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# Strongly Singular Double Phase Problems

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**Abstract.** We study double phase singular problems with strong singularity and unbounded coefficient (that is, in the singular term $u \mapsto \frac{g(z)}{u(z)^\eta}$, where $\eta \geqslant 1$ and $g(\cdot)$ is not bounded). First we deal with the purely singular problem. We consider two distinct cases. In the first one, we assume that $\eta = 1$ and the double phase operator ($(p, q)$-Laplacian with weight) exhibits unbalanced growth. Using modular spaces we prove the existence of a unique positive solution. The second case is when $\eta > 1$ and this is examined in the context of double phase problems with balanced growth. Again we prove the existence of a unique positive solution. Finally, for the second case, we introduce also a superlinear perturbation of the singular term and we prove an existence theorem.

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**Keywords.** Unbalanced growth, balanced growth, double phase energy, Musielak–Orlicz–Sobolev spaces, strongly singular problem, positive solution.

## 1. Introduction

In this paper, we are concerned with the study of a nonlinear problem whose features are the following:

(a) the presence of several differential operators with different growth, which generates a *double phase* associated energy;
(b) the reaction combines the multiple effects generated by a singular term and a nonlinearity with subcritical growth;
(c) we establish global existence properties, which describe an exhaustive picture of *strongly singular double phase* problems.

Summarizing, this paper is concerned with the refined qualitative analysis of solutions for a class of *singular* problems driven by differential operators with *unbalanced* or *balanced* growth.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The aim of this paper is to study strongly singular double phase problems. The

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name "strongly singular", refers to equations in which the singular part of the reaction $u \mapsto u^{-\eta}$ has exponent $\eta \geqslant 1$. Weakly singular problems (that is, the case $0 < \eta < 1$), were studied recently in a general framework by Papageorgiou et al. [23].

The problem under consideration is the following:

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \dfrac{g(z)}{u(z)^\eta} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 1 < q < p,\ 1 \leqslant \eta. \end{array} \right\} \tag{1}$$

In this problem, $\Delta_p^a$ denotes the weighted $p$-Laplacian defined by

$$\Delta_p^a u = \operatorname{div}\left(a(z)|Du|^{p-2}Du\right) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We denote by $\Delta_q$ the usual $q$-Laplace differential operator, namely

$$\Delta_q u = \operatorname{div}\left(|Du|^{q-2}Du\right) \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

In the reaction (right-hand side of (1)), the coefficient $g(\cdot)$ of the singular term is not bounded and the perturbation $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z,x)$ is continuous), which exhibits $(p-1)$-superlinear growth as $x \to +\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short).

First we will deal with the purely singular problem, that is, $f \equiv 0$ (no perturbation). For this problem, we consider two distinct cases: $(a)$ $\eta = 1$ and $(b)$ $\eta > 1$. For the first case ($\eta = 1$), we assume that the coefficient $a \in C^{0,1}(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$, but need not be bounded away from zero. In this way, the differential operator $u \mapsto -\Delta_p^a u - \Delta_q u$, exhibits "unbalanced growth". More precisely, if we consider the integrated $\xi : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ associated with the energy functional of this operator, then we have

$$\xi(z,x) = a(z)x^p + x^q \quad \text{for all } z \in \Omega,\ \text{all } x \geqslant 0,$$

where the modulating coefficient $a(z) \geqslant 0$ dictates the geometry of the composite made by two different materials, with hardening exponents $p$ and $q$, respectively.

This is a phase transition model and in the region where $a$ is positive, the $p$-material is present; otherwise the $q$-material is the only one making the composite. The anisotropic case corresponds to a composite that changes its hardening exponents according to the point; in this case, the exponents $p$ and $q$ are no longer constant and they change their values for $z \in \Omega$.

We observe that

$$x^q \leqslant \xi(z,x) \leqslant c_0(1 + x^p) \quad \text{for all } z \in \Omega,\ \text{all } x \geqslant 0,\ \text{some } c_0 > 0.$$

Integral functionals defined by such integrands, were first investigated by Marcellini [19] and Zhikov [30,31] in connection with problems from nonlinear elasticity theory. Recently, the interest for such problems was revived by Mingione and his co-workers, who produced important local regularity results for local minimizers (see Baroni et al. [3] and Colombo and Mingione [11,12]). However, there are not yet any global regularity results for such problems and as we will see, this prevents us from dealing with the cases $\eta > 1$ and $f \not\equiv 0$ in the context of unbalanced double phase equations. For recent developments in problems with nonstandard growth and nonuniform ellipticity we refer to Beck and Mingione [4] and Mingione and Rădulescu [20].

We mention that problems involving the sum of two differential operators of different nature (double phase problems), arise in many mathematical models of physical processes. We already mentioned the problems in elasticity theory (see [19,31]) and in addition we have the works of Bahrouni et al. [2] (transonic flow problems), Benci et al. [5] (quantum physics), Cherfils and Il'yasov [8] (reaction-diffusion systems).

Strongly singular problems, were first investigated by Lazer and McKenna [17], who considered semilinear equations driven by the Laplace operator. More precisely, they considered the following problem

$$-\Delta u(z) = u(z)^{-\eta} \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 0 < \eta.$$

They proved that this problem has a solution in $H_0^1(\Omega)$ if and only if $0 < \eta < 3$. Moreover, the solution is not in $C_0^1(\overline{\Omega})$ if $1 < \eta$. Extensions of their work were obtained by Boccardo and Orsina [7], Coclite [9], Lair and Shaker [16], and Pucci and Vitillaro [28]. More on semilinear singular problems and a rich bibliography, can be found in the book of Ghergu and Rădulescu [14]. For double phase problems there are only the recent works of Papageorgiou et al. [23] and Papageorgiou et al. [25]. Both papers deal with weakly singular problems (that is, $0 < \eta < 1$).

## 2. Mathematical Background

To treat the unbalanced growth case described in Sect. 1, we need to consider Musielak–Orlicz–Sobolev spaces. So, let $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be the Carathéodory function defined by

$$\xi(z, x) = a(z)x^p + x^q \text{ for all } z \in \Omega,\ x \geqslant 0$$

(recall that $a \in C^{0,1}(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$). Then $\xi(\cdot, \cdot)$ is a generalized N-function (see Musielak [21, p. 82]) and it satisfies

$$\xi(z, 2x) \leqslant 2^p \xi(z, x) \quad \text{for all } z \in \Omega,\ x \geqslant 0.$$

This is known as the $(\Delta_2)$-condition (see Musielak [21, p. 52]). Let $M(\Omega)$ be the space of all measurable functions $u : \Omega \mapsto \mathbb{R}$. As always, we identify two such functions which differ only on a Lebesgue-null set. The Musielak–Orlicz space $L^\xi(\Omega)$ is defined by

$$L^\xi(\Omega) = \left\{ u : u \in M(\Omega), \int_\Omega \xi(z, |u|)\mathrm{d}z < +\infty \right\}.$$

This space is equipped with the so-called "Luxemburg norm" defined by

$$\|u\|_\xi = \inf \left\{ \lambda > 0 : \int_\Omega \xi\left(z, \frac{|u|}{\lambda}\right) \mathrm{d}z \leqslant 1 \right\}.$$

Using $L^\xi(\Omega)$, we can define the Musielak–Orlicz–Sobolev space $W^{1,\xi}(\Omega)$ as follows

$$W^{1,\xi}(\Omega) = \left\{u : u \in L^\xi(\Omega),\, |Du| \in L^\xi(\Omega)\right\},$$

where $D$ denotes the gradient in the weak sense. We furnish this space with the norm

$$\|u\|_{1,\xi} = \|u\|_\xi + \|Du\|_\xi.$$

Also, we set $W_0^{1,\xi}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\xi}}$. All these spaces ($L^\xi(\Omega)$, $W^{1,\xi}(\Omega)$, $W_0^{1,\xi}(\Omega)$) are separable and uniformly convex (thus, also reflexive) Banach spaces. Moreover, if $\frac{p}{q} < 1 + \frac{1}{N}$, then from Colasuonno and Squassina [10, Proposition 2.18], we have that there exists $c_1 > 0$ such that

$$\|u\|_\xi \leqslant c_1\|Du\|_\xi \quad \text{for all } u \in W_0^{1,\xi}(\Omega).$$

This is the "Poincaré inequality" for the space $W_0^{1,\xi}(\Omega)$ and it implies that on $W_0^{1,\xi}(\Omega)$ we can use the equivalent norm

$$\|u\| = \|Du\|_\xi \quad \text{for all } u \in W_0^{1,\xi}(\Omega).$$

We have the following embeddings, which generalize the classical Sobolev embedding theorem (see [10,21]):

(a) $q \neq N \Rightarrow W_0^{1,\xi}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leqslant r \leqslant q^*$,

$$q^* = \begin{cases} \dfrac{Nq}{N-q} & \text{if } q < N \\ +\infty & \text{if } q \geqslant N \end{cases};$$

(b) $q = N \Rightarrow W_0^{1,\xi}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leqslant r < +\infty$;

(c) $q \leqslant N \Rightarrow W_0^{1,\xi}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leqslant r < q^*$;

(d) $q > N \Rightarrow W_0^{1,\xi}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly;

(e) $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$.

By $L_a^p(\Omega)$ we denote the weighted Lebesgue space defined by

$$L_a^p(\Omega) = \left\{u : u \in M(\Omega), \int_\Omega a(z)|u|^p \mathrm{d}z < +\infty\right\}.$$

The norm of this space is given by

$$\|u\|_{p,a} = \left(\int_\Omega a(z)|u|^p \mathrm{d}z\right)^{1/p}.$$

We have

$$L^p(\Omega) \hookrightarrow L^\xi(\Omega) \hookrightarrow L_a^p(\Omega) \cap L^q(\Omega).$$

In the study of the Musielak–Orlicz and of the Musielak–Orlicz–Sobolev spaces, the following modular function is important

$$\rho_\xi(v) = \int_\Omega \xi(z, |v|)\mathrm{d}z = \int_\Omega \left(a(z)|v|^p + |v|^q\right) \mathrm{d}z.$$

This modular function is closely related to the norms of these spaces.

**Proposition 1.** (a) *For $v \neq 0$, $\|v\|_\xi = \lambda$ if and only if $\rho_\xi\left(\frac{v}{\lambda}\right) = 1$.*

(b) $\|v\|_\xi < 1$ (*resp.* $= 1, > 1$) *if and only if* $\rho_\xi(v) < 1$ (*resp.* $= 1, > 1$).
(c) $\|v\|_\xi < 1 \Rightarrow \|v\|_\xi^p \leqslant \rho_\xi(v) \leqslant \|v\|_\xi^q$.
(d) $\|v\|_\xi > 1 \Rightarrow \|v\|_\xi^q \leqslant \rho_\xi(v) \leqslant \|v\|_\xi^p$.
(e) $\|v\|_\xi \to 0 \Leftrightarrow \rho_\xi(v) \to 0$.
(f) $\|v\|_\xi \to +\infty \Leftrightarrow \rho_\xi(v) \to +\infty$.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W_0^{1,\xi}(\Omega)^*, W_0^{1,\xi}(\Omega))$ and let $A_p^a : W_0^{1,\xi}(\Omega) \mapsto W_0^{1,\xi}(\Omega)^*$ be the nonlinear map defined by

$$\langle A_p^a(u), h\rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \quad \text{for all } u, h \in W_0^{1,\xi}(\Omega).$$

Also let $\langle \cdot, \cdot \rangle_{1,q}$ denote the duality brackets for the pair $(W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^*, W_0^{1,q}(\Omega))$ $\left(\frac{1}{q} + \frac{1}{q'} = 1\right)$ and let $A_q : W_0^{1,q}(\Omega) \mapsto W^{-1,q'}(\Omega)$ be the nonlinear map defined by

$$\langle A_q(u), h\rangle_{1,q} = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \quad \text{for all } u, h \in W_0^{1,q}(\Omega).$$

We know that

$$\langle A_q(u), h\rangle_{1,q} = \langle A_q(u), h\rangle \text{ for all } u, h \in W_0^{1,\xi}(\Omega)$$

(recall that $W_0^{1,\xi}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$). Both maps are bounded (that is, they map bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$. Recall that if $X$ is a reflexive Banach space and $A : X \mapsto X^*$, then we say that $A(\cdot)$ is of type $(S)_+$ if it has the following property:

$$\text{“}x_n \xrightarrow{w} x \text{ in } X \text{ and } \limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle_X \leqslant 0$$
$$\Downarrow$$
$$x_n \to x \text{ in } X \text{ as } n \to \infty\text{''}$$

( see Definition 3.2.55(b) of Gasinski and Papageorgiou [13, p. 338] ).

Suppose $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the “C-condition” if it has the following property:

“Every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(x_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|x_n\|_X)\varphi'(x_n) \to 0$ in $X^*$ as $n \to \infty$, admits a strongly convergent subsequence”.

This is a compactness type condition on the functional $\varphi(\cdot)$. Since the ambient space $X$ in most cases of interest is infinite dimensional, it is not locally compact and so the burden of compactness is passed on the functional $\varphi(\cdot)$, as it happens in the Leray–Schauder degree theory.

Given $u \in M(\Omega)$, we set $u^\pm = \max\{\pm u, 0\}$. We know that if $u \in W_0^{1,\xi}(\Omega)$ then

$$u^\pm \in W_0^{1,\xi}(\Omega),\ u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

We will also use the Banach space $C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}$. This is an ordered Banach space with positive cone $C_+ = \left\{u \in C_0^1(\overline{\Omega}) :\right.$

$u(z) \geqslant 0$ for all $z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0 \right\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Since $\partial\Omega$ is by hypothesis Lipschitz, by Rademacher's theorem, $n(\cdot)$ exists at almost all points of $\partial\Omega$.

Given $u$, $v \in M(\Omega)$ with $u \leqslant v$, we define

$$[u, v] = \left\{ h \in W_0^{1,\xi}(\Omega) : u(z) \leqslant h(z) \leqslant v(z) \text{ for a.a. } z \in \Omega \right\}.$$

Also by $\hat{u}_1^*(p) \in W_0^{1,p}(\Omega)$, we denote the positive, $L^p$-normalized (that is, $\|\hat{u}_1^*(p)\|_p = 1$) eigenfunction corresponding to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. We know that $\hat{u}_1^*(p) \in \operatorname{int} C_+$ (see Gasinski and Papageorgiou [13, p. 739]).

Finally if $p < N$, then $p^* = \frac{Np}{N-p}$ (the critical Sobolev exponent corresponding to $p$) and if $r \in (1, +\infty)$, then $r' \in (1, +\infty)$ is the conjugate exponent corresponding to $r$, that is, $\frac{1}{r} + \frac{1}{r'} = 1$.

## 3. Purely Singular Problem: Case $\eta = 1$

In this section we deal with the purely singular problem, with $\eta = 1$. So, the problem under consideration is

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \dfrac{g(z)}{u(z)} \ \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 1 < q < p. \end{array} \right\} \tag{2}$$

Our hypotheses on the data of this problem are the following.

$H$: $a \in C^{0,1}(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$, $\frac{p}{q} < 1 + \frac{1}{N}$, $g \in L^1(\Omega)$, $g(z) \geqslant 0$ for a.a. $z \in \Omega$ and $g \not\equiv 0$.

*Remark* 1. We stress that the coefficient $g(\cdot)$ is not bounded. Since the coefficient $g(\cdot)$ is unbounded, following Boccardo and Orsina [7], we understand by a positive solution of problem (2) a function $u \in W_0^{1,\xi}(\Omega)$ such that

$$\text{for all } K \subseteq \Omega \text{ compact, } 0 < c_K \leqslant u(z) \text{ for a.a. } z \in K,$$
$$\langle A_p^a(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega \frac{g(z)}{u(z)} h \mathrm{d}z \quad \text{for all } h \in C_c^1(\Omega).$$

Evidently, since $h(\cdot)$ has compact support, the right-hand side of the last equation is well-defined.

For $\varepsilon > 0$, we define $g_\varepsilon = \min\left\{g, \frac{1}{\varepsilon}\right\}$. Evidently $g_\varepsilon \in L^\infty(\Omega)$. Then given $\beta \in L^q(\Omega)$, we consider the following auxiliary Dirichlet problem

$$-\Delta_p^a u(z) - \Delta_q u(z) = \frac{g_\varepsilon(z)}{|\beta(z)| + \varepsilon} \ \text{ in } \Omega,\ u|_{\partial\Omega} = 0,\ u \geqslant 0. \tag{3}$$

**Proposition 2.** *If hypotheses $H$ hold, then problem (3) admits a unique solution $\overline{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$, $\overline{u}_\varepsilon \geqslant 0$, $\overline{u}_\varepsilon \neq 0$.*

*Proof.* We consider the operator $V : W_0^{1,\xi}(\Omega) \mapsto W_0^{1,\xi}(\Omega)^*$ defined by

$$V(u) = A_p^a(u) + A_q(u) \quad \text{for all } u \in W_0^{1,\xi}(\Omega).$$

This operator is continuous, strictly monotone (hence maximal monotone, too) and coercive. It follows that it is surjective (see Corollary 2.8.7 of Papageorgiou et al. [22, p. 135]). Since $\frac{g_\varepsilon(\cdot)}{|\beta(\cdot)|+\varepsilon} \in L^\infty(\Omega) \hookrightarrow W_0^{1,\xi}(\Omega)^*$, we can find $\overline{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$ such that

$$V(\overline{u}_\varepsilon) = \frac{g_\varepsilon(\cdot)}{|\beta(\cdot)| + \varepsilon}.$$

The strict monotonicity of $V(\cdot)$ implies that this solution is unique. Finally, acting with $-\overline{u}_\varepsilon^- \in W_0^{1,\xi}(\Omega)$ we obtain

$$\begin{aligned} &\rho_\varepsilon(D\overline{u}_\varepsilon^-) = 0, \\ &\quad \Rightarrow \overline{u}_\varepsilon \geqslant 0,\ \overline{u}_\varepsilon \neq 0 \text{ (see Proposition 1).} \end{aligned}$$

This proof is now complete. □

So, we can define the solution map $\sigma_\varepsilon : L^q(\Omega) \mapsto L^q(\Omega)$ for problem (3) by

$$\sigma_\varepsilon(\beta) = \overline{u}_\varepsilon \text{ (recall that } W_0^{1,\xi}(\Omega) \hookrightarrow L^q(\Omega)).$$

Clearly this map is continuous.

Next, we consider the following truncation-perturbation of problem (2):

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \dfrac{g_\varepsilon(z)}{u(z) + \varepsilon} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 1 < q < p. \end{array} \right\} \tag{4}$$

**Proposition 3.** *If hypotheses $H$ hold, then problem (4) admits a unique solution $\hat{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$.*

*Proof.* Note that the fixed points of $\sigma_\varepsilon(\cdot)$ are solutions of problem (4).

We know that $\sigma_\varepsilon(\cdot)$ is continuous. Also, if $\overline{u}_\varepsilon = \sigma_\varepsilon(\beta)$, then we have

$$\langle A_p^a(\overline{u}_\varepsilon), h\rangle + \langle A_q(\overline{u}_\varepsilon), h\rangle = \int_\Omega \frac{g_\varepsilon h}{|\beta| + \varepsilon} \mathrm{d}z \quad \text{for all } h \in W_0^{1,\xi}(\Omega). \tag{5}$$

In (5) we choose $h = \overline{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$. Then

$$\begin{aligned} &\rho_\varepsilon(D\overline{u}_\varepsilon) = \int_\Omega \frac{g_\varepsilon \overline{u}_\varepsilon}{|\beta| + \varepsilon} \mathrm{d}z \leqslant \frac{c_2}{\varepsilon^2} \|\overline{u}_\varepsilon\| \text{ for some } c_2 > 0, \\ &\quad \Rightarrow \sigma_\varepsilon(L^q(\Omega)) \subseteq W_0^{1,\xi}(\Omega) \text{ is a bounded set (see Proposition 1).} \end{aligned}$$

Recall that $W_0^{1,\xi}(\Omega) \hookrightarrow L^q(\Omega)$ compactly. So, we can use the Schauder-Tychonov fixed point theorem (see Theorem 4.3.21 of Papageorgiou et al. [22, p. 298]) and produce $\hat{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$ such that

$$\sigma_\varepsilon(\hat{u}_\varepsilon) = \hat{u}_\varepsilon.$$

Evidently $\hat{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$ is a solution of problem (4) and $\hat{u}_\varepsilon \neq 0$.

We show that this solution is unique. Suppose that $\hat{v}_\varepsilon \in W_0^{1,\xi}(\Omega)$ is another solution of problem (4). We have

$$\begin{aligned} 0 &\leqslant \langle A_p^a(\hat{u}_\varepsilon) - A_p^a(\hat{v}_\varepsilon), \hat{u}_\varepsilon - \hat{v}_\varepsilon\rangle + \langle A_q(\hat{u}_\varepsilon) - A_q(\hat{v}_\varepsilon), \hat{u}_\varepsilon - \hat{v}_\varepsilon\rangle \\ &= \int_\Omega g_\varepsilon(z)\left(\frac{1}{\hat{u}_\varepsilon+\varepsilon} - \frac{1}{\hat{v}_\varepsilon+\varepsilon}\right)(\hat{u}_\varepsilon - \hat{v}_\varepsilon)\mathrm{d}z \leqslant 0, \\ \Rightarrow \ \hat{u}_\varepsilon &= \hat{v}_\varepsilon \text{ (on account of the strict monotonicity of } u \mapsto V(u) \\ &= A_p^a(u) + A_q(u)). \end{aligned}$$

This proof is now complete. □

From the maximum principle for unbalanced double phase problems of Papageorgiou et al. [26, Proposition 2.4]), we know that for every $K \subseteq \Omega$ compact subset, we have

$$0 < c_K \leqslant \hat{u}_\varepsilon(z) \text{ for a.a. } z \in K. \tag{6}$$

**Proposition 4.** *If hypotheses $H$ hold, then $\{\hat{u}_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W_0^{1,\xi}(\Omega)$ is nonincreasing.*

*Proof.* Suppose that $0 < \varepsilon' < \varepsilon \leqslant 1$. We have

$$-\Delta_p^a \hat{u}_{\varepsilon'} - \Delta_q \hat{u}_{\varepsilon'} = \frac{g_{\varepsilon'}}{\hat{u}_{\varepsilon'}+\varepsilon'} \geqslant \frac{g_\varepsilon}{\hat{u}_{\varepsilon'}+\varepsilon} \quad \text{in } \Omega. \tag{7}$$

We introduce the Carathéodory function $k_\varepsilon : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$k_\varepsilon(z,x) = \begin{cases} \frac{g_\varepsilon(z)}{x^+ + \varepsilon} & \text{if } x \leqslant \hat{u}_{\varepsilon'}(z), \\ \frac{g_\varepsilon(z)}{\hat{u}_{\varepsilon'}(z)+\varepsilon} & \text{if } \hat{u}_{\varepsilon'}(z) < x \end{cases} \tag{8}$$

(recall that $x^+ = \max\{x, 0\}$). We set $K_\varepsilon(z,x) = \int_0^x k_\varepsilon(z,s)ds$ and consider the $C^1$-functional $\Psi_\varepsilon : W_0^{1,\xi}(\Omega) \mapsto \mathbb{R}$ defined by

$$\Psi_\varepsilon(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p\mathrm{d}z + \frac{1}{q}\|Du\|_q^q - \int_\Omega K_\varepsilon(z,u)\mathrm{d}z \quad \text{for all } u \in W_0^{1,\xi}(\Omega).$$

Note that

$$\begin{aligned} &\Psi_\varepsilon(u) \geqslant \frac{1}{p}\rho_\varepsilon(Du) - c_3 \text{ for some } c_3 > 0, \text{ all } u \in W_0^{1,\xi}(\Omega) \\ &\quad \text{(see (8) and recall that } q < p), \\ &\quad \Rightarrow \Psi_\varepsilon(\cdot) \text{ is a coercive (see Proposition 1).} \end{aligned}$$

Also, $\Psi_\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass–Tonelli theorem, we can find $\widetilde{u}_\varepsilon \in W_0^{1,\xi}(\Omega)$ such that

$$\Psi_\varepsilon(\widetilde{u}_\varepsilon) = \min\left\{\Psi_\varepsilon(u) : u \in W_0^{1,\xi}(\Omega)\right\}. \tag{9}$$

Let $u \in \operatorname{int} C_+$. On account of (8) we see that for $t \in (0,1)$ small, we have

$$\begin{aligned} &\Psi_\varepsilon(tu) < 0, \\ &\quad \Rightarrow \Psi_\varepsilon(\widetilde{u}_\varepsilon) < 0 = \Psi_\varepsilon(0) \text{ (see (9))}, \\ &\quad \Rightarrow \widetilde{u}_\varepsilon \neq 0. \end{aligned}$$

From (9) we have

$$\Psi'_\varepsilon(\widetilde{u}_\varepsilon)=0,$$
$$\Rightarrow \langle A_p^a(\widetilde{u}_\varepsilon),h\rangle+\langle A_q(\widetilde{u}_\varepsilon),h\rangle=\int_\Omega k_\varepsilon(z,\widetilde{u}_\varepsilon)h\mathrm{d}z \quad \text{for all } h\in W_0^{1,\xi}(\Omega). \tag{10}$$

First we test (10) with $h=-\widetilde{u}_\varepsilon^-\in W_0^{1,\xi}(\Omega)$. Then

$$\rho_\varepsilon(D\widetilde{u}_\varepsilon^-)\leqslant 0 \text{ (see (8))},$$
$$\Rightarrow \widetilde{u}_\varepsilon\geqslant 0,\ \widetilde{u}_\varepsilon\neq 0 \text{ (see Proposition 1)}.$$

Next, we test (10) with $h=(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\in W_0^{1,\xi}(\Omega)$. We have

$$\begin{aligned}&\langle A_p^a(\widetilde{u}_\varepsilon),(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\rangle+\langle A_q(\widetilde{u}_\varepsilon),(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\rangle\\ &=\int_\Omega \frac{g_\varepsilon}{\hat{u}_\varepsilon+\varepsilon}(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\mathrm{d}z \text{ (see (8))}\\ &\leqslant \langle A_p^a(\hat{u}_{\varepsilon'}),(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\rangle+\langle A_q(\hat{u}_{\varepsilon'}),(\widetilde{u}_\varepsilon-\hat{u}_{\varepsilon'})^+\rangle \text{ (see (7))},\\ &\Rightarrow\ \widetilde{u}_\varepsilon\leqslant\hat{u}_{\varepsilon'}.\end{aligned}$$

So, we have proved that

$$\widetilde{u}_\varepsilon\in[0,\hat{u}_{\varepsilon'}],\ \widetilde{u}_\varepsilon\neq 0. \tag{11}$$

From (11), (8), (10) and Proposition 3, we infer that

$$\widetilde{u}_\varepsilon=\hat{u}_\varepsilon,$$
$$\Rightarrow \hat{u}_\varepsilon\leqslant\hat{u}_{\varepsilon'} \text{ (see (11))}.$$

This proof is now complete. □

Now we are ready to produce a positive solution for problem (2) (purely singular, double phase unbalanced growth case).

**Theorem 5.** *If hypotheses $H$ hold, then problem (2) admits a unique positive solution $u_0\in W_0^{1,\xi}(\Omega)$.*

*Proof.* Let $\varepsilon_n=\frac{1}{n}$ and $\hat{u}_n=\hat{u}_{\varepsilon_n}\in W_0^{1,\xi}(\Omega)$ be the unique positive solution of problem (3) (with $\varepsilon=\frac{1}{n}$), see Proposition 3. We have

$$\langle A_p^a(\hat{u}_n),h\rangle+\langle A_q(\hat{u}_n),h\rangle=\int_\Omega\frac{g_n}{\hat{u}_n+\frac{1}{n}}h\mathrm{d}z \tag{12}$$

for every $h\in W_0^{1,\xi}(\Omega)$, all $n\in\mathbb{N}$ (recall that $g_n=g_{\varepsilon_n}=\min\{g,n\}\in L^\infty(\Omega)$, $g_n\geqslant 0$, $g_n\neq 0$ for all $n\in\mathbb{N}$). In (12) we choose $h=\hat{u}_n\in W_0^{1,\xi}(\Omega)$ and obtain

$$\rho_\varepsilon(D\hat{u}_n)=\int_\Omega\frac{g_n\hat{u}_n}{\hat{u}_n+\frac{1}{n}}\mathrm{d}z\leqslant\int_\Omega g_n\mathrm{d}z\leqslant\|g\|_1,$$
$$\Rightarrow \{\hat{u}_n\}_{n\in\mathbb{N}}\subseteq W_0^{1,\xi}(\Omega) \text{ is bounded}.$$

So, by passing to suitable subsequence if necessary, we may assume that

$$\left\{\begin{array}{l}\hat{u}_n\xrightarrow{w}\hat{u} \text{ in } W_0^{1,\xi}(\Omega),\ \hat{u}_n\to\hat{u} \text{ in } L^p(\Omega) \text{ (by hypotheses } H,\ p<q^*),\\ \hat{u}_n(z)\to\hat{u}(z) \text{ for a.a. } z\in\Omega,\\ 0\leqslant\hat{u}_n(z)\leqslant\gamma(z) \text{ for a.a. } z\in\Omega,\ \text{all } n\in\mathbb{N} \text{ and with } \gamma\in L^p(\Omega).\end{array}\right\} \tag{13}$$

Consider a test function $h \in C_c^1(\Omega)$ and let $K = \operatorname{supp} h \subseteq \Omega$. Using Proposition 4 and (6), we have

$$0 < c_K \leqslant \hat{u}_1(z) \leqslant \hat{u}_n(z) \text{ for a.a. } z \in K, \text{ all } n \in \mathbb{N}. \tag{14}$$

We have

$$0 \leqslant \frac{g_n(z)\hat{u}_n(z)|h(z)|}{\hat{u}_n(z) + \frac{1}{n}} \leqslant g_n(z)|h(z)| \leqslant g(z)\|h\|_\infty$$

for a.a. $z \in \Omega$, all $n \in \mathbb{N}$.

We see that $g_n(z) \to g(z)$ for a.a. $z \in \Omega$. So, we have

$$\frac{g_n(z)\hat{u}_n(z)h(z)}{\hat{u}_n(z) + \frac{1}{n}} \to \frac{g(z)\hat{u}(z)h(z)}{\hat{u}(z)} \text{ for a.a. } z \in \Omega \text{ (see (13))}.$$

Then using the Lebesgue dominated convergence theorem, we obtain

$$\frac{g_n\hat{u}_n h}{\hat{u}_n + \frac{1}{n}} \to \frac{g\hat{u}h}{\hat{u}} \text{ in } L^1(\Omega) \text{ as } n \to \infty. \tag{15}$$

Moreover, from (13) and Theorem 2.1 of Boccardo and Murat [6], we know that we may assume that

$$D\hat{u}_n(z) \to D\hat{u}(z) \text{ in } \mathbb{R}^N \text{ for a.a. } z \in \Omega.$$

It follows that

$$a(z)|D\hat{u}_n|^{p-2}(D\hat{u}_n, Dh)_{\mathbb{R}^N} \to a(z)|D\hat{u}|^{p-2}(D\hat{u}, Dh)_{\mathbb{R}^N} \text{ for a.a. } z \in \Omega. \tag{16}$$

From (13) we see that

$$\begin{aligned} &\left\{|D\hat{u}_n|^{p-2}(D\hat{u}_n, Dh)_{\mathbb{R}^N}\right\}_{n\in\mathbb{N}} \subseteq L_a^p(\Omega) \text{ is bounded (recall that } h \in C_c^1(\Omega)),\\ &\Rightarrow \left\{a(\cdot)|D\hat{u}_n|^{p-2}(D\hat{u}_n, Dh)_{\mathbb{R}^N}\right\}_{n\in\mathbb{N}} \text{ is uniformly integrable.} \end{aligned} \tag{17}$$

From (16), (17) and Vitali's theorem (see, for example, Theorem 2.3.44 of Papageorgiou and Winkert [27, p. 124]), we have that

$$\int_\Omega a(z)|D\hat{u}_n|^{p-2}(D\hat{u}_n, Dh)_{\mathbb{R}^N}\mathrm{d}z \to \int_\Omega a(z)|D\hat{u}|^{p-2}(D\hat{u}, Dh)_{\mathbb{R}^N}\mathrm{d}z. \tag{18}$$

In a similar fashion, we show that

$$\int_\Omega |D\hat{u}_n|^{q-2}(D\hat{u}_n, Dh)_{\mathbb{R}^N}\mathrm{d}z \to \int_\Omega |D\hat{u}|^{q-2}(D\hat{u}, Dh)_{\mathbb{R}^N}\mathrm{d}z. \tag{19}$$

We return to (12), pass to the limit as $n \to \infty$ and use (15), (18) and (19). We obtain

$$\begin{aligned} \langle A_p^a(\hat{u}), h\rangle + \langle A_q(\hat{u}), h\rangle &= \int_\Omega \frac{gh}{\hat{u}}\mathrm{d}z \quad \text{ for all } h \in C_c^1(\Omega),\\ &\Rightarrow \overline{u}_1 \leqslant \hat{u}. \end{aligned}$$

Therefore $\hat{u} \in W_0^{1,p}(\Omega)$ is a positive solution of problem (2) and this solution is unique on account of the strict monotonicity of $V(\cdot)$.

This proof is now complete. □

## 4. Purely Singular Problem: Case $\eta > 1$

In this section, we deal with the purely singular problem (that is, $f \equiv 0$) and when the exponent satisfies $\eta > 1$. We will do this in the context of double phase problems with balanced growth. It is an open problem whether we can have such an existence result for the unbalanced growth case. As it will be evident from the proof, the lack of a global regularity theory, prevents us from extending the result to unbalanced growth equations.

So, now the problem under consideration is the following:

$$\left\{ \begin{array}{ll} -\Delta_p^a u(z) - \Delta_q u(z) = \dfrac{g(z)}{u(z)^\eta} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 1 < q < p,\ 1 < \eta. & \end{array} \right\} \tag{20}$$

Now we assume that $\partial\Omega$ is of class $C^2$ and the following conditions hold:

$\hat{H}$: $a \in C^{0,1}(\overline{\Omega})$ with $a(z) \geqslant \hat{c} > 0$ for all $z \in \overline{\Omega}$, $g \in L^\theta(\Omega)$, $(p^*)' < \theta$, $1 < \eta < 2 - \frac{1}{\theta}$, $g(z) \geqslant 0$ for a.a. $z \in \Omega$, $g \not\equiv 0$ and $1 < q < p < N$.

The weight $a(\cdot)$ is bounded away from zero and so the differential operator of problem (20) exhibits balanced growth. Also we have restricted further the coefficient $g(\cdot)$. So, in the present setting by a positive solution of problem (20), we understand a function $u \in W_0^{1,p}(\Omega)$ such that

$$\text{for all } K \subseteq \Omega \text{ compact},\ 0 < c_K \leqslant u(z) \text{ for a.a. } z \in K,$$
$$\langle A_p^a(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega \frac{g(z)}{u(z)^\eta} h \mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

The conditions on $g(\cdot)$ imply that the right-hand side of this equation is well defined. Note that now on account of the balanced growth of $\xi(z,\cdot)$, problem (20) will be studied on the classical Sobolev space $W_0^{1,p}(\Omega)$.

**Theorem 6.** *If hypotheses $\hat{H}$ hold, then problem (20) admits a unique positive solution $\hat{u} \in W_0^{1,p}(\Omega)$.*

*Proof.* As before (see Sect. 3), we generate a nondecreasing sequence $\{\hat{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ consisting of solutions of problem (4) with $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. In the present case on account of the hypothesis on the weight $a(\cdot)$ (see hypotheses $\hat{H}$), using the nonlinear regularity theory of Lieberman [18] and the nonlinear maximum principle of Papageorgiou et al. [26] (see also Papageorgiou et al. [24], Proposition $A2$ and Zhang [29, Theorem 1.2]), we have that $\hat{u}_n \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$.

Consider the Banach space $C_0(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}) : \frac{u}{\hat{d}} \in C(\overline{\Omega}) \right\}$, where $\hat{d}(z) = \mathrm{d}\,(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. This is an ordered Banach space with positive cone

$$K_+ = \left\{ u \in C_0(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \overline{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \left\{ u \in C_0(\overline{\Omega}) : c_u \hat{d} \leqslant u \text{ for some } c_u > 0 \right\}.$$

By Lemma 14.16 of Gilbarg and Trudinger [15, p. 355], there exists $\delta_0 > 0$ such that $\hat{d} \in C^2(\Omega_{\delta_0})$ with $\Omega_{\delta_0} = \left\{z \in \overline{\Omega} : \hat{d}(z) < \delta_0\right\}$. It follows that $\hat{d} \in \operatorname{int} C_+$ and so by Proposition 4.1.22 of Papageorgiou et al. [22, p. 274], we can find $0 < c_3 < c_4$ such that

$$\begin{aligned} & c_3\hat{d} \leqslant \hat{u}_1 \leqslant c_4\hat{d} \text{ (recall that } \hat{u}_1 \in \operatorname{int} C_+), \\ & \quad \Rightarrow (c_3\hat{d})^{(\eta-1)\theta'} \leqslant \hat{u}_1^{(\eta-1)\theta'} \quad \left(\frac{1}{\theta} + \frac{1}{\theta'} = 1\right). \end{aligned} \tag{21}$$

Hypotheses $\hat{H}$ imply that $(\eta - 1)\theta' < 1$ and so it follows that $\hat{d}^{(\eta-1)\theta'} \in \operatorname{int} K_+$. Using again Proposition 4.1.22 of Papageorgiou et al. [22, p. 274], we can find $c_5 > 0$ such that

$$\begin{aligned} & \hat{u}_1^*(p)^{(\eta-1)\theta'} \leqslant c_5\hat{d}^{(\eta-1)\theta'}, \\ & \quad \Rightarrow \hat{d}^{(1-\eta)\theta'} \leqslant c_6\hat{u}_1^*(p)^{(1-\eta)\theta'} \text{ for some } c_6 > 0. \end{aligned} \tag{22}$$

Since $(\eta - 1)\theta' < 1$, using the Lemma in Lazer and McKenna [17], we have

$$\begin{aligned} & \hat{u}_1^*(p)^{(1-\eta)\theta'} \in L^1(\Omega), \\ & \quad \Rightarrow \hat{d}^{(1-\eta)\theta'} \in L^1(\Omega) \text{ (see (22))}, \\ & \quad \Rightarrow \hat{u}_1^{(1-\eta)\theta'} \in L^1(\Omega) \text{ (see (21))}. \end{aligned} \tag{23}$$

We have that

$$\langle A_p^a(\hat{u}_n), h\rangle + \langle A_q(\hat{u}_n), h\rangle = \int_\Omega \frac{g_n h}{\left(\hat{u}_n + \frac{1}{n}\right)^\eta}\mathrm{d}z \tag{24}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (24) we choose $h = \hat{u}_n \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|D\hat{u}_n\|_p^p \leqslant \int_\Omega \frac{g_n\hat{u}_n}{\left(\hat{u}_n + \frac{1}{n}\right)^\eta}\mathrm{d}z \leqslant \int_\Omega \frac{g_n}{\hat{u}_n^{\eta-1}}\mathrm{d}z \leqslant \int_\Omega g\hat{u}_1^{1-\eta}\mathrm{d}z$$

(recall that $\hat{u}_1 \leqslant \hat{u}_n$ for all $n \in \mathbb{N}$).

From (23) we see that $\hat{u}_1^{1-\eta} \in L^{\theta'}(\Omega)$. Hence using Hölder's inequality, we have

$$\begin{aligned} & \hat{c}\|D\hat{u}_n\|_p^p \leqslant \|g\|_\theta\|\hat{u}_n^{1-\eta}\|_{\theta'} \quad \text{for all } n \in \mathbb{N}, \\ & \quad \Rightarrow \{\hat{u}\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

So, we may assume that

$$\hat{u}_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,p}(\Omega) \text{ and } \hat{u}_n \to \hat{u} \text{ in } L^p(\Omega). \tag{25}$$

In (24) we choose $h = \hat{u}_n - \hat{u} \in W_0^{1,p}(\Omega)$. Note that

$$0 \leqslant \frac{g_n(\hat{u}_n - \hat{u})}{\left(\hat{u}_n + \frac{1}{n}\right)^\eta} \leqslant \frac{g_n}{\hat{u}_n^{\eta-1}} \leqslant g\hat{u}_1^{1-\eta} \in L^1(\Omega).$$

So, by the dominated convergence theorem, we have

$$\int_\Omega \frac{g_n(\hat{u}_n-\hat{u})}{\left(\hat{u}_n+\frac{1}{n}\right)^\eta}\mathrm{d}z \to 0 \text{ (see (25))},$$
$$\Rightarrow \lim_{n\to\infty}\left[\langle A_p^a(\hat{u}_n),\hat{u}_n-\hat{u}\rangle + \langle A_q(\hat{u}_n),\hat{u}_n-\hat{u}\rangle\right]=0,$$
$$\Rightarrow \limsup_{n\to\infty}\left[\langle A_p^a(\hat{u}_n),\hat{u}_n-\hat{u}\rangle + \langle A_q(\hat{u}),\hat{u}_n-\hat{u}\rangle\right]\leqslant 0$$
$$\text{(since } A_q(\cdot) \text{ is monotone)},$$
$$\Rightarrow \limsup_{n\to\infty}\langle A_p^a(\hat{u}_n),\hat{u}_n-\hat{u}\rangle\leqslant 0 \text{ (see (25))},$$
$$\Rightarrow \hat{u}_n\to\hat{u} \text{ in } W_0^{1,p}(\Omega) \text{ (since } A_p^a(\cdot) \text{ is of type } (S)_+). \tag{26}$$

On account of the hypothesis on $\theta$ (see hypotheses $\hat{H}$), we can find $s>1$ big such that $\frac{1}{\theta}+\frac{1}{p^*}+\frac{1}{s}<1$. Since $\hat{u}_1\in \operatorname{int} C_+$, we can find $c_7>0$ such that $\hat{u}_1^*(p)^{\frac{1}{s\eta^2}}\leqslant c_7\hat{u}_1$, hence $\hat{u}_1^{-\eta}\leqslant c_8\hat{u}_1^*(p)^{-\frac{1}{s\eta}}$. But since $\eta>1$, $\hat{u}_1^*(p)^{-\frac{1}{s\eta}}\in L^s(\Omega)$, hence $\hat{u}_1^{-\eta}\in L^s(\Omega)$. Note that by the generalized Hölder inequality (see Proposition 2.3.16 of Papageorgiou and Winkert [27, p. 115]), we have

$$0\leqslant \frac{g_n|h|}{\left(\hat{u}_n+\frac{1}{n}\right)^\eta}\leqslant \frac{g|h|}{\hat{u}_1^\eta}\in L^1(\Omega) \quad \text{for all } h\in W_0^{1,p}(\Omega).$$

Therefore passing to the limit as $n\to\infty$ in (24) and using the dominated convergence theorem, we obtain

$$\langle A_p^a(\hat{u}),h\rangle+\langle A_q(\hat{u}),h\rangle=\int_\Omega \frac{gh}{\hat{u}^\eta}\mathrm{d}z \text{ for all } h\in W_0^{1,p}(\Omega),$$
$$\Rightarrow \hat{u}_1\leqslant\hat{u}.$$

It follows that $\hat{u}\in W_0^{1,p}(\Omega)$ is a positive solution of problem (20). The strict monotonicity of $V(\cdot)$ implies the uniqueness of this positive solution.

This proof is now complete. □

## 5. Superlinear Perturbation

In this section we deal with the following parametric and perturbed version of the purely singular problem:

$$\left\{\begin{array}{l} -\Delta_p^a u(z)-\Delta_q u(z)=\dfrac{\lambda g(z)}{u(z)^\eta}+f(z,u(z)) \text{ in } \Omega,\\ u|_{\partial\Omega}=0,\ u\geqslant 0,\ 1<q<p,\ 1<\eta,\ \lambda>0. \end{array}\right\} \tag{27}$$

In this problem, $\lambda>0$ is the parameter and $f(z,x)$ is the perturbation of the singularity. We impose the following conditions on $f(z,x)$.

$H_0$: $f:\Omega\times\mathbb{R}\mapsto\mathbb{R}$ is Carathéodory function, $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) $0\leqslant f(z,x)\leqslant a(z)\left(1+x^{r-1}\right)$ for a.a. $z\in\Omega$, all $x\geqslant 0$, with $a\in L^\infty(\Omega)$, $p<r<p^*$;

(ii) if $F(z,x)=\int_0^x f(z,s)ds$, then $\lim_{x\to+\infty}\frac{F(z,x)}{x^p}=+\infty$ uniformly for a.a. $z\in\Omega$;

(iii) there exists $\tau \in \left((r-p)\frac{N}{p}, p^*\right)$ such that

$$0 < \hat{\beta}_0 \leqslant \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} = 0$ uniformly for a.a. $z \in \Omega$.

*Remark* 2. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we may assume that

$$f(z,x) = 0 \text{ for a.a. } z \in \Omega, \text{ all } x \leqslant 0. \tag{28}$$

Hypotheses $H_0$(ii), (iii) imply that $f(z,\cdot)$ is $(p-1)$-superlinear, that is, we have

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

However, $f(z,x)$ need not satisfy the AR-condition (see [1]), which is common in the literature, when we study superlinear problems. For example, consider the function $f(x) = x^{p-1}\ln(1+x)$ for all $x \geqslant 0$ (for the sake of simplicity of have dropped the $z$-dependence). This function satisfies hypotheses $H_0$ but fails to satisfy the AR-condition.

As before, by a positive solution of problem (27) we understand a function $u \in W_0^{1,p}(\Omega)$ such that

$$\text{for all } K \subseteq \Omega \text{ compact, } 0 < c_K \leqslant u(z) \text{ for a.a. } z \in K,$$
$$\langle A_p^a(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega \frac{g(z)h}{u^\eta}\mathrm{d}z + \int_\Omega f(z,u)h\mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Under hypotheses $\hat{H}$, the first integral of the right-hand side is well defined. Similarly, hypotheses $H_0$ imply that the second integral of the right-hand side is well defined.

**Theorem 7.** *If hypotheses $\hat{H}$, $H_0$ hold, then for all $\lambda > 0$ problem (27) admits a positive solution.*

*Proof.* First we consider the following auxiliary purely singular problem

$$\left\{\begin{array}{ll} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda \dfrac{g(z)}{u(z)^\eta} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0,\ u \geqslant 0,\ 1 < q < p,\ 1 < \eta,\ \lambda > 0. & \end{array}\right\}$$

According to Theorem 6, for every $\lambda > 0$ this problem admits a unique positive solution $\hat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$v_\lambda \leqslant \hat{u}_\lambda \text{ with } v_\lambda \in \operatorname{int} C_+. \tag{29}$$

As in the proof Theorem 6, we can find $s > 1$ big such that

$$v_\lambda{}^{-\eta} \in L^s(\Omega)$$

and then via the generalized Hölder inequality, we have

$$g v_\lambda{}^{-\eta} \in L^{p'}(\Omega). \tag{30}$$

We introduce the Carathéodory function $j_\lambda : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$j_\lambda(z,x) = \begin{cases} \lambda g(z) v_\lambda(z)^{-\eta} + f(z,x) & \text{if } x \leqslant v_\lambda(z) \\ \lambda g(z) x^{-\eta} + f(z,x) & \text{if } v_\lambda(z) < x \end{cases} \quad \text{(see (28))}. \tag{31}$$

We set $J_\lambda(z,x) = \int_0^x j_\lambda(z,s)ds$ and consider the functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega a(z)|Du|^p \mathrm{d}z + \frac{1}{q}\|Du\|^q - \int_\Omega J_\lambda(z,u)\mathrm{d}z$$

for all $u \in W_0^{1,p}(\Omega)$.

On account of (30), we have that $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$.

Hypotheses $H_0$(i), (iv) imply that given $\varepsilon > 0$, we can find $c_9 = c_9(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{q} x^q + c_9 x^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ (see (28))}. \tag{32}$$

From (30) we see that

$$\lambda \int_\Omega g v_\lambda{}^{-\eta} u \mathrm{d}z \leqslant \lambda c_{10}\|u\| \text{ for some } c_{10} > 0, \text{ all } u \in W_0^{1,p}(\Omega). \tag{33}$$

Using (31), (32) and (33), we have that

$$\varphi_\lambda(u) \geqslant \frac{\hat{c}}{p}\|Du\|_p^p + \frac{1}{q}\left(\|Du\|_q^q - \varepsilon\|u\|_q^q\right) - c_{11}\left(\|u\|^r + \lambda\|u\|\right) \tag{34}$$

for some $c_{11} > 0$.

Let $\hat{\lambda}_1(q) > 0$ be the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. Then from the variational characterization of $\hat{\lambda}_1(q) > 0$ (see Gasinski and Papageorgiou [13, p. 732]), we have

$$\|Du\|_q^q - \varepsilon\|u\|_q^q \geqslant \left(1 - \frac{\varepsilon}{\hat{\lambda}_1(q)}\right)\|Du\|_q^q.$$

Choosing $\varepsilon \in \left(0, \hat{\lambda}_1(q)\right)$, from (34) we have

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \frac{\hat{c}}{p}\|Du\|_p^p - c_{11}\left(\|u\|^r + \lambda\|u\|\right) \\ &= \left(\frac{\hat{c}}{p} - c_{11}\left(\|u\|^{r-p} + \lambda\|u\|^{1-p}\right)\right)\|u\|^p. \end{aligned} \tag{35}$$

Consider the function $\gamma_\lambda(t) = t^{r-p} + \lambda t^{1-p}$, $t > 0$. Since $1 < p < r$, we see that $\gamma_\lambda(t) \to +\infty$ as $t \to +\infty$. So, we can find $t_0 > 0$ such that

$$\begin{aligned}\gamma_\lambda(t_0) &= \min_{t>0} \gamma(t),\\ &\Rightarrow \gamma_\lambda'(t_0) = 0,\\ &\Rightarrow t_0 = \left(\frac{\lambda(p-1)}{r-p}\right)^{\frac{1}{r-1}},\\ &\Rightarrow \gamma_\lambda(t_0) = \lambda^{\frac{r-p}{r-1}} \left(\frac{p-1}{r-p}\right)^{\frac{r-p}{r-1}} + \lambda^{\frac{r-p}{r-1}} \left(\frac{r-p}{p-1}\right)^{\frac{p-1}{r-1}},\\ &\Rightarrow \gamma_\lambda(t_0) \to 0^+ \text{ as } \lambda \to 0^+ \text{ (recall that } 1 < p < r).\end{aligned}$$

So, we can find $\lambda^* > 0$ such that

$$\gamma_\lambda(t_0) < \frac{\hat{c}}{c_{11} p} \quad \text{for all } \lambda \in (0, \lambda^*). \tag{36}$$

From (35) and (36) it follows that

$$\varphi_\lambda(0) = 0 < \inf\{\varphi_\lambda(u) : \|u\| = t_0(\lambda) = \rho_\lambda\} = m_\lambda \quad \text{for all } \lambda \in (0, \lambda^*). \tag{37}$$

Let $u \in \operatorname{int} C_+$. Using hypothesis $H_0$(ii), we deduce that

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{38}$$

*Claim*: $\varphi_\lambda(\cdot)$ satisfies the C-condition for every $\lambda > 0$.

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\varphi_\lambda(u_n)| \leqslant c_{12} \text{ for some } c_{12} > 0, \text{ all } n \in \mathbb{N}, \tag{39}$$

$$(1 + \|u_n\|)\varphi_\lambda'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{40}$$

From (40) we have

$$\left|\langle A_p^a(u_n), h\rangle + \langle A_q(u_n), h\rangle - \int_\Omega j_\lambda(z, u_n) h \mathrm{d}z\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \tag{41}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$.

In (41) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Using (28), (30) and (31) we obtain

$$\hat{c}\|Du_n^-\|_p^p \leqslant c_{13} \text{ for some } c_{13} > 0, \text{ all } n \in \mathbb{N}. \tag{42}$$

From (39) and (42), we have

$$\begin{aligned}&\int_\Omega a(z)|Du_n^+|^p \mathrm{d}z + \frac{p}{q}\|Du_n^+\|_q^q - \int_{\{u_n \leqslant v_\lambda\}} p\lambda g(z) {v_\lambda}^{-\eta} u_n^+ \mathrm{d}z\\ &\quad - \frac{1}{1-\eta} \int_{\{v_\lambda < u_n\}} p\lambda g(z) u_n^{1-\eta} \mathrm{d}z\\ &\quad - \int_\Omega pF(z, u_n^+) \mathrm{d}z \leqslant c_{14}\end{aligned} \tag{43}$$

for some $c_{14} > 0$, all $n \in \mathbb{N}$.

Also if in (41) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$, then we have

$$-\int_\Omega a(z)|Du_n^+|^p \mathrm{d}z - \|Du_n^+\|_q^q + \int_{\{u_n \leqslant v_\lambda\}} \lambda g(z) v_\lambda{}^{-\eta} u_n^+ \mathrm{d}z$$
$$+ \int_{\{v_\lambda < u_n\}} \lambda g(z) u_n^{1-\eta} \mathrm{d}z$$
$$+ \int_\Omega f(z, u_n^+) u_n^+ \mathrm{d}z \leqslant \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{44}$$

Adding (43) and (44) and recalling that $q < p$, $1 < \eta$, we obtain

$$\int_\Omega \left( f(z, u_n^+) u_n^+ - pF(z, u_n^+) \right) \mathrm{d}z \leqslant c_{15} \text{ for some } c_{15} > 0, \text{ all } n \in \mathbb{N}. \tag{45}$$

On account of hypotheses $H_0$(i), (iii), given $\hat{\beta}_1 \in \left(0, \hat{\beta}_0\right)$, we can find $c_{16} = c_{16}(\hat{\beta}_1) > 0$ such that

$$\hat{\beta}_1 x^\tau - c_{16} \leqslant f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{46}$$

Using (46) in (45), we obtain

$$\{u_n^+\}_{n \in \mathbb{N}} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{47}$$

From hypothesis $H_0$(iii), it is clear that we can always assume that $\tau < r < p^*$. So, we can find $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{48}$$

Invoking the interpolation inequality (see Proposition 2.3.17 of Papageorgiou and Winkert [27, p. 116]), we have

$$\|u_n^+\|_r \leqslant \|u_n^+|_\tau^{1-t} \|u_n^+|_{p^*}^t \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \|u_n^+\|_r^r \leqslant c_{17} \|u_n^+|^{tr} \text{ for some } c_{17} > 0, \text{ all } n \in \mathbb{N}. \tag{49}$$

Note that hypothesis $H_0$(i) implies that

$$0 \leqslant f(z,x)x \leqslant c_{18}\left(1 + x^r\right) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0, \text{ some } c_{18} > 0. \tag{50}$$

In (41) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|Du_n^+\|_p^p \leqslant c_{19} + \int_\Omega f(z, u_n^+) u_n^+ \mathrm{d}z \text{ for some } c_{19} > 0$$
$$\leqslant c_{20}\left(1 + \|u_n^+\|^{tr}\right) \text{ for some } c_{20} > 0, \text{ all } n \in \mathbb{N} \text{ (see (50), (49))},$$
$$\Rightarrow \|u_n^+\|^p \leqslant c_{21}\left(1 + \|u_n^+\|^{tr}\right) \text{ with } c_{21} = \frac{c_{20}}{\hat{c}}, \text{ all } n \in \mathbb{N}. \tag{51}$$

From (48) and hypothesis $H_0$(iii) it follows that

$$tr < p.$$

Therefore from (51) we infer that

$$\{u_n^+\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{52}$$

From (42) and (52) we infer that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega). \tag{53}$$

In (41) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Then as in the proof of Theorem 6, using (53) and the $(S)_+$-property of $A_p^a(\cdot)$, we obtain that

$$\begin{aligned} &u_n \to u \text{ in } W_0^{1,p}(\Omega), \\ &\Rightarrow \varphi_\lambda(\cdot) \text{ satisfies the C-condition.} \end{aligned}$$

This proves the Claim.

Then (37), (38) and the Claim, permit the use of the mountain pass theorem. So, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda'(u_0) = 0 \text{ and } m_\lambda \leqslant \varphi_\lambda(u_0) \quad \text{for all } \lambda \in (0, \lambda^*). \tag{54}$$

From (54) and (37), we see that $u_0 \neq 0$ and

$$\langle A_p^a(u_0), h\rangle + \langle A_p(u_0), h\rangle = \int_\Omega j_\lambda(z, u_0) h \mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{55}$$

In (55) we choose $h = (v_\lambda - u_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A_p^a(u_0), (v_\lambda - u_0)^+\rangle + \langle A_q(u_0), (v_\lambda - u_0)^+\rangle \\ &\quad = \int_\Omega \left(\lambda g(z) v_\lambda{}^{-\eta} + f(z, u_0)\right) (v_\lambda - u_0)^+ \,\mathrm{d}z \text{ (see (31))} \\ &\quad \geqslant \int_\Omega \lambda g(z) v_\lambda{}^{-\eta} (v_\lambda - u_0)^+ \,\mathrm{d}z \text{ (since } f \geqslant 0, \text{ see hypothesis } H_0(\mathrm{i})) \\ &\quad = \langle A_p^a(v_\lambda), (v_\lambda - u_0)^+\rangle + \langle A_q(v_\lambda), (v_\lambda - u_0)^+\rangle \\ &\qquad \text{(since } v_\lambda \in \operatorname{int} C_+ \text{ is a solution of problem (4)),} \\ &\quad \Rightarrow v_\lambda \leqslant u_0. \end{aligned}$$

Therefore from (31) and (55) we conclude that $u_0$ is a positive solution of problem (20). The proof is now complete. □

*Remark* 3. In the unbalanced double phase case, due to the lack of a global regularity theory, we do not control the integrability properties of the singular term $v_\lambda{}^{-\eta}$. Hence the technique of truncation at $v_\lambda(z)$ to bypass the singularity, does not work. So, for unbalanced double phase problems with a superlinear perturbation, the existence of positive solutions is an open problem. Another interesting open problem, is what can be said when $\eta = 1$. For this case, may be a promising approach, is to consider a sequence $\{\eta_n\}_{n\in\mathbb{N}} \subseteq (1, +\infty)$ such that $\eta_n \downarrow 1$ and use the solution of the approximating problems to produce in the limit as $n \to \infty$, a solution of the original problem.

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