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# Multiplicity and concentration of solutions to fractional anisotropic Schrödinger equations with exponential growth

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**Abstract.** In this paper, we consider the Schrödinger equation involving the fractional $(p, p_1, \dots, p_m)$-Laplacian as follows

$$(-\Delta)_p^s u + \sum_{i=1}^{m} (-\Delta)_{p_i}^s u + V(\varepsilon x)(|u|^{(N-2s)/2s} u + \sum_{i=1}^{m} |u|^{p_i - 2} u) = f(u) \text{ in } \mathbb{R}^N,$$

where $\varepsilon$ is a positive parameter, $N = ps, s \in (0, 1), 2 \le p < p_1 < \cdots < p_m < +\infty, m \ge 1$. The nonlinear function $f$ has the exponential growth and potential function $V$ is continuous function satisfying some suitable conditions. Using the penalization method and Ljusternik–Schnirelmann theory, we study the existence, multiplicity and concentration of nontrivial nonnegative solutions for small values of the parameter. In our best knowledge, it is the first time that the above problem is studied.

## 1. Introduction and main results

Let $\Omega$ be a bounded, open domain of $\mathbb{R}^N$ $(N \ge 2)$. The standard Sobolev space $W_0^{k,p}(\Omega)$ is defined by the completion of $C_0^\infty(\Omega)$ equipped with the norm

$$||u||_{W_0^{k,p}(\Omega)} = \left( ||u||_{L^p(\Omega)}^p + \sum_{j=1}^{k} ||\nabla^j u||_{L^p(\Omega)}^p \right)^{1/p}.$$

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The well-known Sobolev embedding theorem states that $W_0^{k,p}(\Omega)$ embeds continuously into $L^{Np/(N-kp)}(\Omega)$ for a positive integer $k < N$ and $1 \le p < \frac{N}{k}$. When $p = \frac{N}{k}$, the embedding $W_0^{k,N/k}(\Omega) \subset L^\infty(\Omega)$ fails. To overcome this difficulty, Trudinger [55] proved that functions in $W_0^{1,N}(\Omega)$ has property

$$W_0^{1,N}(\Omega) \subset \left\{ u \in L^1(\Omega) : E_\beta(u) := \int_\Omega e^{\beta |u|^{N/(N-1)}} dx < +\infty \right\} \text{ for any } \beta < \infty.$$

Furthermore, the function $E_\beta$ is continuous on $W_0^{1,N}(\Omega)$. In 1970, Moser [41] gave the optimal $\beta$ and proved that $\beta \le \alpha_N = N\omega_{N-1}^{1/(N-1)}$, where $\omega_{N-1}$ is the area of the surface of the unit ball. From this work, many works are done and made the research direction about Trudinger–Moser type inequality and applications. Special, In 2007, Adimurthi-Sandeep [2] extended the work of Trudinger–Moser for singular case on bounded domain. When $\Omega$ is unbounded, Adachi and Tanach [1] and do Ó [23] gave a subcritical Trudinger–Moser-type inequality as follows: For $0 < \alpha < \alpha_N$, there exists a positive constant $C_N$ such that

$$\sup_{u \in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |\nabla u|^N dx \le 1} \int_{\mathbb{R}^N} \Phi\left(\alpha |u(x)|^{N/(N-1)}\right) dx \le C_N \int_{\mathbb{R}^N} |u(x)|^N dx,$$

where $\Phi(t) = e^t - \sum_{i=0}^{N-2} \frac{t^i}{i!}$. Moreover, the constant $\alpha_N$ is sharp in the sense that if $\alpha \ge \alpha_N$, the supremum will become infinite. In 2010, Adimurthi-Yang [3] extended the result of Adachi and Tanach [1] and do Ó [23] for singular case. In 2019, Parini and Ruf [43] extended the result of Trudinger–Moser to fractional Sobolev-Slobodeckij spaces and obtained the following result: Let $\Omega$ be a bounded open domain of $\mathbb{R}^N$, $(N \ge 2)$ with Lipschitz boundary, and let $s \in (0,1)$, $N = ps$. Then there exists an exponent $\alpha$ of the fractional Trudinger–Moser inequality such that

$$\sup_{u \in \widetilde{W}_0^{s,p}(\Omega),\, [u]_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_\Omega \exp(\alpha |u|^{N/(N-s)}) dx < +\infty.$$

Set

$$\alpha_* = \alpha_*(s,\Omega) = \sup\left\{ \alpha : \sup_{u \in \widetilde{W}_0^{s,p}(\Omega),\, [u]_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_\Omega \exp(\alpha |u|^{N/(N-s)}) dx < +\infty \right\}.$$

Moreover, $\alpha_* \le \alpha^*_{s,N}$, where

$$\alpha^*_{s,N} = N \left( \frac{2(N\omega_N)^2 \Gamma(p+1)}{N!} \sum_{k=0}^{+\infty} \frac{(N+k-1)!}{k!} \frac{1}{(N+2k)^p} \right)^{s/(N-s)}.$$

By replacing the norm $[u]_{W^{s,p}(\mathbb{R}^N)}$ by $||u||_{W^{s,p}(\mathbb{R}^N)}$, Iula [33] proved that the result of Parini and Ruf is still true in $\mathbb{R}$. In 2019, Zhang [61] has been extended the that result of Parini and Ruf, and Iula to $\mathbb{R}^N$ and get a fractional Trudinger–Moser type imequality. Using that result, Zhang studied the existence of weak solution to Schrödinger equation involving the fractional $p$-Laplacian. For some more results and the applications of Trudinger–Moser inequality and fractional Trudinger–Moser type inequality, we refer the readers to [4,24–27,31,36,37,39,45,59] and the references therein for more details. On singular Trudinger–Moser type inequality in fractional Sobolev space and its application, we recommend the readers to [52] for more details.

Using the fractional Trudinger–Moser type inequality, in this paper, we study the existence and concentration of nontrivial nonnegative solution for the following Schrödinger equation involving fractional $(p, p_1, \ldots, p_m)$-Laplacian:

$$(-\Delta)^s_{N/s}u(x) + \sum_{i=1}^{m}(-\Delta)^s_{p_i}u + V(x)(|u|^{\frac{N}{s}-2}u + \sum_{i=1}^{m}|u|^{p_i-2}u) = f(u) \text{ in } \mathbb{R}^N, \qquad (P_\varepsilon) \tag{1.1}$$

where $\varepsilon$ is small positive parameter, $0 < s < 1, 2 \leq p < p_1 < \cdots < p_m < +\infty, m \geq 1, N = ps$, the potential $V$ is bounded below by $V_0 > 0$, the nonlinearity $f$ has exponential critical growth, and $(-\Delta)^s_t$ $(t \in \{p, p_1, \ldots, p_m\})$ is the fractional $t$-Laplace operator which may be defined along a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ (up to a normalization constant) as

$$(-\Delta)^s_t\varphi(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{t-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+ts}}dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x)$ is a ball with center $x$ and radius $\varepsilon$.

Assume that the continuous function $V$ verifies the following conditions:
$(V_1)$ There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;
$(V_2)$ There exists a bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 = \min_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, we assume that the nonlinear function $f$ satisfying the following conditions:
$(f_1)$ The nonlinearity $f \in C^1(\mathbb{R})$ such that $f(t) = 0$ for all $t \in (-\infty, 0]$, $f(t) > 0$ for all $t > 0$ and there exist constants $\alpha_0 \in (0, \alpha_*)$, $b_1, b_2 > 0$ such that for any $t \in \mathbb{R}$,

$$|f(t)| \leq b_1|t|^{p_m-1} + b_2|t|^{p-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}),$$

where $\Phi_{N,s}(y) = e^y - \sum_{i=0}^{j_p-2} \dfrac{y^j}{j!}$, $j_p = \min\{j \in \mathbb{N} : j \geq p\}$ and $\alpha_* \leq \alpha^*_{s,N}$ (see Lemma 1).

$(f_2)$ There exists $\mu > p_m$ such that

$$f(t)t - \mu F(t) \geq 0$$

for all $t \in \mathbb{R}$, where $F(t) = \int\limits_0^t f(\tau)d\tau$.

$(f_3)$

$$\lim_{t\to 0^+} \frac{f(t)}{t^{p_m-1}} = 0.$$

$(f_4)$ There exists $\gamma_1 > 0$ large enough such that $F(t) \geq \gamma_1|t|^\mu$ for all $t \geq 0$.

$(f_5)$ $\dfrac{f(t)}{t^{p_m-1}}$ is a strictly increasing function in $\mathbb{R}^+$.

Recently, Alves–Ambrosio–Isernia [7], Ambrosio–Radulescu [8] studied the fractional $(p, q)$-Laplacian as follows:

$$(-\Delta)^s_p u + (-\Delta)^s_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$, $1 < p < q < \dfrac{N}{s}$ and $f$ has the subcritical growth and satisfies some suitable conditions. For more results on fractional $(p, q)$-Laplace or $(p, q)$-Laplace, we refer the readers to [9–11]. When $s \to 1^{-1}$, the Eq. (1.2) becomes the following equation

$$-\Delta_p u - \Delta_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \text{ in } \mathbb{R}^N, \tag{1.3}$$

where $\Delta_r u = \text{div}(|\nabla u|^{r-2}\nabla u)$, $r \in \{p, q\}$. The study of Eq. (1.3) is connected to more general reaction-diffusion equation

$$u_t = \text{div}((|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla(u)) + c(x, u) \tag{1.4}$$

which has many applications in biophysics, physics of plasmas and chemical reaction design [13,21]. In that equation, $c(x, u)$ is related to source and loss process. The multiple phases quation is motivated from the following Born–Infeld equation [18–20] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density

$$-\text{div}\Big(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{1/2}}\Big) = h(u) \text{ in } \mathbb{R}^N.$$

We refer the readers to the work of Zhang–Tang–Radulescu [62] for more information and motivation as well as application of double-phases equation.

In 2021, Ambrosio–Repovs [12] have been studied the problem (1.3) when $1 < p < q < N$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying the global Rabinowitz condition, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth. Using suitable variational arguments and Ljusternik–Schnirelmann category theory, they study the relation between the number of positive solutions and the topology of the set where $V$ attains its minimum for small $\varepsilon$.

When $p = q$ and $\varepsilon = 1$, the Eq. (1.2) becomes

$$(-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = g(x,u) \text{ in } \mathbb{R}^N, \tag{1.5}$$

where $V$ and $f$ satisfy some suitable assumptions. Many works were achieved on that equation such as [14–16,25,28,29]. In particular, when $p = 2$, the Eq. (1.5) becomes

$$(-\Delta)^s u + V(\varepsilon x)|u|^{p-2}u = g(x,u) \text{ in } \mathbb{R}^N, \tag{1.6}$$

which has been proposed by Laskin [34,35] as a result of expanding the Feynman path integral, from the Brownian like to the Lévy quantum mechanical paths. We refer the readers to [5,6,30,49–51] for more results about Eq. (1.6). Recently, many authors studied the existence of multiple solution to (1.5) in subcritical growth, exponential growth and Kirchhoff type problem involving fractional $p$-Laplace such as Xiang, Zhang and [58], Zhang, Fiscella and Liang [60], Wang and Xiang [63]. In that works, they use Krasnoselskii's genus theory to study their problems. Motivate by above works, we study the problem (1.1) with exponential growth. We point out that as far as we know, in the literature appears only few papers on fractional $(p,q)$-Laplace problems, and there are no results on the multiplicity and concentration of solutions to the problem (1.1). So the aim of this work is to give the first result in this direction. We use the Ljusternik–Schnirelmann category theory instead of Krasnoselskii's genus theory as in some previous works.

Before starting our results, we recall some useful notations. Suppose that $N = ps$ or $N > ps$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where $[u]_{s,p}$ denotes by the seminorm Gagliardo, that is

$$[u]_{s,p} = \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\Big)^{1/p}.$$

$W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space (similar to [46]) with norm

$$||u|| = \Big(||u||^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\Big)^{1/p}.$$

Set $\eta > 0$, we denote another norm on $W^{s,p}(\mathbb{R}^N)$ as follows

$$||u||_{\eta, W^{s,p}(\mathbb{R}^N)} = \Big(\eta||u||^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\Big)^{1/p}.$$

Then $||.||$ and $||.||_{\eta, W^{s,p}(\mathbb{R}^N)}$ are two norms equivalent on $W^{s,p}(\mathbb{R}^N)$. For each $\varepsilon > 0$, let $W_\varepsilon$ denote by the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm

$$||u||_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} = \Big([u]^p_{s,p} + ||u||^p_{p,V,\varepsilon}\Big)^{1/p}, \; ||u||^p_{p,V,\varepsilon} = \int_{\mathbb{R}^N} V(\varepsilon x)|u(x)|^p dx.$$

Then $W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)$ is uniformly convex Banach space (similar to [46], Lemma 10), and then $W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)$ is a reflexive space. By the condition $(V)$ and Theorem 6.9 [42], we have the embedding from $W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)$ into $L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [\frac{N}{s}, +\infty)$. Similarly, we can define the space $W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)$, $i = 1, \dots, m$. We denote $W_\varepsilon = W^{s,p}_{V,\varepsilon}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)$ endowed with the norm

$$||u||_{W_\varepsilon} = ||u||_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||u||_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)}.$$

Then $W_\varepsilon$ is uniformly convex Banach space (similar to [46], Lemma 10) and we have the embeddings

$$W_\varepsilon = W^{s,p}_{V,\varepsilon}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N) \hookrightarrow W^{s,p}_{V,\varepsilon}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$$

are continuous for any $\nu \in [\frac{N}{s}, +\infty)$. Hence, there exists a best constant $S_{\nu,\varepsilon} > 0$ for all $\nu \in [\frac{N}{s}, +\infty)$ as follows:

$$S_{\nu,\varepsilon} = \inf_{u \neq 0, u \in W_\varepsilon} \frac{||u||_{W_\varepsilon}}{||u||_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$||u||_{L^\nu(\mathbb{R}^N)} \leq S^{-1}_{\nu,\varepsilon} ||u||_{W_\varepsilon} \text{ for all } u \in W_\varepsilon. \tag{1.7}$$

**Definition 1.** We say that $u \in W_\varepsilon$ is a weak solution of problem (1.1) if

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} dxdy \\
&+ \sum_{i=1}^m \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p_i s}} dxdy \\
&+ \int_{\mathbb{R}^N} V(\varepsilon x)(|u(x)|^{\frac{N}{s}-2} u(x) \\
&+ \sum_{i=1}^m |u(x)|^{p_i-2} u(x))\varphi(x) dx = \int_{\mathbb{R}^N} f(u(x))\varphi(x) dx
\end{aligned}$$

for any $\varphi \in W_\varepsilon$.

We denote $\text{cat}_B(A)$ by the category of $A$ with respect to $B$, namely the least integer $k$ such that $A \subset A_1 \cup \cdots \cup A_k$, where $A_i$ $(i = 1, \dots, k)$ is closed and contractible in $B$. We set $\text{cat}_B(\emptyset) = 0$ and $\text{cat}_B(A) = +\infty$ if there is no integer with above property. We refer the reader to [57] for more details on Ljusternik–Schnirelmann theory. Now, we state the main result in this paper.

**Theorem 2.** *Let* $(V_1)$, $(V_2)$ *and* $(f_1)-(f_5)$ *hold. Then for any* $\delta > 0$ *such that*

$$M_\delta = \{x \in \mathbb{R}^3 : dist(x, M) \leq \delta\} \subset \Lambda,$$

*there exists* $\varepsilon_\delta > 0$ *such that problem* $(P_\varepsilon)$ *has at least* $cat_{M_\delta}(M)$ *nontrivial non-negative weak solutions for any* $0 < \varepsilon < \varepsilon_\delta$. *Moreover, if* $u_\varepsilon$ *denotes one of these solutions and* $\eta_\varepsilon$ *is its global maximum, then*

$$\lim_{\varepsilon \to 0^+} V(\eta_\varepsilon) = V_0.$$

*Remark 3.* We use the Nehari manifold, penalization method, concentration compactness principle and Ljusternik–Schnirelmann theory to prove the main result. There are some difficulties in proving our theorem. The first difficulty is that the nonlinearity $f$ has exponential critical growth. The second is that the fractional Sobolev embedding is the lack of compactness. Furthermore, our problem cannot transfer to local problem via to Caffarelli–Silvestre's method. Compare with subcritical case due to Ambrosio–Radulescu [8] as $m = 1$, we need estimate the Mountain pass level due to the Trudinger–Moser nonlinearity and all our steps need focus it. Then our duties are complex and they are not the same in the work of Ambrosio–Radulescu. We emphase that the work Ambrosio–Radulescu studied the Eq. (1.1) when $m = 1$ and $0 < N < ps$. In this case we have the continuous embedding from $W^{s,p}(\Omega)$ into $L^{Np/(N-sp)}(\Omega)$. In our work, $N = ps$, then we do not have the previous embedding. Hence, our work is independent with work of Ambrosio–Radulescu [8]. Furthermore, our problem is more complicated than the problem in [8] due to many phases, not only double phases.

The paper is organized as follows. In Sect. 2, we study the autonomous problem associated. In Sect. 3, we study the modified problem. We prove the Palais-Smale condition for the energy functional and provide some tools which are useful to establish a multiplicity result. This allows us to show that the modified problem has multiple solutions. In Sect. 4, we prove the existence of ground state solution to modified problem. In the final part of this paper, we complete the paper with the proof of Theorem 2.

## 2. Autonomous problem

In this section, we study the autonomous problem associated to (1.1) as following

$$(-\Delta)^s_{N/s} u + \sum_{i=1}^m (-\Delta)^s_{p_i} u + \eta \left( |u|^{\frac{N}{s}-2} u + \sum_{i=1}^m |u|^{p_i - 2} u \right) = f(u) \text{ in } \mathbb{R}^N, \qquad (P_\eta) \tag{2.1}$$

where $\eta > 0$ is a constant. Set $W = W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$. We denote $J_\eta : W \to \mathbb{R}$ by the corresponding energy functional for problem (2.1)

$$J_\eta(u) = \frac{1}{p} ||u||^p_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i} ||u||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u) dx.$$

From the condition $(f_3)$, there exist $\tau > 0$ and $\delta > 0$ such that for all $|t| \leq \delta$, we have

$$|f(t)| \leq \tau |t|^{p_m - 1}. \tag{2.2}$$

Moreover from the condition $(f_1)$ and $f$ is a continuous function, for each $q \geq \frac{N}{s}$, we can find a constant $C = C(q, \delta) > 0$ such that

$$|f(t)| \leq C|t|^{q-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}) \tag{2.3}$$

for all $|t| \geq \delta$. Combine (2.2) and (2.3), we get

$$|f(t)| \leq \tau|t|^{p_m-1} + C|t|^{q-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}) \tag{2.4}$$

for all $t \geq 0$ and

$$|F(t)| \leq \int\limits_0^t |f(s)|ds \leq \tau|t|^{p_m} + C|t|^q\Phi_{N,s}(\alpha_0|t|^{N/(N-s)}) \tag{2.5}$$

for all $t \geq 0$.

**Definition 4.** We said that $u \in W$ is a weak solution of (2.1) if

$$\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}dxdy$$
$$+\sum_{i=1}^m \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p_is}}dxdy$$
$$+\int\limits_{\mathbb{R}^N} \eta(|u(x)|^{\frac{N}{s}-2}u(x) + \sum_{i=1}^m |u(x)|^{p_i-2}u(x))\varphi(x)dx = \int\limits_{\mathbb{R}^N} f(u(x))\varphi(x)dx$$

for any $\varphi \in W$.

In order to prove the result in this paper, we need the following result:

**Lemma 1.** ([61]) *Let* $s \in (0, 1)$ *and* $sp = N$. *Then for every* $0 \leq \alpha < \alpha_* \leq \alpha^*_{s,N}$, *the following inequality holds:*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), ||u||_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})dx < +\infty,$$

*where* $\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p-2} \frac{t^j}{j!}$, $j_p = \min\{j \in \mathbb{N} : j \geq p\}$. *Moreover, for* $\alpha > \alpha^*_{s,N}$,

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), ||u||_{W^{s,p}(\mathbb{R}^N)} \leq 1} \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})dx = +\infty.$$

*Remark 5.* From Lemma 1, if we use the norm $||.||_\eta$ on $W^{s,N/s}(\mathbb{R}^N)$, then we have

$$(\max\{1,\eta\})^{-1/p}||u||_{\eta,W^{s,p}(\mathbb{R}^N)} \le ||u||_{W^{s,p}(\mathbb{R}^N)} \le (\min\{1,\eta\})^{-1/p}||u||_{\eta,W^{s,p}(\mathbb{R}^N)},$$

then we get

$$\sup_{u\in W^{s,p}(\mathbb{R}^N),||u||_{\eta,W^{s,p}(\mathbb{R}^N)}\le(\min\{1,\eta\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha|u|^{N/(N-s)})dx < +\infty$$

for all $0 \le \alpha < \alpha_* \le \alpha^*_{s,N}$.

Using Lemma 1 and note that $C_0^\infty(\mathbb{R}^N)$ is a density subspace of $W^{s,p}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$, we see that $J_\eta$ is well defined on $W^{s,N/s}(\mathbb{R}^N)\cap\cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$. Furthermore, we have

$$\begin{aligned}< J_\eta^{'}(u),\varphi > = &\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}dxdy \\ &+\sum_{i=1}^m \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p_is}}dxdy \\ &+\eta\int_{\mathbb{R}^N}\left(|u|^{\frac{N}{s}-2}u+\sum_{i=1}^m |u|^{p_i-2}u\right)\varphi dx - \int_{\mathbb{R}^N} f(u)\varphi dx.\end{aligned}$$

We know that $W$ is uniformly convex with norm

$$||u||_W = ||u||_{W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u||_{W^{s,p_i}(\mathbb{R}^N)}.$$

Another norm is

$$||u||_{\eta,W} = ||u||_{\eta,W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u||_{\eta,W^{s,p_i}(\mathbb{R}^N)}.$$

By Theorem 6.9 [42], we have the embedding from $W^{s,N/s}(\mathbb{R}^N)$ into $L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [\frac{N}{s},+\infty)$ and $W = W^{s,p}(\mathbb{R}^N)\cap\cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$ is continuously embedded into $W^{s,p}(\mathbb{R}^N)$. Hence, $W$ is continuously embedded into $L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [\frac{N}{s},+\infty)$. Then there exists a best constant $A_{\nu,\eta} > 0$ for all $\nu \in [\frac{N}{s},+\infty)$ as follows:

$$A_{\nu,\eta} = \inf_{u\neq 0,u\in W} \frac{||u||_{\eta,W}}{||u||_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$||u||_{L^\nu(\mathbb{R}^N)} \le A_{\nu,\eta}^{-1}||u||_{\eta,W} \text{ for all } u \in W. \tag{2.6}$$

We can check that $J_\eta$ satisfies the geometry condition of Mountain Pass Theorem. Indeed, we have the following result:

**Lemma 2.** *Suppose that* $(f_1)$ *and* $(f_3)$ *hold. Then there exist constants positive* $t_0, \rho_0$ *such that* $J_\eta(u) \ge \rho_0$ *for all* $u \in W$, *with* $||u||_{\eta,W} = t_0$.

*Proof.* From (2.4), for some $q > p_m$, we have

$$|F(t)| \le \tau|t|^{p_m} + C|t|^q \Phi_{N,s}(\alpha_0|t|^{N/(N-s)})$$

for all $t \in \mathbb{R}$. Then we get

$$\begin{aligned} J_\eta(u) &= \frac{s}{N}||u||^{N/s}_{\eta,W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i}||u||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u)dx \\ &\ge \frac{s}{N}||u||^{N/s}_{\eta,W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i}||u||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} - \tau\int_{\mathbb{R}^N} |u|^{p_m}dx \\ &\quad - C\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})dx. \end{aligned} \tag{2.7}$$

Using Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})dx \\ &\le \left( \int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t dx \right)^{1/t} ||u||^q_{L^{qt'}(\mathbb{R}^N)}, \end{aligned} \tag{2.8}$$

where $t > 1, t' > 1$ such that $\frac{1}{t} + \frac{1}{t'} = 1$. By Lemma 2.3 [38], for any $\mathfrak{b} > t$, there exist a constant $C(\mathfrak{b}) > 0$ such that

$$\left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t \le C(\mathfrak{b})\Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)}) \tag{2.9}$$

on $\mathbb{R}^N$. Denote by $\mathfrak{d} = \min\{1, \eta\}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t dx \le C(\mathfrak{b})\int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)})dx \\ &= C(\mathfrak{b})\int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0\mathfrak{d}^{-s/(N-s)}||u||^{N/(N-s)}_{\eta,W^{s,p}(\mathbb{R}^N)}|\mathfrak{d}^{s/N}u/||u||_{\eta,W^{s,p}(\mathbb{R}^N)}|^{N/(N-s)})dx. \end{aligned} \tag{2.10}$$

We know that $||u||_{\eta, W^{s,p}(\mathbb{R}^N)} \leq ||u||_{\eta, W}$, then $||u||_{\eta, W}$ is small enough implies that $||u||_{\eta, W^{s,p}(\mathbb{R}^N)}$ is also small enough. Therefore, when $\mathfrak{b}$ near $t$, we have

$$\mathfrak{b}\alpha_0 \mathfrak{d}^{-s/(N-s)} ||u||^{N/(N-s)}_{\eta, W^{s,p}(\mathbb{R}^N)} < \alpha_*, \tag{2.11}$$

by Remark 5, (2.10) and (2.11), there exists a constant $D > 0$ such that

$$\left( \int_{\mathbb{R}^N} \left( \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) \right)^t dx \right)^{1/t} \leq D.$$

Since the embedding from $W \to L^{qt'}(\mathbb{R}^N)$ is continuous, we get

$$\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) dx \leq D A^{-q}_{qt',\eta} ||u||^q_{\eta, W} < +\infty. \tag{2.12}$$

From (2.6), we have

$$||u||_{L^{p_m}(\mathbb{R}^N)} \leq A^{-1}_{p_m,\eta} ||u||_{\eta, W} \text{ for all } u \in W. \tag{2.13}$$

Note that the function $f(t) = t^{p_m}$ is convex, then

$$\left( \frac{a_1 + \cdots + a_{m+1}}{m+1} \right)^{p_m} \leq \frac{a_1^{p_m} + \cdots + a_m^{p_m}}{m+1}$$

for all $a_i \geq 0, i = 1, \ldots, m+1$. Hence apply above inequality, combine (2.7), (2.12) and (2.13), when $||u||_{\eta, W}$ is small enough, we obtain

$$\begin{aligned} J_\eta(u) \geq & \frac{(m+1)^{1-p_m}}{p_m} (||u||_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u||_{\eta, W^{s,p_i}(\mathbb{R}^N)})^{p_m} \\ & - \tau A^{-p_m}_{p_m,\eta} ||u||^{p_m}_{\eta, W} - C D A^{-q}_{qt',\eta} ||u||^q_{\eta, W} \\ = & ||u||^{p_m}_{\eta, W} \Big[ \Big( \frac{(m+1)^{1-p_m}}{p_m} - \tau A^{-p_m}_{p_m,\eta} \Big) - C D A^{-q}_{qt',\eta} ||u||^{q-p_m}_{\eta, W} \Big]. \end{aligned} \tag{2.14}$$

We see $\dfrac{(m+1)^{1-p_m}}{p_m} - \tau A^{-p_m}_{p_m,\eta} > 0$ for $\tau$ small enough. Let

$$h(t) = \frac{(m+1)^{1-p_m}}{p_m} - \tau A^{-p_m}_{p_m,\eta} - C D A^{-q}_{qt',\eta} t^{q-p_m}, t \geq 0.$$

We now prove there exists $t_0 > 0$ small satisfying $h(t_0) \geq \dfrac{1}{2}(\dfrac{(m+1)^{1-p_m}}{p_m} - \tau A^{-p_m}_{p_m,\eta})$. We see that $h$ is continuous function on $[0, +\infty)$ and $\lim_{t \to 0^+} h(t) = \dfrac{(m+1)^{1-p_m}}{p_m} - \tau A^{-p_m}_{p_m,\eta}$, then there exists $t_0$ such that $h(t) \geq \dfrac{(m+1)^{1-p_m}}{p_m} -$

$\tau A_{p_m,\eta}^{-p_m} - \varepsilon_1$ for all $0 \leq t \leq t_0$, $t_0$ is small enough such that $||u||_{\eta,W} = t_0$ satisfies (2.11). If we choose $\varepsilon_1 = \frac{1}{2}(\frac{(m+1)^{1-p_m}}{p_m} - \tau A_{p_m,\eta}^{-p_m})$, we have

$$h(t) \geq \frac{1}{2}\left(\frac{(m+1)^{1-p_m}}{p_m} - \tau A_{p_m,\eta}^{-p_m}\right)$$

for all $0 \leq t \leq t_0$. Especialy,

$$h(t_0) \geq \frac{1}{2}\left(\frac{(m+1)^{1-p_m}}{p_m} - \tau A_{p_m,\eta}^{-p_m}\right). \tag{2.15}$$

From (2.14) and (2.15), for $||u||_{\eta,W} = t_0$, we have

$$J_\eta(u) \geq \frac{t_0^{p_m}}{2} \cdot \left(\frac{(m+1)^{1-p_m}}{p_m} - \tau A_{p_m,\eta}^{-p_m}\right) = \rho_0.$$

□

**Lemma 3.** *Suppose that* $(f_4)$ *holds. Then there exists a function* $v \in C_0^\infty(\mathbb{R}^N)$ *with* $||v||_{\eta,W} > t_0$, *such that* $J_\eta(v) < 0$, *where* $t_0 > 0$ *is the number given in Lemma 3.*

*Proof.* For all $u \in C_0^\infty(\mathbb{R}^N)$ with $||u||_{\eta,W} = 1$, from the condition $(f_4)$ and all $t > 0$, we obtain

$$\begin{aligned} J_\eta(tu) &= \frac{st^{N/s}}{N}||u||_{\eta,W^{s,p}(\mathbb{R}^N)}^{N/s} + \sum_{i=1}^m \frac{t^{p_i}}{p_i}||u||_{\eta,W^{s,p_i}(\mathbb{R}^N)}^{p_i} - \int_{\mathbb{R}^N} F(tu)dx \\ &\leq \frac{st^{N/s}}{N}||u||_{\eta,W}^{N/s} + \sum_{i=1}^m \frac{t^{p_i}}{p_i}||u||_{\eta,W^{s,p_i}(\mathbb{R}^N)}^{p_i} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |u(x)|^\mu dx \\ &\leq \frac{st^{N/s}}{N} + \sum_{i=1}^m \frac{t^{p_i}}{p_i} - \gamma_1 t^\mu \int_{\mathbb{R}^N} |u(x)|^\mu dx. \end{aligned}$$

By (2.6), for all $\nu \geq \frac{N}{s}$, we have

$$0 < \frac{1}{A_{\nu,\eta} + \varepsilon} = \frac{||u||_{\eta,W}}{A_{\nu,\eta} + \varepsilon} \leq ||u||_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,\eta}^{-1}||u||_{\eta,W} = A_{\nu,\eta}^{-1} < +\infty,$$

where $\varepsilon > 0$. Since $\mu > p_m$, we have $J_\eta(tu) \to -\infty$ as $t \to +\infty$. Taking $v = \rho_1 u$, $\rho_1 > t_0 > 0$ large enough, we have $J_\eta(v) < 0$, $||v||_{\eta,W} > t_0$. □

Using the version of Mountain Pass Theorem without the Palais-Smale condition, we get a sequence $\{u_n\} \subset W$ such that

$$J_\eta(u_n) \to c_\eta \text{ and } J_\eta'(u_n) \to 0 \text{ as } n \to \infty,$$

where the level $c_\eta$ is characterized by

$$c_\eta = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\eta(\gamma(t))$$

and $\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0, J_\eta(\gamma(1)) < 0\}$.

**Lemma 4.** *Let $\{u_n\}$ be $(PS)_{c_\eta}$ sequence for $J_\eta$. Then there exists a constant $C_{\gamma_1}$ such that $\rho_0 \le c_\eta \le C_{\gamma_1}$.*

*Proof.* We choose a function $w \in W\backslash\{0\}$ such that $||w||_{L^\mu(\mathbb{R}^N)} = 1$ and $||w||_{\eta,W} \le A_{\mu,\eta} + \varepsilon$ for some $\varepsilon > 0$ small enough. We see that

$$\begin{aligned} c &\le \max_{t\ge 0} J_\eta(tw) \\ &= \max_{t\ge 0}\left\{\frac{st^{N/s}}{N}||w||^{N/s}_{\eta,W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m}\frac{t^{p_i}}{p_i}||w||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} - \gamma_1 t^\mu \int_{\mathbb{R}^N}|w(x)|^\mu dx\right\} \\ &\le \max_{t\ge 0}\left\{\frac{s(A_{\mu,\eta}+\varepsilon)^{N/s}t^{N/s}}{N} + \sum_{i=1}^{m}\frac{(A_{\mu,\eta}+\varepsilon)^{p_i}t^{p_i}}{p_i} - \gamma_1 t^\mu\right\}. \end{aligned} \tag{2.16}$$

Set $g(t) = \sum_{i=1}^m \dfrac{(A_{\mu,\eta}+\varepsilon)^{p_i}t^{p_i}}{p_i} + \dfrac{s(A_{\mu,\eta}+\varepsilon)^{N/s}t^{N/s}}{N} - \gamma_1 t^\mu$ on $[0,+\infty)$. We have

$$c \le \max_{t\in[0,1]} g(t) + \max_{t\ge 1} g(t). \tag{2.17}$$

When $t \in [0,1]$, we get

$$g(t) \le h(t) = \left(\sum_{i=1}^{m}\frac{(A_{\mu,\eta}+\varepsilon)^{p_i}}{p_i} + \frac{s(A_{\mu,\eta}+\varepsilon)^{N/s}}{N}\right) t^{\frac{N}{s}} - \gamma_1 t^\mu.$$

We denote $a = \dfrac{s(A_{\mu,\eta}+\varepsilon)^{N/s}}{N} + \sum_{i=1}^m \dfrac{(A_{\mu,\eta}+\varepsilon)^{p_i}}{p_i}$, $b = \gamma_1$. Compute directly, we have

$$\max_{t\in[0,1]} g(t) \le h(\theta_{\gamma_1}) = C_{\gamma_1}, \tag{2.18}$$

where

$$\theta_{\gamma_1} = \left(\frac{aN}{s\gamma_1\mu}\right)^{s/(\mu s-N)} \le 1$$

as $\gamma_1 \ge \dfrac{aN}{s\mu} = \gamma^*$. Compute directly, we get

$$C_{\gamma_1} = h(\theta_{\gamma_1}) = a\left(1 - \frac{N}{s\mu}\right)\left(\frac{aN}{sb\mu}\right)^{N/(\mu s-N)}. \tag{2.19}$$

We see that $\lim_{\gamma_1\to+\infty}\theta_{\gamma_1} = 0$, then $\lim_{\gamma_1\to+\infty} h(\theta_{\gamma_1}) = 0$. By arguments as above, for all $t \ge 1$, we get

$$g(t) \le h_*(t) = \left(\sum_{i=1}^{m}\frac{(A_{\mu,\eta}+\varepsilon)^{p_i}}{p_i} + \frac{s(A_{\mu,\eta}+\varepsilon)^{N/s}}{N}\right) t^{p_m} - \gamma_1 t^\mu$$

and $h_*$ has uniqueness local maximum point at $\beta_{\gamma_1} = \left(\frac{ap_m}{\gamma_1 \mu}\right)^{1/(\mu - p_m)}$ on $(0, +\infty)$. Note that if we choose $\gamma_1 \geq \gamma_*$, where $\gamma_*$ satisfies

$$\left(\frac{ap_m}{\gamma_* \mu}\right)^{1/(\mu - p_m)} \leq 1,$$

we deduce

$$\max_{t \geq 1} g(t) \leq h_*(1) = \sum_{i=1}^{m} \frac{(A_{\mu,\eta} + \varepsilon)^{p_i}}{p_i} + \frac{s(A_{\mu,\eta} + \varepsilon)^{N/s}}{N} - \gamma_1.$$

Set $\gamma_{**} = \sum_{i=1}^{m} \frac{(A_{\mu,\eta} + \varepsilon)^{p_i}}{p_i} + \frac{s(A_{\mu,\eta} + \varepsilon)^{N/s}}{N}$. We have

$$\max_{t \geq 1} g(t) \leq 0 \text{ for all } \gamma_1 \geq \max\{\gamma_*, \gamma_{**}\}. \tag{2.20}$$

Combine (2.17), (2.18), (2.19) and (2.20), we obtain

$$c \leq C_{\gamma_1} = a\left(1 - \frac{N}{s\mu}\right)\left(\frac{aN}{bs\mu}\right)^{N/(\mu s - N)} \tag{2.21}$$

for $\gamma_1 \geq \max\{\gamma^*, \gamma_*, \gamma_{**}\}$. Therefore, the Mountain Pass level $c$ is small enough when $\gamma_1$ is large enough, which will be used later. Combine Lemma 2, (2.16) and (2.21), we get $\rho_0 \leq c_\eta \leq C_{\gamma_1}$. □

The following result is a version of Lions's result:

**Lemma 5.** ([54]) *If $\{u_n\}$ is a bounded sequence in $W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^{m} W^{s,p_i}(\mathbb{R}^N)$ and*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^t dx = 0$$

*for some $R > 0$, $t \geq \frac{N}{s}$, then $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for all $q \in (t, +\infty)$.*

**Lemma 6.** *Let $\{u_n\}$ be a sequence in $W$ converging weakly to 0 verifying*

$$\limsup_{n \to \infty} ||u_n||_{\eta,W}^{N/(N-s)} < \frac{\alpha_* \mathfrak{d}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*where $\mathfrak{c} > 1$ is a suitable constant and assume that $(f_1)$ holds and $\lim_{t \to 0^+} \frac{f(t)}{t^{p_m - 1}} = 0$. If there exists $R > 0$ such that $\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^{p_m} dx = 0$, it follows that*

$$\int_{\mathbb{R}^N} f(u_n) u_n dx \to 0 \text{ and } \int_{\mathbb{R}^N} F(u_n) dx \to 0.$$

*Proof.* Since $\liminf_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{B_R(y)}|u_n|^{p_m}dx=0$, by Lemma 5, we get $u_n\to 0$ strongly in $L^t(\mathbb{R}^N)$ for all $t\in(p_m,+\infty)$. From the condition $(f_1)$ and

$$\lim_{t\to 0^+}\frac{f(t)}{t^{p_m-1}}=0,$$

then for any $\varepsilon>0$ and $q>p_m$, there exists $C(q,\varepsilon)>0$ such that

$$|f(u_n)u_n|\le\varepsilon|u_n|^{p_m}+C(q,\varepsilon)|u_n|^q\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}). \tag{2.22}$$

For $t>1$, $t'>1$ and $t'$ near 1 such that $\frac{1}{t}+\frac{1}{t'}=1$, using Hölder inequality, we get

$$\int\limits_{\mathbb{R}^N}|u_n|^q\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})dx$$
$$\le\left(\int\limits_{\mathbb{R}^N}|u_n|^{qt}dx\right)^{1/t}\left(\int\limits_{\mathbb{R}^N}(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)}))^{t'}dx\right)^{1/t'}. \tag{2.23}$$

Then by Lemma 2.3 [38], for any $\mathfrak{c}>t'$ and near $t'$, there exist a constant $C(\mathfrak{c})>0$ such that

$$\left(\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})\right)^{t'}\le C(\mathfrak{c})\Phi_{N,s}\left(\mathfrak{c}\alpha_0|u_n|^{N/(N-s)}\right) \tag{2.24}$$

on $\mathbb{R}^N$ and all $n$. We have

$$\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{c}\alpha_0|u_n|^{N/(N-s)})dx$$
$$=\int\limits_{\mathbb{R}^N}\Phi_{N,s}\left(\mathfrak{c}\alpha_0\mathfrak{d}^{-s/(N-s)}||u_n||^{N/(N-s)}_{\eta,W^{s,p}(\mathbb{R}^N)}\mathfrak{d}^{s/(N-s)}\left(\frac{|u_n|}{||u_n||_{\eta,W^{s,p}(\mathbb{R}^N)}}\right)^{N/(N-s)}\right)dx. \tag{2.25}$$

Since $||u_n||_{\eta,W^{s,p}(\mathbb{R}^N)}\le||u||_{\eta,W}$, from Remark 5, we get

$$\sup_n\int\limits_{\mathbb{R}^N}\Phi_{N,s}(\mathfrak{c}\alpha_0|u_n|^{N/(N-s)})dx<+\infty. \tag{2.26}$$

Combine (2.23)-(2.26) and the fact that $u_n\to 0$ in $L^{qt}(\mathbb{R}^N)$, we obtain

$$\int\limits_{\mathbb{R}^N}|f(u_n)u_n|dx\le\varepsilon\int\limits_{\mathbb{R}^N}|u_n|^{p_m}dx+C(q,\varepsilon)\int\limits_{\mathbb{R}^N}|u_n|^q\Phi_{N,s}(\alpha_0|u_n|^{N/(N-s)})dx\to 0 \tag{2.27}$$

as $n\to\infty$ since $\{u_n\}$ is a bounded sequence in $L^{p_1}(\mathbb{R}^N)$. Similarly as (2.27), we also get $\int\limits_{\mathbb{R}^N}|F(u_n)|dx\to 0$ as $n\to\infty$. □

**Proposition 1.** *Assume that the conditions* $(f_1)-(f_5)$ *satisfies. Then problem* (2.1) *admits a nontrivial nonnegative weak solution.*

*Proof.* From Lemma 2, Lemma 3 and a version of Mountain Pass Theorem without the Palais–Smale condition [47,57], we get a sequence $\{u_n\} \subset W$ such that

$$J_\eta(u_n) \to c_\eta \text{ and } J'_\eta(u_n) \to 0 \text{ as } n \to \infty,$$

where the level $c_\eta$ is characterized by

$$0 < c_\eta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\eta(\gamma(t)).$$

By the assumption $(f_5)$, using the idea in [43] and Lemma 3.2 [7], we can get

$$c_\eta = \inf_{u \in W \setminus \{0\}} \sup_{t \geq 0} J_\eta(tu) = \inf_{u \in \mathcal{N}_\eta} J_\eta(u),$$

where $\mathcal{N}_\eta$ is Nehari manifold for $J_\eta$.

Note that $\{u_n\}$ is a $(PS)$ sequence with level $c_\eta \in \mathbb{R}$ in $W$. This means

$$J_\eta(u_n) \to c_\eta \text{ and } \sup_{||\varphi||_{\eta,W}=1} |< J'_\eta(u_n), \varphi >| \to 0 \tag{2.28}$$

as $n \to \infty$. We show that the sequence $\{u_n\}$ is bounded in $W$. From (2.28), we have

$$< J'_\eta(u_n), \frac{u_n}{||u_n||_{\eta,W}} >= o_n(1) \text{ and } J_\eta(u_n) = c_\eta + o_n(1)$$

when $n$ large enough. It implies

$$J_\eta(u_n) - \frac{1}{\mu} < J'_\eta(u_n), u_n >= c_\eta + o_n(1) + o_n(1)||u_n||_{\eta,W}, \tag{2.29}$$

where $\mu$ is a parameter in the condition $(f_2)$. We have

$$\begin{aligned}
J_\eta(u_n) - \frac{1}{\mu} < J'_\eta(u_n), u_n >&= \frac{s}{N}||u_n||^{N/s}_{\eta,W^{s,p}(\mathbb{R}^N)} \\
&+ \sum_{i=1}^m \frac{1}{p_i}||u_n||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u_n)dx \\
&- \frac{1}{\mu}\Big[||u_n||^{N/s}_{\eta,W^{s,N/s}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} f(u_n)u_n dx\Big] \\
&= \Big(\frac{s}{N} - \frac{1}{\mu}\Big)||u_n||^{N/s}_{\eta,W^{s,p}(\mathbb{R}^N)} \\
&+ \sum_{i=1}^m \Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)||u_n||^{p_i}_{\eta,W^{s,p_i}(\mathbb{R}^N)} + \frac{1}{\mu}\int_{\mathbb{R}^N} (f(u_n)u_n - \mu F(u_n))\,dx.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&J_\eta(u_n) - \frac{1}{\mu} < J'_\eta(u_n), u_n > \\
&\geq \Big(\frac{s}{N} - \frac{1}{\mu}\Big)||u_n||^{N/s}_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m}\Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)}.
\end{aligned} \tag{2.30}
$$

Combine (2.29) and (2.30), we get

$$
\begin{aligned}
&\Big(\frac{s}{N} - \frac{1}{\mu}\Big)||u_n||^{N/s}_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m}\Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} \\
&\leq c_\eta + o_n(1) + o_n(1)||u_n||_{\eta, W}.
\end{aligned} \tag{2.31}
$$

Note that

$$
\lim_{x\to+\infty, x_1\to+\infty, \ldots, x_m\to+\infty} \frac{\mathfrak{a}x^{N/s} + \mathfrak{b}_1 x_1^{p_1} \cdots + \mathfrak{b}_m x_m^{p_m}}{x + x_1 + \cdots + x_m} = +\infty,
$$

where $\mathfrak{a} > 0$, $\mathfrak{b}_1 > 0, \ldots, \mathfrak{b}_m > 0$. Then from (2.31), we conclude that the sequence $\{u_n\}$ is bounded in $W$. Since

$$
J_\eta(u_n) - \frac{1}{\mu} < J'_\eta(u_n), u_n > \to c_\eta
$$

as $n \to \infty$, then

$$
\limsup_{n\to\infty} ||u_n||^{N/s}_{\eta, W^{s,p}(\mathbb{R}^N)} \leq \frac{c_\eta}{\frac{s}{N} - \frac{1}{\mu}} \leq \frac{C_{\gamma_1}}{\frac{s}{N} - \frac{1}{\mu}} \tag{2.32}
$$

and

$$
\limsup_{n\to\infty} ||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} \leq \frac{c_\eta}{\frac{1}{p_i} - \frac{1}{\mu}} \leq \frac{C_{\gamma_1}}{\frac{1}{p_i} - \frac{1}{\mu}} \tag{2.33}
$$

for all $i = 1, \ldots, m$. Hence, we deduce

$$
\limsup_{n\to\infty} ||u_n||_{\eta, W} \leq \left(\frac{C_{\gamma_1}}{\frac{s}{N} - \frac{1}{\mu}}\right)^{s/N} + \sum_{i=1}^{m}\left(\frac{C_{\gamma_1}}{\frac{1}{p_i} - \frac{1}{\mu}}\right)^{\frac{1}{p_i}}. \tag{2.34}
$$

Moreover, we claim that there exists $R > 0$, $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$
\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^{p_m} dx \geq \delta. \tag{2.35}
$$

If the above inequality doesnot hold, it means that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^{p_m} dx = 0$$

for some $R > 0$, then from (2.21) and (2.34), when $\gamma_1$ large enough, we get

$$\limsup_{n\to\infty} ||u_n||_{\eta,W}^{N/(N-s)} < \frac{\alpha_* \mathfrak{d}^{s/(N-s)}}{\mathfrak{c}\alpha_0}.$$

Using Lemma 6, we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n dx \to 0$ as $n \to \infty$. Then

$$o(1) =< J_\eta'(u_n), u_n >= ||u_n||^p_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} f(u_n)u_n dx$$
$$= ||u_n||^p_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} + o(1)$$

as $n \to \infty$. Hence $u_n \to 0$ strongly in $W$. It implies that

$$J_\eta(u_n) = \frac{1}{p}||u_n||^p_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i}||u_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(u_n)dx \to 0$$

as $n \to \infty$. It contradicts with $c_\eta > 0$. Therefore (2.35) holds. We denote $v_n(x) = u_n(x + y_n)$, then from (2.35) we get

$$\int_{B_R(0)} |v_n|^{p_m} dx \geq \delta/2. \tag{2.36}$$

Because $J_\eta$ and $J_\eta'$ are both invariant by the translation, it implies that

$$J_\eta(v_n) \to c_\eta \text{ and } J_\eta'(v_n) \to 0 \text{ in } W^*.$$

Because $||v_n||_{\eta,W} = ||u_n||_{\eta,W}$, then $\{v_n\}$ is also bounded in $W$, then exists $v \in W$ such that $v_n \to v$ weak in $W$, $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ $(q \in (p_m, +\infty))$ and $v_n(x) \to v(x)$ almost everywhere in $\mathbb{R}^N$. From (2.36), we get $\int_{B_R(0)} |v|^{p_m} dx \geq \delta/2 > 0$, then $v \not\equiv 0$. By arguments as in [53,54], we get $J_\eta'(v) = 0$. Furthermore, from the condition $f(t) = 0$ for all $t \in (-\infty, 0]$, we can get $v \geq 0$.

By Fatou's lemma, we have

$$
\begin{aligned}
c_\eta \le J_\eta(v) &= J_\eta(v) - \frac{1}{\mu} < J_\eta'(v), v > \\
&= \left(\frac{s}{N} - \frac{1}{\mu}\right) ||v||^p_{\eta, W^{s,p}(\mathbb{R}^N)} \\
&\quad + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||v||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} + \frac{1}{\mu} \int\limits_{\mathbb{R}^N} (f(v)v - \mu F(v))dx \\
&\le \liminf_{n\to\infty} \left\{ \left(\frac{s}{N} - \frac{1}{\mu}\right) ||v_n||^p_{\eta, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||v_n||^{p_i}_{\eta, W^{s,p_i}(\mathbb{R}^N)} \right. \\
&\quad \left. + \frac{1}{\mu} \int\limits_{\mathbb{R}^N} (f(v_n)v_n - \mu F(v_n))dx \right\} = \liminf_{n\to\infty} \left\{ J_\eta(v_n) - \frac{1}{\mu} < J_\eta'(v_n), v_n > \right\} = c_\eta.
\end{aligned}
$$

Hence $v$ is a ground state solution to the problem (2.1). □

## 3. The modified problem

Now, we introduce a penalized function in the spirit of [44] which will be fundamental to get our main result. First of all, without loss of generality, we may assume that

$$0 \in \Lambda \text{ and } V(0) = V_0.$$

Let us choose $k > \dfrac{\mu}{\mu - p_m} > 1$ and $a > 0$ such that

$$\frac{f(a)}{a^{p_m - 1}} = \frac{V_0}{k}.$$

We define

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \le a \\ \dfrac{V_0}{k} t^{p_m - 1} & \text{if } t > a \end{cases},$$

and

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x)) \tilde{f}(t) \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

We show that if $u_\varepsilon$ is a solution in $W$ to

$$
\begin{aligned}
(-\Delta)^s_p u + \sum_{i=1}^m (-\Delta)^s_{p_i} u + V(\varepsilon x) \left( |u|^{p-2}u + \sum_{i=1}^m |u|^{p_i - 2} u \right) \\
= g(\varepsilon x, u) \text{ in } \mathbb{R}^N \quad (P_\varepsilon^*) \qquad (2.37)
\end{aligned}
$$

with $u_\varepsilon(x) \le a$ for all $x \in \Lambda^c_\varepsilon = \mathbb{R}^N \backslash \Lambda_\varepsilon$, where $\Lambda_\varepsilon := \{\mathbb{R}^N : \varepsilon x \in \Lambda\}$, then $g(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$. Hence $u_\varepsilon$ is a solution of (1.1).

**Definition 6.** We say that $u \in W_\varepsilon$ is a weak solution of problem (2.37) if

$$\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{\frac{N}{s}-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}}dxdy$$
$$+\sum_{i=1}^{m}\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p_is}}dxdy$$
$$+\int\limits_{\mathbb{R}^N} V(\varepsilon x)(|u(x)|^{\frac{N}{s}-2}u(x)+\sum_{i=1}^{m}|u(x)|^{p_i-2}u(x))\varphi(x)dx$$
$$=\int\limits_{\mathbb{R}^N} g(\varepsilon x,u(x))\varphi(x)dx$$

for any $\varphi \in W_\varepsilon$.

We have that $g$ satisfies the following properties [40]:
$(g_1)$ $g(x,t)=0$ for all $t \le 0$ and $g(x,t)>0$ for all $t>0$ and $x \in \mathbb{R}^N$;
$(g_2)$ $\lim_{t\to 0^+}\dfrac{g(x,t)}{t^{p_m-1}}=0$ uniformly with respect to $x \in \mathbb{R}^N$;
$(g_3)$ $g(x,t) \le f(t)$ for all $t \ge 0$ and $x \in \mathbb{R}^N$;
$(g_4)$ $0<\mu G(x,t) \le g(x,t)t$ for all $x \in \Lambda$ and $t>0$, where $G(x,t)=\int\limits_0^t g(x,\tau)d\tau$;
$(g_5)$ $0<p_m G(x,t) \le g(x,t)t \le \dfrac{V_0}{k}t^{p_m}$ for all $x \in \Lambda^c$ and $t>0$.
$(g_6)$ for each $x \in \Lambda$, the function $\dfrac{g(x,t)}{t^{p_m-1}}$ is a strictly increasing of $t$ in $(0,+\infty)$;
$(g_7)$ for each $x \in \Lambda^c$, the function $\dfrac{g(x,t)}{t^{p_m-1}}$ is a strictly increasing of $t$ in $(0,a)$. Further, if $t \ge a$, we have $\dfrac{g(x,t)}{t^{p_m-1}}=\dfrac{V_0}{k}$.

In order to study the Eq. (2.37), we consider the energy functional $I_\varepsilon : W_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u)=\frac{1}{p}||u||^p_{W^{s,p}_{V,\varepsilon}}+\sum_{i=1}^{m}\frac{1}{p_i}||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}}-\int\limits_{\mathbb{R}^N} G(\varepsilon x,u)dx.$$

By the condition $(f_1)$ and $(g_3)$, $I_\varepsilon$ is well defined on $W_\varepsilon$, $I_\varepsilon \in C^2(W_\varepsilon,\mathbb{R})$ and its critical points are weak solution of problem (2.37). Associated to $I_\varepsilon$, we consider the Nehari manifold $\mathcal{N}_\varepsilon$ given by

$$\mathcal{N}_\varepsilon=\{u \in W_\varepsilon\backslash\{0\} :< I'_\varepsilon(u),u>=0\},$$

where

$$< I_\varepsilon'(u), \varphi > = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy$$
$$+ \sum_{i=1}^{m} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+p_i s}} dxdy$$
$$+ \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^{p-2}u + \sum_{i=1}^{m} |u|^{p_i-2}u)\varphi dx - \int_{\mathbb{R}^N} g(\varepsilon x, u)\varphi dx$$

for any $u, \varphi \in W_\varepsilon$.

**Proposition 2.** *There exists $r_* > 0$ such that*

$$||u||_{W_\varepsilon} \geq r_* > 0 \textit{ for all } u \in \mathcal{N}_\varepsilon.$$

*Proof.* We are easy to get the inequality

$$||u||_{W^{s,p}(\mathbb{R}^N)} \leq \min\{1, V_0\}^{-1/p} ||u||_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} \leq \min\{1, V_0\}^{-1/p} ||u||_{W_\varepsilon}. \quad (2.38)$$

Then from Lemma 1 and (2.38), we have

$$\sup_{u \in W_\varepsilon, ||u||_{W_\varepsilon} \leq (\min\{1, V_0\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty \quad (2.39)$$

and

$$\sup_{u \in W^{s,p}_{V,\varepsilon}(\mathbb{R}^N), ||u||_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} \leq (\min\{1, V_0\})^{s/N}} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{N/(N-s)}) dx < +\infty \quad (2.40)$$

for all $0 \leq \alpha < \alpha_*$. From the condition $(f_1)$, $(f_3)$ and $(g_3)$, for any $\varepsilon_* > 0$ and $q > p_m$, there exists $C_{q,\varepsilon_*} > 0$ such that

$$|g(\varepsilon x, t)t| \leq |f(t)t| \leq \varepsilon_* |t|^{p_m} + C_{q,\varepsilon_*} |t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \quad (2.41)$$

for all $t \geq 0$. Combining (2.39) and (2.41), by arguments as Proposition 2 in [54], we can get the result of Proposition 2. We omit the details at here. □

**Lemma 7.** *The functional $I_\varepsilon$ satisfies the following conditions:*
*$(i)$ There exists $\alpha > 0, \rho > 0$ such that $I_\varepsilon(u) \geq \alpha$ for all $u \in W_\varepsilon$ with $||u||_{W_\varepsilon} = \rho$.*
*$(ii)$ There exists $e \in W_\varepsilon$ with $||e||_{W_\varepsilon} > \rho$ such that $I_\varepsilon(e) < 0$.*

*Proof.* First we prove the statement $(i)$. From (2.41), for any $\tau > 0$ and some $q > p_m$, there exists $C > 0$ such that

$$|G(\varepsilon x, t)| \leq |g(\varepsilon x, t)t| \leq |f(t)t| \leq \tau |t|^{p_m} + C|t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)})$$

for all $t \in \mathbb{R}$. Then for all $u \in W_\varepsilon$ such that $||u||_{W_\varepsilon} \in (0, 1)$, we have

$$I_\varepsilon(u) = \frac{1}{p}||u||^p_{W^{s,p}_{V,\varepsilon}} + \sum_{i=1}^{m} \frac{1}{p_i}||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}} - \int_{\mathbb{R}^N} G(\varepsilon x, u)dx$$
$$\geq \frac{(m+1)^{1-p_m}}{p_m}||u||^{p_m}_{W_\varepsilon} - \tau \int_{\mathbb{R}^N} |u|^{p_m} dx - C \int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)})dx. \tag{2.42}$$

Using Hölder inequality, we have

$$\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})dx \leq \left( \int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t dx \right)^{1/t} ||u||^q_{L^{qt'}(\mathbb{R}^N)}, \tag{2.43}$$

where $t > 1, t' > 1$ such that $\frac{1}{t} + \frac{1}{t'} = 1$. By Lemma 2.3 [38], for any $\mathfrak{b} > t$, there exist a constant $C(\mathfrak{b}) > 0$ such that

$$\left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t \leq C(\mathfrak{b})\Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)}) \tag{2.44}$$

on $\mathbb{R}^N$. We get

$$\int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t \leq C(\mathfrak{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0|u|^{N/(N-s)})dx$$
$$= C(\mathfrak{b}) \int_{\mathbb{R}^N} \Phi_{N,s}(\mathfrak{b}\alpha_0\mathfrak{d}^{-s/(N-s)}||u||^{N/(N-s)}_{W_\varepsilon}|\mathfrak{d}^{s/N}u/||u||_{W_\varepsilon}|^{N/(N-s)})dx. \tag{2.45}$$

When $||u||_{W_\varepsilon}$ is small enough and $\mathfrak{b}$ near $t$, we have

$$\mathfrak{b}\alpha_0\mathfrak{d}^{-s/(N-s)}||u||^{N/(N-s)}_{W_\varepsilon} < \alpha_*, \tag{2.46}$$

From (2.45) and (2.46), there exists a constant $D > 0$ such that

$$\left( \int_{\mathbb{R}^N} \left(\Phi_{N,s}(\alpha_0|u|^{N/(N-s)})\right)^t dx \right)^{1/t} \leq D.$$

Since the embedding from $W_\varepsilon \to L^{qt'}(\mathbb{R}^N)$ is continuous, we get

$$\int_{\mathbb{R}^N} |u|^q \Phi_{N,s}(\alpha_0|u|^{N/(N-s)})dx \leq DS^{-q}_{qt',\varepsilon}||u||^q_{W_\varepsilon} < +\infty. \tag{2.47}$$

From (1.7), we have

$$||u||_{L^{p_m}(\mathbb{R}^N} \leq S^{-1}_{p_m,\varepsilon}||u||_{W_\varepsilon} \text{ for all } u \in W_\varepsilon. \tag{2.48}$$

Hence, combine (2.42), (2.47) and (2.48), we obtain

$$\begin{aligned} I_\varepsilon(u) &\geq \frac{(m+1)^{1-p_m}}{p_m}||u||^{p_m}_{W_\varepsilon} - \tau S^{-p_m}_{p_m,\varepsilon}||u||^{p_m}_{W_\varepsilon} - CDS^{-q}_{qt',\varepsilon}||u||^q_{W_\varepsilon} \\ &= ||u||^{p_m}_{W_\varepsilon}\Big[\Big(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon}\Big) - CDS^{-q}_{qt',\varepsilon}||u||^{q-p_m}_{W_\varepsilon}\Big]. \end{aligned} \tag{2.49}$$

We see $\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon} > 0$ for $\tau$ small enough. Let

$$h(t) = \frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon} - CDS^{-q}_{qt'}t^{q-p_m}, t \geq 0.$$

We now prove there exists $t_0 > 0$ small satisfying $h(t_0) \geq \frac{1}{2}(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon})$. We see that $h$ is continuous function on $[0, +\infty)$ and $\lim_{t\to 0^+} h(t) = \frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon}$, then there exists $t_0$ such that $h(t) \geq \frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon} - \varepsilon_1$ for all $0 \leq t \leq t_0$, $t_0$ is small enough such that $||u||_{W_\varepsilon} = t_0$ satisfies (2.46). If we choose $\varepsilon_1 = \frac{1}{2}(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon})$, we have

$$h(t) \geq \frac{1}{2}\left(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon}\right)$$

for all $0 \leq t \leq t_0$. Especialy,

$$h(t_0) \geq \frac{1}{2}\left(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon}\right). \tag{2.50}$$

From (2.49) and (2.50), for $||u||_{W_\varepsilon} = t_0$, we have

$$I_\varepsilon(u) \geq \frac{t_0^{p_m}}{2} \cdot \left(\frac{(m+1)^{1-p_m}}{p_m} - \tau S^{-p_m}_{p_m,\varepsilon}\right) = \rho_0.$$

Second, we prove the statement $(ii)$. Set $u \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}$ such that $\text{supp}(u) \subset \Lambda_\varepsilon$. From the condition $(f_4)$ and all $t > 0$, we obtain

$$\begin{aligned} I_\varepsilon(tu) &= \frac{t^{N/s}}{p}||u||^{N/s}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{t^{p_i}}{p_i}||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} F(tu)dx \\ &\leq \frac{t^{N/s}}{p}||u||^{N/s}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{t^{p_i}}{p_i}||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \gamma_1 t^\mu \int_{\text{supp}(u)} |u(x)|^\mu dx. \end{aligned}$$

Since $\mu > p_m > \frac{N}{s}$, we have $I_\varepsilon(tu) \to -\infty$ as $t \to +\infty$. Taking $v = \rho_1 u$, $\rho_1 > t_0 > 0$ large enough, we have $I_\varepsilon(v) < 0$, $||v||_\eta > t_0$. □

From Lemma 7 and the version of Mountain Pass Theorem, there exists a $(PS)_{c_\varepsilon}$ sequence $\{u_n\} \subset W_\varepsilon$, that is,

$$I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I_\varepsilon^{'}(u_n) \to 0,$$

where

$$c_\varepsilon = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

and $\Gamma = \{\gamma \in C([0, 1], W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}$.

The following result is the characteristic of Mountain Pass level which the original idea comes from [43]:

**Proposition 3.** *We have* $c_\varepsilon = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$.

*Proof.* We denote $c_\varepsilon^* = \inf_{u\in W_\varepsilon\setminus\{0\}} \sup_{t\geq 0} I_\varepsilon(tu)$ and $c_\varepsilon^{**} = \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u)$. For each $u \in \mathcal{N}_\varepsilon\setminus\{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\varepsilon$ and the maximum of $I_\varepsilon(tu)$ for all $t \geq 0$ is achieved at $t = t(u)$. Indeed, by Lemma 7, $h_u(t) = I_\varepsilon(tu) > 0$ when $t > 0$ is small enough and $h_u(t) = I_\varepsilon(tu) < 0$ when $t > 0$ is large enough. Then there exists $t(u) > 0$ such that $h_u(t(u)) = I_\varepsilon(t(u)u) = max_{t\geq 0} I_\varepsilon(tu)$. By Fermat's Theorem, we have $h_u'(t(u)) = 0$ iff $t(u)u \in \mathcal{N}_\varepsilon$. From $g(\varepsilon x, t) = 0$ for all $t \leq 0$, it follows that

$$\begin{aligned}\frac{||u||^p_{W^{s,p}_{V,\varepsilon}}}{t^{p_m-p}} + \cdots + \frac{||u||^{p_1}_{W^{s,p_1}_{V,\varepsilon}}}{t^{p_m-p_1}} + ||u||^{p_m}_{W^{s,p_m}_{V,\varepsilon}} &= \int\limits_{\mathbb{R}^N} \frac{ug(\varepsilon x, tu)}{t^{p_m-1}}dx\\ &= \int\limits_{\{x\in\mathbb{R}^N : tu(x)>0\}} (u^+)^{p_m} \frac{g(\varepsilon x, tu^+)}{(tu^+)^{p_m-1}}dx.\end{aligned}$$

We conisder the case $m \geq 2$, the case $m = 1$ is proved similarly. Arguing by a contradiction, there exists two positive numbers $t_1 > t_2 > 0$ such that $t_1u, t_2u \in \mathcal{N}_\varepsilon$, from $(g_6)$, we get

$$\begin{aligned}&\left(\frac{1}{t_1^{p_m-p}} - \frac{1}{t_2^{p_m-p}}\right)[u]^p_{s,p} + \left(\frac{1}{t_1^{p_m-p}} - \frac{1}{t_2^{p_m-p}}\right)\int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^p dx + \cdots\\ &+ \left(\frac{1}{t_1^{p_m-p_{m-1}}} - \frac{1}{t_2^{p_m-p_{m-1}}}\right)[u]^{p_{m-1}}_{s,p_{m-1}}\\ &+ \left(\frac{1}{t_1^{p_m-p_{m-1}}} - \frac{1}{t_2^{p_m-p_{m-1}}}\right)\int\limits_{\mathbb{R}^N} V(\varepsilon x)|u|^{p_{m-1}}dx\\ &= \int\limits_{\mathbb{R}^N} (u^+)^{p_m}\left[\frac{g(\varepsilon x, t_1u^+)}{(t_1u^+)^{p_m-1}} - \frac{g(\varepsilon x, t_2u^+)}{(t_2u^+)^{p_m-1}}\right]dx\\ &= \int\limits_{\mathbb{R}^N\setminus\Lambda_\varepsilon} (u^+)^{p_m}\left[\frac{g(\varepsilon x, t_1u^+)}{(t_1u^+)^{p_m-1}} - \frac{g(\varepsilon x, t_2u^+)}{(t_2u^+)^{p_m-1}}\right]dx\end{aligned}$$

$$+\int_{\Lambda_\varepsilon}(u^+)^{p_m}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx$$
$$\geq \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(u^+)^{p_m}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx. \tag{2.51}$$

We have

$$\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}(u^+)^{p_m}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx$$
$$=\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_2u>a\}}(u^+)^{p_m}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx$$
$$+\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_2u\leq a<t_1u\}}(u^+)^{p_1}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx$$
$$+\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_1u<a\}}(u^+)^{p_m}\Big[\frac{g(\varepsilon x,t_1u^+)}{(t_1u^+)^{p_m-1}}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx$$
$$:=I+II+III.$$

By the definition of $g$, we have $I=0$. Since $g(\varepsilon x,t)=\tilde{f}(t)=\frac{V_0}{k}t^{p_m-1}$ for all $x\in\Lambda_\varepsilon^c$ and $t>a$, we get

$$II=\int_{(\mathbb{R}^N\setminus\Lambda_\varepsilon)\cap\{t_2u\leq a<t_1u\}}(u^+)^{p_m}\Big[\frac{V_0}{k}-\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}\Big]dx.$$

We have $\frac{g(\varepsilon x,t_2u^+)}{(t_2u^+)^{p_m-1}}=\frac{f(t_2u^+)}{(t_2u^+)^{p_m-1}}\leq\frac{f(a)}{a^{p_m-1}}=\frac{V_0}{k}$ since $\frac{f(t)}{t^{p_m-1}}$ is an increasing function. Therefore $II\geq 0$. By the condition $(g_7)$ and $t_1u^+>t_2u^+$, we have $III>0$. Since $t_1>t_2$, then we have $\frac{1}{t_1^{p_m-p}}-\frac{1}{t_2^{p_m-p}}<0$ and $\frac{1}{t_1^{p_m-p_i}}-\frac{1}{t_2^{p_m-p_i}}<0$ for all $i=1,\ldots,m-1$. Combine that property, (2.51) and $I+II+III>0$, we get a contradiction. Thus $t(u)$ is uniqueness. Therefore, we see that

$$\sup_{t\geq 0}I_\varepsilon(tu)=I(t(u)u)$$

and $t(u)u\in\mathcal{N}_\varepsilon$. It implies that $c_\varepsilon^*=c_\varepsilon^{**}$. On the other hand, for fixed $u\in W_\varepsilon\setminus\{0\}$, we have $I_\varepsilon(tu)<0$ when $t$ large enough. Then there exist $t_0>>0$ such that $I_\varepsilon(tu)<0$ for all $t\geq t_0$. We consider the curve $g_u:[0,1]\to W_\varepsilon$ such that $g_u(t)=tt_0u$ for all $t\in[0,1]$ and $g_u\in\Gamma$. Hence, we obtain $\max_{t\geq 0}I_\varepsilon(tu)=\max_{t\in[0,1]}I(g_u(t))$ and it implies that

$$c_\varepsilon^*=\inf_{u\in W_\varepsilon\setminus\{0\}}\max_{t\geq 0}I_\varepsilon(tu)=\inf_{u\in W_\varepsilon\setminus\{0\}}\max_{t\in[0,1]}I(g_u(t))\geq\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))=c_\varepsilon.$$

Next we prove that $c_\varepsilon \geq c_\varepsilon^{**}$. Indeed, we only need show that every path $\gamma \in \Gamma$ has to cross $\mathcal{N}_\varepsilon$. Conversly, if $\gamma \cap \mathcal{N}_\varepsilon = \emptyset$, then $< I_\varepsilon'(\gamma(t)), \gamma(t) >> 0$ or $< I'(\varepsilon)(\gamma(t)), \gamma(t) >< 0$ for all $t \in [0, 1]$. We have

$$< I_\varepsilon'(\gamma(t)), \gamma(t) >= ||\gamma(t)||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||\gamma(t)||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} g(\varepsilon x, \gamma(t))\gamma(t)dx.$$

Using Trudinger–Moser inequality we get

$$< I_\varepsilon'(\gamma(t)), \gamma(t) >> 0$$

when $||\gamma(t)||_{W_\varepsilon}$ is small enough. Then the case $< I'(\varepsilon)(\gamma(t)), \gamma(t) >< 0$ for all $t \in [0, 1]$ is not true. Next, we prove that $< I_\varepsilon'(\gamma(t)), \gamma(t) >> 0$ for all $t \in [0, 1]$ can not occur.

From the assumptions $(g_4)$ and $(g_5)$, we have

$$\begin{aligned} \int\limits_{\mathbb{R}^N} g(\varepsilon x, \gamma(t))\gamma(t)dx &= \int\limits_{\Lambda_\varepsilon} g(\varepsilon x, \gamma(t))\gamma(t)dx + \int\limits_{\Lambda_\varepsilon^c} g(\varepsilon x, \gamma(t))\gamma(t)dx \\ &\geq \mu \int\limits_{\Lambda_\varepsilon} G(\varepsilon x, \gamma(t))dx + p_m \int\limits_{\Lambda_\varepsilon^c} G(\varepsilon x, \gamma(t))dx \geq p_m \int\limits_{\mathbb{R}^N} G(\varepsilon x, \gamma(t))dx. \end{aligned}$$

Then, we get

$$\begin{aligned} 0 << I_\varepsilon'(\gamma(t)), \gamma(t) > &\leq ||\gamma(t)||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} \\ &+ \sum_{i=1}^m ||\gamma(t)||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - p_m \int\limits_{\mathbb{R}^N} G(\varepsilon x, \gamma(t))dx \end{aligned}$$

for all $t \in [0, 1]$. By the definition of $\gamma$, when $t$ near 1, we have $I_\varepsilon(\gamma(t)) < 0$ due to the continuous of $I_\varepsilon$ on $W_\varepsilon$. Then we get

$$\begin{aligned} \int\limits_{\mathbb{R}^N} G(\varepsilon x, \gamma(t))dx &< \frac{1}{p_m}(||\gamma(t)||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||\gamma(t)||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)}) \\ &< \frac{1}{p}||\gamma(t)||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i}||\gamma(t)||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} < \int\limits_{\mathbb{R}^N} G(\varepsilon x, \gamma(t))dx. \end{aligned}$$

It is a contradiction. Hence $\gamma \cap \mathcal{N}_\varepsilon \neq \emptyset$ and then $c_\varepsilon \geq c_\varepsilon^{**}$. □

**Lemma 8.** *Assume that* $\{u_n\} \subset W_\varepsilon$ *is a* $(PS)_d$ *sequence for the functional* $I_\varepsilon$ *and* $k > \dfrac{\mu}{\mu - p_m}$. *If* $0 < d$ *and* $d$ *satisfies the condition*

$$\Big[\Big(\frac{s}{N} - \frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \sum_{i=1}^{m-1} \Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)^{\frac{-1}{p_i}} d^{\frac{1}{p_i}} \quad + \Big(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\Big)^{\frac{-1}{p_m}} d^{\frac{1}{p_m}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*if* $m \geq 2$ *and*

$$\Big[\Big(\frac{s}{N} - \frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \Big(\frac{1}{p_1} - \frac{1}{\mu} - \frac{1}{p_1 k}\Big)^{\frac{-1}{p_1}} d^{\frac{1}{p_1}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

*if* $m = 1$, *then* $\{u_n\}$ *is a bounded sequence in* $W_\varepsilon$ *and*

$$\limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*where* $\mathfrak{c} > 1$ *is a suitable constant and* $\mathfrak{d}_* = \min\{1, V_0\}$.

*Proof.* We only consider the case $m \geq 2$. The case $m = 1$ is proved similarly as $m \geq 2$. We omit the details. First, we see that

$$\begin{aligned}
d + o_n(1) + o_n(1)||u_n||_{W_\varepsilon} &\geq I_\varepsilon(u_n) - \frac{1}{\mu} < I_\varepsilon^{'}(u_n), u_n > \\
&= \left(\frac{1}{p} - \frac{1}{\mu}\right) ||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} \\
&\quad + \int_{\mathbb{R}^N} (\frac{1}{\mu} g(\varepsilon x, u_n)u_n - G(\varepsilon x, u_n))dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) ||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} \\
&\quad + \int_{\Lambda^c} \left(\frac{1}{\mu} g(\varepsilon x, u_n)u_n - G(\varepsilon x, u_n)\right) dx.
\end{aligned}$$

Therefore, we get

$$d + o_n(1) + o_n(1)||u_n||_{W_\varepsilon} \geq \int_{\Lambda^c} \left(\frac{1}{\mu} g(\varepsilon x, u_n)u_n - G(\varepsilon x, u_n)\right) dx$$

$$\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) ||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \int_{\Lambda^c} G(\varepsilon x, u_n) dx$$

$$\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) ||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \int_{\Lambda^c} \frac{V_0}{kp_m} |u_n|^{p_m} dx$$

$$\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) ||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \frac{1}{p_m k} \int_{\Lambda^c} V(\varepsilon x) |u_n|^{p_m} dx$$

$$\geq \left(\frac{s}{N} - \frac{1}{\mu}\right) ||u_n||^{N/s}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^{m-1} \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)}$$
$$+ \left(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\right) ||u_n||^{p_m}_{W^{s,p_m}_{V,\varepsilon}(\mathbb{R}^N)}.$$

Since $k > \dfrac{\mu}{\mu - p_m}$, using the property

$$\lim_{x\to+\infty, x_1\to+\infty,\ldots,x_m\to+\infty} \frac{ax^p + a_1 x_1^{p_1} + \cdots + a_m x_m^{p_m}}{x + x_1 + \cdots + x_m} = +\infty,$$

where $a > 0, a_1 > 0, \ldots, a_m > 0$, we have $\{u_n\}$ is a bounded sequence in $W_\varepsilon$. Then, we deduce

$$\limsup_{n\to\infty} ||u_n||^{N/s}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{d}{\frac{s}{N} - \frac{1}{\mu}}, \quad \limsup_{n\to\infty} ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{d}{\frac{1}{p_i} - \frac{1}{\mu}}$$

for all $i = 1, \ldots, m-1$ and

$$\limsup_{n\to\infty} ||u_n||^{p_m}_{W^{s,p_m}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{d}{\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}}.$$

From the assumption of $d$, we get

$$\limsup_{n\to\infty} ||u_n||^{N/(N-s)}_{W_\varepsilon} = \limsup_{n\to\infty} \left(||u_n||_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)}\right)^{N/(N-s)}$$
$$\leq \left[\left(\frac{s}{N} - \frac{1}{\mu}\right)^{-s/N} d^{s/N} + \sum_{i=1}^{m-1} \left(\frac{1}{p_i} - \frac{1}{\mu}\right)^{\frac{-1}{p_i}} d^{\frac{1}{p_i}}\right.$$
$$\left. + \left(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\right)^{\frac{-1}{p_m}} d^{\frac{1}{p_m}}\right]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}.$$

□

**Lemma 9.** *Let $d > 0$ and $d$ satisfies the condition*

$$\Big[\Big(\frac{s}{N}-\frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \sum_{i=1}^{m-1}\Big(\frac{1}{p_i}-\frac{1}{\mu}\Big)^{\frac{-1}{p_i}} d^{\frac{1}{p_i}} + \Big(\frac{1}{p_m}-\frac{1}{\mu}-\frac{1}{p_m k}\Big)^{\frac{-1}{p_m}} d^{\frac{1}{p_m}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*if $m \geq 2$ and*

$$\Big[\Big(\frac{s}{N}-\frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \Big(\frac{1}{p_1}-\frac{1}{\mu}-\frac{1}{p_1 k}\Big)^{\frac{-1}{p_1}} d^{\frac{1}{p_1}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

*if $m = 1$, and $\{u_n\} \subset W_\varepsilon$ be a $(PS)_d$ sequence for $I_\varepsilon$ such that $u_n \to 0$ weak in $W_\varepsilon$. Then we have either:*
*$(i)$ $u_n \to 0$ in $W_\varepsilon$ or*
*$(ii)$ there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R > 0, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^{p_m} dx \geq \beta > 0.$$

*Proof.* Suppose that $(ii)$ doesnot occur. By Lemma 8, we have

$$\limsup_{n\to\infty} ||u_n||^{N/(N-s)}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}.$$

Since the embeddings $W_\varepsilon \to W^{s,N/s}_{V,\varepsilon}(\mathbb{R}^N) \to W^{s,p}(\mathbb{R}^N)$ are continuous, then we can apply Lemma 5 and get $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (p_m, +\infty)$. By arguments as Lemma 6, from the conditions $(g_2)$ and $(g_3)$, using the inequality (2.40), we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)u_n dx = 0$. Recalling that $< I'_\varepsilon(u_n), u_n >\to 0$ as $n \to \infty$, then we deduce $u_n \to 0$ strongly in $W_\varepsilon$. The proof of Lemma 9 is completed. □

**Lemma 10.** *The number $c_\varepsilon$ and $c_{V_0}$ satisfy the following inequality*

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq c_{V_0} \leq a\left(1-\frac{N}{s\mu}\right)\left(\frac{aN}{\gamma_1 s\mu}\right)^{N/(\mu s-N)}$$

*for all $\gamma_1 \geq a$,*

$$a = \frac{s(A_{\mu,\eta}+\varepsilon_*)^{N/s}}{N} + \sum_{i=1}^{m} \frac{(A_{\mu,\eta}+\varepsilon_*)^{p_i}}{p_i}$$

*for some $\varepsilon_* > 0$.*

*Proof.* First, we consider the case $m \geq 2$. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\phi \equiv 1$ on $B_{\delta/2}(0)$, $\mathrm{supp}(\phi) \subset B_\delta(0) \subset \Lambda$ for some $\delta > 0$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B_\delta(0)$. For each $\varepsilon > 0$, let us define $v_\varepsilon(x) = \phi(\varepsilon x)w(x)$, where $w$ is a ground state solution of the problem $(P_{V_0})$ given in Proposition 1. Then $v_\varepsilon \to w$ strong in $W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$ (see Lemma 2.4 [14]). We see that support of $v_\varepsilon$ is contained in $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$. For each $v_\varepsilon$, there exists $t_\varepsilon > 0$ such that $t_\varepsilon v_\varepsilon \in \mathcal{N}_\varepsilon$, and we have

$$\begin{aligned}
c_\varepsilon \leq I_\varepsilon(t_\varepsilon v_\varepsilon) &= \frac{t_\varepsilon^p}{p} \int\limits_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^p}{|x - y|^{2N}} dxdy + \frac{t_\varepsilon^p}{p} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|v_\varepsilon(x)|^p dx \\
&\quad + \sum_{i=1}^m \Big( \frac{t_\varepsilon^{p_i}}{p_i} \int\limits_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^{p_i}}{|x - y|^{N+p_i s}} dxdy + \frac{t_\varepsilon^{p_i}}{p_i} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|v_\varepsilon(x)|^{p_i} dx \Big) \\
&\quad - \int\limits_{\mathbb{R}^N} G(\varepsilon x, t_\varepsilon v_\varepsilon) dx \\
&= \frac{t_\varepsilon^p}{p} \int\limits_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^p}{|x - y|^{2N}} dxdy + \frac{t_\varepsilon^p}{p} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|v_\varepsilon(x)|^p dx \\
&\quad + \sum_{i=1}^m \Big( \frac{t_\varepsilon^{p_i}}{p_i} \int\limits_{\mathbb{R}^{2N}} \frac{|v_\varepsilon(x) - v_\varepsilon(y)|^{p_i}}{|x - y|^{N+p_i s}} dxdy + \frac{t_\varepsilon^{p_i}}{p_i} \int\limits_{\mathbb{R}^N} V(\varepsilon x)|v_\varepsilon(x)|^{p_i} dx \Big) \\
&\quad - \int\limits_{\mathbb{R}^N} F(t_\varepsilon v_\varepsilon) dx
\end{aligned}$$

Since $t_\varepsilon v_\varepsilon \in \mathcal{N}_\varepsilon$, we have

$$||t_\varepsilon v_\varepsilon||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||t_\varepsilon v_\varepsilon||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} g(\varepsilon x, t_\varepsilon v_\varepsilon) t_\varepsilon v_\varepsilon dx = \int\limits_{\mathbb{R}^N} f(t_\varepsilon v_\varepsilon) t_\varepsilon v_\varepsilon dx. \tag{2.52}$$

Then we get

$$\begin{aligned}
I_\varepsilon(t_\varepsilon v_\varepsilon) &= \frac{1}{p}||t_\varepsilon v_\varepsilon||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{1}{p_i}||t_\varepsilon v_\varepsilon||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \int\limits_{\mathbb{R}^N} F(t_\varepsilon v_\varepsilon) dx \\
&= \Big(\frac{1}{p} - \frac{1}{p_m}\Big)||t_\varepsilon v_\varepsilon||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^{m-1} \Big(\frac{1}{p_i} - \frac{1}{p_m}\Big)||t_\varepsilon v_\varepsilon||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} \\
&\quad - \int\limits_{\mathbb{R}^N} \left(\frac{1}{p_m} f(t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon - F(t_\varepsilon u_\varepsilon)\right) dx \geq 0.
\end{aligned} \tag{2.53}$$

From (2.53), we see that the sequence $\{t_\varepsilon\}$ must be bounded as $\varepsilon \to 0^+$. Indeed, if $t_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$, then using the condition $(f_4)$, we have

$$I_\varepsilon(t_\varepsilon v_\varepsilon) \geq \frac{t_\varepsilon^p}{p}||v_\varepsilon||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{t_\varepsilon^{p_i}}{p_i}||v_\varepsilon||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} - \gamma_1 t_\varepsilon^\mu ||v_\varepsilon||^\mu_{L^\mu(\mathbb{R}^N)} \to -\infty,$$

which is a contradiction with (2.53). Thus, we can assume that $t_\varepsilon \to t_0$ as $\varepsilon \to 0^+$. Then we get

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} c_\varepsilon \le \frac{t_0^p}{p}\int\limits_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{2N}}dxdy + \frac{t_0^p}{p}\int\limits_{\mathbb{R}^N} V_0|w|^p dx \\ + \sum_{i=1}^{m}\Big(\frac{t_0^{p_i}}{p_i}\int\limits_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^{p_i}}{|x-y|^{N+p_i s}}dxdy \\ + \frac{t_0^{p_i}}{p_i}\int\limits_{\mathbb{R}^N} V_0|w|^{p_i}dx\Big) - \int\limits_{\mathbb{R}^N} F(t_0 w)dx \\ = J_{V_0}(t_0 w)\end{aligned}$$

via to Vitali's theorem. If $t_0 = 0$, by the condition $(f_1)$ and $(f_3)$, we have

$$|f(t)| \le \varepsilon_*|t|^{p_m-1} + C(\varepsilon_*)|t|^{q-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})$$

for all $t \ge 0$ and some constants $q > p_m$. Then from (2.52), we get

$$\begin{aligned}t_\varepsilon^{p-p_1}||v_\varepsilon||^p_{W^{s,p}_{V,\varepsilon}} + \sum_{i=1}^{m-1} t_\varepsilon^{p-p_i}||v_\varepsilon||^{p_i}_{W^{s,p_i}_{V,\varepsilon}}) + ||v_\varepsilon||^{p_m}_{W^{s,p_m}_{V,\varepsilon}} = \int\limits_{\mathbb{R}^N} \frac{f(t_\varepsilon v_\varepsilon)}{t_\varepsilon^{p_m-1}} v_\varepsilon dx \\ \le \varepsilon_* \int\limits_{\mathbb{R}^N} |v_\varepsilon|^{p_m}dx + t_\varepsilon^{q-p_m}C(\varepsilon_*)\int\limits_{\mathbb{R}^N} |v_\varepsilon|^q \Phi_{N,s}(\alpha_0|t_\varepsilon v_\varepsilon|^{N/(N-s)})dx. \quad (2.54)\end{aligned}$$

Choose $\varepsilon_* > 0$ is small enough, using Trudinger–Moser inequality and note that $v_\varepsilon \to w$ strong $W^{s,t}(\mathbb{R}^N)$ $(t \ge \dfrac{N}{s})$ from (2.54), we get a contradiction since the left side tends to $\infty$ and the right side tends to zero. Hence $t_0 > 0$. Using Vitali's theorem and take limit of (2.52) as $\varepsilon \to 0^+$, we deduce

$$\begin{aligned}t_0^{p-p_1}||w||^p_{W_{V_0,W^{s,p}(\mathbb{R}^N)}} \\ + \sum_{i=1}^{m-1} t_0^{p-p_i}||w||^{p_i}_{W_{V_0,W^{s,p_i}(\mathbb{R}^N)}} + ||w||^{p_m}_{W_{V_0,W^{s,p_m}(\mathbb{R}^N)}} = \int\limits_{\mathbb{R}^N} \frac{f(t_0 w)}{t_0^{p_m-1}} w dx.\end{aligned}$$

Note that $w \in \mathcal{N}_{V_0}$ and using the condition $(f_5)$, we obtain $t_0 = 1$. Therefore

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \le J_{V_0}(w) = c_{V_0}.$$

By Lemma 4, we get $c_{V_0} \le C_{\gamma_1} = a(1 - \dfrac{N}{s\mu})(\dfrac{aN}{\gamma_1 s\mu})^{N/(\mu s-N)}$ for all $\gamma_1 \ge a$. In the case $m = 1$, we can proved similarly as above. We omit the details. □

**Lemma 11.** *The functional $I_\varepsilon$ satisfies the $(PS)_d$ condition at any level $d > 0$ and $d$ satisfies the condition*

$$\Big[\Big(\frac{s}{N}-\frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \sum_{i=1}^{m-1}\Big(\frac{1}{p_i}-\frac{1}{\mu}\Big)^{\frac{-1}{p_i}} d^{\frac{1}{p_i}} + \Big(\frac{1}{p_m}-\frac{1}{\mu}-\frac{1}{p_m k}\Big)^{\frac{-1}{p_m}} d^{\frac{1}{p_m}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*if $m \geq 2$ and*

$$\Big[\Big(\frac{s}{N}-\frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \Big(\frac{1}{p_1}-\frac{1}{\mu}-\frac{1}{p_1 k}\Big)^{\frac{-1}{p_1}} d^{\frac{1}{p_1}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

*if $m = 1$, where $\mathfrak{c} > 1$ is a suitable constant and near* 1.

*Proof.* Let $\{u_n\}$ be a $(PS)_d$ sequence of $I_\varepsilon$, then by Lemma 8, $\{u_n\}$ is a bounded sequence in $W_\varepsilon$ and

$$\limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}, \tag{2.55}$$

where $\mathfrak{c} > 1$ is a suitable constant and $\mathfrak{c}$ near 1. Therefore, up to a subsequence, we can assume that $u_n \to u$ weak in $W_\varepsilon$, $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ for all $q \in [\frac{N}{s}, +\infty)$ and $u_n(x) \to u(x)$ almost everywhere on $\mathbb{R}^N$. By arguments as Lemma 2.5 [8], for any $\varepsilon_* > 0$, there exists $R = R(\varepsilon_*) > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \Big(\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{2N}} + \sum_{i=1}^{m} \frac{|u_n(x)-u_n(y)|^{p_i}}{|x-y|^{N+p_i s}} + V(\varepsilon x)(|u_n|^p + \sum_{i=1}^{m}|u_n|^{p_i})\Big) dx < \varepsilon_*.$$

Then, we obtain

$$\int_{\mathbb{R}^N\setminus B_R(0)} |u_n|^{N/s} dx < \frac{\varepsilon_*}{V_0} \text{ and } \int_{\mathbb{R}^N\setminus B_R(0)} |u_n|^{p_m} dx < \frac{\varepsilon_*}{V_0} \tag{2.56}$$

for all $n$ large enough. From the condition $(f_1)$, $(f_3)$ and $(g_3)$, we get

$$|g(x,t)t| \leq \delta |t|^{p_m} + C_\delta |t|^q \Phi_{N,s}(\alpha_0 |t|^{N/(N-s)}) \tag{2.57}$$

for all $t \in \mathbb{R}, x \in \mathbb{R}^N$ and some $\delta > 0, q > p_m$. Using (2.55), (2.57) and Trudinger–Moser inequality, Hölder inequality, there exists $D > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |g(\varepsilon x, u_n)u_n|dx \leq \delta \int_{\mathbb{R}^N\setminus B_R(0)} |u_n|^{p_m}dx + D\left(\int_{\mathbb{R}^N\setminus B_R(0)} |u_n|^{qt}dx\right)^{1/t} \tag{2.58}$$

for some constant $t > 1$.

For any $\nu \in (\frac{N}{s}, +\infty)$, choose $\mathfrak{a} > \frac{N}{s}$ such that $\nu \in (\frac{N}{s}, \mathfrak{a})$, there exists $\sigma_1 \in (0, 1)$ such that $\frac{1}{\nu} = \frac{s\sigma_1}{N} + \frac{1-\sigma_1}{\mathfrak{a}}$. Apply the Hölder inequality to estimate $\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^\nu dx$, and we get

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^\nu dx &= \int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^{\nu\sigma_1}|u_n(x)|^{(1-\sigma_1)\nu}dx \\ &\leq \Big(\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^{N/s}dx\Big)^{\sigma_1\nu s/N}\Big(\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^{\mathfrak{a}}dx\Big)^{(1-\sigma_1)\nu/\mathfrak{a}}.\end{aligned} \tag{2.59}$$

From (2.48), we have

$$||u_n||_{L^{\mathfrak{a}}(\mathbb{R}^N\setminus B_R(0))} \leq S_{\mathfrak{a},\varepsilon}^{-1}||u_n||_{W_\varepsilon}.$$

On combining that inequality with (2.59), we deduce

$$\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^\nu dx \leq S_{\mathfrak{a},\varepsilon}^{-(1-\sigma_1)\nu}||u_n||^{\sigma_1\nu}_{L^{N/s}(\mathbb{R}^N\setminus B_R(0))}||u_n||^{(1-\sigma_1)\nu}_{W_\varepsilon}. \tag{2.60}$$

From (2.55), (2.56) and (2.60), there exists constant $\mathcal{D} > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)} |u_n(x)|^\nu dx \leq \mathcal{D}\varepsilon_*. \tag{2.61}$$

Join (2.56), (2.58) and apply (2.61) to $\nu = qt$, we get

$$\int_{\mathbb{R}^N\setminus B_R(0)} |g(\varepsilon x, u_n)u_n|dx \leq \kappa^*\varepsilon_*$$

for all $n$ large enough and $\kappa^* > 0$ is a suitable constant. Hence, we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} |g(\varepsilon x, u_n)u_n|dx = 0. \tag{2.62}$$

Note that $\Lambda_\varepsilon \subset B_R(0)$, and the embedding from $W_\varepsilon$ into $L^{\mathfrak{q}}(B_R(0))$ is compact for any $\mathfrak{q} \in [\frac{N}{s}, +\infty)$, we have

$$\lim_{n\to\infty} \int_{B_R(0)} |g(\varepsilon x, u_n)u_n| dx = \lim_{n\to\infty} \int_{B_R(0)} |g(\varepsilon x, u)u| dx \tag{2.63}$$

by the Lebesgue Dominated convergence theorem or Vitali's theorem. Using Trudinger–Moser inequality, we get $g(\varepsilon x, u)u \in L^1(\mathbb{R}^N)$, then can choose $R$ large enough such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |g(\varepsilon x, u)u| dx < \varepsilon_*. \tag{2.64}$$

From (2.62), (2.63) and (2.64), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(\varepsilon x, u_n)u_n dx = \int_{\mathbb{R}^N} g(\varepsilon x, u)u dx. \tag{2.65}$$

By arguments as in [54], we get $< I_\varepsilon'(u), \varphi >= 0$ for all $\varphi \in W_\varepsilon$. Consequently, we get $< I_\varepsilon'(u), u >= 0$, or equivalently

$$||u||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} g(\varepsilon x, u)u dx. \tag{2.66}$$

Since $\{u_n\}$ is $(PS)$ sequence, then $< I_\varepsilon'(u_n), u_n >= o_n(1)$ as $n \to \infty$.

$$||u_n||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} g(\varepsilon x, u_n)u_n dx + o_n(1). \tag{2.67}$$

Apply Brezis–Lieb lemma, (2.66) and (2.67), we obtain $u_n \to u$ strong in $W_\varepsilon$. We finish the proof of Lemma 11. □

**Lemma 12.** *The functional $I_\varepsilon$ restricted to $\mathcal{N}_\varepsilon$ satisfies the $(PS)_d$ condition at any level $d > 0$ and $d$ verifies*

$$\Big[\Big(\frac{s}{N} - \frac{1}{\mu}\Big)^{-s/N} d^{s/N} + \sum_{i=1}^{m-1} \Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)^{\frac{-1}{p_i}} d^{\frac{1}{p_i}}$$
$$+ \Big(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\Big)^{\frac{-1}{p_m}} d^{\frac{1}{p_m}}\Big]^{N/(N-s)}$$
$$< \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*if* $m \geq 2$ *and*

$$\Big[\Big(\frac{s}{N}-\frac{1}{\mu}\Big)^{-s/N} d^{s/N}+\Big(\frac{1}{p_1}-\frac{1}{\mu}-\frac{1}{p_1 k}\Big)^{\frac{-1}{p_1}} d^{\frac{1}{p_1}}\Big]^{N/(N-s)}<\frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

*if* $m=1$, *where* $\mathfrak{c}>1$ *is a suitable constant and near* 1.

*Proof.* Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be such that $I_\varepsilon(u_n) \to d$ and $||I_\varepsilon^{'}(u_n)|_{\mathcal{N}_\varepsilon}||_{W_\varepsilon^*} = o_n(1)$ as $n \to \infty$, where $W_\varepsilon^*$ is the dual space of $W_\varepsilon$. Then there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$I_\varepsilon^{'}(u_n)=\lambda_n T_\varepsilon^{'}(u_n)+o_n(1), \tag{2.68}$$

where

$$T_\varepsilon(u)=||u||^p_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)}+\sum_{i=1}^m ||u||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)}-\int\limits_{\mathbb{R}^N} g(\varepsilon x,u)u\,dx.$$

Taking into account $< I_\varepsilon^{'}(u_n), u_n >= 0$, we have

$$\begin{aligned}
< T_\varepsilon^{'}(u_n),u_n > &= p\int\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{2N}}dxdy+p\int\limits_{\mathbb{R}^{2N}} V(\varepsilon x)|u_n|^p dx\\
&+\sum_{i=1}^m\Big(p_i\int\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p_i}}{|x-y|^{N+p_i s}}dxdy+p_i\int\limits_{\mathbb{R}^{2N}} V(\varepsilon x)|u_n|^{p_i}dx\Big)\\
&-\int\limits_{\mathbb{R}^N} g(\varepsilon x,u_n)u_n dx-\int\limits_{\mathbb{R}^N} g_t^{'}(\varepsilon x,u_n)u_n^2 dx\\
&\leq \int\limits_{\mathbb{R}^N}((p_m-1)g(\varepsilon x,u_n)u_n-g_t'(\varepsilon x,u_n)u_n^2)dx\\
&=\int\limits_{\Lambda_\varepsilon}((p_m-1)g(\varepsilon x,u_n)u_n-g_t'(\varepsilon x,u_n)u_n^2)dx\\
&+\int\limits_{\Lambda_\varepsilon^c\cap\{x:u_n(x)<a\}}((p_m-1)g(\varepsilon x,u_n)u_n-g_t'(\varepsilon x,u_n)u_n^2)dx\\
&+\int\limits_{\Lambda_\varepsilon^c\cap\{x:u_n(x)\geq a\}}((p_m-1)g(\varepsilon x,u_n)u_n-g_t'(\varepsilon x,u_n)u_n^2)dx.
\end{aligned}$$

When $x \in \Lambda_\varepsilon^s$ and $t > a$, we have $g(\varepsilon x,t)=\frac{V_0}{k}t^{p_m-1}$. It implies that

$$(p_m-1)g(\varepsilon x,t)t-g_t^{'}(\varepsilon x,t)t^2=0.$$

Therefore, we get

$$-<T_\varepsilon^{'}(u_n),u_n>\ \geq \int_{\Lambda_\varepsilon}(g_t'(\varepsilon x,u_n)u_n^2-(p_m-1)g(\varepsilon x,u_n)u_n)dx + \int_{\Lambda_\varepsilon^c\cap\{x:u_n(x)<a\}}(g_t'(\varepsilon x,u_n)u_n^2-(p_m-1)g(\varepsilon x,u_n)u_n)dx\geq 0 \tag{2.69}$$

via to the conditions $(g_6)$ and $(g_7)$. By arguments as Lemma 8, for $\gamma_1$ large enough, we have $\{u_n\}$ is a bounded sequence in $W_\varepsilon$ and

$$\limsup_{n\to\infty}||u_n||_{W_\varepsilon}^{N/(N-s)}<\frac{\beta_*\mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}, \tag{2.70}$$

where $\mathfrak{c}>1$ is a suitable constant and $\mathfrak{c}$ near 1. Therefore, up to a subsequence, we can assume that $u_n\to u$ weak in $W_\varepsilon$, $u_n\to u$ in $L_{loc}^q(\mathbb{R}^N)$ for all $q\in[\frac{N}{s},+\infty)$ and $u_n(x)\to u(x)$ almost everywhere on $\mathbb{R}^N$. We prove that $\sup_{n\in\mathbb{N}}<T_\varepsilon^{'}(u_n),u_n><0$. Conversly, if $\sup_{n\in\mathbb{N}}<T_\varepsilon^{'}(u_n),u_n>=0$, then up to a subsequence, we can assume that $\lim_{n\to\infty}<T_\varepsilon^{'}(u_n),u_n>=0$. Using Fatou's lemma and (2.69), we have

$$\begin{aligned}0&\geq\liminf_{n\to\infty}\int_{\Lambda_\varepsilon}(g_t'(\varepsilon x,u_n)u_n^2-(p_m-1)g(\varepsilon x,u_n)u_n)dx\\&\geq\int_{\Lambda_\varepsilon}(g_t'(\varepsilon x,u)u^2-(p_m-1)g(\varepsilon x,u)u)dx\geq 0\end{aligned} \tag{2.71}$$

due to the condition $(g_7)$. Hence $u\equiv 0$ in $\Lambda_\varepsilon$. Then $u_n\to 0$ in $L^q(\Lambda_\varepsilon)$. Using Trudinger–Moser inequality and (2.70), we get

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon}g(\varepsilon x,u_n)u_ndx=\lim_{n\to\infty}\int_{\Lambda_\varepsilon}f(u_n)u_ndx=0.$$

Hence, we obtain

$$\begin{aligned}||u_n||_{W_{V,\varepsilon}^{s,p}(\mathbb{R}^N)}^p+\sum_{i=1}^m||u_n||_{W_{V,\varepsilon}^{s,p_i}(\mathbb{R}^N)}^{p_i}&=\int_{\Lambda_\varepsilon}g(\varepsilon x,u_n)u_ndx+\int_{\Lambda_\varepsilon^c}g(\varepsilon x,u_n)u_ndx\\&=\int_{\Lambda_\varepsilon^c}g(\varepsilon x,u_n)u_ndx+o_n(1)\\&\leq\frac{1}{k}\int_{\Lambda_\varepsilon^c}V(\varepsilon x)|u_n|^{p_m}dx+o_n(1),\end{aligned}$$

thanks to the condition $(g_5)$. Then, we deduce

$$||u_n||_{W_\varepsilon} \to 0$$

as $n \to \infty$, it is a contradiction with the fact that $||u_n||_{W_\varepsilon} \geq r_* > 0$ for all $n$. In conclusion, we get $\sup_{n\in\mathbb{N}} < T_\varepsilon^{'}(u_n), u_n >< 0$, and (2.68) implies $\lambda_n = o_n(1)$ as $n \to \infty$. Therefore, $\{u_n\}$ is a $(PS)_c$ sequence of $I_\varepsilon$ and Lemma 12 is obtained from Lemma 11. □

**Corollary 1.** *The critical points of $I_\varepsilon$ on $\mathcal{N}_\varepsilon$ are critical points of $I_\varepsilon$ in $W_\varepsilon$.*

Now, we prove the existence of a ground state solution for problem $(P_\varepsilon^*)$. That is a critical point $u_\varepsilon$ of $I_\varepsilon$ satisfying $I_\varepsilon(u_\varepsilon) = c_\varepsilon$.

**Theorem 7.** *Assume that $(f_1) - (f_5)$ and $(V)$ hold. Then there exists $\overline{\varepsilon} > 0$ such that $(P_\varepsilon^*)$ has a ground state solution for all $0 < \varepsilon < \overline{\varepsilon}$.*

*Proof.* By Lemma 10 and Lemma 11, there exists $\overline{\varepsilon} > 0$ such that $c_\varepsilon \leq c_{V_0}$ for all $\varepsilon \in (0, \overline{\varepsilon})$. We can choose $d = c_{V_0} \leq a(1 - \dfrac{N}{s\mu})(\dfrac{aN}{\gamma_1 s\mu})^{N/(\mu s - N)}$ and $\gamma_1 \geq \max\{a, \gamma_3\}$ where $\gamma_3$ satisfies the condition

$$\Big[\Big(\frac{s}{N} - \frac{1}{\mu}\Big)^{-s/N} \mathfrak{b}^{s/N} + \Big(\frac{1}{p_1} - \frac{1}{\mu} - \frac{1}{p_1 k}\Big)^{\frac{-1}{p_1}} \mathfrak{b}^{\frac{1}{p_1}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

in which $\mathfrak{b} = a(1 - \dfrac{N}{s\mu})(\dfrac{aN}{\gamma_3 s\mu})^{N/(\mu s - N)}$ and $m = 1$. When $m \geq 2$, $\gamma_3$ satisfies the condition

$$\Big[\Big(\frac{s}{N} - \frac{1}{\mu}\Big)^{-s/N} \mathfrak{b}^{s/N} + \sum_{i=1}^{m-1} \Big(\frac{1}{p_i} - \frac{1}{\mu}\Big)^{\frac{-1}{p_i}} \mathfrak{b}^{\frac{1}{p_i}}\Big]^{N/(N-s)}$$
$$+ \Big(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\Big)^{\frac{-1}{p_m}} \mathfrak{b}^{\frac{1}{p_m}}\Big]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}.$$

Lemma 11 implies that $I_\varepsilon$ satisfies the $(PS)_{c_\varepsilon}$ condition. Combine that result with Lemma 7, $I_\varepsilon$ has a critical point at level $c_\varepsilon$. □

## 4. Multiplicity of solutions to $(P_\varepsilon^*)$

In this section, we show that the existence of multiple weak solutions and study the behavior of its maximum points related with the set $M$. The main result of this section is equivalent to Theorem 2 and it is stated as follows:

**Theorem 8.** *Assume that $(f_1) - (f_5)$ and $(V)$ hold. Then for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that $(P_\varepsilon^*)$ has at least $cat_{M_\delta}(M)$ nontrival nonnegative solutions, for any $0 < \varepsilon < \varepsilon_\delta$. Moreover, if $u_\varepsilon$ denotes one of these solutions and $z_\varepsilon$ is its global maximum, then*

$$\lim_{\varepsilon \to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

*Proof.* We consider the energy function

$$J_{V_0}(u) = \frac{1}{p}\left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}} + \int\limits_{\mathbb{R}^N} V_0|u|^p dx\right) + \sum_{i=1}^{m}\frac{1}{p_i}\left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p_i}}{|x-y|^{N+p_i s}} + \int\limits_{\mathbb{R}^N} V_0|u|^{p_i} dx\right) - \int\limits_{\mathbb{R}^N} F(u)dx$$

of problem $(P_{V_0})$. We recall that $c_{V_0}$ is the minimax level related to $J_{V_0}$ and $\mathcal{N}_{V_0}$ is the Nehari manifold related to $J_{V_0}$ is given by

$$\mathcal{N}_{V_0} = \{u \in W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)\backslash\{0\} :< J'_{V_0}(u), u >= 0\}.$$

Let $\delta > 0$ be a fixed and $w$ be a ground state solution of problem $(P_{V_0})$. It means that $J_{V_0}(w) = c_{V_0}$ and $J'_{V_0}(w) = 0$. Let $\eta$ be a smooth nonincreasing cut-off function in $[0, +\infty)$ such that $\eta(s) = 1$ if $0 \le s \le \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \ge \delta$. For any $y \in M$, we denote

$$\psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

and $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ which is defined by $\Phi_\varepsilon(y) = t_\varepsilon\psi_{\varepsilon,y}$, where $t_\varepsilon > 0$ satisfies

$$\max_{t\ge 0} I_\varepsilon(t\psi_{\varepsilon,y}) = I_\varepsilon(t_\varepsilon\psi_{\varepsilon,y}).$$

From the construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$. □

**Lemma 13.** *The function $\Phi_\varepsilon$ satisfies the following limit*

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M.$$

*Proof.* Suppose that the statement of Lemma 13 doesnot hold, then there exists $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \ge \delta_0. \tag{4.1}$$

By Lemma 2.2 [14], we have

$$\lim_{n\to\infty} ||\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} = ||w||^p_{W_{V_0,W^{s,p}(\mathbb{R}^N)}}. \tag{4.2}$$

and

$$\lim_{n\to\infty} ||\psi_{\varepsilon_n,y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} = ||w||^{p_i}_{W_{V_0,W^{s,p_i}(\mathbb{R}^N)}} \tag{4.3}$$

for all $i = 1, \ldots, m$. Since $< I'_{\varepsilon_n}(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}), t_{\varepsilon_n}\psi_{\varepsilon_n,y_n} >= 0$, using the change of variable $z = \dfrac{\varepsilon_n x - y_n}{\varepsilon_n}$, then we get

$$||t_\varepsilon \psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^{m} ||t_\varepsilon \psi_{\varepsilon_n,y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} g(\varepsilon_n x, t_\varepsilon \psi_{\varepsilon_n,y_n}) t_{\varepsilon_n} \psi_{\varepsilon_n,y_n} dx$$
$$= \int_{\mathbb{R}^N} g(\varepsilon_n z + y_n, t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z)) t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z)dz. \tag{4.4}$$

We observe that if $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Then

$$g(\varepsilon_n z + y_n, t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z)) = f(t_{\varepsilon_n}\psi(|\varepsilon_n z|)w(z)).$$

Now we prove that $t_{\varepsilon_n} \to 1$. First we show that $t_{\varepsilon_n} \to t_0 < +\infty$. Conversly if $t_{\varepsilon_n} \to +\infty$, from (4.4) we have

$$t_{\varepsilon_n}^{p-p_1} ||\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + ||\psi_{\varepsilon_n,y_n}||^{p_1}_{W^{s,p_1}_{V,\varepsilon_n}(\mathbb{R}^N)} \geq \int_{|z| \leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n} w(z))w(z)}{t_{\varepsilon_n}^{p_1-1}} dz \tag{4.5}$$

if $m = 1$, and

$$t_{\varepsilon_n}^{p-p_m} ||\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^{m-1} t_{\varepsilon_n}^{p_i-p_m} ||\psi_{\varepsilon_n,y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} + ||\psi_{\varepsilon_n,y_n}||^{p_m}_{W^{s,p_m}_{V,\varepsilon_n}(\mathbb{R}^N)} \tag{4.6}$$

$$\geq \int_{|z| \leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n} w(z))w(z)}{t_{\varepsilon_n}^{p_m-1}} dz \tag{4.7}$$

if $m \geq 2$. From the condition $(f_2)$ and $(f_4)$, we have $f(t) \geq \gamma_1 \mu |t|^{\mu-1}$ for all $t \geq 0$. If $m = 1$, combine that property and (4.4), we deduce

$$t_{\varepsilon_n}^{p-p_1} ||\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + ||\psi_{\varepsilon_n,y_n}||^{p_1}_{W^{s,p_1}_{V,\varepsilon_n}} \geq \int_{|z| \leq \frac{\delta}{2\varepsilon_n}} \frac{f(t_{\varepsilon_n} w(z))w(z)}{t_{\varepsilon_n}^{p_1-1}} dz$$
$$\geq \gamma_1 \mu t_{\varepsilon_n}^{\mu-p_1} \int_{|z| < \frac{\delta}{2\varepsilon_n}} w^\mu dx \to +\infty$$

as $n \to \infty$. It is a contradiction with (4.2) and (4.3). Similarly, we get a contradiction in the case $m = 2$. Therefore, up to a subsequence, we may assume that $t_{\varepsilon_n} \to t_0 \geq 0$ as $n \to \infty$. We consider the case that $t_0 = 0$. From (2.41), we have

$$\begin{aligned} & f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)| \\ & \quad \leq \tau|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^{p_m} \\ & \quad + C|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^q \Phi_{N,s}(\alpha_0|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^{N/(N-s)})) \\ & \quad \leq \tau|t_{\varepsilon_n}w(z)|^{p_m} + C|t_{\varepsilon_n}w(z)|^q \Phi_{N,s}(\alpha_0|t_{\varepsilon_n}w(z)|^{N/(N-s)})) \end{aligned} \tag{4.8}$$

due to $\Phi_{N,s}(t)$ is an increasing function on $[0, +\infty)$, where $\tau > 0$ is small enough and $q > p_m$. Combine (4.6) and (4.8), we get

$$\begin{aligned} & ||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^m ||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} \\ & \quad \leq \tau \int_{\mathbb{R}^N} |t_{\varepsilon_n}w(z)|^{p_m}dx + Ct^q_{\varepsilon_n}\int_{\mathbb{R}^N} |w(z)|^q \Phi_{N,s}(\alpha_0|t_{\varepsilon_n}w(z)|^{N/(N-s)}))dx. \end{aligned} \tag{4.9}$$

Since $||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} \to 0$ and $||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ for all $i = 1, \ldots, m$, then

$$\begin{aligned} & ||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^m ||t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} \\ & \quad \geq (m+1)^{1-p_m}.t^{p_m}_{\varepsilon_n}(||\psi_{\varepsilon_n,y_n}||_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^m ||\psi_{\varepsilon_n,y_n}||_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)})^{p_m}. \end{aligned} \tag{4.10}$$

Using Trudinger–Moser inequality and note that $t_{\varepsilon_n} \to 0$ as $n \to \infty$, take $\tau > 0$ is small enough such that $(m+1)^{1-p_m} - \tau A^{-p_m}_{p_m,V_0} > 0$, from (4.9) and (4.10), we obtain $((m+1)^{1-p_m} - \tau A^{-p_m}_{p_m,V_0})||w||^{p_m}_{V_0,W} \leq o_n(1)$ as $n \to \infty$ due to

$$\begin{aligned} & ||\psi_{\varepsilon_n,y_n}||_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} \\ & + \sum_{i=1}^m ||\psi_{\varepsilon_n,y_n}||_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} \to ||w||_{W_{V_0,W^{s,p}(\mathbb{R}^N)}} + \sum_{i=1}^m ||w||_{W_{V_0,W^{s,p_i}(\mathbb{R}^N)}} > 0 \end{aligned}$$

as $n \to \infty$. It is a contradiction. Hence, $t_0 > 0$. Now we prove that $t_0 = 1$. From (4.6), using Lebesgue Dominated convergence theorem, we have

$$t_0^{p-p_1}||w||^p_{W_{V_0,W^{s,p}(\mathbb{R}^N)}} + ||w||^{p_1}_{W_{V_0,W^{s,p_1}(\mathbb{R}^N)}} = \int_{\mathbb{R}^N} \frac{f(t_0w)w}{t_0^{p_1-1}}dx \quad \text{if} \quad m = 1$$

and

$$t_0^{p-p_m}||w||^p_{W_{V_0,W^{s,p}(\mathbb{R}^N)}} + \sum_{i=1}^{m-1} t_0^{p_i-p_m}||w||^{p_i}_{W_{V_0,W^{s,p_i}(\mathbb{R}^N)}} + ||w||^{p_m}_{W_{V_0,W^{s,p_m}(\mathbb{R}^N)}} = \int_{\mathbb{R}^N} \frac{f(t_0w)w}{t_0^{p_m-1}}dx$$

if $m \geq 2$. Note that $w \in \mathcal{N}_{V_0}$, then the condition $(f_5)$ implies $t_0 = 1$. Still using Lebesgue Dominated convergence theorem or Vitali's theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(t_\varepsilon \psi_{\varepsilon_n, y_n}(x))dx = \int_{\mathbb{R}^N} F(w)dx.$$

Hence, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) \\
&= \lim_{n\to\infty} \Big[ \frac{t_{\varepsilon_n}^p}{p} ||\psi_{\varepsilon_n, y_n}||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^m \frac{t_{\varepsilon_n}^{p_i}}{p_i} ||\psi_{\varepsilon_n, y_n}||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}} - \int_{\mathbb{R}^N} F(t_{\varepsilon_n}\psi_{\varepsilon_n, y_n})dx \Big] \\
&= \frac{||w||^p_{W_{V_0, W^{s,p}(\mathbb{R}^N)}}}{p} + \sum_{i=1}^m \frac{||w||^{p_i}_{W_{V_0, W^{s,p_i}(\mathbb{R}^N)}}}{p_i} - \int_{\mathbb{R}^N} F(w)dx = J_{V_0}(w) = c_{V_0}
\end{aligned}$$

which contradicts with (4.1). □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be define as

$$\chi(x) = \begin{cases} x & \text{if } |x| < \rho \\ \dfrac{\rho x}{|x|} & \text{if } |x| \geq \rho \end{cases}.$$

Next, we define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)(|u(x)|^p + \sum_{i=1}^m |u(x)|^{p_i})dx}{\int_{\mathbb{R}^N} (|u(x)|^p + \sum_{i=1}^m |u(x)|^{p_i})dx}.$$

**Lemma 14.** ([54]) *The functional $\Phi_\varepsilon$ satisfies the following limit*

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \textit{ uniformly in } y \in M. \tag{4.11}$$

*Proof.* For the convenience to the readers, we present a proof to above lemma. Suppose by a contradiction that there exists $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0 \tag{4.12}$$

for all $n$ large enough. Using the definitions of $\Phi_{\varepsilon_n}(y_n)$, $\beta_{\varepsilon_n}$, $\eta$ and the change of variable $z = \dfrac{\varepsilon_n x - y_n}{\varepsilon_n}$, we have

$$\begin{aligned}
&\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n \\
&+ \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + y_n) - y_n]([\eta(|\varepsilon_n z|)|w(z)|]^p + \sum_{i=1}^m [\eta(|\varepsilon_n z|)|w(z)|]^{p_i})dz}{\int_{\mathbb{R}^N} ([\eta(|\varepsilon_n z|)|w(z)|]^p + \sum_{i=1}^m [\eta(|\varepsilon_n z|)|w(z)|]^{p_i})dz}.
\end{aligned} \tag{4.13}$$

From the assumptions $\{y_n\} \subset M \subset B_\rho(0)$ and $|\chi(x)| \leq \rho$ for all $x \in \mathbb{R}^N$, use the Dominated convergence theorem by taking $n \to \infty$ in (4.13), we get

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = 0,$$

which contradicts with (4.12). □

**Lemma 15.** *Let $\varepsilon_n \to 0^+$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to c_{V_0}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that the translation sequence $v_n(x) = u_n(x + \tilde{y}_n)$ has a subsequence which converges in $W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$. Moreover, up to a subsequence, $\{y_n\} : y_n = \varepsilon \tilde{y}_n \to y \in M$.*

*Proof.* Since $< I'_{\varepsilon_n}(u_n), u_n >= 0$ and $I_{\varepsilon_n}(u_n) \to c_{V_0}$, by arguments Lemma 8 and Lemma 10, $\{||u_n||_{W_{\varepsilon_n}}\}$ is a bounded sequence and when $\gamma_1$ is choosen such that $\gamma_1 \geq \max\{a, \gamma_3\}$ and

$$c_{V_0} \leq a\left(1 - \frac{N}{s\mu}\right)\left(\frac{aN}{\gamma_3 s\mu}\right)^{N/(\mu s - N)} = \mathfrak{b},$$

$$a = \frac{s\left(A_{\mu,\eta} + \varepsilon_*\right)^{N/s}}{N} + \sum_{i=1}^{m} \frac{\left(A_{\mu,\eta} + \varepsilon_*\right)^{p_i}}{p_i}$$

for some $\varepsilon_* > 0$ and $\gamma_3$ satisfies

$$\left[\left(\frac{s}{N} - \frac{1}{\mu}\right)^{-s/N} \mathfrak{b}^{s/N} + \left(\frac{1}{p_1} - \frac{1}{\mu} - \frac{1}{p_1 k}\right)^{\frac{-1}{p_1}} \mathfrak{b}^{\frac{1}{p_1}}\right]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

if $m = 1$ and

$$\left[\left(\frac{s}{N} - \frac{1}{\mu}\right)^{-s/N} \mathfrak{b}^{s/N} + \sum_{i=1}^{m-1}\left(\frac{1}{p_i} - \frac{1}{\mu}\right)^{\frac{-1}{p_i}} \mathfrak{b}^{\frac{1}{p_i}} + \left(\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}\right)^{\frac{-1}{p_m}} \mathfrak{b}^{\frac{1}{p_m}}\right]^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

if $m \geq 2$. Then, we deduce

$$\limsup_{n\to\infty} ||u_n||^{N/s}_{W^{s,p}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{c_{V_0}}{\frac{s}{N} - \frac{1}{\mu}}$$

and

$$\limsup_{n\to\infty} ||u_n||^{p_1}_{W^{s,p_1}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{c_{V_0}}{\frac{1}{p_1} - \frac{1}{\mu} - \frac{1}{p_1 k}}$$

if $m = 1$ and

$$\limsup_{n\to\infty} ||u_n||^{p_m}_{W^{s,p_m}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{c_{V_0}}{\frac{1}{p_m} - \frac{1}{\mu} - \frac{1}{p_m k}}, \quad \limsup_{n\to\infty} ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon}(\mathbb{R}^N)} \leq \frac{c_{V_0}}{\frac{1}{p_i} - \frac{1}{\mu}},$$

$i = 1, \ldots, m-1$ if $m \geq 2$. Hence, we obtain

$$\limsup_{n\to\infty} ||u_n||_{W_\varepsilon}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}. \tag{4.14}$$

We also get

$$\limsup_{n\to\infty} ||u_n||_{V_0,W}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c}\alpha_0}$$

due to the continuous embedding from $W_\varepsilon$ into $W$. Now, we show that there exist a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $r > 0$, $\beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(\tilde{y}_n)} |u_n|^{p_m} dx \geq \beta > 0. \tag{4.15}$$

Indeed, if (4.15) is false, then for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^{p_m} dx = 0.$$

By Lemma 5, we have $u_n \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $q \in (p_m, +\infty)$. Using Trudinger–Moser inequality and (4.14), we deduce

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n dx = 0.$$

Combine that result and $u_n \in \mathcal{N}_{\varepsilon_n}$, we obtain $||u_n||_{W_{\varepsilon_n}} \to 0$ as $n \to \infty$. It is a contradiction with $I_{\varepsilon_n}(u_n) \to c_{V_0} > 0$. Therefore, (4.15) holds. Let us define $v_n := u_n(x + \tilde{y}_n)$. Since the $||.||_{V_0}$ is invariant with the translation, then $\{v_n\}$ is a bounded sequence in $W$, thus up to a subsequence, we can assume that there exists $v \in W$ such that $v_n \to v$ weak in $W$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$ and $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ for any $q \in [\frac{N}{s}, +\infty)$. From that result and (4.15), we get $v \not\equiv 0$. Let $t_n > 0$ such that $w_n = t_n v_n \in \mathcal{N}_{V_0}$ and we set $y_n := \varepsilon_n \tilde{y}_n$. Thus, using the change of the variable $z = x + \tilde{y}_n$, $V(\varepsilon_n(x + \tilde{y}_n)) \geq V_0$ and the invariance by translation, we can see that

$$\begin{aligned} c_{V_0} \leq J_{V_0}(w_n) &\leq \frac{1}{p}[w_n]_{s,p}^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^p dx - \int_{\mathbb{R}^N} F(w_n) dx \\ &\quad + \sum_{i=1}^m \Big(\frac{1}{p_i}[w_n]_{s,p_i}^{p_i} + \frac{1}{p_i}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^{p_i} dx\Big) \\ &\leq \frac{1}{p}[w_n]_{s,p}^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^p dx \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\Big(\frac{1}{p_i}[w_n]_{s,p_i}^{p_i}+\frac{1}{p_i}\int\limits_{\mathbb{R}^N}V(\varepsilon_n x+y_n)|w_n|^{p_i}dx\Big)\\
&-\int\limits_{\mathbb{R}^N}G(\varepsilon_n x+y_n,w_n)dx\\
&=I_{\varepsilon_n}(t_nu_n)\le I_{\varepsilon_n}(u_n)\le c_{V_0}+o_n(1)
\end{aligned}
$$

due to the condition $(g_3)$. Then we get $J_{V_0}(w_n)\to c_{V_0}$. Since $\{w_n\}\subset\mathcal{N}_{V_0}$, using the condition $(f_2)$, there exists a constant $K>0$ such that $||w_n||_{W,V_0}\le K$ for all $n$. We have $v_n\not\to 0$ strongly in $W$. Indeed, if $v_n\to 0$ in $W$, then $v_n\to v$ weak in $W$, it contradicts with $v_n\to v\not\equiv 0$ in $W$. Hence, there exists $\alpha>0$ such that $||v_n||_{V_0,W}\ge\alpha>0$ for all $n$. Consequently, we have

$$
t_n\alpha\le||t_nv_n||_{V_0,W}=||w_n||_{V_0,W}\le K,
$$

which yields $t_n\le\dfrac{K}{\alpha}$ for all $n\in\mathbb{N}$. Therefore, up to a subsequence, we can assume that $t_n\to t_0\ge 0$. We prove that $t_0>0$. If $t_0=0$, then $||w_n||_{V_0,W}\to 0$, it is a contradiction with $w_n\in\mathcal{N}_{V_0}$. Up to a subsequence, we suppose that $w_n\to w:=t_0v\not\equiv 0$ weak in $W$ and $w_n(x)\to w(x)$ a.e. on $\mathbb{R}^N$. By arguments as in Proposition 1 (also see [54]), we can get $J'_{V_0}(w)=0$. Now we prove that

$$
\lim_{n\to\infty}||w_n||^p_{V_0,W^{s,p}(\mathbb{R}^N)}=||w||^p_{V_0,W^{s,p}(\mathbb{R}^N)}\tag{4.16}
$$

and

$$
\lim_{n\to\infty}||w_n||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)}=||w||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)},\ i=1,\dots,m.\tag{4.17}
$$

Using Brezis–Lieb's lemma, (4.16) and (4.17), we obtain $w_n\to w$ strong in $W$. By Fatou's lemma, we have

$$
||w||^p_{V_0,W^{s,p}(\mathbb{R}^N)}\le\liminf_{n\to\infty}||w_n||^p_{V_0,W^{s,p}(\mathbb{R}^N)}\tag{4.18}
$$

and

$$
||w||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)}\le\liminf_{n\to\infty}||w_n||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)},\ i=1,\dots,m.\tag{4.19}
$$

Assume that by contradiction that

$$
||w||^p_{V_0,W^{s,p}(\mathbb{R}^N)}<\limsup_{n\to\infty}||w_n||^p_{V_0,W^{s,p}(\mathbb{R}^N)}.
$$

or

$$
||w||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)}<\limsup_{n\to\infty}||w_n||^{p_i}_{V_0,W^{s,p_i}(\mathbb{R}^N)}
$$

for some $i \in \{1, \ldots, m\}$. We see that

$$c_{V_0} + o_n(1) = J_{V_0}(w_n) - \frac{1}{\mu} < J'_{V_0}(w_n), w_n >$$
$$= \left(\frac{1}{p} - \frac{1}{\mu}\right) ||w_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m} \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||w_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)}$$
$$+ \int_{\mathbb{R}^N} \left[\frac{1}{\mu} f(w_n) w_n - F(w_n)\right] dx.$$

Using the condition $(f_2)$, and Fatou's lemma, we get

$$c_{V_0} \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \limsup_{n\to\infty} ||w_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m} \left(\frac{1}{p_i} - \frac{1}{\mu}\right) \limsup_{n\to\infty} ||w_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)}$$
$$+ \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\mu} f(w_n) w_n - F(w_n)\right] dx$$
$$> \left(\frac{1}{p} - \frac{1}{\mu}\right) ||w||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^{m} \left(\frac{1}{p_i} - \frac{1}{\mu}\right) ||w||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)}$$
$$+ \int_{\mathbb{R}^N} \left[\frac{1}{\mu} f(w) w - F(w)\right] dx$$
$$= J_{V_0}(w) - \frac{1}{\mu} < J'_{V_0}(w), w >= J_{V_0}(w) \geq c_{V_0},$$

which is a contradiction. Then

$$||w||^p_{V_0, W^{s,p}(\mathbb{R}^N)} \geq \limsup_{n\to\infty} ||w_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)}. \tag{4.20}$$

and

$$||w||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)} \geq \limsup_{n\to\infty} ||w_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)}, \; i = 1, \ldots, m. \tag{4.21}$$

Combine (4.18) and (4.20), (4.19) and (4.21), we get (4.16). Since $t_n \to t_0$ as $n \to \infty$, then $v_n \to v$ in $W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$ as $n \to \infty$. Now we prove that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Indeed, if $\{y_n\}$ is not bounded, that is there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for all $n$ large enough, we have $|y_n| > 2R$, and for any $x \in B_{R/\varepsilon_n}(0)$, we have

$$\varepsilon_n x + y_n \geq |y_n| - \varepsilon_n |x| > R.$$

From the condition $(V_1)$, $u_n \in \mathcal{N}_{\varepsilon_n}$ and the definition of $g$ we have

$$||u_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)}$$
$$\le ||u_n||^p_{W^{s,p}_{V,\varepsilon_n}(\mathbb{R}^N)} + \sum_{i=1}^m ||u_n||^{p_i}_{W^{s,p_i}_{V,\varepsilon_n}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n)u_n dx. \tag{4.22}$$

Using the change of variable $z = x + \tilde{y}_n$, from (4.22), we get

$$||v_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||v_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n)v_n dx$$
$$= \int_{B_{R/\varepsilon_n}(0)} g(\varepsilon_n x + y_n, v_n)v_n dx + \int_{B^c_{R/\varepsilon_n}(0)} g(\varepsilon_n x + y_n, v_n)v_n dx$$
$$= \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(v_n)v_n dx + \int_{B^c_{R/\varepsilon_n}(0)} g(\varepsilon_n x + y_n, v_n)v_n dx. \tag{4.23}$$

Note that $\tilde{f}(t) \le \dfrac{V_0}{k}|t|^{p_m-1}$. Then (4.23) implies

$$||v_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} + \sum_{i=1}^m ||v_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)} \le \frac{1}{k}\int_{B_{R/\varepsilon_n}(0)} V_0|v_n|^{p_m} dx$$
$$+ \int_{B^c_{R/\varepsilon_n}(0)} g(\varepsilon_n x + y_n, v_n)v_n dx. \tag{4.24}$$

Since $v_n \to v$ strong in $W$, then $v_n \to v$ strong $L^q(\mathbb{R}^N)$ for all $q \ge \dfrac{N}{s}$, then for any $\varepsilon_* > 0$, we can choose $R$ as above large enough such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^{p_m} dx < \varepsilon^{p_m} \text{ and } \int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^q dx < \varepsilon^q$$

for some $q > p_m$. Using the condition $(g_3)$ and Trudinger–Moser inequality, we get

$$\int_{B^c_{R/\varepsilon_n}(0)} |g(\varepsilon_n x + y_n, v_n)v_n| dx < \kappa\varepsilon_*, \tag{4.25}$$

where $\kappa_* > 0$ is a suitable constant and $n$ large enough. Combine (4.24) and (4.25), we have

$$\left(1 - \frac{1}{k}\right) ||v_n||^{p_m}_{V_0, W^{s,p_m}(\mathbb{R}^N)} + \sum_{i=1}^{m-1} ||v_n||^{p_i}_{V_0, W^{s,p_i}(\mathbb{R}^N)} + ||v_n||^p_{V_0, W^{s,p}(\mathbb{R}^N)} = o_n(1)$$

if $m \geq 2$ and

$$\left(1 - \frac{1}{k}\right) ||v_n||^{p_1}_{V_0, W^{s,p_1}(\mathbb{R}^N)} + ||v_n||^{p}_{V_0, W^{s,p}(\mathbb{R}^N)} = o_n(1)$$

if $m = 1$. That is $v_n \to 0$ strong in $W^{s,N/s}(\mathbb{R}^N) \cap \cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N)$ which contradicts with $v_n \to v \not\equiv 0$. Therefore, we may assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every $n$ large enough, we have $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Thus if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$. By arguments as above, we get a contradiction. Hence, $y_0 \in \overline{\Lambda}$. We now prove $V(y_0) = V_0$. Indeed, if $V(y_0) > V_0$, using the Fatou's lemma and the change of variable $z = x + \tilde{y}_n$, then we have

$$\begin{aligned}
c_{V_0} &= J_{V_0}(w) < J_{V(y_0)}(w) \\
&\leq \liminf_{n\to\infty} \Bigg[ \frac{1}{p} \left( [w_n]^p_{s,p} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^p dx \right) \\
&\quad + \sum_{i=1}^m \frac{1}{p_i} \left( [w_n]^{p_i}_{s,p_i} + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|w_n|^{p_i} dx \right) - \int_{\mathbb{R}^N} F(w_n) dx \Bigg] \\
&= \liminf_{n\to\infty} \Bigg[ \frac{t_n^p}{p}[u_n]^p_{s,p} + \frac{t_n^p}{p} \int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^p dz \\
&\quad + \sum_{i=1}^m \left( \frac{t_n^{p_i}}{p_i}[u_n]^{p_i}_{s,p_i} + \frac{t_n^{p_i}}{p_i} \int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^{p_i} dz \right) - \int_{\mathbb{R}^N} F(t_n u_n) dz \Bigg].
\end{aligned}$$

From above inequality, we deduce

$$\begin{aligned}
c_{V_0} &= J_{V_0}(w) < J_{V(y_0)}(w) \\
&\leq \liminf_{n\to\infty} \Bigg[ \frac{t_n^p}{p}[u_n]^p_{s,p} + \frac{t_n^p}{p} \int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^p dz \\
&\quad + \sum_{i=1}^m \left( \frac{t_n^{p_i}}{p_i}[u_n]^{p_i}_{s,p_i} + \frac{t_n^{p_i}}{p_i} \int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^{p_i} dz \right) - \int_{\mathbb{R}^N} G(\varepsilon_n z, t_n u_n) dz \Bigg] \\
&= \liminf_{n\to\infty} I_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = c_{V_0},
\end{aligned} \tag{4.26}$$

which is an absurd. □

Let $\mathbb{R}^+ \to \mathbb{R}^+$ be a positive function such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and let

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\}.$$

By Lemma 14, we have $h(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Hence $\Phi_\varepsilon(y) \in \mathcal{N}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$. Moreover, we have the following result:

**Lemma 16.** ([7]) *For any* $\delta > 0$, *it holds that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), M_\delta) = 0.$$

**Lemma 17.** *Assume that* $(V)$ *and* $(f_1) - (f_5)$ *hold and let* $v_n$ *be a nontrivial nonnegative solution of the following problem*

$$\begin{aligned}(-\Delta)^s_p v_n + \sum_{i=1}^m (-\Delta)^s_{p_i} v_n + V_n(x)\left(|v_n|^{p-2}v_n + \sum_{i=1}^m |v_n|^{p_i-2}v_n\right)\\ = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n) \text{ in } \mathbb{R}^N,\end{aligned} \tag{4.27}$$

*where* $V_n(x) = V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ *and* $\varepsilon_n \tilde{y}_n \to y \in M$. *If* $\{v_n\}$ *is a bounded sequence in* $W$ *verifying*

$$\limsup_{n\to\infty} ||v_n||^{N/(N-s)}_{V_0,W} < \frac{\beta_* \mathfrak{d}_*{}^{s/(N-s)}}{\mathfrak{c}\alpha_0},$$

*where* $\mathfrak{c} > 1$ *is a suitable constant and* $v_n \to v$ *strong in* $W$, *then* $v_n \in L^\infty(\mathbb{R}^N)$ *and there exists* $C > 0$ *such that* $||v_n||_{L^\infty(\mathbb{R}^N)} \le C$ *for all* $n \in \mathbb{N}$. *Furthermore*

$$\lim_{|x|\to+\infty} v_n(x) = 0 \text{ uniformly in } n.$$

*Proof.* For any $L > 0$ and $\beta > 1$, let us to consider the function $\gamma(t) = t(\min\{t, L\})^{p(\beta-1)}$ and

$$\gamma(v_n) = \gamma_{L,\beta}(v_n) = v_n v_{L,n}^{p(\beta-1)} \in W,\ \ v_{L,n} = \min\{v_n, L\}.$$

Set

$$\Lambda(t) = \frac{|t|^p}{p} \text{ and } \Gamma(t) = \int_0^t (\gamma'(t))^{\frac{1}{p}} d\tau.$$

Then we have [14]

$$\Lambda'(a-b)(\gamma(a)-\gamma(b)) \ge |\Gamma(a)-\Gamma(b)|^p \text{ for any } a, b \in \mathbb{R}. \tag{4.28}$$

From (4.28), we get

$$\begin{aligned}&|\Gamma(v_n(x)) - \Gamma(v_n(y))|^p\\ &\quad\le |v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x) - (v_n v_{L,n}^{p(\beta-1)})(y)).\end{aligned} \tag{4.29}$$

Therefore, taking $\gamma(v_n) = v_n v_{L,n}^{p(\beta-1)}$ as a test function in (4.27), we have

$$\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x)-(v_n v_{L,n}^{p(\beta-1)})(y))}{|x-y|^{2N}}dxdy$$
$$+\sum_{i=1}^{m}\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p_i-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x)-(v_n v_{L,n}^{p(\beta-1)})(y))}{|x-y|^{N+p_i s}}dxdy$$
$$+\int_{\mathbb{R}^N} V_n(x)\left(|v_n|^p+\sum_{i=1}^{m}|v_n|^{p_i}\right)v_{L,n}^{p(\beta-1)}dx = \int_{\mathbb{R}^N} g(\varepsilon_n x+\varepsilon_n \tilde{y}_n, v_n)v_n v_{L,n}^{p(\beta-1)}dx.$$

From the condition $(f_1)$, $(f_3)$ and $(g_3)$, for any $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ such that

$$g(x,t) \le f(t) \le \varepsilon|t|^{p-1} + C(\varepsilon)|t|^{p-1}\Phi_{N,s}(\alpha_0|t|^{N/(N-s)})$$

for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. By arguments as [7], we have

$$\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p_i-2}(v_n(x)-v_n(y))((v_n v_{L,n}^{p(\beta-1)})(x)-(v_n v_{L,n}^{p(\beta-1)})(y))}{|x-y|^{2N}}dxdy$$
$$\ge 0$$

for all $i = 1, \ldots, m$. Combine that inequality with (4.29), we have

$$[\Gamma(v_n)]_{s,p}^p + \int_{\mathbb{R}^N} V_n(x)|v_n|^p v_{L,n}^{p(\beta-1)}dx \le \int_{\mathbb{R}^N} f(v_n)v_n v_{L,n}^{p(\beta-1)}dx.$$

Since $\Gamma(v_n) \ge \frac{1}{\beta} v_n v_{L,n}^{\beta-1}$, $v_n v_{L,n}^{\beta-1} \ge \Gamma(v_n)$ and the embedding from $W^{s,N/s}(\mathbb{R}^N) \to L^{N^*}(\mathbb{R}^N)$ $(N^* > \frac{N}{s})$ is continuous, then there exists a suitable constant $S_* > 0$ such that

$$||\Gamma(v_n)||_{V_0,W^{s,p}(\mathbb{R}^N)}^p \ge S_*||\Gamma(v_n)||_{L^{N^*}(\mathbb{R}^N)}^p \ge \frac{1}{\beta^p}S_*||v_n v_{L,n}^{\beta-1}||_{L^{N^*}(\mathbb{R}^N)}^p. \quad (4.30)$$

We know that the embedding from $W^{s,N/s}(\mathbb{R}^N)\cap\cap_{i=1}^m W^{s,p_i}(\mathbb{R}^N) \to W^{s,N/s}(\mathbb{R}^N) \to L^{\nu}(\mathbb{R}^N)$ $(\nu \ge \frac{N}{s})$ is continuous, then there exists a best constant

$$\mathcal{S}_\nu = \inf_{u\neq 0, u\in W^{s,N/s}(\mathbb{R}^N)} \frac{||u||_{V_0,W^{s,p}(\mathbb{R}^N)}}{||u||_{L^\nu(\mathbb{R}^N)}}, \nu \ge \frac{N}{s}.$$

This implies

$$||u||_{L^p(\mathbb{R}^N)} \le \mathcal{S}_p^{-1}||u||_{V_0,W^{s,p}(\mathbb{R}^N)} \text{ for all } u \in W^{s,p}(\mathbb{R}^N). \quad (4.31)$$

Then we obtain

$$\begin{aligned} ||\Gamma(v_n)||^p_{V_0,W^{s,p}(\mathbb{R}^N)} &\le \varepsilon \int\limits_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^p dx + C(\varepsilon) \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}) |v_n v_{L,n}^{\beta-1}|^p dx \\ &\le \varepsilon \beta^p \int\limits_{\mathbb{R}^N} |\Gamma(v_n)|^p dx + C(\varepsilon) \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}) |v_n v_{L,n}^{\beta-1}|^p dx \\ &\le \varepsilon \beta^p \mathcal{S}_p^{-p} ||\Gamma(v_n)||^p_{V_0,W^{s,p}(\mathbb{R}^N)} + C(\varepsilon) \int\limits_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}) |v_n v_{L,n}^{\beta-1}|^p dx. \end{aligned} \tag{4.32}$$

Choose $0 < \varepsilon < \beta^{-p} \mathcal{S}_p^p$, then (4.32) implies

$$\begin{aligned} &\frac{1}{\beta^p} S_*(1 - \varepsilon \beta^p \mathcal{S}_p^{-p}) ||v_n v_{L,n}^{\beta-1}||^p_{L^{N^*}(\mathbb{R}^N)} \\ &\quad \le C(\varepsilon) \left( \int\limits_{\mathbb{R}^N} (\Phi_{N,s}(\alpha_0 |v_n|^{N/(N-s)}))^{q'} dx \right)^{\frac{1}{q'}} \left( \int\limits_{\mathbb{R}^N} |v_n v_{L,n}^{\beta-1}|^{qp} dx \right)^{\frac{1}{q}}. \end{aligned}$$

Using Trudinger–Moser inequality in $W^{s,N/s}(\mathbb{R}^N)$ with $q >> \dfrac{N}{s}$ such that $N^{**} = qp < N^*$, $q' > 1$ and $q'$ near 1, then there exists a constant $D > 0$ such that

$$||v_n v_{L,n}^{\beta-1}||^p_{L^{N^*}(\mathbb{R}^N)} \le D\beta^p ||v_n v_{L,n}^{\beta-1}||^p_{L^{qp}(\mathbb{R}^N)}.$$

Let $L \to +\infty$ in above inequality, we deduce

$$||v_n||_{L^{N^*\beta}} \le D^{\frac{1}{p\beta}} \beta^{\frac{1}{\beta}} ||v_n||_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{4.33}$$

Now, we set $\beta = \dfrac{N^*}{N^{**}} > 1$. Then $\beta^2 N^{**} = \beta N^*$ and (4.33) holds with $\beta$ replaced by $\beta^2$. Therefore, we obtain

$$\begin{aligned} ||v_n||_{L^{N^*\beta^2}} &\le D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} ||v_n||_{L^{N^{**}\beta^2}(\mathbb{R}^N)} \\ &= D^{\frac{1}{p\beta^2}} \beta^{\frac{2}{\beta^2}} ||v_n||_{L^{N^*\beta}(\mathbb{R}^N)} \\ &\le D^{\frac{1}{p}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2}} ||v_n||_{L^{N^{**}\beta}(\mathbb{R}^N)}. \end{aligned} \tag{4.34}$$

Iterating this process as in (4.34), we can infer that for any positive integer $m$,

$$||v_n||_{L^{N^*\beta^m}} \le D^{\sum_{j=1}^m \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^m j\beta^{-j}} ||v_n||_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{4.35}$$

Taking the limit in (4.35) as $m \to \infty$, we get

$$||v_n||_{L^\infty(\mathbb{R}^N)} \leq C$$

for all $n$, where $C = D^{\sum_{j=1}^{\infty} \frac{1}{p\beta^j}} \beta^{\sum_{j=1}^{\infty} j\beta^{-j}} \sup_n ||v_n||_{L^{N^{**}\beta}(\mathbb{R}^N)} < +\infty$. Since $v_n \to v$ strong in $W$, then $\lim_{|x|\to+\infty} v_n(x) = 0$ uniformly in $n$. □

Let $\delta > 0$ be small enough such that $M_\delta \subset \Lambda$. By Lemma 14 and Lemma 16, there exists $\overline{\varepsilon} = \overline{\varepsilon}_\delta > 0$ such that the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \widetilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well-defined for any $\varepsilon \in (0, \overline{\varepsilon})$. Thanks to Lemma 14 and by decreasing $\overline{\varepsilon}$ if necessary, we obtain that

$$\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$$

for all $y \in M$, for some function $\theta = \theta(\varepsilon, y)$ satisfying $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$, and for all $\varepsilon \in (0, \overline{\varepsilon})$. Therefore, $H(t, y) := y + (1 - t)\theta(\varepsilon, y)$, with $(t, y) \in [0, 1] \times M$, is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon$ and the inclusion map $\mathrm{id} : M \to M_\delta$. By [17, Lemma 4.3] (see also Lemma [22, Lemma 2.2]), we get

$$\mathrm{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M).$$

Since the functional $I_\varepsilon$ satisfies the $(PS)_{c_\varepsilon}$ condition on $\mathcal{N}_\varepsilon$ with $0 < c_\varepsilon \leq c_{V_0} + h(\varepsilon))$, then by Lusternik-Schnirelmann's theory of critical points (see [57, Theorem 5.20]), $I_\varepsilon$ has at least $\mathrm{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geq \mathrm{cat}_{M_\delta}(M)$ critical points on $\widetilde{\mathcal{N}}_\varepsilon \subset \mathcal{N}_\varepsilon$. By Corollary 1, $I_\varepsilon$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points restricted to $\widetilde{\mathcal{N}}_\varepsilon$ which are critical points of $I_\varepsilon$ in $W_\varepsilon$. This means that $(P_\varepsilon)^*$ has at least $\mathrm{cat}_{M_\delta}(M)$ solutions.

Now, we show that there exists $\hat{\varepsilon} = \hat{\varepsilon}_\delta$ such that, for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.37), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a. \tag{4.36}$$

Assuming (4.36) to be false, then there exists a sequence $\varepsilon_n \to 0$ and a sequence $\{u_{\varepsilon_n}\} \subseteq \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $I'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_{\varepsilon_n}|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geq a. \tag{4.37}$$

Since $V(\varepsilon_n x) \geq V_0$ for all $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$, then

$$c_{V_0} \leq \max_{t\geq 0} J_{V_0}(tu_n) \leq \max_{t\geq 0} I_{\varepsilon_n}(tu_n) = I_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n),$$

and $h(\varepsilon_n) \to 0$. It implies that $I_{\varepsilon_n}(u_{\varepsilon_n}) \to c_{V_0}$. By Lemmas 15 and 17, we can find a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $v_n(\cdot) = u_{\varepsilon_n}(\cdot + \tilde{y}_n) \to v$ in $W$ and $y_n = \varepsilon_n \tilde{y}_n \to y \in M$. Then, we can find $r > 0$ such that $B_r(y) \subset B_{2r}(y) \subset \Lambda$ and so $B_{r/\varepsilon_n}(y/\varepsilon_n) \subset \Lambda_{\varepsilon_n}$, for all $n$ large enough. In particular, for any $y \in B_{r/\varepsilon_n}(\tilde{y}_n)$, we have

$$\left| y - \frac{y}{\varepsilon_n} \right| \leq |y - \tilde{y}_n| + \left| \tilde{y}_n - \frac{y}{\varepsilon_n} \right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{n}$$

and $\Lambda_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}^c(\tilde{y}_n)$ for $n$ large enough. Since $v_n \to v$ in $W$, we deduce that $v_n(x) \to 0$ as $|x| \to +\infty$ uniformly in $n \in \mathbb{N}$, and hence there exist $R, n_0 > 0$ such that $v_n(x) < a$ for all $|x| \geq R$ and $n \geq n_0$. Consequently,

$$u_{\varepsilon_n}(x) < a \quad \text{for all } x \in B_R^c(\tilde{y}_n) \text{ and } n \geq n_0. \tag{4.38}$$

Increasing $n_0$ if necessary, we can assume that $\frac{r}{\varepsilon_n} > R$, and we get $\Lambda_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n)$. So,

$$u_{\varepsilon_n}(x) < a \quad \text{for all } x \in \Lambda_{\varepsilon_n}^c \text{ and } n \geq n_0, \tag{4.39}$$

which contradicts (4.37). Hence (4.36) holds.

Setting $\varepsilon_\delta = \min\{\bar{\varepsilon}_\delta, \hat{\varepsilon}_\delta\}$, we can then guarantee that problem (2.37) admits at least $\mathrm{cat}_{M_\delta}(M)$ non-trivial solutions. If $u_\varepsilon \in \mathcal{N}_\varepsilon$ is one of these solutions, in the light of (4.36) and the definition of $g$, $u_\varepsilon$ is a solution of (2.37) and $\hat{u}_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution of problem (1.1).

Final we consider the behavior of maximum points of $\hat{u}_\varepsilon(x)$ as $\varepsilon \to 0$. Take $\varepsilon_n \to 0^+$ and the sequence $\{u_{\varepsilon_n}\}$ of solutions of (2.37) for $\varepsilon = \varepsilon_n$. By $(g_1)$ we can find $\gamma > 0$ small enough such that

$$g(\varepsilon x, t)t \leq \frac{V_0}{k} t^{p_m} \quad \text{for all } x \in \mathbb{R}^N,\ 0 < t \leq \gamma. \tag{4.40}$$

Arguing as before, we can take $R > 0$ such that, for $n$ large enough,

$$\left\| u_{\varepsilon_n} \right\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{4.41}$$

Up to a subsequence, we may assume that, for $n$ large enough,

$$\left\| u_{\varepsilon_n} \right\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma, \tag{4.42}$$

otherwise we would get $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$. Since $I'_{\varepsilon_n}(u_n)(u_n) = 0$, we obtain

$$\begin{aligned} \left\| u_{\varepsilon_n} \right\|_{W_{V,\varepsilon_n}^{s,p}(\mathbb{R}^N)}^p + \sum_{i=1}^m \left\| u_{\varepsilon_n} \right\|_{W_{V,\varepsilon_n}^{s,p_i}(\mathbb{R}^N)}^{p_i} &= \int_{\mathbb{R}^N} g(\varepsilon_n x, u_{\varepsilon_n}) dx \leq \frac{V_0}{k} \int_{\mathbb{R}^N} |u_{\varepsilon_n}|^{p_m} dx \\ &\leq \frac{1}{k} \left\| u_{\varepsilon_n} \right\|_{W_{V,\varepsilon_n}^{s,p_m}(\mathbb{R}^N)}^{p_m}, \end{aligned}$$

and hence $\left\| u_{\varepsilon_n} \right\|_{W_{\varepsilon_n}} \to 0$ as $n \to \infty$, in contrast with $I_{\varepsilon_n}(u_{\varepsilon_n}) \to c_{V_0} > 0$. From (4.41) and (4.42), we deduce that the global maximum points $p_{\varepsilon_n}$ of $u_{\varepsilon_n}$ belong to $B_R(\tilde{y}_n)$, that is $p_{\varepsilon_n} = q_n + \tilde{y}_n$ for some $q_n \in B_R(0)$. Recalling that $\hat{u}_n(x) = u_n(x/\varepsilon_n)$ solves (1.1), then the maximum points $\eta_{\varepsilon_n}$ of $\hat{u}_n$ are $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$.

Noting that $q_n \in B_R(0)$, $\varepsilon_n \tilde{y}_n \to y \in M$, we get $V(y) = V_0 = \lim_{n\to\infty} V(\eta_{\varepsilon_n})$. Then, we deduce

$$\lim_{\varepsilon\to 0^+} V(\eta_\varepsilon) = \lim_{n\to+\infty} V(\varepsilon_n p_{\varepsilon_n}) = V_0.$$

and the proof is concluded. □

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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