \frac{1}{q}\left[\|Dv\|_q^q - \hat{\lambda}_V \|v\|_q^q\right] - C_{18}\|v\|^r \\ &\ \text{for some } C_{18} > 0 \text{(see (4.11))} \\ \geq &\ \frac{1}{p}\|v\|^p - C_{18}\|v\|^r \\ &\ \text{(see Propositions 2 and 12).} \end{aligned}$$

Since $p < r$, we can find $\delta \in (0, 1)$ small such that

$$\varphi(v) > 0 \text{ for all } v \in \hat{V} \text{ with } 0 < \|v\| \leq \delta. \tag{4.12}$$

From (4.10), (4.12) we conclude that $\varphi(\cdot)$ has local linking at 0 with respect to (4.8). □

Now we are ready for our first multiplicity theorem for problem (1.1) (coercive case).

Theorem 1 *If hypotheses H_0', H_1 hold, then problem* (1.1) *has at least two nontrivial solutions $u_0, \hat{u} \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$.*

Proof We already know that $\varphi(\cdot)$ has local linking at 0 with respect to (4.8). Also, by Proposition 13 we know that $\varphi(\cdot)$ is coercive. Therefore it satisfies the Palais-Smale condition (see [21, Proposition 5.1.15, p.369]). Moreover, from (4.10), we see that

$$-\infty < \inf\left\{\varphi(u) : u \in W_0^{1,\xi}(\Omega)\right\} < 0.$$

Hence we can apply Theorem 5.4.17 in [21, p.410] and find two nontrivial critical points $u_0, \hat{u}$ of φ. Then these are nontrivial solutions of problem (1.1) and

$$\begin{aligned} &u_0, \hat{u} \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega) \\ &\text{(see Gasinski \&Winkert [11, Theorem 3.1]).} \end{aligned}$$

This completes the proof. □

5 Noncoercive problems

In this section we study noncoercive problems. First we deal with the nonresonant case (nonuniform nonresonance) and then with the resonant case (resonance with respect to the principal eigenvalue $\hat{\lambda}_1 > 0$ from the right).

For the nonresonant case the hypotheses on the reaction $f(z, x)$ are the following (see [23]):

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \le a(z)\left[1 + |x|^{p-1}\right]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$;

(ii) there exist $\eta_0 \in L^\infty(\Omega)$ and $\eta_1 \in \left(\hat{\lambda}_1, \hat{\lambda}_2\right)$ such that

$$\hat{\lambda}_1 \le \eta_0(z) \text{ for a.a. } z \in \Omega,\ \ \eta_0 \not\equiv \hat{\lambda}_1$$
$$\eta_0(z) \le \liminf_{x\to\pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \le \limsup_{x\to\pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \le \eta_1$$

uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to 0} \frac{f(z, x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$, there exists $s \in (1, q)$ such that

$$\lim_{x\to 0} \frac{f(z, x)}{|x|^{s-2}x} = 0 \text{ uniformly for a.a. } z \in \Omega$$
$$0 \le \liminf_{x\to 0} \frac{sF(z, x) - f(z, x)x}{|x|^p} \text{ uniformly for a.a. } z \in \Omega.$$

Similar conditions concerning the asymptotic behavior as $x \to +\infty$ can be found in Papageorgiou & Scapelatto [23] in the context of parametric Robin problems with an indefinite potential term.

Remark 4 Hypothesis $H_2(iii)$ implies the presence of a concave term near zero. Consider for example the following functions (for the sake of simplicity we drop the z-dependence)

$$f_1(x) = \begin{cases} |x|^{p-2}x - |x|^{\tau-2}x, & \text{if } |x| \le 1 \\ \eta_0\left[|x|^{p-2} - |x|^{q-2}\right], & \text{if } 1 < |x| \end{cases} \qquad 1 < \tau < q,\ \hat{\lambda}_1 < \eta_0$$
$$f_2(x) = \begin{cases} \eta_0|x|^{q-2}x - |x|^{s-2}x \ln|x|, & \text{if } |x| \le 1 \\ \eta_0|x|^{p-2}x, & \text{if } 1 < x \end{cases} \qquad 1 < s < q,\ \hat{\lambda}_1 < \eta_0.$$

Both functions satisfy hypotheses H_2.

Recall that $\varphi : W_0^{1,\xi}(\Omega) \to \mathbb{R}$ is the energy functional for problem (1.1), defined by

$$\varphi(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega F(z, u)dz$$

for all $u \in W_0^{1,\xi}(\Omega)$.

We know that $\varphi \in C^1\left(W_0^{1,\xi}(\Omega)\right)$.

Proposition 15 *If hypotheses H_0', H_2 hold, then φ satisfies the C-condition (see [21, p.336]).*

Proof We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ such that $\{\varphi(u_n)\} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W_0^{1,\xi}(\Omega)^* \text{ as } n \to \infty. \tag{5.1}$$

From (5.1) we have

$$\left|\langle A_p^a(u_n), h\rangle + \langle A_q(u_n), h\rangle - \int_\Omega f(z, u_n)h dz\right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}$$
$$\text{for all } h \in W_0^{1,\xi}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{5.2}$$

Suppose that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that

$$\|u_n\| \to \infty \text{ as } n \to \infty. \tag{5.3}$$

We set $v_n = \frac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,\xi}(\Omega) \text{ and } v_n \to v \text{ in } L^p(\Omega)$$
$$\text{(see Proposition 1 and } H_0'). \tag{5.4}$$

From (5.2) we have

$$\left|\langle A_p^a(v_n), h\rangle + \frac{1}{\|u_n\|^{p-q}}\langle A_q(v_n), h\rangle - \int_\Omega \frac{f(z, u_n)}{\|u_n\|^{p-1}} h dz\right| \le \frac{\varepsilon_n' \|h\|}{(1 + \|u_n\|)} \tag{5.5}$$

for all $h \in W_0^{1,\xi}(\Omega)$, all $n \in \mathbb{N}$, $\varepsilon_n' = \frac{\varepsilon_n}{\|u_n\|^{p-1}}$

Hypothesis $H_2(i)$ implies that

$$\left\{\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{5.6}$$

If in (5.5) we choose $h = v_n - v \in W_0^{1,\xi}(\Omega)$, pass to the limit as $n \to \infty$ and use (5.3), (5.4), (5.6), we obtain

$$\lim_{n\to\infty} \langle A_p^a(u_n), u_n - u\rangle = 0,$$
$$\Rightarrow u_n \to u \text{ in } W_0^{1,\xi}(\Omega) \text{ (see Proposition 3), so} \|v\| = 1. \tag{5.7}$$

On account of (5.6) and hypothesis $H_2(iv)$, we have

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} \tilde{\eta}(\cdot)a(\cdot)|v|^{p-2}v \text{ in } L^{p'}(\Omega) \tag{5.8}$$

with $\tilde{\eta} \in L^\infty(\Omega)$, $\eta_0(z) \leq \tilde{\eta}(z) \leq \eta_1$ for a.a. $z \in \Omega$ (see Aizicovici, Papageorgiou & Staicu [2], proof of Proposition 16).

So, if in (5.5) we pass to the limit as $n \to \infty$ and use (5.3), (5.7), (5.8), we obtain

$$\begin{aligned} &\langle A_p^a(v), h\rangle = \int_\Omega \tilde{\eta}(z)a(z)|v|^{p-2}hdz \text{ for all } h \in W_0^{1,\xi}(\Omega) \\ &\Rightarrow -\Delta_p^a v = \tilde{\eta}(z)a(z)|v|^{p-2}v \text{ in } \Omega,\ v\Big|_{\partial\Omega} = 0. \end{aligned} \tag{5.9}$$

From Proposition 6 we know that

$$\hat{\lambda}_1^a(p, \tilde{\eta}) < \hat{\lambda}_1^a(p, \hat{\lambda}_1) = 1. \tag{5.10}$$

Moreover, we have

$$\begin{aligned} &\tilde{\eta}(z) \leq \eta_1 < \hat{\lambda}_2 \text{ for a.a. } z \in \Omega \\ &\text{(see (5.8) and} H_2(iv)). \end{aligned} \tag{5.11}$$

From (5.9), (5.10), (5.11) it follows that

$$v = 0,$$

which contradicts (5.7).

This proves that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\xi}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{5.12}$$

In (5.2) we choose $h = u_n - u \in W_0^{1,\xi}(\Omega)$, pass to the limit as $n \to \infty$ and use (5.12) and hypothesis $H_2(i)$. We obtain

$$\begin{aligned} &\lim_{n\to\infty}\Big[\langle A_p^a(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle\Big] = 0, \\ &\Rightarrow \limsup_{n\to\infty}\Big[\langle A_p^a(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle\Big] \leq 0 \\ &\quad \text{(using the monotonicity of} A_q(\cdot)), \\ &\Rightarrow \limsup_{n\to\infty}\langle A_p^a(u_n), u_n - u\rangle \leq 0 \text{ (see (5.12))}, \\ &\Rightarrow u_n \to u \text{ in } W_0^{1,\xi}(\Omega) \text{ (see Proposition 10)} \end{aligned} \tag{5.13}$$

Similarly, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle A_q(u_n), u_n - u\rangle \le 0,\\
\Rightarrow\ & u_n \to u \text{ in } W_0^{1,q}(\Omega)\\
&(\text{since } W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,q}(\Omega) \text{ continuously}).
\end{aligned} \tag{5.14}$$

From (5.13) and (5.14) we conclude that

$$\begin{aligned}
&u_n \to u \text{ in } W_0^{1,\xi}(\Omega)\\
\Rightarrow\ & \varphi(\cdot) \text{ satisfies the } C\text{-condition.}
\end{aligned}$$

The proof is now complete. □

The next proposition shows that $\varphi(\cdot)$ is not bounded below (so the problem is not coercive).

Proposition 16 *If hypotheses H_0', H_2 hold, then $\varphi(t\hat{u}_1) \to -\infty$ as $t \to \pm\infty$.*

Proof On account of hypothesis $H_2(ii)$, given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that

$$F(z, x) \ge \frac{1}{p}[\eta(z) - \varepsilon]\, a(z)|x|^p \text{ for a.a. } z \in \Omega,\ \text{all } |x| \ge M. \tag{5.15}$$

We have

$$\begin{aligned}
\varphi(t\hat{u}_1) &= \frac{|t|^p}{p}\rho_a(D\hat{u}_1) + \frac{|t|^q}{q}\|D\hat{u}_1\|_q^q - \int_\Omega F(z, t\hat{u}_1)dz\\
&\le \frac{|t|^p}{p}\rho_a(D\hat{u}_1) + \frac{|t|^q}{q}\|D\hat{u}_1\|_q^q - \frac{|t|^p}{p}\int_{\{|t\hat{u}_1|\ge M\}}[\eta(z) - \varepsilon]a(z)\hat{u}_1^p dz\\
&\quad - \int_{\{|t\hat{u}_1|<M\}} F(z, t\hat{u}_1)dz \text{ (see (5.15))}\\
&\le \frac{|t|^p}{p}\left(\int_{\{|t\hat{u}_1|\ge M\}}[\hat{\lambda}_1 - \eta(z)]a(z)\hat{u}_1^p dz + C_{19}\varepsilon|\Omega|_N\right) + C_{20}\left|\{t\hat{u}_1 < M\}\right|_N\\
&\quad \text{for some } C_{19}, C_{20} > 0 \text{(see hypothesis } H_2(i)).
\end{aligned} \tag{5.16}$$

Note that

$$\int_\Omega [\eta(z) - \hat{\lambda}_1]a(z)\hat{u}_1^p dz = C^* > 0 \text{ (see } H_2(iv))$$

and $\left|\{t\hat{u}_1 < M\}\right|_N \to 0$ as $t \to \pm\infty$.

Then $C_t = \int_{\{t\hat{u}_1 \geq M\}} [\eta(z) - \hat{\lambda}_1] a(z) |t\hat{u}_1|^p dz \to 0$ as $t \to \pm\infty$. Therefore from (5.16) and by choosing $\varepsilon > 0$ small we have

$$\varphi(t\hat{u}_1) \to -\infty \text{ as } t \to \pm\infty.$$

This completes the proof. □

Consider the following set in $W_0^{1,\xi}(\Omega)$

$$S = \left\{ u \in W_0^{1,\xi}(\Omega) : \ \rho_a(Du) = \hat{\lambda}_2 \rho_a(u) \right\}.$$

Proposition 17 *If hypotheses H_0', H_2 hold, then $\varphi\big|_S$ is coercive, hence bounded below.*

Proof Hypotheses $H_2(i), (ii)$ imply that we can find $\eta_2 \in (\eta_1, \hat{\lambda}_2)$ and $C_{21} > 0$ such that

$$F(z, x) \leq \frac{\eta_2}{p} a(z)|x|^p + C_{21} \text{ for a.a. } z \in \Omega, \ \text{all } x \in \mathbb{R}. \tag{5.17}$$

Let $u \in S$ with $\|u\| \geq 1$. We have

$$\begin{aligned}
&\varphi(u) \geq \frac{1}{p}\left[1 - \frac{\eta_2}{\hat{\lambda}_2}\right] \rho_a(Du) - C_{21}|\Omega|_N \text{ (see (5.17))} \\
\Rightarrow\ &\varphi(u) \geq C_{22}\|u\|^q - C_{21}|\Omega|_N \text{ (see Proposition 2)}, \\
\Rightarrow\ &\varphi\big|_S \text{ is coercive, thus bounded below.}
\end{aligned}$$

The proof is now complete. □

Propositions 16 and 17 imply that we can find $t_0 > 1$ big such that

$$\varphi(\pm t_0 \hat{u}_1) < \inf_S \varphi = m_S. \tag{5.18}$$

In addition to S above, we also consider the following two subsets of $W_0^{1,\xi}(\Omega)$

$$\begin{aligned}
D_0 &= \left\{ t_0\hat{u}_1, -t_0\hat{u}_1 \right\}, \\
D &= \left\{ u \in W_0^{1,\xi}(\Omega) : \ u = s(-t_0\hat{u}_1) + (1-s)t_0\hat{u}_1, \ 0 \leq s \leq 1 \right\}.
\end{aligned}$$

Proposition 18 *If hypotheses H_0', H_2 hold, then the pair $\{D_0, D\}$ is linking with S in $W_0^{1,\xi}(\Omega)$ (see [21, p.397]).*

Proof Consider the set

$$K = \left\{ u \in W_0^{1,\xi}(\Omega) : \ \rho_a(Du) \leq \hat{\lambda}_2 \rho_a(u) \right\}.$$

Evidently, $\pm t_0\hat{u}_1 \in K$. We claim that K is not path connected and $t\hat{u}_1$, $-t\hat{u}_1$ belong in different components of K. To this end, we consider a path $\gamma_0 \in C\left([-1,1], W_0^{1,\xi}(\Omega)\right)$ which connects $-t_0\hat{u}_1$ and $t_0\hat{u}_1$ and $\gamma_0(t) \neq 0$ for all $t \in [-1,1]$. We have

$$\gamma_0(-1) = -t_0\hat{u}_1, \gamma_0(1) = t_0\hat{u}_1.$$

Consider the line segments

$$L^- = \operatorname{conv}\{-t_0\hat{u}_1, -\hat{u}_1\}, \quad L^+ = \operatorname{conv}\{t_0\hat{u}_1, \hat{u}_1\}.$$

We extend $\gamma_0(\cdot)$ in a linear way along them and obtain continuous path $\hat{\gamma}_0(\cdot)$ connecting $-\hat{u}_1$ and $\hat{u}_1$. Let

$$\hat{\gamma}_0^*(t) = \frac{\hat{\gamma}_0(t)}{\|\hat{\gamma}_0(t)\|_{\xi_0}}, \ t \in [-1,1].$$

Then $\hat{\gamma}_0^* \in \Gamma$ (see Proposition 11) and so

$$\hat{\lambda}_2 \leq \max_{-1\leq t\leq 1} \frac{\rho_a(D\hat{\gamma}_0^*(t))}{\rho_a(\hat{\gamma}_0^*)} \text{ (see Proposition 11).}$$

This means that there exists $\hat{t} \in (-1,1)$ such that

$$\hat{\gamma}_0^*(\hat{t}) \notin K.$$

Therefore K cannot be path connected and $-t_0\hat{u}_1$, $t_0\hat{u}_1$ belong to different path components.

Note that $D_0 \cap S = \emptyset$. Also let $\gamma \in C\left(D, W_0^{1,\xi}(\Omega)\right)$ be such that $\gamma\Big|_{D_0} = id\Big|_{D_0}$. We have

$$\gamma(-t_0\hat{u}_1) = -t\hat{u}_1, \quad \gamma(t_0\hat{u}_1) = t_0\hat{u}_1.$$

From the first part of the proof we know that these two elements belong to different path components of K. Hence

$$\gamma(D) \cap \partial K \neq \emptyset.$$

But $\partial K \subseteq S$. Therefore

$$\begin{aligned} &\gamma(D) \cap S \neq \emptyset, \\ \Rightarrow &\{D_0, D\} \text{ links with } S \text{ in } W_0^{1,\xi}(\Omega) \text{(see [21, p.397]).} \end{aligned}$$

The proof is now complete. $\square$

Now we are ready for the existence theorem in the noncoercive, nonresonant case.

Theorem 2 *If hypotheses H_0', H_2 hold, then problem (1.1) has a nontrivial solution*

$$u_0 \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega).$$

Proof In Proposition 18 we have established that the closed sets $\{D_0, D, K\}$ are linking (in the sense of Definition 5.4.1 in [21, p.397]). Also we have

$$\varphi(\pm t_0 \hat{u}_1) < \inf_S \varphi = m_S \text{ (see (5.18)).}$$

Finally, from Proposition 15 we know that

$$\varphi(\cdot) \text{ satisfies the } C\text{-condition.}$$

The above facts permit the use of Theorem 5.4.4 of Papageorgiou, Rădulescu & Repovš [21, p.399]. Therefore we can find $u_0 \in W_0^{1,\xi}(\Omega)$ such that

$$u_0 \in K_\varphi \text{ and } m_S \leq \varphi(u_0).$$

So u_0 is a solution of problem (1.1) and $u_0 \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$. Moreover, Corollary 6.6.12 of [21, p.534] implies that the first critical group of φ at u_0 is nontrivial, that is,

$$C_1(\varphi, u_0) \neq 0. \tag{5.19}$$

On the other hand, hypothesis $H_2(iii)$ and Proposition 6 of Leonardi & Papageorgiou [14] imply that

$$C_k(\varphi, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{5.20}$$

Comparing (5.19) and (5.20) we conclude that $u_0 \neq 0$. □

Next, we will consider the noncoercive resonant case. To deal with this case we need to strengthen the hypotheses on the reaction.

The new hypotheses on $f(z, x)$ are the following.

H_3: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exist $\eta_* < \hat{\lambda}_2$ and $\tau \in (q, p]$ such that

$$\hat{\lambda}_1 \leq \liminf_{x \to \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \limsup_{x \to \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \eta_*$$

uniformly for a.a. $z \in \Omega$;

$$0 < \hat{\beta} \leq \liminf_{x \to \pm\infty} \frac{pF(z,x) - f(z,x)x}{|x|^\tau}$$

uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x \to 0} \frac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$, there exists $s \in (1, q)$ such that

$$\lim_{x \to 0} \frac{f(z,x)}{|x|^{s-2}x} = 0 \text{ uniformly for a.a. } z \in \Omega$$
$$0 \leq \liminf_{x \to 0} \frac{sF(z,x) - f(z,x)s}{|x|^p} \text{ uniformly for a.a. } z \in \Omega.$$

Remark 5 Hypothesis $H_3(ii)$ implies that we can have resonance with respect to $\hat{\lambda}_1 = \hat{\lambda}_1^a(p, 1) > 0$. In this case the resonance occurs form the right of $\hat{\lambda}_1$ in the sense that $\hat{\lambda}_1 a(z)|x|^p - pF(z,x) \to -\infty$ as $x \to \pm\infty$.

The following function satisfies hypotheses H_3 above. For the sake of simplicity we drop the z-dependence

$$f(x) = \begin{cases} |x|^{s-2}x - |x|^{\eta-2}x, & \text{if } |x| \leq 1 \\ \hat{\lambda}_1 \left[|x|^{p-2}x + |x|^{\tau-2}x - 2|x|^{\vartheta-2}x \right], & \text{if } 1 < |x|. \end{cases}$$

with $1 < s < q \leq \eta$ and $1 < \vartheta < q < \tau < p$.

The approach is similar to that for nonresonant case. Our aim is to apply Theorem 5.4.4 of [21, p.399].

Proposition 19 *If hypotheses H_0', H_3 hold, then the energy functional $\varphi(\cdot)$ satisfies the C-condition.*

Proof Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ such that

$$|\varphi(u_n)| \leq C_{23} \text{ for some } C_{23} > 0, \text{ all } n \in \mathbb{N} \tag{5.21}$$
$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W_0^{1,\xi}(\Omega)^* \text{ as } n \to \infty. \tag{5.22}$$

Suppose that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\xi_0}(\Omega)$ is not bounded in $W_0^{1,\xi_0}(\Omega)$. We may assume that

$$\|u_n\|_{1,\xi_0} \to \infty \text{ as } n \to \infty. \tag{5.23}$$

Let $v_n = \frac{u_n}{\|u_n\|_{1,\xi_0}}$ for all $n \in \mathbb{N}$. Then $\|v_n\|_{1,\xi_0} = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,\xi_0}(\Omega) \text{ and } v_n \to v \text{ in } L^{\xi_0}(\Omega). \tag{5.24}$$

From (5.22) we have

$$\begin{aligned}
&\langle A_p^a(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega f(z, u_n) h dz \\
&\text{for all } h \in W_0^{1,\xi}(\Omega) \\
\Rightarrow\ &\langle A_p^a(v_n), h\rangle \frac{1}{\|u_n\|_{1,\xi_0}^{p-q}} \langle A_q(v_n), h\rangle = \int_\Omega \frac{f(z, u_n)}{\|u_n\|_{1,\xi_0}^{p-1}} h dz \\
&\text{for all } h \in W_0^{1,\xi_0}(\Omega).
\end{aligned} \tag{5.25}$$

From hypotheses $H_3(i), (ii)$ and Hölder's inequality, we see that

$$\left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|_{1,\xi_0}^{p-1}} \right\} \subseteq L^{\xi_0}(\Omega)^* \text{ is bounded.} \tag{5.26}$$

In (5.25) we use $h = v_n - v$, pass to the limit as $n \to \infty$ and use (5.23), (5.24), (5.26). We obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A_p^a(v_n), v_n - v\rangle = 0, \\
\Rightarrow\ &v_n \to v \text{ in } W_0^{1,\xi_0}(\Omega) \text{ (see Proposition 10), so } \|v\|_{1,\xi_0} = 1.
\end{aligned} \tag{5.27}$$

On account of (5.26) and hypothesis $H_3(iii)$, we have

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|_{1,\xi_0}^{p-1}} \xrightarrow{w} \tilde{\eta}(\cdot) a(\cdot) |v|^{p-2} v \text{ in } L^{\xi_0}(\Omega)^*, \tag{5.28}$$

$$\hat{\lambda}_1 \le \tilde{\eta}(z) \le \eta_* \text{ for a.a. } z \in \Omega. \tag{5.29}$$

So, if in (5.25) we pass to the limit as $n \to \infty$ and use (5.23), (5.27), (5.28), we obtain

$$\begin{aligned}
&\langle A_p^a(v), h\rangle = \int_\Omega \tilde{\eta}(z) a(z) |v|^{p-2} v h dz \\
&\text{for all } h \in W_0^{1,\xi}(\Omega) \\
\Rightarrow\ &-\Delta_p^a v = \tilde{\eta}(z) a(z) |v|^{p-2} v \text{ in } \Omega,\ v\Big|_{\partial\Omega} = 0 \\
&\text{(recall that } W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,\xi_0}(\Omega) \text{ continuously and densely).}
\end{aligned} \tag{5.30}$$

If $\tilde{\eta} \not\equiv \hat{\lambda}_1$ (nonresonance), then as in the proof of Proposition 15, we have that $v = 0$, a contradiction to (5.27).

If $\tilde{\eta}(z) = \hat{\lambda}_1$ for a.a. $z \in \Omega$, then from (5.30) we see that we can say that

$$v = \hat{u}_1. \tag{5.31}$$

From (5.31) it follows that

$$|u_n(z)| \to +\infty \text{ for a.a. } z \in \Omega. \tag{5.32}$$

From (5.21) we have

$$-\rho_a(Du_n) - \frac{p}{q}\|Du_n\|_q^q + \int_\Omega pF(z,u_n)dz \leq pC_{23} \tag{5.33}$$

for all $n \in \mathbb{N}$.

On the other hand from (5.22), we have

$$\rho_a(Du_n) + \|Du_n\|_q^q - \int_\Omega f(z,u_n)u_n dz \leq \varepsilon_n \tag{5.34}$$

for all $n \in \mathbb{N}$, with $\varepsilon_n \to 0^+$.

Adding (5.33) and (5.34), we obtain

$$\begin{aligned} &\int_\Omega [pF(z,u_n) - f(z,u_n)u_n]\,dz \leq \left[\frac{p}{q} - 1\right]\|Du_n\|_q^q + C_{24} \\ &\text{for some } C_{24} > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow &\int_\Omega \frac{pF(z,u_n) - f(z,u_n)u_n}{\|u_n\|^\tau}dz \leq \left(\frac{p}{q} - 1\right)\frac{\|Dv_n\|_q^q}{\|u_n\|^{\tau-q}} + \frac{C_{24}}{\|u_n\|^\tau} \\ &\text{for all } n \in \mathbb{N}. \end{aligned}$$

The right-hand side goes to 0 as $n \to \infty$ (see H_0'), while on account of hypothesis $H_3(ii)$ and Fatou's lemma, we have

$$0 < \liminf_{n\to\infty} \int_\Omega \frac{pF(z,u_n) - f(z,u_n)u_n}{\|u_n\|^\tau}dz,$$

a contradiction.

Therefore we have that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\xi}(\Omega)$ is bounded and then continuing as in the proof of Proposition 15, we conclude that $\varphi(\cdot)$ satisfies the C-condition. $\square$

Proposition 20 *If hypotheses H_0', H_3 hold, then $\varphi(t\hat{u}_1) \to -\infty$ as $t \to \pm\infty$.*

Proof On account of hypothesis $H_3(ii)$, we can find $\hat{\beta}_0 \in (0, \hat{\beta})$ and $M_0 > 0$ such that

$$-\hat{\beta}_0|x|^\tau \leq pF(z,x) - f(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M_0. \tag{5.35}$$

We have

$$\frac{d}{dx}\left[\frac{F(z,x)}{x^p}\right] = \frac{f(z,x)x - pF(z,x)}{x^{p+1}}$$

$$
\begin{aligned}
&\leq -\beta_0 x^{\tau-p-1} \text{ for a.a. } z \in \Omega,\ \text{all } x \geq M_0,\\
\Rightarrow \frac{F(z,y)}{y^p} - \frac{F(z,x)}{x^p} &\leq \frac{\hat{\beta}_0}{p-\tau}\left[\frac{1}{y^{p-\tau}} - \frac{1}{x^{p-\tau}}\right]\\
&\quad \text{for a.a. } z \in \Omega,\ \text{all } y \geq x \geq M_0.
\end{aligned}
$$

Let $y \to +\infty$. Using hypothesis $H_3(ii)$, we have

$$
\begin{aligned}
&\frac{\hat{\lambda}_1 a(z)}{p} - \frac{F(z,x)}{x^p} \leq -\frac{\hat{\beta}}{p-\tau}\frac{1}{x^{p-\tau}},\\
&\Rightarrow \hat{\lambda}_1 a(z) x^p - pF(z,x) \leq -\hat{\beta}_0 \frac{p}{p-\tau}\\
&\text{for a.a. } z \in \Omega,\ \text{all } x \geq M_0.
\end{aligned} \tag{5.36}
$$

For $t > 0$, we have

$$
\begin{aligned}
\varphi(t\hat{u}_1) &= \frac{t^p}{p}\rho_a(D\hat{u}_1) + \frac{t^q}{q}\|D\hat{u}_1\|_q^q - \int_\Omega F(z,t\hat{u}_1)dz\\
&= \frac{t^p}{p}\hat{\lambda}_1\rho_a(\hat{u}_1) + \frac{t^q}{q}\|D\hat{u}_1\|_q^q - \int_\Omega F(z,t\hat{u}_1)dz\\
&\leq -\hat{\beta}_0 t^\tau \|\hat{u}_1\|_\tau^\tau + \frac{t^q}{q}\|D\hat{u}_1\|_q^q + C_{25}\\
&\quad \text{for some} C_{25} > 0 \text{(see (5.36))},\\
&\Rightarrow \varphi(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty \text{ (recall that } q < \tau).
\end{aligned}
$$

Similarly we show that

$$
\varphi(t\hat{u}_1) \to -\infty \text{ as } t \to -\infty.
$$

The proof is now complete. □

Now we proceed as in the nonresonant case. Namely, note that by hypothesis $H_3(ii)$ we can find $\hat{\eta} \in (\hat{\lambda}_1, \hat{\lambda}_2)$ and $C_{22} > 0$ such that

$$
F(z,x) \leq \frac{\hat{\eta}}{\rho} a(z)|x|^p + C_{22} \text{ for a.a. } z \in \Omega,\ \text{all} x \in \mathbb{R}.
$$

As before, let $S = \{u \in W_0^{1,\xi}(\Omega) : \ \rho_a(Du) = \hat{\lambda}_2\rho_a(u)\}$. Then for all $u \in S$ with $\|u\| \geq 1$, we have

$$
\begin{aligned}
&\varphi(u) \geq \frac{1}{p}\left(1 - \frac{\hat{\eta}}{\hat{\lambda}_2}\right)\rho_a(Du) - C_{22}|\Omega|_N,\\
&\Rightarrow \varphi(u) \geq C_{23}\|u\|^q - C_{22}|\Omega|_N \text{ for some } C_{23} > 0,\\
&\Rightarrow \ \varphi|_S \ \text{ is coercive.}
\end{aligned}
$$

Therefore we can find $\gamma > 0$ such that

$$\varphi|_S \geq -\gamma.$$

On the other hand, on account of Proposition 20 we can find $t_0 > 0$ such that

$$\varphi(t_0\hat{u}_1),\ \varphi(-t_0\hat{u}_1) < -\gamma.$$

Hence we see that

$$\varphi(\pm t_0\hat{u}_1) < -\gamma \leq \inf_S \varphi. \tag{5.37}$$

From Proposition 19 we know that

$$\varphi(\cdot) \text{ satisfies the } C\text{-condition.} \tag{5.38}$$

Let

$$D_0 = \{\pm t_0\hat{u}_1\},\ \ D = \{u \in W_0^{1,\xi}(\Omega) :\ u = s(-\hat{u}_1) + (1-s)\hat{u}_1,\ \ s \in [0,1]\}$$

and S as above. From Proposition 18 we know that

$$\{D_0, D\} \text{ is linking with} S \text{in } W_0^{1,\xi}(\Omega). \tag{5.39}$$

Finally, relations (5.37), (5.38), (5.39) and Theorem 5.4.4 of [21, p.399] produce $u_0 \in K_\varphi$ with $-\gamma \leq \varphi(u_0)$. Then $u_0 \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$ is a solution of problem (1.1) and since $C_1(\varphi, u_0) \neq 0$, while $C_k(\varphi, 0) = 0$ for all $k \in \mathbb{N}_0$, we conclude that $u_0 \neq 0$. So, we can state the following theorem.

Theorem 3 *If hypotheses H_0', H_3 hold, then problem* (1.1) *has a nontrivial solution*

$$u_0 \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega).$$

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Declarations

Conflict of interest The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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