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Existence implies uniqueness for a class of singular anisotropic elliptic boundary value problems

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SUMMARY

We consider a singular anisotropic quasilinear problem with Dirichlet boundary condition and we establish two sufficient conditions for the uniqueness of the solution, provided such a solution exists. The proofs use elementary tools and they are based on a general comparison lemma combined with the generalized mean value theorem. Copyright © 2001 John Wiley & Sons, Ltd.

1. INTRODUCTION AND THE MAIN RESULTS

Singular anisotropic boundary value problems arise naturally when studying many concrete situations. We refer to Čanić–Keyfitz [1] for the study of self-similar solutions of conservation laws in two dimensions. We also mention Ding–Liu [2], where another anisotropic problem in the plane is studied. Their model is closely related to the phase transition problem in anisotropic superconductivity with 'thermal noise' term.

In [3], Choi *et al.* studied a problem that is linked to an equation arising in fluid dynamics. They proved that the singular elliptic boundary value problem

$$\begin{aligned} &u^a u_{xx} + u^b u_{yy} + p(x,y) = 0, \quad (x,y)\in\Omega \\ &u = 0, \quad (x,y)\in\partial\Omega \end{aligned} \tag{1}$$

has a positive classical solution, where $\Omega\subset\mathbf{R}^2$ is a bounded convex domain with smooth boundary, p is a positive Hölder continuous function and the constants a, b satisfy $a>b\geqslant 0$. Choi *et al.* also developed a new comparison principle for quasilinear problems that is based on the method of sub- and super-solutions.

Recently, Choi and McKenna [4] removed the assumption that the dimension be restricted to two, but they also retained the convexity assumption which is crucial in the construction

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of a super-solution ψ, satisfying the boundary conditions. More precisely, they showed that the boundary value problem

$$
\begin{aligned}
&\sum_{i=1}^{N} u^{a_i} u_{x_i x_i} + p(x) = 0, \quad x \in \Omega \\
&u = 0, \quad x \in \partial\Omega
\end{aligned} \tag{2}
$$

has at least one positive classical solution u, such that $u(x) \leqslant \psi(x)$ for all $x \in \Omega$, where $\Omega \subset \mathbf{R}^N$ $(N \geqslant 1)$ is a bounded convex domain with smooth boundary and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N \geqslant 0$, with $a_1 > a_N$. Choi and McKenna point out that the most significant omission of their paper is the absence of any information on the uniqueness of solutions. In this direction very few results are known which hold only for the two-dimensional case.

Lair and Shaker proved in [5] a uniqueness result related to (1) and they required neither the domain Ω to be convex nor the function p to be as smooth as in [3]. They made only the assumption that there is some solution u for which u_{xx} is bounded above appropriately. In their paper, two different situations are distinguished: $a - b \geqslant 1$, resp., $a - b < 1$.

Reichel [6] established that problem (1) has at most one positive classical solution. It is assumed that

$$
p(\tau_1 x, \tau_2 y) \geqslant p(x, y) \quad \text{for all } (x, y) \in \Omega, \ \tau_i \in [0, 1]
$$

and the bounded domain Ω (with $0 \in \Omega$) satisfies an interior rectangle condition, i.e. for each $(x, y) \in \partial\Omega$ the rectangle $\{(\tau_1 x, \tau_2 y) : \tau_i \in [0, 1)\}$ is a subset of Ω.

It is natural to ask if it is possible to give a uniqueness result which holds for more general degenerate quasilinear operators and for a larger class of functions p, with no assumption on the geometry of the domain or the dimension of the space.

For this aim, we consider the singular anisotropic elliptic boundary value problem

$$
\begin{aligned}
&\sum_{i=1}^{N-1} f_i(u) u_{x_i x_i} + u_{yy} + p(x) g(u) = 0, \quad x \in \Omega \\
&u = 0, \quad x \in \partial\Omega
\end{aligned} \tag{3}
$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \geqslant 2$ and p is a positive continuous function on $\bar{\Omega}$. We have denoted the last co-ordinate x_N by y and we shall use notation x' for the first $(N-1)$ co-ordinates.

Throughout this paper, we assume that the following hypotheses are fulfilled:

(H_1) $f_i, g : (0, \infty) \to (0, \infty)$, $i = \overline{1, N-1}$ are C^1-functions;
(H_2) f_i, $i = \overline{1, N-1}$ is non-decreasing on $(0, \infty)$ and g is non-increasing on $(0, \infty)$.

Since Ω is bounded, we can make a translation of the domain so that it lies in the interior of the strip $\mathbf{R}^{N-1} \times [0, \ell]$ for some $\ell > 0$. The fact that $p \in C(\bar{\Omega})$ is a positive function implies the existence of $\alpha \geqslant 0$ and $\beta > 0$ such that $p(x) \in [\alpha, \beta]$ for each $x \in \bar{\Omega}$.

Set

$$D = \{ y \in [0, \ell] : \exists x' \text{ such that } (x', y) \in \bar{\Omega} \}$$

We can suppose, without loss of generality, that $\ell \notin D$.

Let ψ be the unique positive function defined by

$$\int_0^{\psi(y)} \frac{1}{g(t)} \, \mathrm{d}t = \frac{\beta}{2}(\ell y - y^2) \quad \text{for any } y \in [0, \ell] \tag{4}$$

It is obvious that

$$\max_{y \in D} \psi(y) \leqslant \max_{y \in [0, \ell]} \psi(y) = A \tag{5}$$

where $A > 0$ is uniquely defined by

$$\int_0^{A} \frac{1}{g(t)} \, \mathrm{d}t = \frac{\beta}{8} \ell^2 \tag{6}$$

We also assume

(H_3) $f_1' > 0$ on $(0, A]$.

In the first result of this paper we impose the condition

(C_1) there exists and is finite $\lim_{x \searrow 0} f_1 f_i'/f_1'(x)$, for all $i = \overline{2, N-1}$.

In view of this hypothesis we observe that for any $i = \overline{2, N-1}$ it makes sense to define

$$m_i = \min_{[0,A]} \frac{(f_i/f_1)'}{(1/f_1)'} = \min_{[0,A]} \left(f_i - \frac{f_i' f_1}{f_1'} \right) \quad \text{and} \quad M_i = \max_{[0,A]} \frac{(f_i/f_1)'}{(1/f_1)'} = \max_{[0,A]} \left(f_i - \frac{f_i' f_1}{f_1'} \right)$$

For any $x \in \Omega$ we define the sets

$$P_x = \{ 2 \leqslant i \leqslant N-1; \; u_{x_i x_i}(x) \geqslant 0 \} \quad \text{and} \quad N_x = \{ 2 \leqslant i \leqslant N-1; \; u_{x_i x_i}(x) < 0 \}$$

Our first result asserts that the existence of a positive solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ of (3) ensures its uniqueness, provided that the expression $\sum_{i \in P_x} m_i u_{x_i x_i} + \sum_{i \in N_x} M_i u_{x_i x_i} + u_{yy}$ is bounded below appropriately.

Theorem 1. Assume (H_1)–(H_3) and (C_1) hold. There exists a positive constant K_1, depending on f_1, g, p and Ω, such that if u is a positive solution of (3) satisfying

$$\sum_{i \in P_x} m_i u_{x_i x_i} + \sum_{i \in N_x} M_i u_{x_i x_i} + u_{yy} > -K_1 \quad \text{in } \Omega \tag{7}$$

then u is the unique solution of (3).

We now drop the assumption (C_1) but we require

(C_2) f_i/f_1, $i = \overline{2, N-1}$ is non-increasing on $(0, \infty)$.

Our next theorem shows that the uniqueness of the solution to (3) is assured if we find a positive solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ with the property that $u_{x_1 x_1} + \sum_{i \in P_x} f_i(u)/f_1(u) u_{x_i x_i} + \sum_{i \in N_x} (\inf_{(0,A)} f_i'/f_1') u_{x_i x_i}$ is bounded above appropriately.

Theorem 2. Assume (H_1)–(H_3) and (C_2) hold. There exists a non-negative constant K_2, depending on f_1, g, p and Ω, such that if u is a positive solution of problem (3) satisfying

$$u_{x_1x_1} + \sum_{i\in P_x} \frac{f_i(u)}{f_1(u)} u_{x_ix_i} + \sum_{i\in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_ix_i} < K_2 \quad \text{in } \Omega \tag{8}$$

then u is the unique solution of (3).

We point out that hypotheses (7) and (8) should be understood as sufficient conditions that guarantee the uniqueness of the solution, provided such a solution exists. Problems related to uniqueness for singular anisotropic quasilinear boundary value problems have been recently studied by Hill *et al.* in [7]. In [7] the authors impose a topological constraint to the boundary and the proof of the uniqueness of the solution uses essentially the fact that Ω satisfies a weighted starshapedness condition. In order to illustrate our above stated results, let us consider the problem

$$\begin{aligned} &\sum_{i=1}^{N} u^{a_i} u_{x_ix_i} + 2\sum_{i=1}^{N} (1-|x|^2)^{a_i} = 0 \quad \text{if } x \in B(0,1) \subset \mathbf{R}^N \\ &u = 0 \quad \text{if } |x| = 1 \end{aligned} \tag{9}$$

where $a_1 \geqslant \cdots \geqslant a_N \geqslant 0$ and $a_1 > a_N$. By Theorem 4.3 in [7], this problem has a unique solution. The same conclusion follows from our results. Indeed, let us observe that the functions $f_i(t) = t^{a_i - a_N}$ and $g(t) = t^{-a_N}$ fulfill conditions (H_1)–(H_3) and (C_1), with $\alpha = 0$, $\beta = 2N$, $A = [N(a_N + 1)]^{1/(a_N+1)}$, $m_i = 0$ and $M_i = (a_1 - a_i)(a_1 - a_N)^{-1} A^{a_i - a_N}$, for $1 \leqslant i \leqslant N-1$. Choosing

$$K_1 > 2 + \frac{2}{a_1 - a_N} \sum_{i=2}^{N-1} (a_1 - a_i) A^{a_i - a_N}$$

it follows from Theorem 1 that $u(x) = 1 - |x|^2$ is the unique solution of problem (9).

The main difficulty in the treatment of (3) is the lack of the usual comparison principle between sub- and super-solution, due to the anisotropic character of the equation. To this end, using a result of Choi and McKenna, we will state in Section 2 a comparison principle which is suitable for (3).

2. AN AUXILIARY RESULT

In this section we prove that the number A given by (6) is an upper bound for every positive classical solution of problem (3). To this end, we make use of a comparison lemma on a class of quasilinear elliptic equations established by Choi–McKenna [4]. In view of this result we can obtain L^∞ bounds on the solutions to this class of equations using the method of suband super-solutions. Consider the problem

$$\begin{aligned} &\sum_{i=1}^{N-1} f_i(x,u)u_{x_i x_i} + u_{yy} + p(x)g(x,u) = 0 \quad \text{in } \Omega \\ &u = u_0 \quad \text{on } \partial\Omega \end{aligned} \tag{10}$$

with $u_0|_{\partial\Omega} \geqslant 0$, where the functions f_i, g and p satisfy the assumptions:

(A_1) $f_i : \Omega \times [0,\infty) \to [0,\infty)$ is continuous and $f_i(x,\cdot)$ is non-decreasing for each $x \in \Omega$;
(A_2) $g : \Omega \times (0,\infty) \to (0,\infty)$ is continuous, and $g(x,\cdot)$ is non-increasing for each $x \in \Omega$;
(A_3) $p : \bar{\Omega} \to \mathbf{R}$ is continuous, and there exist positive constants α and β such that

$$0 < \alpha \leqslant p(x) \leqslant \beta \quad \text{for all } x \in \bar{\Omega}$$

Assume that
(L) There exists a sub-solution $\varphi \in C(\bar{\Omega}) \cap C^2(\Omega)$ with $\varphi > 0$ on Ω satisfying

$$\begin{aligned} &\sum_{i=1}^{N-1} f_i(x,\varphi)\varphi_{x_i x_i} + \varphi_{yy} + p(x)g(x,\varphi) > 0 \quad \text{in } \Omega \\ &\varphi_{x_i x_i} \leqslant 0 \quad \text{in } \Omega \quad \text{for any } i = 1,2,\ldots,N-1 \end{aligned}$$

and $\varphi \leqslant u_0$ on $\partial\Omega$.
(U) There exists a super-solution $\psi \in C(\bar{\Omega}) \cap C^2(\Omega)$ with $\psi > 0$ in Ω satisfying

$$\begin{aligned} &\sum_{i=1}^{N-1} f_i(x,\psi)\psi_{x_i x_i} + \psi_{yy} + p(x)g(x,\psi) \leqslant 0 \quad \text{in } \Omega \\ &\psi_{x_i x_i} \leqslant 0 \quad \text{in } \Omega \quad \text{for any } i = 1,2,\ldots,N-1 \end{aligned}$$

and $\psi \geqslant u_0$ on $\partial\Omega$.

Lemma 1. Assume (A_1)–(A_3), (L) and (U) hold. Then any positive solution u of (3) satisfies $u \leqslant A$ in $\bar{\Omega}$, where A is defined in (6).

Proof. Under the above hypotheses, Choi and McKenna proved in [4] that every solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of problem (10), with $u > 0$ in Ω, satisfies

$$\varphi \leqslant u \leqslant \psi \quad \text{in } \bar{\Omega}$$

Moreover, if only conditions (A_1)–(A_3) and (U) hold, then $u \leqslant \psi$ in $\bar{\Omega}$.

It is easy to check whether the function ψ defined in (4) satisfies condition (U) considered for our problem (3). Therefore, by the Choi–McKenna comparison lemma and (5), we find that every positive classical solution of (3) is bounded above by the same number A defined in (6). □

3. PROOF OF THEOREM 1

Let u and v be solutions of (3) and let u satisfy (7), where

$$K_1 = \frac{\pi^2}{\ell^2}\frac{1}{f_1(A)}\inf_{(0,A)}\frac{f_1^2}{f_1'} + \alpha\inf_{(0,A)}\frac{(g/f_1)'}{(1/f_1)'}$$

We prove in what follows that $u = v$ in $\bar{\Omega}$. Set

$$w(x) = \frac{u(x', y)}{s(y)}, \quad z(x) = \frac{v(x', y)}{s(y)}$$

where

$$s(y) = \sin\frac{\pi y}{\ell}, \quad c(y) = \cos\frac{\pi y}{\ell}, \quad y \in (0, \ell)$$

Since $s > 0$ and $s \in C^\infty$, it follows that w and z are well-defined and that they are as smooth as u and v, respectively, on Ω. A simple computation shows that w satisfies the boundary value problem

$$\begin{aligned}&\sum_{i=1}^{N-1} s f_i(u) w_{x_i x_i} + \frac{2\pi c}{\ell} w_y + s w_{yy} - \frac{\pi^2 s}{\ell^2} w + p(x) g(u) = 0 \quad \text{in } \Omega \\ &w = 0 \quad \text{on } \partial\Omega\end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned}&\sum_{i=1}^{N-1} s f_i(v) z_{x_i x_i} + \frac{2\pi c}{\ell} z_y + s z_{yy} - \frac{\pi^2 s}{\ell^2} z + p(x) g(v) = 0 \quad \text{in } \Omega \\ &z = 0 \quad \text{on } \partial\Omega\end{aligned} \tag{12}$$

Relations (11) and (12) yield

$$\begin{aligned}&\sum_{i=1}^{N-1} s \frac{f_i(v)}{f_1(v)}(z - w)_{x_i x_i} + \sum_{i=2}^{N-1} s\left[\left(\frac{f_i}{f_1}\right)(v) - \left(\frac{f_i}{f_1}\right)(u)\right] w_{x_i x_i} + \frac{2\pi c}{\ell}\frac{1}{f_1(v)}(z - w)_y \\ &+ \frac{2\pi c}{\ell}\left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right) w_y + s\frac{1}{f_1(v)}(z - w)_{yy} + s\left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right) w_{yy} \\ &- \frac{\pi^2 s}{\ell^2}\frac{1}{f_1(v)}(z - w) - \frac{\pi^2 s}{\ell^2}\left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right) w + p(x)\left[\left(\frac{g}{f_1}\right)(v) - \left(\frac{g}{f_1}\right)(u)\right] = 0\end{aligned}$$

Whenever $z \neq w$ we can rewrite the above equation as follows:

$$\sum_{i=1}^{N-1} s \frac{f_i(v)}{f_1(v)}(z-w)_{x_i x_i} + s\frac{1}{f_1(v)}(z-w)_{yy} + \frac{2\pi c}{\ell}\frac{1}{f_1(v)}(z-w)_y + \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right) Q(z,w) = 0 \tag{13}$$

where

$$Q(z,w) = u_{yy} + \sum_{i=2}^{N-1} \frac{(f_i/f_1)(v) - (f_i/f_1)(u)}{(1/f_1)(v) - (1/f_1)(u)} u_{x_i x_i} - \frac{\pi^2}{\ell^2}\frac{1}{f_1(v)}\frac{v-u}{(1/f_1)(v) - (1/f_1)(u)} + p(x)\frac{(g/f_1)(v) - (g/f_1)(u)}{(1/f_1)(v) - (1/f_1)(u)}$$

In order to conclude the proof it is enough to show that

$$Q(z,w) > 0 \quad \text{whenever } z \neq w \tag{14}$$

Indeed, if $(z-w) > 0$ at some point in Ω, then $\max_{\bar\Omega}(z-w)$ is achieved in Ω, since $z = w = 0$ on $\partial\Omega$. At that point we have

$$(z-w)_{x_i x_i} \leqslant 0 \quad (z-w)_{yy} \leqslant 0, \quad (z-w)_y = 0 \quad \text{and} \quad \left(\frac{1}{f_1(v)} - \frac{1}{f_1(u)}\right) Q(z,w) < 0$$

which contradicts (13). A similar argument shows that $(z - w)$ cannot be negative at any point in Ω. Hence, $z = w$ in Ω which implies $u = v$ on $\bar\Omega$.

For every $x \in \Omega$, let us define

$$\mu(x) = \min(u(x), v(x)) \quad \text{and} \quad \nu(x) = \max(u(x), v(x))$$

Thus, by Lemma 1, $\nu \leqslant A$ in Ω.

In (13) we apply the Cauchy generalized mean value theorem on every interval $[\mu(x), \nu(x)]$, where $x \in \Omega$ is taken such that $z(x) \neq w(x)$. Hence, for all $i = \overline{2, N-1}$ we obtain the existence of $\xi_i(x)$, $\sigma(x)$, $\lambda(x) \in (\mu(x), \nu(x)) \subset (0, A)$ such that

$$m_i \leqslant \frac{(f_i/f_1)(v(x)) - (f_i/f_1)(u(x))}{(1/f_1)(v(x)) - (1/f_1)(u(x))} = \frac{(f_i/f_1)'}{(1/f_1)'}(\xi_i(x)) \leqslant M_i \tag{15}$$

$$-\frac{v(x) - u(x)}{(1/f_1)(v(x)) - (1/f_1)(u(x))} = \frac{f_1^2}{f_1'}(\sigma(x)) \geqslant \inf_{(0,A)} \frac{f_1^2}{f_1'} \tag{16}$$

$$\frac{(g/f_1)(v(x)) - (g/f_1)(u(x))}{(1/f_1)(v(x)) - (1/f_1)(u(x))} = \frac{(g/f_1)'}{(1/f_1)'}(\lambda(x)) \geqslant \inf_{(0,A)} \frac{(g/f_1)'}{(1/f_1)'} \tag{17}$$

Using (15)–(17) we find

$$Q(z,w) \geqslant u_{yy} + \sum_{i\in P_x} m_i u_{x_i x_i} + \sum_{i\in N_x} M_i u_{x_i x_i} + \frac{\pi^2}{\ell^2}\frac{1}{f_1(A)} \inf_{(0,A)} \frac{f_1^2}{f_1'} + \alpha \inf_{(0,A)} \frac{(g/f_1)'}{(1/f_1)'}$$

$$= u_{yy} + \sum_{i\in P_x} m_i u_{x_i x_i} + \sum_{i\in N_x} M_i u_{x_i x_i} + K_1$$

Since the solution u satisfies (7) we deduce that relation (14) is true. This completes the proof. □

4. PROOF OF THEOREM 2

Let u and v be two solutions of (3) and set

$$K_2 = -\alpha \sup_{(0,A)} \frac{g'}{f_1'} \geqslant 0 \tag{18}$$

The functions w, z, μ and v will have the same signification as in the above proof.

By (11) and (12) it follows that

$$\sum_{i=1}^{N-1} s f_i(v)(z-w)_{x_i x_i} + \sum_{i=1}^{N-1} s[f_i(v) - f_i(u)] w_{x_i x_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy}$$
$$- \frac{\pi^2 s}{\ell^2}(z-w) + p(x)[g(v) - g(u)] = 0 \tag{19}$$

Whenever $z \neq w$, relation (19) may be rewritten in the following form:

$$\sum_{i=1}^{N-1} s f_i(v)(z-w)_{x_i x_i} + \frac{2\pi c}{\ell}(z-w)_y + s(z-w)_{yy} + [f_1(v) - f_1(u)] R(z,w) = 0$$

where

$$R(z,w) = u_{x_1 x_1} + \sum_{i=2}^{N-1} \frac{f_i(v) - f_i(u)}{f_1(v) - f_1(u)} u_{x_i x_i} - \frac{\pi^2}{\ell^2} \frac{v-u}{f_1(v) - f_1(u)} + p(x) \frac{g(v) - g(u)}{f_1(v) - f_1(u)}$$

Using the maximum principle (as we did in the proof of Theorem 1) we see that the proof will be concluded if we prove that

$$R(z,w) < 0 \quad \text{whenever } z \neq w$$

From now on, we shall consider only the points $x \in \Omega$ with the property that $z(x) \neq w(x)$. For these points, we again apply the Cauchy generalized mean value theorem on $[\mu(x), v(x)]$ and we obtain $\eta_i(x)$, $\theta(x), \zeta(x) \in (\mu(x), v(x)) \subset (0, A)$ such that

$$\frac{f_i(v(x)) - f_i(u(x))}{f_1(v(x)) - f_1(u(x))} = \frac{f_i'}{f_1'}(\eta_i(x)) \geqslant \inf_{(0,A)} \frac{f_i'}{f_1'}, \quad i = \overline{2, N-1} \tag{20}$$

$$\frac{v(x) - u(x)}{f_1(v(x)) - f_1(u(x))} = \frac{1}{f_1'(\theta(x))} \tag{21}$$

$$\frac{g(v(x)) - g(u(x))}{f_1(v(x)) - f_1(u(x))} = \frac{g'}{f_1'}(\zeta(x)) \leqslant \sup_{(0,A)} \frac{g'}{f_1'} \leqslant 0 \tag{22}$$

It is easy to verify that hypothesis (C_2) implies

$$\frac{f_i(v(x)) - f_i(u(x))}{f_1(v(x)) - f_1(u(x))} \leqslant \frac{f_i(u(x))}{f_1(u(x))} \quad \text{for all } i = \overline{2, N-1} \tag{23}$$

On the other hand, since f_1 is increasing on $(0, A)$,

$$\frac{v(x) - u(x)}{f_1(v(x)) - f_1(u(x))} > 0 \tag{24}$$

Combining relations (20), (22), (23) and (24) with the expression of $R(z,w)$ we deduce that

$$R(z,w) < u_{x_1 x_1} + \sum_{i \in P_x} \frac{f_i(u)}{f_1(u)} u_{x_i x_i} + \sum_{i \in N_x} \left(\inf_{(0,A)} \frac{f_i'}{f_1'} \right) u_{x_i x_i} + \alpha \sup_{(0,A)} \frac{g'}{f_1'}$$

Since u is a solution of (3) satisfying (8) we deduce that $R(z,w)$ is negative. This completes our proof.

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