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Nonlocal Kirchhoff Problems with Singular Exponential Nonlinearity

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Abstract

In this paper, we first develop the fractional Trudinger–Moser inequality in singular case and then we use it to study the existence and multiplicity of solutions for a class of perturbed fractional Kirchhoff type problems with singular exponential nonlinearity. Under some suitable assumptions, the existence of two nontrivial and nonnegative solutions is obtained by using the mountain pass theorem and Ekeland's variational principle as the nonlinear term satisfies critical or subcritical exponential growth conditions. Moreover, the existence of ground state solutions for the aforementioned problems without perturbation and without the Ambrosetti–Rabinowitz condition is investigated.

Keywords Fractional Kirchhoff problems · Singular exponent nonlinearity · Multiple solutions

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1 Introduction and Main Results

Let $N \geq 2$ and assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and $0 \in \Omega$. Given $s \in (0, 1)$, we study the following fractional Kirchhoff type problem with exponential growth:

$$\begin{cases} M\big(\|u\|^{N/s}\big)\mathcal{L}_{\mathcal{K}}u = \dfrac{f(x,u)}{|x|^{\beta}} + \lambda h(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where

$$\|u\| = \left(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^{N/s}\mathcal{K}(x-y)dxdy\right)^{s/N},$$

$M : [0, \infty) \to [0, \infty)$ is a continuous function, $\beta \in [0, N)$, $\lambda > 0$ is a parameter, $h : \mathbb{R}^N \to [0, \infty)$ is a perturbed function which belongs to the dual space $(W^{s,N/s}_{0,\mathcal{K}}(\Omega))^*$ of $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ (see Sect. 2), $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\mathcal{L}_K$ is the associated nonlocal integro-differential operator which, up to a normalization constant, is defined as

$$\mathcal{L}_{\mathcal{K}}\varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |\varphi(x) - \varphi(y)|^{N/s-2}(\varphi(x) - \varphi(y))\mathcal{K}(x-y)\, dy, \quad x \in \mathbb{R}^N,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^N)$. Henceforward $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. Throughout the paper, we always assume that *the singular kernel* $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ *satisfies the following properties*:

(K_1) $m\mathcal{K} \in L^1(\mathbb{R}^N)$, *where* $m(x) = \min\{1, |x|^{N/s}\}$;
(K_2) *there exists* $\mathcal{K}_0 > 0$ *such that* $\mathcal{K}(x) \geq \mathcal{K}_0|x|^{-2N}$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$.

Obviously, if $\mathcal{K}(x) = |x|^{-2N}$, then $\mathcal{L}_{\mathcal{K}}$ reduces to the fractional N/s-Laplacian $(-\Delta)^s_{N/s}$.

Equations of the type (1.1) are important in many fields of science, notably continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and anomalous diffusion, since they are the typical outcome of stochastically stabilization of Lévy processes, see [2,8,25] and the references therein. Moreover, such equations and the associated fractional operators allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media, for more details see [2,8] and the references therein. Indeed, the nonlocal fractional operators have been extensively studied by many authors in many different cases: bounded domains and unbounded domains, different behavior of the nonlinearity, and so on. In particular, many works focus on the subcritical and critical growth of the nonlinearity which allow us to treat the problem variationally using general critical point theory.

Recently, some authors have paid considerable attention in the limiting case of the fractional Sobolev embedding. Let ω_{N-1} be the $N-1$-dimensional measure of the unit sphere in $\mathbb{R}^N$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain and define $W_0^{s,N/s}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_{s,N/s}$ which is defined as

$$[u]_{s,N/s} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{N/s}}{|x-y|^{2N}} dxdy \right)^{s/N}.$$

In [29], Martinazzi obtained that there exist positive constants

$$\alpha_{N,s} = \frac{N}{\omega_{N-1}} \left(\frac{\Gamma((N-s)/2)}{\Gamma(s/2)2^s \pi^{N/2}} \right)^{-\frac{N}{N-s}}$$

and $C_{N,s}$ depending only on N and s such that

$$\sup_{\substack{u \in W_0^{s,N/s}(\Omega) \\ [u]_{s,N/s} \le 1}} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-s}}) dx \le C_{N,s} |\Omega|, \tag{1.2}$$

for all $\alpha \in [0, \alpha_{N,s}]$ and there exists $\alpha^*_{N,s} \ge \alpha_{N,s}$ such that the supremum in (1.2) is ∞ for $\alpha > \alpha^*_{N,s}$. However, it still an open problem whether or not $\alpha_{N,s} = \alpha^*_{N,s}$? For more details about Trudinger–Moser inequality, we also refer to [23] and [39].

On one hand, in the setting of the fractional Laplacian, Iannizzotto and Squassina in [22] investigated existence of solutions for the following Dirichlet problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = f(u) & \text{in } (0,1), \\ u = 0 & \text{in } \mathbb{R} \setminus (0,1), \end{cases} \tag{1.3}$$

where $(-\Delta)^{\frac{1}{2}}$ is the fractional Laplacian and $f(u)$ behaves like $\exp(\alpha |u|^2)$ as $u \to \infty$. Using the mountain pass theorem, the authors obtained the existence of solutions for problem (1.3). The existence of ground state solutions for (1.3) was discussed in [16]. Subsequently, Giacomoni, Mishra and Sreenadh in [21] studied the multiplicity of solutions for problems like (1.3) by using the Nehari manifold method. For more recent results for problem (1.3) in the higher dimension case, we refer the interested reader to [41] and the references therein. For the general fractional p-Laplacian in unbounded domains, Souza in [13] considered the following nonhomogeneous fractional p-Laplacian equation

$$(-\Delta)^s_p u + V(x)|u|^{p-2}u = f(x,u) + \lambda h \ \text{ in } \mathbb{R}^N, \tag{1.4}$$

where $(-\Delta)^s_p$ is the fractional p-Laplacian and the nonlinear term f satisfies exponential growth. The author obtained a nontrivial weak solution of the equation (1.4) by using fixed point theory. Li and Yang [27] studied the following equation

$$(-\Delta)_p^{\zeta} u + V(x)|u|^{p-2}u = \lambda A(x)|u|^{q-2}u + f(u) \ \ \mathrm{x} \in \mathbb{R}^{\mathrm{N}},$$

where $p \geq 2, 0 < \zeta < 1, 1 < q < p, \lambda > 0$ is a real parameter, A is a positive function in $L^{\frac{p}{p-q}}(\mathbb{R}^N)$, $(-\Delta)_p^{\zeta}$ is the fractional p-Laplacian and f satisfies exponential growth.

On the other hand, Li and Yang in [26] studied the following Schrödinger–Kirchhoff type equation

$$\left(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N) dx\right)^k (-\Delta_N u + V(x)|u|^{N-2}u) \\ = \lambda A(x)|u|^{p-2}u + f(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplaician, $k > 0$, $V : \mathbb{R}^N \to (0, \infty)$ is continuous, $\lambda > 0$ is a real parameter, A is a positive function in $L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and f satisfies exponential growth. By using the mountain pass theorem and Ekeland's variational principle, the authors obtained two nontrivial solutions of (1.5) as the parameter λ small enough. Mingqi, Rădulescu and Zhang studied the following problem

$$\begin{cases} M\big(\|u\|^{N/s}\big)(-\Delta)_p^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where f behaves like $\exp(\alpha|t|^{N/(N-s)})$ as $t \to \infty$ for some $\alpha > 0$. Under suitable assumption on M and f, the authors obtained the existence of ground state solutions by using the mountain pass lemma combined with the fractional Trudinger–Moser inequality. Actually, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. More precisely, Kirchhoff established a model governed by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.6}$$

for all $x \in (0, L)$, $t \geq 0$, where $u = u(x, t)$ is the lateral displacement at the coordinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and ρ_0 is the initial axial tension. Eq. (1.6) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Recently, Fiscella and Valdinoci in [18] have proposed a stationary Kirchhoff model driven by the fractional Laplacian by taking into account the nonlocal aspect of the tension, see [18, Appendix A] for more details. Related results in the case of critical nonlinearities have been obtained by Fiscella and Pucci [19] and Miyagaki and Pucci [35].

It is worth mentioning that when $s \to 1$ and $M \equiv 1$, the equation in problem (1.1) becomes

$$-\Delta_N u = f(x, u) + \lambda h(x),$$

which studied by many authors by using variational methods, see for example, [1,12, 15,20,24].

Inspired by the above works, we are devoted to the existence and multiplicity of solutions for problem (1.1) and overcome the lack of compactness due to the presence of critical exponential growth terms as well as the degenerate nature of the Kirchhoff coefficient. To the best of our knowledge, there are no results for (1.1) in such a generality.

Throughout the paper, without explicit mention, we assume that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is assumed to be continuous and to verify*

(M_1) *for any* $\tau > 0$ *there exists* $\kappa = \kappa(\tau) > 0$ *such that* $M(t) \geq \kappa$ *for all* $t \geq \tau$;
(M_2) there exists $\theta > 1$ such that

$$\theta \mathscr{M}(t) \geq M(t)t \;\text{ forall } t \geq 0,$$

where $\mathscr{M}(t) = \int_0^\tau M(\tau)d\tau$.

A typical example of M is given by $M(t) = a + b\theta\, t^{\theta-1}$ for $t \geq 0$, where $a, b \geq 0$ and $a+b > 0$. When M is of this type, problem (1.1) is said to be degenerate if $a = 0$, while it is called non-degenerate if $a > 0$. Recently, the fractional Kirchhoff problems have received more and more attention. Some new existence results of solutions for fractional non-degenerate Kirchhoff problems were given, for example, in [42–44,49]. On some recent results concerning the degenerate case of Kirchhoff-type problems, we refer to [3,9,30,45,50] and the references therein. It is worth pointing out that the degenerate case in Kirchhoff theory is rather interesting, for example, it was treated in the seminal paper [11]. In the large literature on degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M(\|u\|^2)$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero. Clearly, assumptions (M_1)–(M_2) cover the degenerate case.

Define

$$\lambda_1 = \inf\left\{\frac{\|u\|^{\theta N/s}}{\int_\Omega \frac{1}{|x|^\beta}|u|^{\theta N/s}dx} : u \in W_{0,\mathcal{K}}^{s,N/s}(\Omega) \setminus \{0\}\right\}.$$

Clearly, by $0 \leq \beta < N$ and the fractional Sobolev embedding, we obtain that $\lambda_1 > 0$.

First in bounded domain Ω, we assume that the nonlinear term $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function, with* $f(x,t) \equiv 0$ *for* $t \leq 0$ *and* $x \in \Omega$. In the following, we also require the following assumptions (f_1)–(f_3).

(f_1) f satisfies subscritical growth, i.e., for any $\alpha > 0$ there holds

$$\lim_{t\to\infty} f(x,t)\exp(-\alpha|t|^{N/(N-s)}) = 0,$$

uniformly in Ω.

(f_1') f satisfies critical growth, i.e., there exists $\alpha_0 > 0$ such that,

$$\lim_{t\to\infty} f(x,t)\exp(-\alpha|t|^{N/(N-s)}) = \begin{cases} 0, & \forall\alpha > \alpha_0, \\ \infty, & \forall\alpha < \alpha_0, \end{cases}$$

uniformly in Ω.

(f_2) There exists $\mu > \theta N/s$ such that

$$0 < \mu F(x,t) \leq f(x,t)t, \quad F(x,t) = \int_0^t f(x,\tau)d\tau,$$

whenever $x \in \Omega$ and $t > 0$, and there exists some $T > 0$ such that $\inf_{x\in\Omega} F(x,T) > 0$.

(f_3) There holds:

$$\limsup_{t\to 0^+} \frac{F(x,t)}{|t|^{\theta N/s}} < \frac{s\mathscr{M}(1)}{N}\lambda_1 \text{ uniformly in } x \in \Omega.$$

(f_4) There exist $q_0 > \theta N/s$ and $\mathcal{C}_0 > 0$ such that

$$F(x,t) \geq \frac{\mathcal{C}_0}{q_0} t^{q_0} \text{ for all } x \in \Omega \text{ and } t \geq 0,$$

where

$$\mathcal{C}_0 > \left(\frac{4\mu(sq_0 - N\theta)}{q(s\mu - N\theta)}\right)^{\frac{q_0 s - N\theta}{N\theta}} \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{(N-s)(q_0 s - N\theta)}{Ns}} C_{q_0}^{\frac{sq_0}{N\theta}}, \tag{1.7}$$

and $C_{q_0} > 0$ is defined by

$$C_{q_0} = \inf_{u\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)\setminus\{0\}} \left\{ \|u\|^{N/s} : \int_\Omega \frac{1}{|x|^\beta}|u|^{q_0}dx = 1 \right\}.$$

A simple example of f, verifying (f_1)–(f_2), is given by

$$f(x,t) = t^{\theta N/s}\left[\exp(|t|^{N/(N-s)}) - 1\right] + \mathcal{C}_0 t^{\theta N/s - 1},$$

where $\mathcal{C}_0$ is a positive constant.

Definition 1.1 We say that $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ is a (weak) *solution* of problem (1.1), if

$$
\begin{aligned}
M(\|u\|^{N/s})\langle u,\varphi\rangle_{s,N/s} &= \int_\Omega \left(\frac{f(x,u)}{|x|^\beta} + \lambda h(x)\right)\varphi dx,\\
\langle u,\varphi\rangle_{s,N/s} &= \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{\frac{N}{s}-2}[u(x)-u(y)]\\
&\quad \cdot\big[\varphi(x)-\varphi(y)\big]K(x-y)dxdy,
\end{aligned}
$$

for all $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, where $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ will be introduced in Sect. 2.

Now we are in a position to state our results concerning the subcritical case.

Theorem 1.1 *Assume that f satisfies (f_1)–(f_3) and M fulfills (M_1)–(M_2). Let $0 \le h \in (W^{s,N/s}_{0,\mathcal{K}}(\Omega))^*$. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda < \lambda^*$, problem (1.1) admits at least two nontrivial and nonnegative solutions in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, where one is a mountain pass type solution and another is a local least energy solution.*

For the critical case, we have the following result.

Theorem 1.2 *Assume that f satisfies (f_1'), (f_2)–(f_4) and $M = a + b\theta t^{\theta-1}$ with $a \ge 0,\ b > 0$ and $\theta > 1$. Let $0 \le h \in (W^{s,N/s}_{0,\mathcal{K}}(\Omega))^*$. Then there exists $\lambda_* > 0$ such that for all $0 < \lambda < \lambda_*$, problem (1.1) admits at least two nontrivial and nonnegative solutions in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, where one is a mountain pass type solution and another is a local least energy solution.*

Let us simply give an sketch of the proofs of Theorems 1.1 and 1.2. Since the problems discussed here satisfies singular exponential growth conditions, the fractional Trudinger–Moser inequality is not available directly. Thus, we first obtain the fractional Trudinger–Moser inequality in singular case. Then, two theorems are proved by using the mountain pass lemma and the Eekand variational principle combined with the singular fractional Trudinger–Moser inequality. To applying the mountain pass theorem and the Ekeland variational principle, we must verify that the associated functional satisfies the Palais–Smale conditions. However, since the nonlinear term satisfies the critical exponential growth, it becomes more difficulty to get the compactness of the energy functional. To overcome the loss of compactness of the energy functional, we have to estimate the range of level value of energy functional. This is the key point to obtain the existence of solutions for the critical case.

Finally, we consider the following problem with critical exponential growth

$$
\begin{cases}
M\big(\|u\|^{N/s}\big)\mathcal{L}_{\mathcal{K}}u = \dfrac{f(x,u)}{|x|^\beta} & \text{in } \Omega,\\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\tag{1.8}
$$

To get the existence of ground state solutions for problem (1.8), we also need the following hypotheses:

(M_3) There exists $\theta > 1$ such that $\dfrac{M(t)}{t^{\theta-1}}$ is nonincreasing for $t > 0$.

(M_4) $\mathscr{M}$ is superadditive, i.e., for any $t_1, t_2 \geq 0$ there holds

$$\mathscr{M}(t_1) + \mathscr{M}(t_2) \leq \mathscr{M}(t_1 + t_2).$$

(f_5) There exists $\beta_0 > \dfrac{M\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right)\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}}{\frac{\omega_{N-1}R_0^{N-\beta}}{N-\beta}}$ such that

$$\lim_{t\to\infty} \frac{f(x,t)t}{\exp\left(\frac{\alpha_0\alpha^*_{N,s}}{\alpha_{N,s}}t^{N/(N-s)}\right)} \geq \beta_0 \quad \text{uniformly in } x \in \Omega,$$

where R_0 is the radius of the largest open ball centered at zero contained in Ω.

(f_6) For each $x \in \Omega$, $\dfrac{f(x,t)}{t^{\frac{\theta N}{s}-1}}$ is increasing for $t > 0$, where $\theta > 1$ is given by (M_3).

Remark 1.1 If M is a nondecreasing function, then (M_4) holds. Indeed, for any $0 \leq t_1 \leq t_2 < \infty$

$$\mathscr{M}(t_1+t_2) = \int_0^{t_1+t_2} M(t)dt = \int_0^{t_1} M(t)dt + \int_{t_1}^{t_1+t_2} M(t)dt \geq \mathscr{M}(t_1) + \mathscr{M}(t_2).$$

In terms of (M_3) and Remark 1.1 of [33], we can obtain that

$$\theta\mathscr{M}(t) - M(t)t \ \text{ is nondecreasing for } \ t > 0.$$

In particular, we have

$$\theta\mathscr{M}(t) - M(t)t \geq 0, \quad \forall t \geq 0. \tag{1.9}$$

Moreover, from (M_3) one can deduce that

$$\lim_{t\to\infty} \mathscr{M}(t) = \infty.$$

Remark 1.2 According to (f_1'), for some $0 < \alpha < \alpha_0$ we have

$$\lim_{t\to\infty} \frac{f(x,t)}{\exp(\alpha t^{\frac{N}{N-s}})} = \infty,$$

uniformly in Ω. Then

$$\lim_{t\to\infty} \frac{f(x,t)}{t^{\theta\frac{N}{s}-1}} = \infty,$$

uniformly in Ω. Furthermore, we deduce

$$\lim_{t\to\infty} \frac{F(x,t)}{t^{\theta\frac{N}{s}}} = \infty, \tag{1.10}$$

uniformly in Ω.

Using (f_6) and the same discussion as [33], one can deduce that for each $x \in \Omega$,

$$tf(x,t) - \frac{N\theta}{s}F(x,t) \text{ is increasing for } t > 0. \tag{1.11}$$

In particular, $tf(x,t) - \frac{N\theta}{s}F(x,t) \geq 0$ for all $(x,t) \in \Omega \times [0,\infty)$.

Theorem 1.3 *Assume that f satisfies (f_1'), (f_3), (f_5) and (f_6), and M fulfills (M_1), (M_3) and (M_4). Then problem (1.8) has a ground state solution in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$.*

To get the existence of ground state solutions for problem (1.8), we first show that problem (1.8) has a nonnegative mountain pass solution, and then prove that the mountain pass solution is a ground state solution. The main difficulty is that how we can get the strong convergence of $(u_n)_n$ and how to prove that the limit of $(u_n)_n$ is the ground state solution of problem (1.8). In the process of proving our main results, some ideas are inspired from papers [17] and [33].

To the best of our knowledge, Theorems 1.1–1.3 are the first results for the Kirchhoff equations involving singular Trudinger–Moser nonlinearities in the fractional setting.

The paper is organized as follows. In Sect. 2, we present the functional setting and show preliminary results. In Sect. 3, by using the mountain pass theorem and Ekeland' variational principle, we obtain the existence of two nontrivial nonnegative solutions for problem (1.1) with subcritical exponential growth conditions as λ small. In Sect. 4, we get the existence of two nonnegative solutions for problem (1.1) with critical exponential nonlinearity. In Sect. 5, we investigate the existence of ground state solutions for problem (1.8) without perturbation term and the Ambrosetti–Rabinowitz condtion.

2 Preliminary Results

In this section, we give the variational framework of problem (1.1) and prove several necessary results which will be used later.

Define $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ as

$$W_{0,\mathcal{K}}^{s,N/s}(\Omega) = \left\{ u \in L^{N/s}(\Omega) : [u]_{s,\mathcal{K}} < \infty,\ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

where

$$[u]_{s,\mathcal{K}} = \left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^{N/s} \mathcal{K}(x-y)\,dx\,dy \right)^{s/N}.$$

Equip $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ with respect to the norm

$$\|u\| = [u]_{s,\mathcal{K}},$$

here we apply (k_1). By a similar discussion as in [44], we know that $(W^{s,N/s}_{0,\mathcal{K}}(\mathbb{R}^N), \|\cdot\|)$ is a reflexive Banach space. Clearly, the embedding $W^{s,N/s}_{0,\mathcal{K}}(\Omega) \hookrightarrow W^{s,N/s}_0(\Omega)$ is continuous, being

$$[u]_{s,N/s} \le \mathcal{K}_0^{-s/N}[u]_{s,\mathcal{K}} \quad \text{for all } u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega),$$

by (k_2).

Theorem 2.1 (see [14, Theorem 6.10]) *Let $s \in (0,1)$ and $N \ge 1$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Then there exists a positive constant $C = C(N, s, \Omega)$ such that for any $u \in W^{s,N/s}_0(\Omega)$ there holds*

$$\|u\|_{L^q(\Omega)} \le C[u]_{s,N/s}$$

for any $q \in [1,\infty)$, i.e. the space $W^{s,N/s}_0(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,\infty)$.

For $\nu \ge 1$ and $\beta < N$, we define

$$L^\nu(\Omega, |x|^{-\beta}) = \left\{ u : \Omega \to \mathbb{R} \ \text{ is measurable} \middle| \int_\Omega \frac{1}{|x|^\beta}|u(x)|^\nu dx < \infty \right\},$$

endowed with the norm

$$|u|_{\nu,\beta} = \left(\int_\Omega \frac{1}{|x|^\beta}|u(x)|^\nu dx \right)^{\frac{1}{\nu}}.$$

To prove the existence of weak solutions for problem (1.1), we shall use the following embedding theorem.

Theorem 2.2 (Compact embedding) *Let $s \in (0,1)$, $N \ge 1$ and $0 \le \beta < N$. Assume that Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. Then, for any $\nu \ge 1$ the embeddings $W^{s,N/s}_0(\Omega) \hookrightarrow L^\nu(\Omega)$ and $W^{s,N/s}_0(\Omega) \hookrightarrow L^\nu(\Omega, |x|^{-\beta})$ are compact.*

Proof By [33], we know that the embedding $W^{s,N/s}_0(\Omega) \hookrightarrow L^\nu(\Omega)$ is compact for any $\nu \in [1,\infty)$.

Next we show that $W^{s,N/s}_0(\Omega) \hookrightarrow L^\nu(\Omega, |x|^{-\beta})$ is compact. To this aim, we choose $t > 1$ close to 1 such that $\beta t < N$. Then for any bounded sequence $(u_n)_n$ in $W^{s,N/s}_0(\Omega)$, we have

$$\int_\Omega \frac{1}{|x|^\beta}|u_n-u|^\nu dx \le \left(\int_\Omega \frac{1}{|x|^{\beta t}}dx\right)^{\frac{1}{t}} \left(\int_\Omega |u_n-u|^{\frac{\nu t}{t-1}}dx\right)^{\frac{t-1}{t}}$$
$$\le C\left(\int_\Omega |u_n-u|^{\frac{\nu t}{t-1}}dx\right)^{\frac{t-1}{t}}.$$

Note that the embedding $W_0^{s,N/s}(\Omega) \hookrightarrow L^{\frac{\nu t}{t-1}}(\Omega)$ is compact. Thus,

$$\int_\Omega \frac{1}{|x|^\beta}|u_n-u|^\nu dx \to 0.$$

This proves the theorem. □

Theorem 2.3 *Let $N \ge 2$ and let Ω be a bounded domain in $\mathbb{R}^N$ containing the origin. Assume $u \in W_0^{s,N/s}(\Omega)$. Then for any $\alpha \ge 0$ and $\beta \in [0,N)$ there holds*

$$\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx < \infty.$$

Moreover, for all $0 \le \alpha < \left(1-\frac{\beta}{N}\right)\alpha_{N,s}$ there holds

$$\sup_{[u]_{s,N/s}\le 1}\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx < \infty,$$

and the supremum is ∞ for $\alpha > \left(1-\frac{\beta}{N}\right)\alpha^*_{N,s}$.

Proof Choose $\sigma > 1$ such that $\sigma\beta < N$. Then by the Hölder inequality and the fractional Trudinger–Moser inequality, we have

$$\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx \le \left(\int_\Omega \exp\left(\alpha\frac{\sigma}{\sigma-1}|u|^{N/(N-s)}\right)dx\right)^{\frac{\sigma-1}{\sigma}} \left(\int_\Omega \frac{1}{|x|^{\sigma\beta}}dx\right)^{\frac{1}{\sigma}} < \infty,$$

being $\beta\sigma < N$. If $\alpha < (1-\frac{\beta}{N})\alpha_{N,s}$, we can choose $\sigma > 1$ is sufficiently close 1 such that $\sigma\alpha < \alpha_{N,s}$ and $\sigma(\sigma-1)^{-1} < \frac{N}{\beta}$. Then by the Hölder inequality and the fractional Trudinger–Moser inequality, we deduce that

$$\sup_{\|u\|\le 1}\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx \le \sup_{\|u\|\le 1}\left(\int_\Omega \exp(\alpha\sigma|u|^{N/(N-s)})dx\right)^{\frac{1}{\sigma}} \left(\int_\Omega \frac{1}{|x|^{\beta\frac{\sigma}{\sigma-1}}}dx\right)^{\frac{\sigma-1}{\sigma}}$$
$$<\infty.$$

Now we define the Moser functions which have been used in [40]:

$$\widetilde{G}_n(x) = \frac{1}{\gamma_{s,N}^{s/N}} \begin{cases} |\ln n|^{\frac{N-s}{N}} & \text{if } |x| \le \frac{1}{n}, \\ \frac{|\ln |x||}{|\ln n|^{\frac{s}{N}}} & \text{if } \frac{1}{n} < |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where

$$\gamma_{s,N} := \frac{2(N\omega_N)^2 \Gamma\left(\frac{N}{s}+1\right)}{N!} \sum_{k=0}^{\infty} \frac{N+k-1}{k!} \frac{1}{(N+2k)^{N+s}}.$$

By the result in [40], we get

$$\lim_{n\to\infty} [\widetilde{G}_n(x)]_{s,N/s}^{N/s} = 1.$$

Choose $R > \varepsilon > 0$ such that $B_R(0) \subset \Omega$ and define

$$G_n(x) = \widetilde{G}_n(x/R),$$

then $G_n(x) \in W_0^{s,N/s}(\Omega)$, the support of $G_n(x)$ is the ball $B_R(0)$ and

$$\lim_{n\to 0} [G_n]_{s,N/s} = 1. \tag{2.1}$$

Consider $\omega_n = \frac{G_n}{[G_n]_{s,N/s}}$, then we can write

$$\omega_n^{N/(N-s)} = \gamma_{s,N}^{-s/(N-s)} \ln n + d_n \ \text{ for } |x| \le \frac{R}{n}.$$

Moreover, we have

$$\frac{d_n}{\ln n} \to 0 \ \text{ as } \ n \to \infty. \tag{2.2}$$

Thus, for $\alpha > (N-\beta)\gamma_{s,N}^{\frac{s}{N-s}}$, we deduce that

$$\begin{aligned} &\int_\Omega \frac{\exp(\alpha|\omega_n|^{N/(N-s)})}{|x|^\beta} dx \\ &\ge \int_{B_{R/n}(0)} \frac{\exp(\alpha|\omega_n|^{N/(N-s)})}{|x|^\beta} dx \\ &= \exp[\alpha(\gamma_{s,N}^{-s/(N-s)} \ln n + d_n)] \frac{(\frac{R}{n})^{N-\beta}\omega_{N-1}}{N-\beta} \\ &= R^{N-\beta}\omega_{N-1} \exp[(\alpha\gamma_{s,N}^{-s/(N-s)} - N + \beta)\ln n + \alpha d_n] \to \infty \ \text{ as } \ n \to \infty, \end{aligned}$$

which together with

$$\sup_{[u]_{s,N/s}\le 1}\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx \ge \int_\Omega \frac{\exp(\alpha|\omega_n|^{N/(N-s)})}{|x|^\beta}dx$$

yields that

$$\sup_{[u]_{s,N/s}\le 1}\int_\Omega \frac{\exp(\alpha|u|^{N/(N-s)})}{|x|^\beta}dx = \infty.$$

It follows from [40] that $\alpha^*_{N,s} = N\gamma_{s,N}^{s/(N-s)}$. In conclusion, the proof is complete. □

We give a singular fractional version of theorem of P.L. Lions ([28]).

Theorem 2.4 *Let $(u_n)_n$ be sequence in $W_0^{s,N/s}(\Omega)$ satisfying $[u_n]_{s,N/s} = 1$ and converging weakly to a nonzero function u. Then for any $\alpha < (1-\frac{\beta}{N})\alpha_{N,s}(1-[u]_{s,N/s}^{N/s})^{-s/(N-s)}$ and $0 \le \beta < N$,*

$$\sup_n \int_\Omega \frac{\exp\left(\alpha|u_n|^{N/(N-s)}\right)}{|x|^\beta}dx < \infty.$$

Proof By the Hölder inequality, we obtain

$$\int_\Omega \frac{\exp\left(\alpha|u_n|^{N/(N-s)}\right)}{|x|^\beta}dx \le \left(\int_\Omega \exp\left(t\alpha|u_n|^{N/(N-s)}\right)dx\right)^{\frac{1}{t}}\left(\int_\Omega \frac{1}{|x|^{\beta\frac{t}{t-1}}}dx\right)^{\frac{t-1}{t}}.$$

where $t > \frac{N}{N-\beta}$ sufficiently close to $\frac{N}{N-\beta}$ such that $\alpha t < \alpha_{N,s}(1-[u]_{s,N/s}^{N/s})^{-s/(N-s)}$. By Theorem 2.2 in [41], we have

$$\sup_n \left(\int_\Omega \exp\left(t\alpha|u_n|^{N/(N-s)}\right)dx\right)^{\frac{1}{t}} < \infty.$$

Clearly, from $t > \frac{N}{N-\beta}$, one can deduce that

$$\int_\Omega \frac{1}{|x|^{\beta\frac{t}{t-1}}}dx < \infty.$$

Therefore, the desired result holds true. □

To study the nonnegative solutions of problems (1.1) and (1.8), we define the associated functionals $I_\lambda, I : W_{0,\mathcal{K}}^{s,N/s}(\Omega) \to \mathbb{R}$ as

$$I_\lambda(u) = \frac{s}{N}\mathscr{M}(\|u\|^{N/s}) - \int_\Omega \frac{1}{|x|^\beta}F(x,u)dx - \lambda\int_\Omega h(x)udx$$

and

$$I(u)=\frac{s}{N}\mathscr{M}(\|u\|^{N/s})-\int_\Omega\frac{1}{|x|^\beta}F(x,u)dx.$$

Since f is continuous and satisfies (f_1) (or (f_1')) and (f_3), for any $0<\varepsilon<\lambda_1,\alpha>\alpha_0$ and $q\geq 0$, there exists $C=C(\varepsilon,\alpha,q)>0$ such that

$$|F(x,t)|\leq\frac{s}{N}\mathscr{M}(1)(\lambda_1-\varepsilon)|t|^{\frac{\theta N}{s}}+C|t|^q\exp(\alpha|t|^{\frac{N}{N-s}})\ \forall(x,t)\in\Omega\times\mathbb{R}. \quad (2.3)$$

If (f_1') holds, then the $\alpha>0$ in (2.3) is arbitrary. Using (2.3), Theorem 2.3 and the assumption on $\mathcal{K}$, one can verify that the functionals I_λ and I are well defined, of class $C^1(W^{s,N/s}_{0,\mathcal{K}}(\Omega),\mathbb{R})$. Moreover,

$$\langle I_\lambda'(u),v\rangle=M(\|u\|^{N/s})\langle u,v\rangle_{s,N/s}-\int_\Omega\frac{f(x,u)}{|x|^\beta}vdx-\lambda\int_\Omega hvdx$$

and

$$\langle I'(u),v\rangle=M(\|u\|^{N/s})\langle u,v\rangle_{s,N/s}-\int_\Omega\frac{f(x,u)}{|x|^\beta}vdx$$

for all $u,v\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. From now on, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\big(W^{s,N/s}_{0,\mathcal{K}}(\Omega)\big)'$ and $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Evidently, the critical points of I_λ and I are exactly the weak solutions of problem (1.1) and problem (1.8), respectively. Moreover, the following lemma shows that any nontrivial weak solution of problem (1.1) or problem (1.8) is nonnegative.

Lemma 2.1 *If $h(x)\geq 0$ for almost every $x\in\Omega$, then for all $\lambda>0$ any nontrivial critical point of functional I_λ is nonnegative. Similarly, any nontrivial critical point of functional I is also nonnegative.*

Proof Fix $\lambda>0$ and let $u_\lambda\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)\setminus\{0\}$ be a critical point of functional I_λ. Clearly, $u_\lambda^-=\max\{-u,0\}\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Then $\langle I_\lambda'(u_\lambda),-u_\lambda^-\rangle=0$, a.e.

$$M(\|u_\lambda\|^{N/s})\langle u_\lambda,-u_\lambda^-\rangle_{s,N/s}=\int_\Omega\frac{1}{|x|^\beta}f(x,u_\lambda)(-u_\lambda^-)dx+\lambda\int_\Omega h(-u_\lambda^-)dx.$$

Observe that for a.e. $x,y\in\Omega$,

$$\begin{aligned}&|u_\lambda(x)-u_\lambda(y)|^{N/s-2}(u_\lambda(x)-u_\lambda(y))(-u_\lambda^-(x)+u_\lambda(y)^-)\\&\quad=|u_\lambda(x)-u_\lambda(y)|^{N/s-2}u_\lambda^+(x)u_\lambda^-(y)\\&\qquad+|u_\lambda(x)-u_\lambda(y)|^{N/s-2}u_\lambda^-(x)u_\lambda^+(y)+[u_\lambda^-(x)-u_\lambda^-(y)]^p\\&\quad\geq|u_\lambda^- -u_\lambda^-(y)|^{N/s},\end{aligned}$$

$f(x,u_\lambda)u_\lambda^- = 0$ a.e. $x \in \Omega$ by assumption and $h(-u_\lambda^-) \le 0$ a.e. in Ω. Hence,

$$M(\|u_\lambda\|^{N/s})\|u_\lambda^-\|^{N/s} \le 0.$$

This, together with $\|u_\lambda\| > 0$ and (M_1), implies that $u_\lambda^- \equiv 0$, that is $u_\lambda \ge 0$ a.e. in Ω.
Similarly, one can verify that any nontrivial critical point of functional I is nonnegative. ☐

3 The Subcritical Case

Let us recall that I_λ satisfies the $(PS)_c$ *condition* in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ at level $c \in \mathbb{R}$, if any $(PS)_c$ sequence $(u_n)_n \subset W_{0,\mathcal{K}}^{s,N/s}(\Omega)$, namely a sequence such that $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$, admits a strongly convergent subsequence in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$.

In the sequel, we shall make use of the well-known mountain pass theorem. For the reader's convenience, we state it as follows (see for example [46]).

Theorem 3.1 *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$ with $J(0) = 0$. Suppose that*

(i) *there exist $\rho, \alpha > 0$ such that $J(u) \ge \alpha$ for all $u \in X$, with $\|u\|_X = \rho$;*
(ii) *there exists $e \in X$ satisfying $\|e\|_X > \rho$ such that $J(e) < 0$.*

Define $\Gamma = \{\gamma \in C^1([0,1]; X) : \gamma(0) = 1, \gamma(1) = e\}$. Then

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)) \ge \alpha$$

and there exists a $(PS)_c$ sequence $(u_n)_n \subset X$.

To find a mountain pass solution of problem (1.1), let us first verify the validity of the conditions of Theorem 3.1.

Lemma 3.1 (Mountain Pass Geometry I) *Assume that (f_1) and (f_4) hold. Then there exist $\Lambda^* > 0$, $\rho > 0$ and $\sigma > 0$ such that $I_\lambda(u) \ge \sigma$ for any $u \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\|u\| = \rho$, and all $0 < \lambda < \Lambda^*$.*

Proof Since f satisfies subcritical growth condition (f_1), for $q > \theta N/s$ and any $\alpha > 0$, we have

$$\begin{aligned}\int_\Omega F(x,u)dx &\le \frac{s}{N}\mathscr{M}(1)\,(\lambda_1-\varepsilon)\int_\Omega \frac{1}{|x|^\beta}|u|^{\theta N/s}dx + C\int_\Omega |u|^q\frac{1}{|x|^\beta}\exp(\alpha|u|^{\frac{N}{N-s}})dx\\ &\le \frac{s}{N}\mathscr{M}(1)\left(1-\frac{\varepsilon}{\lambda_1}\right)\|u\|^{\theta N/s}\\ &\quad + C\left(\int_\Omega \frac{1}{|x|^\beta}|u|^{2q}dx\right)^{1/2}\left(\int_\Omega \frac{1}{|x|^\beta}\exp(2\alpha\|u\|^{\frac{N}{N-s}}(u/\|u\|)^{\frac{N}{N-s}})dx\right)^{1/2},\end{aligned} \tag{3.1}$$

for all $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ and any $\varepsilon \in (0, \lambda_1)$. Since $0 \le \beta < N$, we can choose $\nu > 1$ close to 1 such that $\beta\nu < N$. It follows from Theorem 2.1 and (K_2) that there exists $C > 0$ such that

$$\left(\int_\Omega \frac{1}{|x|^\beta}|u|^{2q}dx\right)^{1/2} \le \left(\int_\Omega \frac{1}{|x|^{\nu\beta}}dx\right)^{\frac{1}{2\nu}} \left(\int_\Omega |u|^{\frac{2q\nu}{\nu-1}}dx\right)^{\frac{\nu-1}{2\nu}} \le C\|u\|^q.$$

Thus, we deduce from (3.1) that

$$\begin{aligned}\int_\Omega F(x,u)dx \le \frac{s}{N}\mathscr{M}(1)\left(1-\frac{\varepsilon}{\lambda_1}\right)\|u\|^{\theta N/s} \\ + C\|u\|^q\left(\int_\Omega \frac{1}{|x|^\beta}\exp(2\alpha\|u\|^{\frac{N}{N-s}}(u/\|u\|)^{\frac{N}{N-s}})dx\right)^{1/2}, \quad (3.2)\end{aligned}$$

for all $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. On the other hand, by (M_2) one can deduce

$$\mathscr{M}(t) \ge \mathscr{M}(1)t^\theta \quad \text{for all } t \in [0,1]. \quad (3.3)$$

Thus, combining (3.2) with (3.3), we obtain

$$\begin{aligned}I_\lambda(u) \ge &\frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1}\|u\|^{\theta N/s} - C\|u\|^q\left(\int_\Omega \frac{1}{|x|^\beta}\exp(2\alpha\rho_1^{\frac{N}{N-s}}(u/\|u\|)^{\frac{N}{N-s}})dx\right)^{1/2} \\ &- \lambda\|h\|_*\|u\|,\end{aligned}$$

for all $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ with $\|u\| \le \rho_1 \le 1$, and $\varepsilon \in (0,\lambda_1)$. Here $\|h\|_*$ denotes $\|h\|_{(W^{s,N/s}_{0,\mathcal{K}}(\Omega))^*}$. Choosing $2\alpha\rho_1^{N/(N-s)} \le (1-\beta/N)\,\alpha_{N,s}$ and using Theorem 2.3, we get

$$I_\lambda(u) \ge \|u\|^{\theta N/s}\frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1} - C\|u\|^q - \lambda\|h\|_*\|u\|.$$

Fix $\varepsilon \in (0,\lambda_1)$ and define

$$g(t) = \frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1}t^{\frac{\theta N}{s}-1} - Ct^{q-1}, \quad \text{for all } t \in [0,\rho_1].$$

Due to $\theta N/s < q$, we can choose $0 < \rho \le \rho_1 < 1$ such that $g(\rho) > 0$. Thus, $I_\lambda(u) \ge \sigma := \rho\,(g(\rho) - \lambda\|h\|_*) > 0$ for all $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ with $\|u\| = \rho$, and $0 < \lambda < \Lambda^* := \frac{g(\rho)}{\|h\|_*}$. □

Lemma 3.2 (Mountain Pass Geometry II) *Assume that (f_1)–(f_2) hold. Then there exists a nonnegative function $e \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ independent of λ, such that $I_\lambda(e) < 0$ and $\|e\| \ge \rho$ for all $\lambda \in \mathbb{R}^+$, where ρ is given in Lemma 3.1.*

Proof By (M_2), one can deduce that

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^{\theta} \quad \text{for all } t \geq 1. \tag{3.4}$$

On the other hand, using (f_2) and the continuity of f, there exist positive constants $C_1, C_2 > 0$ such that

$$F(x,t) \geq C_1 t^{\mu} - C_2 \ \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{3.5}$$

Now, choose nonnegative function $v_0 \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx > 0$ and $\|v_0\| = 1$. Then for all $t \geq 1$, we have

$$\begin{aligned} I_\lambda(tv_0) \leq & \frac{s}{N}\mathscr{M}(1)t^{\theta N/s}\|v_0\|^{\theta N/s} - C_1 t^{\mu}\int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx \\ & + C_2 \int_\Omega \frac{1}{|x|^\beta} dx - t\lambda \int_\Omega h v_0 dx \\ \leq & \frac{s}{N}\mathscr{M}(1)t^{\theta N/s} - C_1 t^{\mu}\int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx + C \quad \to -\infty \ \text{ as } t \to \infty, \end{aligned}$$

thanks to $\theta N/s < \mu$. The lemma is proved by taking $e = T_0 v_0$, with $T_0 > 0$ so large that $\|e\| \geq \rho$ and $I_\lambda(e) < 0$. □

Lemma 3.3 (The $(PS)_c$ condition) *Let* $(M_1) - (M_2)$ *and* (f_1), (f_2), (f_4) *hold. Then the functional* I_λ *satisfies the* $(PS)_c$ *condition for all* $c \in \mathbb{R}$.

Proof Let $(u_n)_n$ be a $(PS)_c$ sequence in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$. Then

$$I_\lambda(u_n) \to c \ \text{ and } I'_\lambda(u_n) \to 0.$$

If $d := \inf_{n\geq 1}\|u_n\| = 0$, either 0 is an isolated point or accumulation point of the sequence $(\|u_n\|)_n$. If 0 is an isolated point, then there is a subsequence $(u_{n_k})_k$ such that

$$\inf_{k\in\mathbb{N}} \|u_{n_k}\| = d > 0.$$

Otherwise, 0 is an accumulation point of the sequence $(\|u_n\|)_n$ and so there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that $u_{n_k} \to 0$ strongly in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ as $k \to \infty$. Thus, we need only consider the case $d := \inf_{n\geq 1}\|u_n\| > 0$.

In the following, we assume that $d := \inf_{n\geq 1} \|u_n\| > 0$. We first show that $(u_n)_n$ is bounded in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Using (M_1), (M_2) and (f_2) with $\mu > \frac{\theta N}{s}$, we get

$$\begin{aligned} C + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\mu}\langle I'_\lambda(u_n), u_n\rangle \\ &\geq \left(\frac{s}{N} - \frac{1}{\mu}\right) M(\|u_n\|^{N/s})\|u_n\|^{N/s} - \left(1 - \frac{1}{\mu}\right)\lambda\|h\|_*\|u_n\| \\ &\geq \left(\frac{s}{N\theta} - \frac{1}{\mu}\right)\kappa\|u_n\|^{N/s} - \left(1 - \frac{1}{\mu}\right)\lambda\|h\|_*\|u_n\|. \end{aligned} \tag{3.6}$$

It follows from (3.6) that $(u_n)_n$ is bounded in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$.

Next we show that $(u_n)_n$ has a convergence subsequence in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Going if necessary to a subsequence, there exists a function $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ weaklyin } W^{s,N/s}_{0,\mathcal{K}}(\Omega), \\ &u_n \to u \ \text{ stronglyin } L^\nu(\Omega)(\nu \geq 1), \\ &u_n \to u \ \text{ a.e.in } \Omega. \end{aligned} \tag{3.7}$$

Here we have used the compact embedding from $W^{s,N/s}_0(\Omega)$ to $L^\nu(\Omega)$ for any $\nu \geq 1$ (see Theorem 2.2) and the embedding $W^{s,N/s}_{0,\mathcal{K}}(\Omega) \hookrightarrow W^{s,N/s}_0(\Omega)$ is continuous.

Next we show that

$$\lim_{n\to\infty} \int_\Omega \frac{1}{|x|^\beta} f(x, u_n)(u_n - u)dx = 0. \tag{3.8}$$

Choose $\nu > 1$ close to 1 and α small enough such that $\nu\alpha\|u_n\|^{N/(N-s)} < \delta < \frac{N-\beta}{N}\alpha_{N,s}$. Thus, it follows from (f_1) and (f_4) that

$$\begin{aligned} &\left|\int_\Omega \frac{1}{|x|^\beta} f(x, u_n)(u_n - u)dx\right| \\ &\leq C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\theta N/s - 1}|u_n - u|dx + \int_\Omega \frac{1}{|x|^\beta}|u_n - u|\exp(\alpha|u_n|^{N/(N-s)})dx\right) \\ &\leq C\left[\left(\int_\Omega \frac{|u_n - u|^{\frac{N\theta}{s}}}{|x|^\beta}dx\right)^{\frac{s}{N\theta}} + \left(\int_\Omega \frac{|u_n - u|^{\frac{\nu}{\nu-1}}}{|x|^\beta}dx\right)^{\frac{\nu-1}{\nu}}\right. \\ &\qquad \left.\left(\int_\Omega \frac{1}{|x|^\beta}\exp[\nu\alpha\|u_n\|^{\frac{N}{N-s}}(\frac{u_n}{\|u_n\|})^{\frac{N}{N-s}}]dx\right)^{\frac{1}{\nu}}\right] \\ &\leq C\left[\left(\int_\Omega \frac{|u_n - u|^{\frac{N\theta}{s}}}{|x|^\beta}dx\right)^{\frac{s}{N\theta}} + \left(\int_\Omega \frac{|u_n - u|^{\frac{\nu}{\nu-1}}}{|x|^\beta}dx\right)^{\frac{\nu-1}{\nu}}\right] \to 0 \end{aligned}$$

as $n \to \infty$, thanks to Theorem 2.2. Thus, (3.8) holds true.

Since $(u_n)_n$ is a bounded $(PS)_c$ sequence, we get as $n \to \infty$

$$\langle I'_\lambda(u_n), u_n - u\rangle = M(\|u_n\|^{N/s})\langle u_n, u_n - u\rangle_{s,N/s} - \int_\Omega \frac{1}{|x|^\beta} f(x, u_n)(u_n - u)dx - \lambda \int_\Omega h(u_n - u)dx \to 0,$$

which implies that

$$M(\|u_n\|^{N/s})\langle u_n, u_n - u\rangle_{s,N/s} \to 0.$$

Moreover, one can prove that $\langle u, u_n - u\rangle_{s,N/s} \to 0$. Hence we obtain that

$$M(\|u_n\|^{N/s})\left[\langle u_n, u_n - u\rangle_{s,N/s} - \langle u, u_n - u\rangle_{s,N/s}\right] \to 0.$$

By using a similar discussion as [33], we have $u_n \to u$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. This ends the proof. □

Proof of Theorem 1.1 By Lemmas 3.1 and 3.2, we know that there exists a threshold $\lambda^* > 0$ such that for all $0 < \lambda < \lambda^*$, I_λ satisfies all the assumptions of Theorem 3.1. Hence there exists a $(PS)_c$ sequence. Moreover, by Lemma 3.3, for all $\lambda < \lambda^*$ the functional I_λ admits a nontrivial critical point $u_1 \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. The critical point u_1 is a nontrivial mountain pass solution of problem (1.1). Furthermore, Lemma 2.1 shows that u_1 is nonnegative.

Next we show that problem has another nontrivial and nonnegative solution. Define

$$c_\lambda = \inf_{u\in\overline{B}_\rho} I_\lambda(u) \quad \text{and} \quad \inf_{x\in\partial B_\rho} I_\lambda(u) > 0,$$

where $\rho > 0$ is given by Lemma 3.1 and $B_\rho = \{u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega) : \|u\| < \rho\}$. Now we claim that $c_\lambda < 0$. Consider the following problem

$$\begin{cases} \mathcal{L}_K v = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By the direct method and $0 \le h \in (W^{s,N/s}_{0,\mathcal{K}}(\Omega))^*$, one can verify that the above problem has a unique nonnegative solution $v \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Moreover, $\|v\|^{N/s} = \int_\Omega h(x)v dx > 0$. Then

$$I_\lambda(tv) \le \left(\max_{0\le\tau\le1} M(\tau)\right)\frac{st^{N/s}}{N} - \lambda t \int_\Omega h(x)v dx$$

for all $0 \le t \le 1$ small enough. Since $N/s > 1$, it follows that $I_\lambda(tv) < 0$ for $t \in (0, 1)$ small enough. Thus, the claim is true. By Ekeland's principle and a standard argument, there exists a sequence $(u_n)_n \subset B_\rho$ such that $I_\lambda(u_n) \to c_\lambda < 0$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Furthermore, Lemma 3.3 yields that $(u_n)_n$ converges to some u_2 strongly in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, and so u_2 is a nontrivial and nonnegative solution of problem (1.1). Clearly, u_1 and u_2 are two distinct solutions. □

4 The Critical Case

In this section, we consider the critical case of problem (1.1). Without further mentioning, we always assume that f satisfies (f_1'), $(f_2)-(f_4)$, and $M(t)=t^{\theta-1}$ with $\theta>1$. To prove Theorem 1.2, we first give several necessary results.

Lemma 4.1 *Under assumptions (f_1'), (f_2), (f_3), the functional I_λ satisfies the conditions of the mountain pass theorem:*

(1) $I(0)=0$;
(2) *there exist $\Lambda_2>0$, $\rho_2>0$ and $\sigma_2>0$ such that for $0<\lambda<\Lambda_2$, $I_\lambda(u)\geq\sigma_2>0$ for any $u\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, with $\|u\|=\rho_2$. Furthermore, ρ_2 can be chosen small enough such that $\rho_2<(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0})^{(N-s)/N}$;*
(3) *there exists a nonnegative function $e\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ independent of λ, such that $I_\lambda(e)<0$ and $\|e\|\geq\rho_2$ for all $\lambda\in\mathbb{R}^+$.*

Proof Clearly $I(0)=0$. The rest of proofs are similar to the proofs of Lemmas 3.1–3.2. □

Lemma 4.2 *There exists $\Lambda_3>0$ such that for all $0<\lambda<\Lambda_3$, the functional I_λ satisfies the $(PS)_c$ condition for $c<\frac{1}{4}(\frac{s}{N\theta}-\frac{1}{\mu})\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{(N-s)\theta}{s}}$.*

Proof Assume $(u_n)_n\subset W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ satisfies

$$I_\lambda(u_n)\to c,\quad I'_\lambda(u_n)\to 0 \text{ as } n\to\infty.$$

We first consider $c>0$. By (f_2) and the assumption on M, we have

$$\begin{aligned} c+o(1)\|u_n\| &\geq I_\lambda(u_n)-\frac{1}{\mu}\langle I'_\lambda(u_n),u_n\rangle \\ &\geq \left(\frac{s}{N\theta}-\frac{1}{\mu}\right)M(\|u_n\|^{N/s})\|u_n\|^{N/s}-\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\|u_n\|, \end{aligned}$$

which means that $(u_n)_n$ is bounded in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Thus, we get

$$\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)\|u_n\|^{\theta\frac{N}{s}}\leq c+o(1)\|u_n\|+\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\|u_n\|. \tag{4.1}$$

For any $\varepsilon > 0$, by the Young inequality we have

$$\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\|u_n\| \leq \varepsilon\|u_n\|^{\frac{N\theta}{s}} + \varepsilon^{-\frac{s}{N\theta-s}}\left(\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\right)^{\frac{N\theta}{N\theta-s}}.$$

Taking $\varepsilon = \frac{1}{2}\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)$ in above inequality and putting it into (4.1), we obtain

$$\frac{1}{2}\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)\|u_n\|^{\theta\frac{N}{s}} \leq c + o(1)\|u_n\| + \left(\frac{1}{2}\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)\right)^{-\frac{s}{N\theta-s}} \times \left(\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\right)^{\frac{N\theta}{N\theta-s}}.$$

It follows that

$$\limsup_{n\to\infty}\|u_n\| \leq \left[\frac{c}{\frac{1}{2}(\frac{s}{N\theta}-\frac{1}{\mu})} + \left(\frac{1}{2}\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)\right)^{-\frac{N\theta}{N\theta-s}}\left(\lambda\left(1-\frac{1}{\mu}\right)\|h\|_*\right)^{\frac{N\theta}{N\theta-s}}\right]^{\frac{s}{N\theta}}$$

Set

$$\Lambda_3' = \frac{(s\mu - N\theta)}{2N\theta(\mu-1)\|h\|_*}\left[\frac{1}{2}\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{(N-s)\theta}{s}}\right]^{\frac{N\theta-s}{N\theta}}.$$

Then for all $0 < \lambda < \Lambda_3'$, we get

$$\limsup_{n\to\infty}\|u_n\| < \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{N-s}{N}}, \tag{4.2}$$

thanks to $c < \frac{1}{4}(\frac{s}{N\theta}-\frac{1}{\mu})\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{(N-s)\theta}{s}}$.

If $c < 0$, then with a similar discussion as above, one can easily get that there exists $\Lambda_3'' > 0$ such that the (PS) sequence satisfies (4.2).

Therefore, there exists $\Lambda_3 = \min\{\Lambda_3', \Lambda_3''\}$ such that (4.2) holds true.

It follows from (4.2) that there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\|u_n\|^{N/(N-s)} < \delta < \frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}$. Choosing $\nu > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that we still have $\nu\alpha\|u_n\|^{N/(N-s)} < \delta < \frac{N-\beta}{N}\alpha_{N,s}$. Thus, it follows from (2.2) with $q = 1$ that

$$\left|\int_\Omega \frac{1}{|x|^\beta} f(x,u_n)(u_n-u)dx\right| \leq C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\theta N/s-1}|u_n-u|dx + \int_\Omega \frac{1}{|x|^\beta}|u_n-u|\exp(\alpha|u_n|^{N/(N-s)})dx\right)$$

$$\leq C\left[\left(\int_\Omega \frac{|u_n-u|^{\frac{N\theta}{s}}}{|x|^\beta}dx\right)^{\frac{s}{N\theta}}+\left(\int_\Omega \frac{|u_n-u|^{\frac{\nu}{\nu-1}}}{|x|^\beta}dx\right)^{\frac{\nu-1}{\nu}}\right.$$
$$\left.\left(\int_\Omega \frac{1}{|x|^\beta}\exp[\nu\alpha\|u_n\|^{\frac{N}{N-s}}(\frac{u_n}{\|u_n\|})^{\frac{N}{N-s}}]dx\right)^{\frac{1}{\nu}}\right]$$
$$\leq C\left[\left(\int_\Omega \frac{|u_n-u|^{\frac{N\theta}{s}}}{|x|^\beta}dx\right)^{\frac{s}{N\theta}}+\left(\int_\Omega \frac{|u_n-u|^{\frac{\nu}{\nu-1}}}{|x|^\beta}dx\right)^{\frac{\nu-1}{\nu}}\right]\to 0$$

as $n\to\infty$, thanks to Theorem 2.2. Then using a similar discussion as Lemma 3.3, one can prove that $u_n\to u$ strongly in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$.

If $\inf_{n\geq 1}\|u_n\|=0$, we can proceed as in Lemma 3.3. □

Proof of Theorem 1.2 By Lemma 4.1 and Theorem 3.1, there exists a sequence $(u_n)_n\subset W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ such that $I_\lambda(u_n)\to c_1$ and $I'_\lambda(u_n)\to 0$, where

$$c_1=\inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I_\lambda(\gamma(t))\geq\alpha$$

and $\Gamma=\left\{\gamma\in C^1([0,1];W^{s,N/s}_{0,\mathcal{K}}(\Omega)):\gamma(0)=1,\gamma(1)=e\right\}$. Next we show that

$$c_1<\frac{1}{4}\left(\frac{s}{N\theta}-\frac{1}{\mu}\right)\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{\frac{(N-s)\theta}{s}}. \tag{4.3}$$

Set

$$C_{q_0}:=\inf_{\varphi\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)\setminus\{0\}}\left\{\|\varphi\|^{N\theta/s}:\int_\Omega\frac{|\varphi|^{q_0}}{|x|^\beta}dx=1\right\}.$$

Clearly, $C_{q_0}>0$. By Theorem 2.2, one can easily verify that there exists a nonnegative function $\varphi_0\in W^{s,N/s}_{0,\mathcal{K}}(\Omega)\setminus\{0\}$ such that

$$\|\varphi_0\|^{N\theta/s}=C_{p_0}\ \text{ and }\ |\varphi_0|^{q_0}_{q_0,\beta}=1.$$

In view of the proof of Lemma 4.1, we take $\gamma(t)=tT\varphi_0$, where $T>0$ is sufficiently large such that $e=T\varphi_0$. Hence, it follows from the definition of c_1 that

$$c_1<\max_{t\geq 0}I_\lambda(t\varphi_0),$$

which implies that

$$c_1\leq\max_{t\geq 0}\left\{\frac{s}{N\theta}\|t\varphi_0\|^{N\theta/s}-\int_\Omega\frac{1}{|x|^\beta}F(x,t\varphi_0)dx\right\}.$$

Furthermore, from (f_4), we obtain

$$\begin{aligned}c_1 &\le \max_{t\ge 0}\left\{t^{\frac{N\theta}{s}}\frac{s}{N\theta}\|\varphi_0\|^{N\theta/s}-t^{q_0}\frac{\mathcal{C}_0}{q_0}\int_\Omega \frac{1}{|x|^\beta}|\varphi_0|^{q_0}dx\right\}\\ &=\max_{t\ge 0}\left\{t^{\frac{N\theta}{s}}\frac{s}{N\theta}C_{p_0}-t^{q_0}\frac{\mathcal{C}_0}{q_0}\right\}\\ &=C_{p_0}^{\frac{q_0}{q_0-\frac{N\theta}{s}}}\mathcal{C}_0^{-\frac{N\theta}{sq_0-N\theta}}\left(\frac{s}{N\theta}-\frac{1}{q_0}\right).\end{aligned}$$

By the assumption on $\mathcal{C}_0$, (4.3) holds.

Thus, it follows from Lemma 4.2 that there exists $\Lambda_4=\min\{\Lambda_2,\Lambda_3\}$ such that problem (1.1) has a nontrivial nonnegative solution.

To show that problem has another solution, we set

$$c_2=\inf_{u\in\overline{B}_{\rho_2}} I_\lambda(u),$$

where $\rho_2>0$ is given by Lemma 4.1 and $B_{\rho_2}=\{u\in W^{s,N/s}_{0,\mathcal{K}}(\Omega):\|u\|<\rho_2\}$. Then $\inf_{x\in\partial B_{\rho_2}} I_\lambda(u)>0$. With a similar discussion as the proof of Theorem 1.1, we can prove that $c_2<0$. By Lemma 4.1, we obtain

$$\rho_2<\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/N}.$$

By Ekeland's variational principle, there exists a sequence $(v_n)_n\subset B_{\rho_2}$ such that $I_\lambda(v_n)\to c_2\le 0$ and $I'_\lambda(v_n)\to 0$, as $n\to\infty$. Observing that

$$\|v_n\|\le\rho_2<\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/N},$$

by Lemma 4.2, for all $\lambda\in(0,\Lambda_4)$, $(v_n)_n$ has a convergent subsequence still denoted by $(v_n)_n$ such that $v_n\to u_\lambda$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Thus, u_λ is a nontrivial nonnegative solution with $I_\lambda(u_\lambda)<0$. Thus, the proof is complete. □

5 Problem (1.1) Without Perturbation

In this section, we consider problem (1.8), i.e. problem (1.1) without perturbation term h and the Ambrosetti-Rabinowitz condition.

The following version of the mountain pass theorem, which will be used later, shows us the existence of a Cerami sequence at the mountain pass level.

Theorem 5.1 (See [10]) *Let X be a real Banach space with its dual space E^* and assume that $J \in C^1(X, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \varrho < \sigma \leq \inf_{\|u\|_X=\rho} J(u),$$

for some $\varrho, \sigma, \rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Let c be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$. Then there exists a Cerami sequence $(u_n)_n$ in X, that is,

$$J(u_n) \to c \geq \sigma, \quad (1 + \|u_n\|_X)\|J'(u_n)\|_{X^*} \to 0,$$

as $n \to \infty$.

To this aim, let us first verify the validity of the conditions of Theorem 5.1.

Lemma 5.1 (Mountain Pass Geometry I) *Assume that (f_1') and (f_3) hold. Then there exist $\rho > 0$ and $\varrho > 0$ such that $I(u) \geq \varrho$ for any $u \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\|u\| = \rho$.*

Proof By (3.2) and (3.3), we obtain

$$I(u) \geq \frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1}\|u\|^{\theta N/s} - C\|u\|^q \left(\int_\Omega \frac{1}{|x|^\beta} \exp(2\alpha\rho_1^{\frac{N}{N-s}}(u/\|u\|)^{\frac{N}{N-s}})dx\right)^{1/2},$$

for all $u \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\|u\| \leq \rho_1 \leq 1$, and $\varepsilon \in (0, \lambda_1)$. Choosing $2\alpha\rho_1^{N/(N-s)} \leq (1 - \beta/N)\,\alpha_{N,s}$ and using Theorem 2.3, we get

$$I(u) \geq \frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1}\|u\|^{\theta N/s} - C\|u\|^q.$$

Fix $\varepsilon \in (0, \lambda_1)$. By virtue of $\theta N/s < q$, we can choose $0 < \rho \leq \rho_1 < 1$ such that $\frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1}\rho^{\theta N/s} - C\rho^q > 0$. Thus, $I(u) \geq \varrho := \varrho^{\theta N/s}\frac{s\mathscr{M}(1)\varepsilon}{N\lambda_1} - C\rho^q > 0$ for all $u \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\|u\| = \rho$. □

Lemma 5.2 (Mountain Pass Geometry II) *Assume that (f_1'), (f_2) and (f_3) hold. Then there exists a nonnegative function $e \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ such that $I(e) < 0$ and $\|e\| \geq \rho$, where ρ is given in Lemma 5.1.*

Proof Choose a nonnegative function $v_0 \in W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ with $\int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx > 0$ and $\|v_0\| = 1$. Then for all $t \geq 1$, we have by (3.4) and (3.5) that

$$\begin{aligned}
I(tv_0) &\leq \frac{s}{N}\mathscr{M}(1)t^{\theta N/s}\|v_0\|^{\theta N/s} - C_1 t^\mu \int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx + C_2 \int_\Omega \frac{1}{|x|^\beta}dx \\
&\leq \frac{s}{N}\mathscr{M}(1)t^{\theta N/s} - C_1 t^\mu \int_\Omega \frac{1}{|x|^\beta}|v_0|^\mu dx + C \quad \to -\infty \text{ as } t \to \infty,
\end{aligned}$$

thanks to $\theta N/s < \mu$. The lemma is proved by taking $e = T_0 v_0$, with $T_0 > 0$ so large that $\|e\| \geq \rho$ and $I(e) < 0$. □

By Theorem 5.1, there exists a Cerami sequence $(u_n)_n \subset W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ such that

$$I(u_n) \to c_* \text{ and } (1 + \|u_n\|)\|I'(u_n)\| \to 0 \text{ as } n \to \infty,$$

where

$$c_* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{5.1}$$

where $\Gamma = \left\{\gamma \in C([0,1]; W^{s,N/s}_{0,\mathcal{K}}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = e\right\}$. Obviously, $c_* > 0$ by Lemma 5.1. To get more precisely estimate of c_*, we first obtain the following result.

Lemma 5.3 *Assume that (f_1'), (f_3) and (f_5) hold. Then there exists $n > 0$ such that*

$$\max_{t \geq 0} I(tG_n) < \frac{s}{N}\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right),$$

where G_n is given by Theorem 2.3.

Proof Arguing by contradiction, we assume that

$$\max_{t \geq 0} I(tG_n) \geq \frac{s}{N}\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right). \tag{5.2}$$

Since I possesses the mountain pass geometry, for each n, $\max_{t \geq 0} I(tG_n)$ is attained at some $t_n > 0$, that is,

$$I(t_n G_n) = \max_{t \geq 0} I(tG_n).$$

Using $F(x,t) \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, one can deduce that

$$\mathscr{M}\left(t_n^{\frac{N}{s}}\|G_n\|^{N/s}\right) \geq \mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right).$$

Since $\mathscr{M} : [0,\infty) \to [0,\infty)$ is a nondecreasing function by (M_1), we get

$$t_n^{\frac{N}{s}}\|G_n\|^{N/s} \geq \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}.$$

It follows from $\|G_n\|^{N/s} \to 1$ that

$$\liminf_{n\to\infty} t_n^{N/s} \geq \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}. \tag{5.3}$$

Due to

$$\frac{d}{dt}I(tG_n)\Big|_{t=t_n} = 0,$$

we deduce

$$\begin{aligned} M(t_n^{N/s}\|G_n\|^{N/s})t_n^{N/s}\|G_n\|^{N/s} &= \int_\Omega \frac{1}{|x|^\beta} f(x, t_nG_n)t_nG_n dx \\ &\geq \int_{B_{R_0}(0)} \frac{1}{|x|^\beta} f(x, t_nG_n).t_nG_n dx. \end{aligned} \tag{5.4}$$

Next we show that $(t_n)_n$ is bounded. Using change of variable, we deduce from (5.4) that

$$\begin{aligned} &M(t_n^{N/s}\|G_n\|^{N/s})t_n^{N/s}\|G_n\|^{N/s} \\ &\geq R_0^N \int_{B_1(0)} \frac{1}{|R_0x|^\beta} f(R_0x, t_n\widetilde{G}_n)t_n\widetilde{G}_n dx \\ &\geq R_0^N \int_{B_{\frac{1}{n}}(0)} \frac{1}{|R_0x|^\beta} f(R_0x, t_n\frac{1}{\gamma_{s,N}^{s/N}}(\ln n)^{(N-s)/N})t_n\frac{1}{\gamma_{s,N}^{s/N}}(\ln n)^{(N-s)/N}dx. \end{aligned}$$

Note that (5.3) implies that

$$\frac{t_n}{\gamma_{s,N}^{s/N}}(\ln n)^{(N-s)/N} \to \infty \text{ as } n \to \infty.$$

It follows from (f_5) that given $\delta > 0$ there exists $t_\delta > 0$ such that

$$f(x,t)t \geq (\beta_0 - \delta)\exp\left(\frac{\alpha_0\alpha^*_{N,s}}{\alpha_{N,s}} t^{N/(N-s)}\right) \ \forall (x,t) \in \Omega \times [t_\delta, \infty). \tag{5.5}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &f(R_0x, t_n\frac{1}{\gamma_{s,N}^{s/N}}(\ln n)^{(N-s)/N})t_n\frac{1}{\gamma_{s,N}^{s/N}}(\ln n)^{(N-s)/N} \\ &\geq (\beta_0 - \delta)\exp\left(\frac{\alpha_0\alpha^*_{N,s}}{\alpha_{N,s}} t_n^{N/(N-s)} \frac{1}{\gamma_{s,N}^{s/(N-s)}} \ln n\right), \end{aligned}$$

for all $n \geq n_0$. Hence,

$$\begin{aligned} & M(t_n^{N/s}\|G_n\|^{N/s})t_n^{N/s}\|G_n\|^{N/s} \\ & \geq (\beta_0-\delta)R_0^{N-\beta}\exp\left(\frac{\alpha_0\alpha_{N,s}^*}{\alpha_{N,s}}t_n^{N/(N-s)}\frac{1}{\gamma_{s,N}^{s/(N-s)}}\ln n\right)\omega_{N-1}\frac{1}{n^{N-\beta}} \\ & = (\beta_0-\delta)\omega_{N-1}R_0^{N-\beta}\exp\left(\frac{\alpha_0\alpha_{N,s}^*}{\alpha_{N,s}}t_n^{N/(N-s)}\frac{1}{\gamma_{s,N}^{s/(N-s)}}\ln n\right)\exp(-(N-\beta)\ln n) \\ & = (\beta_0-\delta)\omega_{N-1}R_0^{N-\beta}\exp\left[\left(\frac{\alpha_0}{\alpha_{N,s}}t_n^{N/(N-s)}N-N+\beta\right)\ln n\right]. \end{aligned} \tag{5.6}$$

From (M_2) and (5.3), we can conclude that

$$\frac{M(t_n^{N/s}\|G_n\|^{N/s})t_n^{N/s}\|G_n\|^{N/s}}{\exp\left[\left(\frac{\alpha_0}{\alpha_{N,s}}t_n^{N/(N-s)}N-N+\beta\right)\ln n\right]} \to 0 \ \text{ as } n\to\infty,$$

which contradicts (5.6). Thus,

$$\limsup_{n\to\infty} t_n^{N/s} \leq \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s},$$

which together with (5.3) yields that

$$\lim_{n\to\infty} t_n^{N/s} = \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s} \tag{5.7}$$

as $n\to\infty$.

Inspired by [12,17,33], we are going to estimate (5.4). In view of (5.5), for $0 < \delta < \beta_0$ and $n \in \mathbb{N}$, we set

$$U_{n,\delta} := \{x\in B_{R_0}(0) : t_nG_n(x) \geq t_\delta\} \ \text{ and } \ V_{n,\delta} := B_{R_0}(0)\setminus U_{n,\delta}.$$

Splitting the integral (5.4) on $U_{n,\delta}$ and $V_{n,\delta}$ and using (5.5), we deduce

$$\begin{aligned} & M(t_n^{N/s}\|G_n\|^{N/s})t_n^{N/s}\|G_n\|^{N/s} \\ & \geq (\beta_0-\delta)\int_{B_{R_0}(0)}\frac{1}{|x|^\beta}\exp\left(\frac{\alpha_0\alpha_{N,s}^*}{\alpha_{N,s}}(t_nG_n)^{N/(N-s)}\right)dx \\ & \quad -(\beta_0-\delta)\int_{V_{n,\delta}}\frac{1}{|x|^\beta}\exp\left(\frac{\alpha_0\alpha_{N,s}^*}{\alpha_{N,s}}(t_nG_n)^{N/(N-s)}\right)dx \\ & \quad +\int_{V_{n,\delta}}\frac{1}{|x|^\beta}f(x,t_nG_n)t_nG_n dx. \end{aligned} \tag{5.8}$$

Since $G_n(x) \to 0$ a.e. in $B_{R_0}(0)$, we deduce that the characteristic functions $\chi_{V_{n,\delta}}$ satisfies

$$\chi_{V_{n,\delta}} \to 1 \text{ a.e. in } B_{R_0}(0) \text{ as } n \to \infty.$$

By $t_n G_n < t_\delta$ and the Lebesgue dominated convergence theorem, we have

$$\int_{V_{n,\delta}} \frac{1}{|x|^\beta} \exp\left(\frac{\alpha_0 \alpha_{N,s}^*}{\alpha_{N,s}}(t_n G_n)^{N/(N-s)}\right) dx \to \frac{\omega_{N-1}}{N-\beta} R_0^{N-\beta} \text{ and}$$
$$\int_{V_{n,\delta}} \frac{1}{|x|^\beta} f(x, t_n G_n) t_n G_n dx \to 0. \tag{5.9}$$

The key point is to estimate the first term on the right hand of (5.8). By (5.3) and the definition of G_n, we have

$$\begin{aligned}
&\int_{B_{R_0}(0)} \frac{1}{|x|^\beta} \exp\left(\frac{\alpha_0 \alpha_{N,s}^*}{\alpha_{N,s}}(t_n G_n)^{N/(N-s)}\right) dx \\
&\geq R_0^{N-\beta} \int_{B_{1/n}(0)} \frac{1}{|x|^\beta} \exp((N-\beta)\ln n) dx \\
&\quad + R_0^{N-\beta} \int_{1/n<|x|<1} \frac{1}{|x|^\beta} \exp\left[(N-\beta)\frac{|\ln|x||^{N/(N-s)}}{(\ln n)^{s/(N-s)}}\right] dx \\
&= \frac{\omega_{N-1} R_0^{N-\beta}}{N-\beta} + R_0^{N-\beta} \int_{1/n<|x|<1} \frac{1}{|x|^\beta} \exp\left[(N-\beta)\frac{|\ln|x||^{N/(N-s)}}{(\ln n)^{s/(N-s)}}\right] dx \\
&\geq \frac{\omega_{N-1} R_0^{N-\beta}}{N-\beta} + \frac{\omega_{N-1} R_0^{N-\beta}}{N-\beta}\left(1 - \frac{1}{n^{N-\beta}}\right).
\end{aligned} \tag{5.10}$$

Inserting (5.9) and (5.10) in (5.8) and using (5.7), we arrive at

$$M\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right)\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s} \geq (\beta_0 - \delta)\frac{\omega_{N-1} R_0^{N-\beta}}{N-\beta}, \quad \forall \delta \in (0, \beta_0).$$

Letting $\delta \to 0^+$, we obtain

$$\beta_0 \leq \frac{M\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right)\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}}{\frac{\omega_{N-1} R_0^{N-\beta}}{N-\beta}},$$

which contradicts (f_5). Therefore, the lemma is proved. □

By Lemma 5.3, we obtain the desired estimate for the level c_*.

Lemma 5.4 *Assume* (M_1), (M_3), (M_4) *and* (f_3) *hold. Then*

$$c_* < \frac{s}{N}\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right).$$

Proof Since $G_n \geq 0$ in Ω and $\|G_n\| \to 1$, as in the proof of Lemma 5.2, we deduce that $I(tG_n) \to -\infty$ as $t \to \infty$. Consequently,

$$c_* \leq \max_{t\geq 0} I(tG_n), \quad \forall n \in \mathbb{N}.$$

Thus, the desired result follows by using Lemma 5.3. □

Consider the Nehari manifold associated to the functional I, that is,

$$\mathcal{N} = \left\{u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega) \setminus \{0\} : \langle I'(u), u\rangle = 0\right\}$$

and define $c^* := \inf_{u\in\mathcal{N}} I(u)$.

The next result is crucial in our arguments to get the existence of a ground state solution for problem (1.8).

Lemma 5.5 *Assume that* (M_3) *and* (f_5) *are satisfied. Then* $c_* \leq c^*$, *where* c_* *is given by* (5.1).

Proof The proof is similar to [17] and [33], so we omit the proof. □

Lemma 5.6 (The $(PS)_c$ condition) *Let* (M_1), (M_3), (M_4) *and* (f_1'), (f_3), (f_5) *and* (f_6) *hold. Then the functional* I *satisfies the* $(PS)_{c_*}$ *condition.*

Proof The proof is similar to Lemma 4.1 of [33]. Let $(u_n)_n$ be a Cerami sequence at level c_* in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Then

$$I(u_n) \to c_* \text{ and } (1 + \|u_n\|)\|I'(u_n)\| \to 0.$$

If $d := \inf_{n\geq 1} \|u_n\| = 0$, we can discuss as Lemma 3.3. Thus, we need only consider the case $d := \inf_{n\geq 1} \|u_n\| > 0$.

In the following, we assume that $d := \inf_{n\geq 1} \|u_n\| > 0$. We first show that $(u_n)_n$ is bounded in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Arguing by contradiction, we assume that

$$\|u_n\| \geq 1 \text{ and } \lim_{n\to\infty} \|u_n\| = \infty.$$

Set

$$v_n = \frac{u_n}{\|u_n\|}.$$

Then $\|v_n\| = 1$. Going if necessary to a subsequence, we can assume that $v_n \rightharpoonup v$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Further, one can show that

$$\begin{cases} v_n^+ \rightharpoonup v^+ \ \text{ in } \ W^{s,N/s}_{0,\mathcal{K}}(\Omega), \\ v_n^+ \to v^+ \ \text{ a.e. in } \Omega, \\ v_n^+ \to v^+ \ \text{ in } L^q(\Omega, |x|^{-\beta})(\forall 1 \le q < \infty). \end{cases}$$

Now we prove that $v^+ = 0$ a.e. in Ω. If the Lebesgue measure of set $U^+ := \{x \in \Omega : v^+(x) > 0\}$ is positive, then we have

$$\lim_{n\to\infty} u_n^+(x) = \lim_{n\to\infty} v_n^+(x)\|u_n\| = \infty \ \text{ in } U^+.$$

Thus, by (1.10), we deduce

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{|x|^\beta |u_n^+|^{N\theta/s}} = \infty \ \text{ a.e. in } U^+,$$

which implies that

$$\lim_{n\to\infty} \frac{F(x, u_n^+(x))}{|x|^\beta |u_n^+|^{N\theta/s}} = \infty \ \text{ a.e. in } U^+.$$

It follows that

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n^+(x))}{|x|^\beta |u_n^+|^{N\theta/s}} dx = \infty. \tag{5.11}$$

Note that $(u_n)_n$ is a Cerami sequence at level c_*. Then

$$\mathscr{M}(\|u_n\|^{N/s}) = \frac{N}{s} c_* + \frac{N}{s} \int_\Omega \frac{1}{|x|^\beta} F(x, u_n^+) dx + o(1),$$

which together with $\lim_{t\to\infty} \mathscr{M}(t) = \infty$ yields that

$$\lim_{n\to\infty} \int_\Omega \frac{1}{|x|^\beta} F(x, u_n^+) dx = \infty.$$

Hence,

$$\liminf_{n\to\infty}\int_\Omega \frac{F(x,u_n^+)}{|x|^\beta |u_n^+|^{N\theta/s}}|v_n^+|^{N\theta/s}dx = \liminf_{n\to\infty}\int_\Omega \frac{F(x,u_n^+)}{|x|^\beta \|u_n\|^{N\theta/s}}dx$$
$$\le \liminf_{n\to\infty} \frac{\mathscr{M}(1)\int_\Omega \frac{1}{|x|^\beta}F(x.u_n^+)dx}{\frac{N}{s}\left(c_* + \int_\Omega \frac{F(x,u_n^+)}{|x|^\beta}dx\right) + o(1)}$$
$$= \frac{s\mathscr{M}(1)}{N}.$$

Here we have used the fact that

$$t^\theta \ge \frac{\mathscr{M}(t)}{\mathscr{M}(1)} \ \ \forall t \ge 1,$$

thanks to (1.9). Note that $F(x,t) \ge 0$. By Fatou's lemma and (5.11) , we get a contradiction. Thus, $v \le 0$ a.e. in Ω and $v_n^+ \rightharpoonup 0$ in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$.

Clearly, there exist $t_n \in [0,1]$ such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n).$$

For any $R \in (0, (\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0})^{\frac{N-s}{N}})$, since f satisfies (f_1'), we choose $\varepsilon = \frac{N-\beta}{N}\frac{\alpha_{N,s}}{R^{N/(N-s)}} - \alpha_0$ and $\alpha_0 < \alpha < \alpha_0 + \varepsilon$ such that

$$F(x,t) \le C(R)|t|^{N\theta/s} + \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{R^{N/(N-s)}} - \alpha_0\right)\exp(\alpha|t|^{N/(N-s)}), \ \ \forall (x,t) \in \Omega \times \mathbb{R}.$$

It follows that

$$I(Rv_n) \ge \frac{s}{N}\mathscr{M}(R^{N/s}) - C(R)R^{N\theta/s}\int_\Omega \frac{|v_n^+|^{\frac{N\theta}{s}}}{|x|^\beta}dx$$
$$- \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{R^{N/(N-s)}} - \alpha_0\right)\int_\Omega \frac{1}{|x|^\beta}\exp(\alpha R^{N/(N-s)}|v_n^+|^{N/(N-s)})dx.$$

Since $v_n^+ \rightharpoonup 0$ in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ and $W_{0,\mathcal{K}}^{s,N/s}(\Omega) \hookrightarrow L^q(\Omega, |x|^{-\beta})$ is compact for any $q \ge 1$, we have

$$\int_\Omega \frac{1}{|x|^\beta}|v_n^+|^{\frac{N\theta}{s}}dx \to 0.$$

By Theorem 2.3 and $\alpha R^{N/(N-s)} < \frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}$, we know that

$$\int_\Omega \frac{1}{|x|^\beta}\exp(\alpha R^{N/(N-s)}|v_n^+|^{N/(N-s)})dx$$

is bounded. Thus,

$$I(Rv_n) \geq \frac{s}{N}\mathscr{M}(R^{N/s}) - C(R)R^{N\theta/s}\int_\Omega \frac{|v_n^+|^{\frac{N\theta}{s}}}{|x|^\beta}dx \\ - C\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{R^{N/(N-s)}} - \alpha_0\right).$$

On the other hand, by $\|u_n\| \to \infty$, we deduce

$$I(t_n u_n) \geq I\left(\frac{R}{\|u_n\|}u_n\right) = I(Rv_n).$$

Thus, letting $n \to \infty$ and then letting $R \to \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/N}$, we obtain

$$\liminf_{n\to\infty} I(t_n u_n) \geq \frac{s}{N}\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right) > c_*. \tag{5.12}$$

Since $I(0) = 0$ and $I(u_n) \to c_*$, we can assume that $t_n \in (0,1)$. Then $\frac{d}{dt}I(tu_n)|_{t_n} = 0$. Thus, we get $\langle I'(t_n u_n), t_n u_n\rangle = 0$, that is,

$$M(t_n^{N/s}\|u_n\|^{N/s})t_n^{N/s}\|u_n\|^{N/s} = \int_\Omega \frac{1}{|x|^\beta}f(x, t_n u_n)t_n u_n dx.$$

From (1.11), it yields that

$$\begin{aligned} I(t_n u_n) &= \frac{s}{N}\mathscr{M}(t_n^{N/s}\|u_n\|^{N/s}) - \int_\Omega \frac{1}{|x|^\beta}F(x, t_n u_n)dx \\ &= \frac{s}{N}\mathscr{M}(t_n^{N/s}\|u_n\|^{N/s}) - \frac{s}{N\theta}M(t_n^{N/s}\|u_n\|^{N/s})t_n^{N/s}\|u_n\|^{N/s} \\ &\quad + \int_\Omega \frac{1}{|x|^\beta}\left[\frac{s}{N\theta}f(x, t_n u_n)t_n u_n - F(x, t_n u_n)\right]dx \\ &\leq \frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) - \frac{s}{N\theta}M(\|u_n\|^{N/s})\|u_n\|^{N/s} \\ &\quad + \int_\Omega \frac{1}{|x|^\beta}\left[\frac{s}{N\theta}f(x, u_n)u_n - F(x, u_n)\right]dx. \end{aligned}$$

Moreover, it follows from $(u_n)_n$ is a Cerami sequence that

$$\frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) - \frac{s}{N\theta}M(\|u_n\|^{N/s})\|u_n\|^{N/s} \\ + \int_\Omega \frac{1}{|x|^\beta}\left[\frac{s}{N\theta}f(x, u_n)u_n - F(x, u_n)\right]d = c_* + o(1).$$

Thus,

$$\limsup_{n\to\infty} I(t_n u_n) \le c_*,$$

which contradicts (5.12). Therefore, $(u_n)_n$ is bounded in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$.

Next we show that $(u_n)_n$ has a convergence subsequence in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Going if necessary to a subsequence, there exist a function $u \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$ and $\xi > 0$ such that

$$\begin{aligned} & u_n \rightharpoonup u \ \text{ weakly in } W^{s,N/s}_{0,\mathcal{K}}(\Omega), \\ & u_n \to u \ \text{ strongly in } L^\nu(\Omega)(\nu \ge 1), \\ & u_n \to u \ \text{ a.e. in } \Omega \\ & \|u_n\| \to \xi. \end{aligned} \tag{5.13}$$

Here we have used the compact embedding from $W^{s,N/s}_0(\Omega)$ to $L^\nu(\Omega)$ for any $\nu \ge 1$ (see Theorem 2.2) and the embedding $W^{s,N/s}_{0,\mathcal{K}}(\Omega) \hookrightarrow W^{s,N/s}_0(\Omega)$ is continuous. Using a similar discussion as [33], we can deduce that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \frac{1}{|x|^\beta} f(x,u_n)dx = \int_\Omega \frac{1}{|x|^\beta} f(x,u)dx \ \text{ and } \lim_{n\to\infty} \int_\Omega \frac{1}{|x|^\beta} F(x,u_n)dx \\ = \int_\Omega \frac{1}{|x|^\beta} F(x,u)dx. \end{aligned} \tag{5.14}$$

Now, we assert that $u \ne 0$. Arguing by contradiction, we assume that $u = 0$. Then, $\int_\Omega \frac{1}{|x|^\beta} F(x,u_n)dx \to 0$ and $I(u_n) \to c$ gives that

$$\frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) \to c < \frac{s}{N}\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right)$$

as $n \to \infty$. Thus, there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $\|u_n\|^{N/(N-s)} < \delta < \frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}$. Choosing $\nu > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that we still have $\nu\alpha\|u_n\|^{N/(N-s)} < \delta < \frac{N-\beta}{N}\alpha_{N,s}$. Thus, it follows from (2.2) with $q = 1$ that

$$\begin{aligned} & \left|\int_\Omega \frac{1}{|x|^\beta} f(x,u_n)u_n dx\right| \\ & \le C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\theta N/s}dx + \int_\Omega \frac{1}{|x|^\beta}|u_n|\exp(\alpha|u_n|^{N/(N-s)})dx\right) \\ & \le C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\theta N/s}dx + \left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\frac{\nu}{\nu-1}}dx\right)^{\frac{\nu-1}{\nu}}\right. \end{aligned}$$

$$\left(\int_\Omega \frac{1}{|x|^\beta}\exp[\nu\alpha\|u_n\|^{\frac{N}{N-s}}(\frac{u_n}{\|u_n\|})^{\frac{N}{N-s}}]dx\right)^{\frac{1}{\nu}}\Bigg)$$

$$\leq C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\theta N/s}dx + \left(\int_\Omega \frac{1}{|x|^\beta}|u_n|^{\frac{\nu}{\nu-1}}dx\right)^{\frac{\nu-1}{\nu}}\right) \to 0$$

as $n\to\infty$. Since $(u_n)_n$ is a bounded Cerami sequence, we get

$$\langle I'(u_n), u_n\rangle = M(\|u_n\|^{N/s})\|u_n\|^{N/s} - \int_\Omega \frac{1}{|x|^\beta} f(x,u_n)u_n dx \to 0,$$

which implies that

$$M(\|u_n\|^{N/s})\|u_n\|^{N/s} \to 0.$$

From this and assumption (M_1), we deduce $\|u_n\|\to 0$, which contradicts the assumption that $\inf_{n\geq 1}\|u_n\| > 0$. Therefore, we must have $u\neq 0$.

We claim that $I(u)\geq 0$. Arguing by contradiction, we assume that $I(u) < 0$. Set $\zeta(t) := I(tu)$ for all $t\geq 0$. Then $\zeta(0) = 0$ and $\zeta(1) < 0$. Arguing as in the proof of Lemma 3.1, we can see that $\zeta(t) > 0$ for $t > 0$ small enough. Hence there exists $t_0\in(0,1)$ such that

$$\zeta(t_0) = \max_{t\in[0,1]}\zeta(t),\quad \zeta'(t_0) = \langle I'(t_0u), u\rangle = 0,$$

which means that $t_0u\in\mathcal{N}$. Therefore, by Remarks 1.1 and 1.2, the semicontinuity of norm and Fatou's lemma, we get

$$\begin{aligned} c_* \leq c^* \leq I(t_0u) &= I(t_0u) - \frac{s}{N\theta}\langle I'(t_0u), t_0u\rangle \\ &= \frac{s}{N}\mathscr{M}(\|t_0u\|^{N/s}) - \frac{s}{N\theta}M(\|t_0u\|^{N/s})\|t_0u\|^{N/s} \\ &\quad + \frac{s}{N\theta}\int_\Omega \frac{1}{|x|^\beta}\left[f(x,t_0u)t_0u - \frac{\theta N}{s}F(x,t_0u)\right]dx \\ &< \frac{s}{N}\mathscr{M}(\|u\|^{N/s}) - \frac{s}{N\theta}M(\|u\|^{N/s})\|u\|^{N/s} \\ &\quad + \frac{s}{N\theta}\int_\Omega \frac{1}{|x|^\beta}\left[f(x,u)u - \frac{\theta N}{s}F(x,u)\right]dx. \end{aligned}$$

By the weak lower semicontinuity of convex functional, we have

$$\|u\|^{N/s} \leq \liminf_{n\to\infty}\|u_n\|^{N/s} = \xi^{N/s}.$$

In view of Remark 1.1 and the continuity of M, we deduce that

$$\begin{aligned}
&\frac{s}{N}\mathscr{M}(\|u\|^{N/s}) - \frac{s}{N\theta}M(\|u\|^{N/s})\|u\|^{N/s} \\
&\quad \leq \frac{s}{N}\mathscr{M}(\xi^{N/s}) - \frac{s}{N\theta}M(\xi^{N/s})\xi^{N/s} \\
&\quad = \lim_{n\to\infty}\left[\frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) - \frac{s}{N\theta}M(\|u_n\|^{N/s})\|u_n\|^{N/s}\right].
\end{aligned}$$

By Fatou's lemma, we get

$$\int_\Omega \frac{1}{|x|^\beta} f(x,u)u dx \leq \liminf_{n\to\infty}\int_\Omega \frac{1}{|x|^\beta} f(x,u_n)u_n dx.$$

It follows from above results and (5.14) that

$$\begin{aligned}
c_* \leq c^* &< \lim_{n\to\infty}\left[\frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) - \frac{s}{N\theta}M(\|u_n\|^{N/s})\|u_n\|^{N/s}\right] \\
&\quad + \frac{s}{N\theta}\liminf_{n\to\infty}\int_\Omega \frac{1}{|x|^\beta}\left[f(x,u_n)u_n - \frac{N\theta}{s}F(x,u_n)\right]dx \\
&\leq \lim_{n\to\infty}\left[I(u_n) - \frac{s}{N\theta}\langle I'(u_n), u_n\rangle\right] = c_*
\end{aligned}$$

which is absurd. Thus the claim holds true.

Now we claim that

$$I(u) = c_*. \tag{5.15}$$

Obviously, by (5.14) and semicontinuity of norm, we have $I(u) \leq c_*$. Next we prove that $I(u_0) < c_*$ can not occur. Actually, if $I(u) < c_*$, then

$$\|u\| < \xi.$$

Note that (5.14) yields that

$$\frac{s}{N}\mathscr{M}(\xi^{N/s}) = \lim_{n\to\infty}\frac{s}{N}\mathscr{M}(\|u_n\|^{N/s}) = c_* + \int_\Omega \frac{1}{|x|^\beta}F(x,u)dx. \tag{5.16}$$

This gives that

$$\xi^{\frac{N}{s}} = \mathscr{M}^{-1}\left(\frac{N}{s}c_* + \frac{N}{s}\int_\Omega \frac{1}{|x|^\beta}F(x,u)dx\right).$$

Set $w_n = u_n/\|u_n\|$. Then $w_n \rightharpoonup w = u/\xi$ in $W_{0,\mathcal{K}}^{s,N/s}(\Omega)$ and $\|w\| < 1$. Thus, it follows from Theorem 2.4 that

$$\sup_n \int_\Omega \frac{\exp\left(\alpha'|w_n|^{N/(N-s)}\right)}{|x|^\beta}dx < \infty, \quad \forall\, \alpha' < \frac{(1-\frac{\beta}{N})\alpha_{N,s}}{(1-\|w\|^{N/s})^{s/(N-s)}}. \tag{5.17}$$

On the other hand, by (5.16), we have

$$\frac{N}{s}c_* - \frac{N}{s}I(u) = \mathscr{M}(\xi^{N/s}) - \mathscr{M}(\|u\|^{N/s}).$$

Thus, it follows from $I(u) \geq 0$ that

$$\mathscr{M}(\xi^{N/s}) \leq \frac{N}{s}c_* + \mathscr{M}(\|u\|^{N/s}) < \mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right) + \mathscr{M}(\|u\|^{N/s}).$$

Furthermore, by (M_1), we get

$$\begin{aligned}\xi^{N/s} <& \mathscr{M}^{-1}\left[\mathscr{M}\left(\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}\right) + \mathscr{M}(\|u\|^{N/s})\right]\\ &\leq \left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s} + \|u\|^{N/s}. \end{aligned} \tag{5.18}$$

Note that

$$\xi^{N/s} = \frac{\xi^{N/s} - \|u\|^{N/s}}{1 - \|w\|^{N/s}}.$$

Hence, it follows from (5.18) that

$$\xi^{N/s} < \frac{\left(\frac{N-\beta}{N}\frac{\alpha_{N,s}}{\alpha_0}\right)^{(N-s)/s}}{1 - \|w\|^{N/s}}.$$

Thus, there exist $n_0 \in \mathbb{N}$ and $\alpha'' > 0$ such that

$$\alpha_0\|u_n\|^{N/(N-s)} < \alpha'' < \frac{\frac{(N-\beta)\alpha_{N,s}}{N}}{(1 - \|w\|^{N/s})^{s/(N-s)}}$$

for all $n \geq n_0$. We choose $\nu > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that

$$\nu\alpha\|u_n\|^{N/(N-s)} \leq \alpha'' < \frac{\frac{(N-\beta)\alpha_{N,s}}{N}}{(1 - \|v\|^{N/s})^{s/(N-s)}}.$$

In view of (5.17), for some $C > 0$ and n large enough, we obtain

$$\int_\Omega \frac{1}{|x|^\beta}\exp(\nu\alpha|u_n|^{N/(N-s)})dx \leq \int_\Omega \frac{1}{|x|^\beta}\exp(\alpha''|w_n|^{N/(N-s)})dx \leq C.$$

Therefore, we deduce from (2.2) that

$$\begin{aligned}
&\left|\int_\Omega \frac{1}{|x|^\beta} f(x,u_n)(u_n-u)dx\right| \\
&\leq C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n-u|^{N\theta/s}dx + \int_\Omega \frac{1}{|x|^\beta}|u_n-u|\exp(\alpha|u_n|^{N/(N-s)})dx\right) \\
&\leq C\int_\Omega \frac{1}{|x|^\beta}|u_n-u|^{N\theta/s}dx + C\left(\int_\Omega \frac{1}{|x|^\beta}|u_n-u|^{\frac{\nu}{\nu-1}}dx\right)^{\frac{\nu-1}{\nu}} \to 0
\end{aligned}$$

as $n\to\infty$.

Since $(u_n)_n$ is a bounded Cerami sequence in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$, we have

$$\begin{aligned}
o(1) &= \langle I'(u_n), u_n-u\rangle \\
&= M(\|u_n\|^{N/s})\langle u_n, u_n-u\rangle_{s,N/s} - \int_\Omega \frac{1}{|x|^\beta} f(x,u_n)(u_n-u)dx. \qquad (5.19)
\end{aligned}$$

Define a functional L as follows:

$$\langle L(v), w\rangle = \langle v, w\rangle_{s,N/s}$$

for all $v, w \in W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. By the Hölder inequality, one can see that

$$|\langle L(v), w\rangle| \leq \|v\|^{\frac{N}{s}-1}\|w\|,$$

which together with the definition of L implies that for each v, $L(v)$ is a bounded linear functional on $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Thus, $\langle L(u), u_n-u\rangle = o(1)$, that is,

$$\langle u, u_n-u\rangle_{s,N/s} = o(1).$$

In conclusion, we can deduce from (5.19) that

$$M(\|u_n\|^{N/s})\left[\langle u_n, u_n-u\rangle_{s,N/s} - \langle u, u_n-u\rangle_{s,N/s}\right] = o(1).$$

In view of the fact that $\|u_n\| \to \xi$ and $\xi > 0$, by using (M_1) and a similar discussion as in [33], we obtain that $u_n \to u$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Furthermore, using (5.14) and the continuity of $\mathscr{M}$, we have $I(u) = c_*$, which is a contradiction. Thus, the assertion (5.15) holds true.

Combining $I(u) = c_*$ with $I(u_n) \to c_*$ and $\|u_n\| \to \xi$, we conclude that

$$\mathscr{M}(\xi^{N/s}) = \mathscr{M}(\|u\|^{N/s}),$$

which implies that $\xi = \|u\|$. By the uniform convexity of norm, we obtain that $u_n \to u$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. This finishes the proof. □

Proof of Theorem 1.3 By Lemmas 5.1 and 5.2, we know that I satisfies all the assumptions of Theorem 5.1. Thus there exists a Cerami sequence $(u_n)_n \subset W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. Moreover, by Lemma 5.6, there exists a subsequence of $(u_n)_n$ (still labeled by $(u_n)_n$) such that $u_n \to u$ in $W^{s,N/s}_{0,\mathcal{K}}(\Omega)$. It follows from $I'(u_n) \to 0$ that

$$M(\|u_n\|^{N/s})\langle u_n, \varphi\rangle_{s,N/s} = \int_\Omega \frac{1}{|x|^\beta} f(x, u_n)\varphi dx, \quad \forall \varphi \in W^{s,N/s}_{0,\mathcal{K}}(\Omega).$$

Furthermore, we have

$$M(\|u\|^{N/s})\langle u, \varphi\rangle_{s,N/s} = \int_\Omega \frac{1}{|x|^\beta} f(x, u)\varphi dx \ \ \forall \varphi \in W^{s,N/s}_{0,\mathcal{K}}(\Omega),$$

which means that u is a nontrivial solution of (1.8) satisfying $I(u) = c_*$, that is, $I'(u) = 0$ and $I(u) = c_*$. Therefore, by the definition of c^* and $c_* \le c^*$, we know that u is a ground state solution of problem (1.8). Moreover, Lemma 2.1 shows that u is nonnegative. □

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