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THE STUDY OF A BIFURCATION PROBLEM ASSOCIATED TO AN ASYMPTOTICALLY LINEAR FUNCTION

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0. INTRODUCTION

In this paper we consider the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth connected bounded open set in $\mathbb{R}^N$, $f: \mathbb{R} \to \mathbb{R}$ is a C^1 convex nonnegative function such that $f(0) > 0$, $f'(0) > (0)$ and f is asymptotically linear, that is

$$\lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, +\infty).$$

In what follows, we suppose that λ is a positive parameter and $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

We point out some well-known facts about the problem (1) (see [1] for details):

(i) there exists $\lambda^* \in (0, +\infty)$ such that (1) has (has no) solution when $\lambda \in (0, \lambda^*)$ ($\lambda \in (\lambda^*, +\infty)$, resp.);

(ii) for $\lambda \in (0, \lambda^*)$, among the solutions of (1) there exists a minimal one, say $u(\lambda)$;

(iii) $\lambda \mapsto u(\lambda)$ is a C^1 convex increasing function;

(iv) $u(\lambda)$ can be characterized as the only solution u of (1) such that the operator $-\Delta - \lambda f'(u)$ is coercive.

In what follows we discuss some natural problems raised by (1):

(i) What can be said when $\lambda = \lambda^*$?

(ii) Which is the behaviour of $u(\lambda)$ when λ approaches λ^*?

(iii) Are there other solutions of (1) excepting $u(\lambda)$?

(iv) If so, which is their behaviour?

Before mentioning our main results, we give some definitions and notations:

(i) let $\lim_{t \to \infty} (f(t) - at) = l \in [-\infty, \infty)$. We say that f obeys *the monotone case* (*the non-monotone case*) if $l \geq 0$ ($l < 0$, resp.);

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(ii) if $\alpha \in L^\infty(\Omega)$ we shall denote by $\varphi_j(\alpha)$ and $\lambda_j(\alpha)$ the jth eigenfunction (eigenvalue, resp.) of $-\Delta - \alpha$. We consider that $\int_\Omega \varphi_j(\alpha)\varphi_k(\alpha) = \delta_{jk}$ and $\varphi_1(\alpha) > 0$. If $\alpha = 0$ we shall write $\varphi_j(\lambda_j$, resp.);

(iii) a solution u of (1) is said to be *stable* if $\lambda_1(\lambda f'(u)) > 0$ and *unstable* otherwise;

(iv) u.c.s.Ω and u.$\bar{\Omega}$ will mean "uniformly on compact subsets of Ω" ("uniformly on Ω", resp.).

All the integrals considered are over Ω, so that we shall omit Ω in writing.

Now we can state the main results.

THEOREM A. If f obeys the monotone case, then:

(i) $\lambda^* = \lambda_1/a$;

(ii) $\lim_{\lambda\to\lambda^*} u(\lambda) = \infty$, u.c.s.$\Omega$;

(iii) $u(\lambda)$ is the only solution of (1) when $\lambda \in (0, \lambda^*)$;

(iv) (1) has no solution when $\lambda = \lambda^*$.

THEOREM B. If f obeys the nonmonotone case, then:

(i) $\lambda^* \in (\lambda_1/a, \lambda_1/\lambda_0)$, where $\lambda_0 = \min_{t>0} f(t)/t$;

(ii) (1) has exactly one solution, say u^*, when $\lambda = \lambda^*$;

(iii) $\lim_{\lambda\to\lambda^*} u(\lambda) = u^*$ u.$\bar{\Omega}$;

(iv) when $\lambda \in (0, \lambda_1/a]$, (1) has no solution but $u(\lambda)$;

(v) when $\lambda \in (\lambda_1/a, \lambda^*)$, (1) has at least an unstable solution, say $v(\lambda)$.

For each choice of $v(\lambda)$ we have:

(vi) $\lim_{\lambda\to\lambda_1/a} v(\lambda) = \infty$ u.c.s.$\bar{\Omega}$;

(vii) $\lim_{\lambda\to\lambda^*} v(\lambda) = u^*$ u.$\bar{\Omega}$.

After we establish these results, we discuss the problem of the order of convergence to ∞ in the theorems A and B.

1. PROOF OF THEOREM A

LEMMA 1. Let $\alpha \in L^\infty(\Omega)$, $w \in H_0^1(\Omega) - \{0\}$, $w \geq 0$, be such that $\lambda_1(\alpha) \leq 0$ and

$$-\Delta w \geq \alpha w. \tag{2}$$

Then:

(i) $\lambda_1(\alpha) = 0$;

(ii) $-\Delta w = \alpha w$;

(iii) $w > 0$ in Ω.

Proof. If we multiply (2) by $\varphi_1(\alpha)$ and integrate by parts, we obtain

$$\int \alpha\varphi_1(\alpha)w + \lambda_1(\alpha)\int \varphi_1(\alpha)w \geq \int \alpha\varphi_1(\alpha)w.$$

Now, this means that $\lambda_1(\alpha) = 0$ and $-\Delta w = \alpha w$. Since $w \geq 0$ and $w \not\equiv 0$, we get $w = c\varphi_1(\alpha)$ for some $c > 0$, which concludes the proof. ∎

LEMMA 2 (The linear case). If $f(t) = at + b$ when $t \geq 0$, with $a, b > 0$, then:

(i) $\lambda^* = \lambda_1/a$;

(ii) (1) has no solution when $\lambda = \lambda^*$.

Proof. (i), (ii) If $\lambda \in (0, \lambda_1/a)$ then the problem

$$\begin{cases} -\Delta u - \lambda a u = \lambda b & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a unique solution in $H_0^1(\Omega)$ which is positive in view of Stampacchia maximum principle (see [1]). Now Ω smooth and $-\Delta u = \lambda a u + \lambda b \in H_0^1(\Omega)$ mean $u \in H^3(\Omega)$ and so on. We get $u \in H^\infty(\Omega)$ and, therefore, $u \in C^\infty(\bar{\Omega})$. We have thus exhibited a smooth solution of (1) when $\lambda \in (0, \lambda_1/a)$.

We claim that (1) has no solution if $\lambda^* = \lambda_1/a$. For if u were such a solution, multiplying (1) by φ_1 and integrating by parts, we get $\int \varphi_1 = 0$, which contradicts $\varphi_1 > 0$. ■

LEMMA 3. (i) $\lambda^* \geq \lambda_1/a$;

(ii) if (1) has solution when $\lambda = \lambda^*$, it is necessarily unstable;

(iii) (1) has at most a solution when $\lambda = \lambda^*$;

(iv) $u(\lambda)$ is the only solution of (1) such that $\lambda_1(\lambda f'(u)) \geq 0$.

Proof. (i) It is enough to exhibit a super and sub solution for $\lambda \in (0, \lambda_1/a)$, that is $\underline{U}, \bar{U} \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $\underline{U} \leq \bar{U}$,

$$\begin{cases} -\Delta \bar{U} \geq \lambda f(\bar{U}) & \text{in } \Omega \\ \bar{U} \geq 0 & \text{on } \partial\Omega \end{cases}$$

and that the reversed inequalities hold for $\underline{U}$ (see [1] for the method of super and subsolutions).

Take some $b > 0$ such that $f(t) \leq at + b$ for nonnegative t. Let $\bar{U}$ be the solution of (3) with $b = f(0)$ and $\underline{U} \equiv 0$. We have $f(t) \leq at + b$ for $t > 0$ and this implies $f(\bar{U}) \leq a\bar{U} + b$ in view of the positivity of $\bar{U}$. The remaining part is trivial.

(ii) Suppose that (1) with $\lambda = \lambda^*$ has a solution u^* with $\lambda_1(\lambda^* f'(u^*)) > 0$. Then by the implicit function theorem applied to

$$G: \{u \in C^{2,1/2}(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\} \times \mathbb{R} \to C^{0,1/2}(\bar{\Omega}), \qquad G(u, \lambda) = -\Delta u - \lambda f(u)$$

it follows that (1) has solution for λ in a neighbourhood of λ^*, contradicting by this the definition of λ^*.

(iii) Let u be such a solution. Then u is a supersolution for (1) when $\lambda \in (0, \lambda^*)$ and, therefore, $u \geq u(\lambda)$ for such λ. This shows that $u(\lambda)$ (which increases with λ) tends in $L^1(\Omega)$ sense to a limit $u^* \leq u$. Since $-\Delta u(\lambda) = \lambda f(\lambda))$ we get $-\Delta u^* = \lambda^* f(u^*)$. In order to conclude that u^* is a solution of (1), it is enough to prove that $u^* \in H_0^1(\Omega)$ and to deduce from this first that either $-\Delta u^* \in L^{2^*}(\Omega)$ and, hence, $u^* \in W^{2,2^*}(\Omega)$ when $N > 2$, or $-\Delta u^* \in L^4(\Omega)$ and, hence, $u^* \in C^{0,1/2}(\bar{\Omega})$ if $N = 1, 2$ (using theorems 8.34 and 9.15 in [2]). The first case is then concluded via a bootstrap argument, while the second one using the theorem 4.3 in [2] (here $2^* = 2N/N - 2$ is the critical Sobolev exponent).

Now we claim that $u(\lambda)$ is bounded in $H_0^1(\Omega)$. Indeed, if we multiply (1) by $u(\lambda)$ and integrate by parts we get

$$\int |\nabla u(\lambda)|^2 = \lambda \int f(u(\lambda))u(\lambda) \leq \lambda^* \int uf(u).$$

Thus, $u(\lambda) \rightharpoonup u^*$ in $H_0^1(\Omega)$ if $\lambda \to \lambda^*$. Indeed, if v is a weak-★ cluster point of $u(\lambda)$ when $\lambda \to \lambda^*$, then, up to a subsequence, $u(\lambda) \to v$ a.e. However, $u(\lambda) \to u$ a.e. We have hence obtained that $u^* \in H_0^1(\Omega)$. The proof will be concluded if we show that $u = u^*$. Let $w = u - u^* \geq 0$. Then

$$-\Delta w = \lambda^*(f(u) - f(u^*)) \geq \lambda^* f'(u^*)w. \tag{4}$$

We also have $\lambda_1(\lambda^* f'(u^*)) \leq 0$, so that lemma 1 implies that either $w = 0$ or $w > 0$, $\lambda_1(\lambda^* f'(u^*)) = 0$ and $-\Delta w = \lambda^* f'(u^*)w$. If we take (4) into account the last equality implies that f is linear in all the intervals $[u^*(x), u(x)]$, $x \in \Omega$. It is easy to see that this forces f to be linear in $[0, \max_\Omega u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta$ and $f(u^*) = \alpha u^* + \beta$. We have

$$0 = \lambda_1(\lambda^* f'(u^*)) = \lambda_1(\lambda^* \alpha) = \lambda_1 - \lambda^* \alpha,$$

that is $\lambda^* = \lambda_1/\alpha$. The last conclusion contradicts lemma 2.

(iv) Suppose (1) has a solution $u \neq u(\lambda)$ with $\lambda_1(\lambda f'(u)) \geq 0$. Then $u > u(\lambda)$ by the strong maximum principle (see the theorem 3.5 in [2]). Let $w = u - u(\lambda) > 0$. Then

$$-\Delta w = \lambda(f(u) - f(u(\lambda))) \leq \lambda f'(u)w. \tag{5}$$

If we multiply (5) by $\varphi = \varphi_1(\lambda f'(u))$ and integrate by parts we get

$$\lambda \int f'(u)\varphi w + \lambda_1(\lambda f'(u)) \int \varphi w \leq \lambda \int f'(u)\varphi w.$$

Thus, $\lambda_1(\lambda f'(u)) = 0$ and in (5) we have equality, that is f is linear in $[0, \max u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta, f(u(\lambda)) = \alpha u(\lambda) + \beta$. Then

$$0 = \lambda_1(\lambda f'(u)) = \lambda_1(\lambda f'(u(\lambda))),$$

contradiction. ■

The following result is a reformulation of the theorem 4.1.9. in [3].

LEMMA 4. Let (u_n) be a sequence of nonnegative superharmonic functions in Ω. Then either:

(i) $\lim_{n \to \infty} u_n = \infty$ u.c.s.Ω; or

(ii) (u_n) contains a subsequence which converges in $L^1_{\text{loc}}(\Omega)$ to some u^*.

LEMMA 5. The following conditions are equivalent:

(i) $\lambda^* = \lambda_1/a$;

(ii) (1) has no solution when $\lambda = \lambda^*$;

(iii) $\lim_{\lambda \to \lambda^*} u(\lambda) = \infty$ u.c.s.Ω.

Proof. (i) ⇒ (ii). Suppose the contrary. Let u be such a solution. As we have already seen, $\lambda_1(\lambda^* f'(u)) \leq 0$. However, $\lambda_1(\lambda^* f'(u)) \geq \lambda_1(\lambda^* a) = 0$.

Hence $\lambda_1(\lambda^* f'(u)) = 0$, that is $f'(u) = a$. As already happened, this contradicts lemma 2.

(ii) ⇒ (iii). Suppose the contrary. We prove first that $u(\lambda)$ are uniformly bounded in $L^2(\Omega)$. Suppose again the contrary. Then, up to a subsequence, $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda) \to \infty$ and $\int w^2(\lambda) = 1$.

Suppose, using again a subsequence if necessary, that $u(\lambda) \to u^*$ in $L^1_{\text{loc}}(\Omega)$. Then $(\lambda/k(\lambda))f(u(\lambda)) \to 0$ in $L^1_{\text{loc}}(\Omega)$, that is

$$-\Delta w(\lambda) \to 0 \qquad \text{in } L^1_{\text{loc}}(\Omega). \tag{6}$$

It is easy to see that $(w(\lambda))$ is bounded in $H^1_0(\Omega)$. Indeed,

$$\int |\nabla w(\lambda)|^2 = \int -\Delta w(\lambda)w(\lambda) = \int \frac{\lambda}{k(\lambda)} f(u(\lambda))w(\lambda) \leq \lambda^* \int \left(aw^2(\lambda) + \frac{f(0)}{k(\lambda)}w(\lambda)\right)$$

$$\leq \lambda^* a + c\int w(\lambda) \leq \lambda^* a + c\int \sqrt{|\Omega|} \qquad \text{(for a suitable } c > 0).$$

Let $w \in H^1_0(\Omega)$ be such that, up to a subsequence,

$$w(\lambda) \to w \text{ weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega). \tag{7}$$

Then, by (6), $-\Delta w = 0$, and by (7), $w \in H^1_0(\Omega)$ and $\int w^2 = 1$. We have obtained the desired contradiction. Hence $(u(\lambda))$ is bounded in $L^2(\Omega)$. As above, $u(\lambda)$ is bounded in $H^1_0(\Omega)$. Let $u \in H^1_0(\Omega)$ be such that, up to a subsequence, $u(\lambda) \to u$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then by (1) we get that u is a $H^1_0(\Omega)$ solution of $-\Delta u = \lambda^* f(u)$. As we have already done, we get that in fact u is a solution of (1) when $\lambda = \lambda^*$. This contradiction concludes the proof.

(iii) ⇒ (ii). As we have seen, if (1) has a solution when $\lambda = \lambda^*$, it is necessarily equal to $\lim_{\lambda \to \lambda^*} u(\lambda)$, which cannot happen in the given context.

[(iii) and (ii)] ⇒ (i) Let $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda)$ and $w(\lambda)$ as above. This time $\lim_{\lambda \to \lambda^*} k(\lambda) = \infty$. As above we get a uniform bound for $(w(\lambda))$ in $H^1_0(\Omega)$. Let $w \in H^1_0(\Omega)$ be such that, up to a subsequence, $w(\lambda) \to w$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then $-\Delta w(\lambda) \to -\Delta w$ in $\mathfrak{D}'(\Omega)$ and $\lambda/k(\lambda) f(u(\lambda)) \to \lambda^* aw$ in $L^2(\Omega)$. (The last statement will be shown out in the proof of lemma 9). So we obtain

$$-\Delta w = \lambda^* aw, \qquad w \in H^1_0(\Omega), \qquad w \geq 0, \qquad \int w^2 = 1.$$

However, this means exactly that $\lambda^* = \lambda_1/a$ (and $w = \varphi_1$). ■

LEMMA 6. The following conditions are equivalent:

(i) $\lambda^* > \lambda_1/a$;

(ii) (1) has exactly a solution, say u^*, when $\lambda = \lambda^*$;

(iii) $u(\lambda)$ is converging $u.\ \bar{\Omega}$ to some u^* which is the unique solution of (1) when $\lambda = \lambda^*$.

Proof. We have already seen that $\lambda^* \geq \lambda_1/a$. This makes this lemma a reformulation of the preceding one apart from the fact that the limit in (iii) is $u.\ \bar{\Omega}$. Since we know that $u(\lambda) \to u^*$ a.e., it is enough to prove that $u(\lambda)$ has a limit in $C(\bar{\Omega})$ when $\lambda \to \lambda^*$. Even less, it is enough to prove that $u(\lambda)$ is relatively compact in $C(\bar{\Omega})$. This will be done via the Arzela–Ascoli theorem if we show that $(u(\lambda))$ is bounded in $C^{0,1/2}(\bar{\Omega})$. Now $0 < u(\lambda) < u^*$ implies $0 < f(u(\lambda)) < f(u^*)$, which offers a uniform bound for $-\Delta u(\lambda)$ in $L^{2N}(\Omega)$. The desired bound is now a consequence of the theorem 8.34 in [2] (see also the remark from the p. 212) and of the closed graph theorem. ■

Proof of theorem A. (i), (ii) and (iv) will follow together if we prove one of them. We shall prove that $\lambda^* = \lambda_1/a$ by showing that (1) has no solution when $\lambda = \lambda_1/a$. For suppose u were such a solution. Then

$$-\Delta u = \lambda f(u) \geq \lambda_1 u. \tag{8}$$

If we multiply (8) by φ_1 and integrate by parts we get $\lambda f(u) = \lambda_1 u$, contradicting the fact that $f(0) > 0$.

(iii) taking into account the lemma 3(iv), it is enough to prove that for $\lambda \in (0, \lambda_1/a)$ any solution u verifies $\lambda_1(\lambda f'(u)) \geq 0$. However,

$$-\Delta - \lambda f'(u) \geq -\Delta - \lambda a,$$

which shows that

$$\lambda_1(\lambda f'(u)) \geq \lambda_1(\lambda a) = \lambda_1 - \lambda a > 0. \quad ■$$

2. PROOF OF THEOREM B

(i) We prove first that $\lambda^* \leq \lambda_1/\lambda_0$. For this aim, we shall see that (1) has no solution when $\lambda = \lambda_1/\lambda_0$. Suppose the contrary and let u be such a solution. Then multiplying (1) by φ_1 and integrating by parts we get

$$\lambda_1 \int \varphi_1 u = \lambda \int \varphi_1 f(u). \tag{9}$$

In our case, (9) it becomes

$$\lambda_1 \int \varphi_1 u = \frac{\lambda_1}{\lambda_0} \int \varphi_1 f(u) \geq \lambda_1 \int \varphi_1 u$$

which forces $f(u) = \lambda_0 u$ and, as above, this contradicts $f(0) > 0$.

The remaining part of (i), (ii) and (iii) are equivalent in view of the lemmas 3(iii) and 6. We shall prove that $\lambda^* > \lambda_1/a$ supposing the contrary. Then $\lim_{\lambda \to \lambda^*} u(\lambda) = \infty$ u.c.s.Ω and $\lambda^* = \lambda_1/a$. If we examine (9) rewritten as

$$\begin{aligned} 0 &= \int \varphi_1[\lambda_1 u(\lambda) - \lambda f(u(\lambda))] \\ &= \int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) - \lambda(f(u)(\lambda)) - au(\lambda))] \geq -\lambda \int \varphi_1[f(u(\lambda)) - au(\lambda))] \end{aligned} \tag{10}$$

we see that the right-hand side integrand converges monotonously to $l\varphi_1$ when $\lambda \to \lambda^*$. Here $l = \lim_{t\to\infty}(f(t) - at) < 0$. Passing to the limit in (10) we obtain the contradictory inequality

$$0 \geq -l\lambda \int \varphi_1 > 0.$$

We have seen that $\lambda^* \leq \lambda_1/\lambda_0$ and we know that (1) has solution when $\lambda = \lambda^*$. This shows that $\lambda^* < \lambda_1/\lambda_0$.

(iv) can be be proved exactly in the same way as (iii) in the theorem A.

Since all the solutions of (1) are positive, we may modify $f(t)$ as we wish for negative t. In what follows we shall suppose, additionally, that f is increasing.

For the proof of (v) we shall use some known results that we point out in what follows.

The Ambrosetti–Rabinowitz theorem. Let E be a Banach space, $J \in C^1(E, \mathbb{R})$, $u_0 \in E$. Suppose that there exist R, $\rho > 0$, $v_0 \in E$ such that

$$J(u) \geq J(u_0) + \rho \qquad \text{if } \|u - u_0\| = R \tag{11}$$

$$J(v_0) \leq J(u_0). \tag{12}$$

Suppose that the following condition is satisfied.

(PS) Every sequence (u_n) in E such that $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ in E^* is relatively compact in E.

Let

$$\mathcal{P} = \{p \in C([0, 1], E) : p(0) = u_0, p(1) = v_0\}$$

and

$$c = \inf_{\mathcal{P}} \max_{[0,1]} F \circ p.$$

Then there exists $u \in E$ such that $J(u) = c$ and $J'(u) = 0$.

Note that $c > J(u_0)$ and that is why $u \neq u_0$ (see [1] for details).

We want to find out solutions of (1) different from $u(\lambda)$, that is critical points, others than $u(\lambda)$, of

$$J : E \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u),$$

where $E = H_0^1(\Omega)$ and $F(t) = \lambda \int_0^t f(s)\,ds$. We take $u(\lambda)$ as u_0 for each $\lambda \in (\lambda_1/a, \lambda^*)$.

We have the following theorem.

LEMMA 7. (i) $J \in C^1(E, \mathbb{R})$;

(ii) for $u, v \in E$ we have $J'(u)v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v$;

(iii) u_0 is a local minimum for J.

The proof can be found in [1].

In order to apply the Ambrosetti–Rabinowitz theorem we transform u_0 into a local strict minimum by modifying J. Let

$$J_\varepsilon : E \to R, \qquad J_\varepsilon(u) = J(u) + \frac{\varepsilon}{2}\int |\nabla(u - u_0)|^2.$$

In view of the preceding lemma we obviously have:

(i) $J \in C^1(E, \mathbb{R})$;

(ii) $J_\varepsilon'(u) \cdot v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v + \varepsilon \int \nabla(u - u_0) \cdot \nabla v$;

(iii) u_0 is a local strict minimum for J_ε if $\varepsilon > 0$ (so that (11) is verified).

We prove first the existence of a v_0 good for all ε near 0.

LEMMA 8. Let $\varepsilon_0 = (\lambda a - \lambda_1)/2\lambda_1$. Then there exists $v_0 \in E$ such that $J_\varepsilon(v_0) < J_\varepsilon(u_0)$ for $\varepsilon \in [0, \varepsilon_0]$.

Proof. Note that $J_\varepsilon(u)$ is bounded by $J_0(u)$ and $J_{\varepsilon_0}(u)$. It suffices to prove that

$$\lim_{t \to \infty} J_{\varepsilon_0}(t\varphi_1) = -\infty.$$

However,

$$J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2}t^2 + \frac{\varepsilon_0}{2}\lambda_1 t^2$$
$$- \varepsilon_0 \lambda_1 t \int \varphi_1 u_0 + \frac{\varepsilon_0}{2}\int |\nabla u_0|^2 - \int F(t\varphi_1). \tag{13}$$

Let $\alpha = (3a\lambda + \lambda_1)/4\lambda$. Since $\alpha < a$, there exists $\beta \in \mathbb{R}$ such that $f(s) \geq \alpha s + \beta$ for all s, which implies that $F(s) \geq \alpha\lambda/2\, s^2 + \beta\lambda s$ when $s \geq 0$. Then (13) shows that

$$\limsup_{t \to \infty} \frac{1}{t^2} J_{\varepsilon_0}(t\varphi_1) \leq \frac{\lambda_1 + \varepsilon_0\lambda_1 - \lambda\alpha}{2} < 0$$

because of the choice of α. ∎

LEMMA 9. The condition (PS) is satisfied uniformly in ε, that is if

$$(J_{\varepsilon_n}(u_n)) \text{ is bounded in } \mathbb{R}, \qquad \varepsilon_n \in [0, \varepsilon_0] \tag{14}$$

and

$$J'_{\varepsilon_n}(u_n) \to 0 \text{ in } E^* \tag{15}$$

then (u_n) is relatively compact in E.

Proof. Since any subsequence of (u_n) verifies (14) and (15), it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E. Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \to u$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e., and $\varepsilon_n \to \varepsilon$. Now (15) gives that

$$-\Delta u_n - \lambda f(u_n) - \varepsilon_n \Delta(u_n - u_0) \to 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \le a|u_n - u|$. This shows that

$$-(1 + \varepsilon_n)\Delta u_n \to \lambda f(u) - \varepsilon \Delta u_0 \qquad \text{in } \mathfrak{D}'(\Omega),$$

that is

$$-\Delta u - \lambda f(u) - \varepsilon\Delta(u - u_0) = 0. \tag{16}$$

The above equality multiplied by u gives

$$(1 + \varepsilon)\int |\nabla u|^2 - \lambda \int uf(u) - \varepsilon\lambda \int uf(u_0) = 0. \tag{17}$$

Now (15) multiplied by (u_n) gives

$$(1 + \varepsilon_n)\int |\nabla u_n|^2 - \lambda \int u_n f(u_n) - \varepsilon_n \lambda \int u_n f(u_0) \to 0 \tag{18}$$

in view of the boundedness of (u_n). The middle term in (18) tends to $-\lambda \int uf(u)$ and the last one to $-\varepsilon\lambda \int uf(u_0)$ in view of the $L^2(\Omega)$-convergence of u_n and $f(u_n)$. Hence, if we compare the first terms in (17) and (18) we get that $\int |\nabla u_n|^2 \to \int |\nabla u|^2$, which insures us that $u_n \to u$ in $H_0^1(\Omega)$. Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$-boundedness of (u_n) implies that $H_0^1(\Omega)$-boundedness of (u_n) as it can be seen by examining (14).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_{L^2(\Omega)} \to \infty$. Let $u_n = k_n w_n$ with $k_n > 0$, $\int w_n^2 = 1$ and $k_n \to \infty$. We may suppose $\varepsilon_n \to \varepsilon$. Then

$$0 = \lim_{n\to\infty} \frac{J_{\varepsilon_n}(u_n)}{k_n^2} = \lim_{n\to\infty}\left[\frac{1}{2}\int |\Delta w_n|^2 - \frac{1}{k_n^2}\int F(u_n) + \frac{\varepsilon_n}{2}\int \left|\nabla\left(w_n - \frac{u_0}{k_n}\right)\right|^2\right]. \tag{19}$$

Now

$$\frac{\varepsilon_n}{2}\int \left|\nabla\left(w_n - \frac{u_0}{k_n}\right)\right|^2 = \frac{\varepsilon_n}{2}\int |\nabla w_n|^2 + \frac{\varepsilon_n}{2k_n^2}\int |\nabla u_0|^2 - \frac{\varepsilon_n \lambda}{k_n}\int w_n f(u_0).$$

Thus (19) can be rewritten

$$\lim_{n\to\infty}\left[\frac{1 + \varepsilon_n}{2}\int |\nabla w_n|^2 - \frac{1}{k_n^2}\int F(u_n)\right] = 0.$$

However,

$$|F(u_n)| = |F(k_n w_n)| \le \frac{\lambda a}{2} k_n^2 w_n^2 + \lambda b |k_n w_n|$$

because $|f(t)| \le a|t| + b$. Here $b = f(0)$. This shows that $((1/k_n^2)\int F(u_n))$ is bounded and this must also be true for $\|w_n\|_{H_0^1(\Omega)}$. Now let $w \in H_0^1(\Omega)$ be such that (up to a subsequence) $w_n \to w$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. Note that $\int w^2 = 1$. We claim that

$$-(1 + \varepsilon)\Delta w = \lambda a w^+. \tag{20}$$

Indeed, (15) divided by k_n gives

$$(1 + \varepsilon_n)\int \nabla w_n \cdot \nabla v - \lambda \int \frac{f(u_n)}{k_n} v - \frac{\varepsilon_n \lambda}{k_n}\int f(u_0) v \to 0 \tag{21}$$

for each $v \in H_0^1(\Omega)$. Now

$$(1+\varepsilon_n)\int \nabla w_n \cdot \nabla v \to (1+\varepsilon)\int \nabla w \cdot \nabla v.$$

Hence (20) can be concluded from (21) if we show that $1/k_n f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $1/k_n f(u_n) = 1/k_n f(k_n w_n)$ and it is easy to see that the required limit is equal to aw^+ in the set

$$\{x \in \Omega : w_n(x) \to w(x) \neq 0\}.$$

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 be such that $|w_n(x)| < \varepsilon$ for $n \geq n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \leq \varepsilon a + \frac{b}{k_n} \qquad \text{for such } n,$$

that is the required limit is 0. Thus, $(f(u_n))/k_n \to aw^+$ a.e. Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and, thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see theorem IV.9 in [4]).

Since $1/k_n f(u_n) \leq a|w_n| + 1/k_n\, b$, it follows that $1/k_n f(u_n)$ is also dominated. Hence (20) is now obtained. Now (20) and the maximum principle imply $w \geq 0$ and (20) becomes

$$\begin{cases} -\Delta w = \dfrac{\lambda a}{1+\varepsilon} w \\ w \geq 0 \\ \displaystyle\int w^2 = 1. \end{cases} \tag{22}$$

Thus $\lambda a/(1+\varepsilon) = \lambda_1$ (and $w = \varphi_1$), which contradicts the fact that $\varepsilon \in [0, \varepsilon_0]$ and the choice of ε_0. This contradiction finishes the proof of the lemma 9. ■

LEMMA 10. c_ε is uniformly bounded.

Proof. The fact that J_ε increases with ε implies $c_\varepsilon \in [c_0, c_{\varepsilon_0}]$. ■

Now we continue the proof of the theorem B(v): for $\varepsilon \in (0, \varepsilon_0]$, let $v_\varepsilon \in H_0^1(\Omega)$ be such that

$$-\Delta v_\varepsilon = \frac{\lambda}{1+\varepsilon} f(v_\varepsilon) + \frac{\lambda\varepsilon}{1+\varepsilon} f(u_0) \tag{23}$$

and

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon. \tag{24}$$

The relation (24) and the lemmas 9 and 10 show that there exists $v \in H_0^1(\Omega)$ such that $v_\varepsilon \to v$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Now (23) implies

$$-\Delta v = \lambda f(v).$$

The last assertions to be proved are that $v \neq u_0 = u(\lambda)$ and $v \in C^2(\Omega) \cap C(\bar{\Omega})$. Note that v_ε is a solution of (23) different from u_0 and, hence, unstable in the sense that

$$\lambda_1\left(\frac{\lambda}{1+\varepsilon} f'(v_\varepsilon)\right) \leq 0.$$

Indeed (23) is an equation of the form

$$-\Delta u = g(u) + h(x),$$

where g is convex and positive and h is positive. Then, if it has solutions, it has a minimal one, say u, with $\lambda_1(g'(u)) \geq 0$ (see [1]). Now the proof of the lemma 3(iv) shows that for all other solutions v we have $\lambda_1(g'(v)) < 0$. In our case, u_0 stands for u and v_ε for v. All we have to prove now is that the limit of a sequence of unstable solutions is also unstable, which will be done in the following lemma.

LEMMA 11. Let $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $\mu_n \to \mu$ be such that $\lambda_1(\mu_n f'(u_n)) \leq 0$.
Then $\lambda_1(\mu f'(u)) \geq 0$.

Proof. The fact that $\lambda_1(\alpha) \leq 0$ is equivalent to the existence of a $\varphi \in H_0^1(\Omega)$ such that

$$\int |\nabla\varphi|^2 \leq \int \alpha\varphi^2 \quad \text{and} \quad \int \varphi^2 = 1$$

follows from the Hilbert-Courant min-max principle.
Let $\varphi_n \in H_0^1(\Omega)$ be such that

$$\int |\nabla\varphi_n|^2 \leq \int \mu_n f'(u_n)\varphi_n^2 \tag{25}$$

and

$$\int \varphi_n^2 = 1. \tag{26}$$

Since $f' \leq a$, (25) shows that (φ_n) is bounded in $H_0^1(\Omega)$. Let $\varphi \in H_0^2(\Omega)$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H_0^1(\Omega)$. Then the right-hand side of (25) converges, up to a subsequence, to $\mu \int f'(u)\varphi^2$. This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega)$ as in the theorem IV.9 in [4]. Since

$$\int \varphi^2 = 1 \quad \text{and} \quad \int |\nabla\varphi|^2 \leq \liminf \int |\nabla\varphi_n|^2,$$

we get the desired result.
The fact that $v \in C^2(\Omega) \cap C(\bar{\Omega})$ follows via a bootstrap argument

$$v \in H_0^1(\Omega) \Rightarrow f(v) \in {}^{,2^*}(\Omega) \Rightarrow v \in W^{2,2^*}(\Omega) \Rightarrow \cdots$$

The key facts are:
(a) if $v \in L^p(\Omega)$ then $f(v) \in L^p(\Omega)$;
(b) an elliptic regularity result (theorem 9.15 in [2]);
(c) the Sobolev embeddings.

(vi) Suppose the contrary. Then there are $\mu_n \to \lambda_1/a$, v_n an unstable solution of (1) with $\lambda = \mu_n$, and $v \in L^1_{loc}(\Omega)$ such that $v_n \to v$ in $L^1_{loc}(\Omega)$.

We claim first that (v_n) cannot be bounded in $H^1_0(\Omega)$. Otherwise, let $w \in H^1_0(\Omega)$ be such that, up to a subsequence, $v_n \to w$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then

$$-\Delta v_n \to -\Delta w \quad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad f(v_n) f(w) \quad \text{in } L^2(\Omega),$$

which shows that $-\Delta w = \lambda_1/a f(w)$.

It follows that $w \in C^2(\Omega) \cap C(\bar{\Omega})$, that is w is a solution of (1). From lemma 11 it follows that

$$\lambda_1\left(\frac{\lambda_1}{a} f'(w)\right) \le 0. \tag{27}$$

Now (27) shows that $w \neq u(\lambda_1/a)$, which contradicts (iv) of the theorem.

The fact that (v_n) is not bounded in $H^1_0(\Omega)$ implies that (v_n) is not bounded in $L^2(\Omega)$. Indeed, we have seen that the $L^2(\Omega)$-boundedness implies the $H^1_0(\Omega)$ one. So, let $v_n = k_n w_n$, where $k_n > 0$, $\int w_n^2 = 1$ and up to a subsequence $k_n \to \infty$.

We have

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n) \to 0 \qquad \text{in } L^1_{loc}(\Omega)$$

(and, hence, we have convergence also in the distribution sense) and (w_n) is seen to be bounded in $H^1_0(\Omega)$ with an already provided argument. If w is a $\star$-cluster point of (w_n) in $H^1_0(\Omega)$, we obtain $-\Delta w = 0$ and $\int w^2 = 1$, the desired contradiction.

(vii) As before, it is enough to prove the $L^2(\Omega)$-boundedness of $v(\lambda)$ near λ^* and to use the uniqueness property of u^*. Suppose the contrary. Let $\mu_n \to \lambda^*$, $\|v_n\|_{L^2(\Omega)} \to \infty$, where v_n are the corresponding solutions of (1). If we write again $v_n = k_n w_n$, then

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n). \tag{28}$$

The fact that the right-hand side of (28) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in $H^1_0(\Omega)$. Let w be such that up to a subsequence $w_n \to w$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. A computation already done shows that

$$-\Delta w = \lambda^* a w, \qquad w \ge 0 \qquad \text{and} \qquad \int w^2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof. ∎

3. SOME FURTHER REMARKS

As we have seen in the proofs of the theorems A and B, we have that:

(i) in the monotone case,

$$\lim_{\lambda \to \lambda_1/a} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \qquad \text{in } H^1_0(\Omega);$$

(ii) in the nonmonotone case,

$$\lim_{\lambda \to \lambda_1/a} \frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1 \qquad \text{in } H_0^1(\Omega).$$

It is natural to try to find out:
(i) if the above limits continue to exist in a more restrictive sense, say in $C(\bar{\Omega})$;
(ii) which is the asymptotic behaviour of $\|u(\lambda)\|_{L^2(\Omega)}$ and $\|v(\lambda)\|_{L^2(\Omega)}$ when λ is near λ_1/a.
It is easy to answer the first question. We have the following proposition.

PROPOSITION 1. (i) in the monotone case,

$$\lim_{\lambda \to \lambda_1/a} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \qquad \text{in } C^1(\bar{\Omega}).$$

(ii) In the nonmonotone case,

$$\lim_{\lambda \to \lambda_1/a} \frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1 \qquad \text{in } C^1(\bar{\Omega}).$$

Proof. (i) The proof is essentially the same as for the lemma 6: it is enough to prove that $(1/(\|u(\lambda)\|_{L^2(\Omega)})\, u(\lambda))$ is relatively compact in $C^1(\bar{\Omega})$ (when λ is near λ_1/a), which can be done by showing that it is bounded in $C^{1,1/2}(\bar{\Omega})$. However, this follows from the fact that the above set is bounded in $H_0^1(\Omega)$ and a bootstrap argument (note that a uniform bound for $w(\lambda) = 1/(\|u(\lambda)\|_{L^2(\Omega)})\, u(\lambda)$ in some $L^p(\Omega)$, $1 < p < \infty$ provides a uniform bound for $-\Delta w(\lambda)$ in $L^p(\Omega)$ for the same p).
(ii) is identical with (i). ∎

Moreover, we have the following proposition.

PROPOSITION 2. If $w(\lambda)$ is either $1/(\|u(\lambda)\|_{L^2(\Omega)})\, u(\lambda)$ or $1/(\|v(\lambda)\|_{L^2(\Omega)})\, v(\lambda)$, then $\varphi_1/w(\lambda)$ is uniformly bounded when λ is near λ_1/a.

Proof. Note that the strong maximum principle implies that $\partial w(\lambda)/\partial \nu < 0$ on $\partial\Omega$ and, hence, $\varphi_1/w(\lambda)$ can be extended to a continuous function on $\bar{\Omega}$ by setting

$$\frac{\varphi_1}{w(\lambda)}(x) = \frac{(\partial \varphi_1/\partial \nu)(x)}{(\partial w(\lambda)/\partial \nu)(x)} \qquad \text{for } x \in \partial\Omega.$$

LEMMA 12. There exists $\varepsilon_0 > 0$ such that if

$$w_0 = \{x \in \mathbb{R}^N : d(x, \partial\Omega) < \varepsilon_0\}$$

then:
(i) for each $x \in w_0$ there is a unique $x_0 \in \partial\Omega$ such that $d(x, \partial\Omega) = |x - x_0|$;
(ii) if $\Pi(x) = x_0$, then $\Pi \in C^1(w_0)$ (x, x_0 are as above);
(iii) if $|x - \Pi(x)| = \varepsilon$ then $x = \Pi(x) - \varepsilon\nu(\Pi(x))$ or $x = \Pi(x) + \varepsilon\nu(\Pi(x))$, according to the case $x \in \Omega$ or $x \notin \Omega$;
(iv) if $x \in \Omega$ then $[x, \Pi(x)) \subset \Omega$.

The proof can be found in [5]. ∎

Let $\omega = \omega_0 \cap \Omega$ and $K = \Omega \backslash \omega$. Since $w(\lambda) \to \varphi_1 u.\ \bar{\Omega}$, for λ close enough to λ_1/a we have $w(\lambda)_{|K} > \frac{1}{2}\min_K \varphi_1$, that is $\varphi_1/w(\lambda) < c$ in K for such λ and a suitable c. If $x \in \omega$, let $x_0 = \Pi(x)$. Then

$$\frac{\varphi_1(x)}{w(\lambda, x)} = \frac{\varphi_1(x) - \varphi_1(x_0)}{w(\lambda, x) - w(\lambda, x_0)} = \frac{-\varepsilon(\partial\varphi_1/\partial\nu(x_0))(x_0 + \tau(x - x_0))}{-\varepsilon(\partial w/\partial\nu(x_0))(\lambda, x_0 + \tau(x - x_0))} \tag{29}$$

for some $\tau \in (0, 1)$. Taking a smaller ε_0, if necessary, we may suppose that $(\partial w/\partial\nu(\Pi(x)))(x) < 0$ on $\bar{\omega}$. Then, as above, the quotient in (29) is smaller than some $c_1 > 0$ for λ near λ_1/a. ■

For the second question the answer is delicate. For example we have the following proposition.

PROPOSITION 3. Suppose f to obey the monotone case, that is $f(t) \geq at$ for all t, and let

$$l = \lim_{t \to \infty}[f(t) - at] \geq 0.$$

Then

$$\lim_{\lambda \to \lambda_1/a} (\lambda_1 - a\lambda)\|u(\lambda)\|_{L^2(\Omega)} = \frac{\lambda_1}{a} l \int \varphi_1 .$$

Proof. Let L_0 be a limit point of $(\lambda_1 - a\lambda)\|u(\lambda)\|_{L^2(\Omega)}$ when $\lambda \to \lambda_1/a$. If we rewrite

$$\int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) - \lambda(f(u(\lambda)) - au(\lambda))] = 0 \tag{10}$$

in the form

$$\int \varphi_1(\lambda_1 - a\lambda)\|u(\lambda)\|_{L^2(\Omega)} w(\lambda) = \int \lambda\varphi_1(f(u(\lambda)) - au(\lambda)) \tag{30}$$

and we note that the right-hand side integrand converges dominated to $(\lambda_1/a)l\varphi_1$ when $\lambda \to \lambda_1/a$, and that the left-hand side integrand tends to $L_0\varphi_1^2 u.\ \bar{\Omega}$ if $L_0 < \infty$ and to ∞ uniformly in Ω if $L_0 = \infty$ (on an appropriate sequence of λ), we get that

$$L_0 = \frac{\lambda_1}{a} l \int \varphi_1 . \quad ■$$

It is obvious that the answer is good only when $l > 0$. If $l = 0$ then it shows only that $\|u(\lambda)\|_{L^2(\Omega)}$ grows slower than $1/(\lambda_1 - a\lambda)$. As we shall see below, in this case the answer depends heavily on f.

Example 1. Let $f(t) = t + 1/(t + 2)$ when $t \geq 0$ (defined no matter how for negative t). Then

$$\lim_{\lambda \to \lambda_1} \sqrt{\lambda_1 - \lambda}\|u(\lambda)\|_{L^2(\Omega)} = \sqrt{\lambda_1 |\Omega|}.$$

Proof. With the usual decomposition $u(\lambda) = k(\lambda)w(\lambda)$, if we divide (10) by $\sqrt{\lambda_1 - \lambda}$ we get

$$\int \varphi_1\sqrt{\lambda_1 - \lambda}k(\lambda)w(\lambda) = \int \frac{\lambda\varphi_1}{\sqrt{\lambda_1 - \lambda}k(\lambda)w(\lambda) + 2\sqrt{\lambda_1 - \lambda}}. \tag{31}$$

We claim first that $\liminf_{\lambda\to\lambda_1}\sqrt{\lambda_1 - \lambda}k(\lambda) > 0$. Otherwise, let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \to 0$. Then

$$\sqrt{\lambda_1 - \mu_n}k(\mu_n)w(\mu_n)\varphi_1 \to 0 \qquad u.\bar{\Omega}$$

and

$$\sqrt{\lambda_1 - \mu_n}k(\mu_n)w(\mu_n) + 2\sqrt{\lambda_1 - \mu_n} \to 0 \qquad u.\bar{\Omega},$$

which contradicts (31) for large n.

We shall also prove that $\limsup_{\lambda\to\lambda_1}\sqrt{\lambda_1 - \lambda}k(\lambda) < \infty$. Suppose the contrary. Let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \to \infty$. Then the left-hand side of (31) tends to ∞ with n. We shall show that the right-hand side remains bounded and the contradiction will conclude the proof. Now $\varphi_1/w(\mu_n)$ is uniformly bounded by some $M > 0$, so that the right-hand side integrand is less than $\lambda_1 M/\sqrt{\lambda_1 - \mu_n}k(\mu_n)$, which is bounded.

Let $c \in (0, +\infty)$ be a limit point of $\sqrt{\lambda_1 - \lambda}k(\lambda)$ when $\lambda \to \lambda_1$. Let $\mu_n \to \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \to c$ and $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \geq c/2$. Then the left-hand side of (31) tends to c, while the right-hand side integrand is dominated by $2\lambda_1 M/c$ and converges a.e. to λ_1/c. Hence $c = \lambda_1/c|\Omega|$ which finishes the proof. ■

Note that a similar computation can be made if $f(t) = \sqrt{t^2 + 1}$.

If $f(t) - at$ decays to ∞ faster than $1/t$ then the behaviour becomes more complicated, as shown in the following example.

Example 2. Let $f(t) = t + 1/(t + 1)^2$. Then $\|u(\lambda)\|_{L^2(\Omega)}$ tends to ∞ like no power of $(\lambda_1 - \lambda)$. More precisely:

(i) $\lim_{\lambda\to\lambda_1}(\lambda_1 - \lambda)^\alpha\|u(\lambda)\|_{L^2(\Omega)} = \infty$ if $\alpha \leq \frac{1}{3}$;

(ii) $\lim_{\lambda\to\lambda_1}(\lambda_1 - \lambda)^\alpha\|u(\lambda)\|_{L^2(\Omega)} = 0$ if $\alpha > \frac{1}{3}$.

Proof. We shall need first some estimations for $\int 1/\varphi_1$ and $\int \mathbf{1}_{\{\varphi_1 > \varepsilon\}} 1/\varphi_1$.

LEMMA 13. (i) There exist positive constants K_1, K_2 and ε_1 such that

$$K_1|\ln\varepsilon| \leq \int \mathbf{1}_{\{\varphi_1 > \varepsilon\}}\frac{1}{\varphi_1} \leq K_2|\ln\varepsilon| \qquad \text{for } \varepsilon \in (0, \varepsilon_1).$$

(ii) $\int 1/\varphi_1 = \infty$.

Proof. (ii) follows obviously from (i).

(i) Let ε_0 and ω_0 as in lemma 12. Let

$$\Phi\colon \omega_0 \to \partial\Omega \times (-\varepsilon_0, \varepsilon_0) \qquad \text{and} \qquad \Psi\colon \partial\Omega \times (-\varepsilon_0, \varepsilon_0) \to \omega_0$$

be defined by

$$\Phi(x) = (\Pi(x), \langle x - \Pi(x), \nu(x)\rangle) \qquad \text{and} \qquad \Psi(x_0, \varepsilon) = x_0 + \varepsilon\nu(x_0).$$

Then Φ, Ψ are smooth and $\Psi = \Phi^{-1}$, so that if we replace if necessary ε_0 with a smaller number, we may suppose that there exist $C_1, C_2 > 0$ such that $0 < C_1 \le |J(\Psi)| \le C_2$ on ω_0.

We claim that there exist $C_3, C_4 > 0$ such that

$$C_3 d(x, \partial\Omega) \le \varphi_1(x) \le C_4 d(x, \partial\Omega)$$

when $x \in \omega$, if we replace, eventually, ε_0 with a smaller number. Indeed, as $\max_{\partial\Omega}(\partial\varphi_1/\partial\nu) < 0$, we obtain that

$$-C_3 = \sup_{x \in \omega} \frac{\partial\varphi_1(x)}{\partial\nu(\Pi(x))} < 0$$

if ε_0 is small enough.

Let $C_4 = \max_{\bar{\Omega}}|\varphi'|$. Then if $x \in \omega$ we get

$$\varphi_1(x) = \varphi_1(x) - \varphi_1(\Pi(x)) = -d(x, \Pi(x)) \frac{\partial\varphi_1(y)}{\partial\nu(\Pi(x))}$$

for some $y \in [x, \Pi(x)]$ and also the desired result.

Take $\varepsilon_1 < \min(\inf_{\Omega\setminus\omega} \varphi_1, C_3\varepsilon_0)$. Now if $\varepsilon < \varepsilon_1$ then

$$\int \mathbf{1}_{\{\varphi_1 > \varepsilon\}} \frac{1}{\varphi_1} = \int \mathbf{1}_{\{\varphi_1 \ge \varepsilon_1\}} \frac{1}{\varphi_1} + \int \mathbf{1}_{\{\varepsilon < \varphi_1 < \varepsilon_1\}} \frac{1}{\varphi_1}.$$

Note that

$$\left\{\frac{\varepsilon}{C_3} < d(x, \partial\Omega) < \frac{\varepsilon_1}{C_4}\right\} \subset \{\varepsilon < \varphi_1 < \varepsilon_1\} \subset \left\{\frac{\varepsilon}{C_4} < d(x, \partial\Omega) < \frac{\varepsilon_1}{C_3}\right\}$$

and

$$\frac{1}{C_4 d(x, \partial\Omega)} \le \frac{1}{\varphi_1(x)} \le \frac{1}{C_3 d(x, \partial\Omega)}$$

there. Then

$$\int \mathbf{1}_{\{\varphi_1 \ge \varepsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_4} \int \mathbf{1}_{\{\varepsilon/C_3 < d(x,\partial\Omega) < \varepsilon_1/C_4\}} \frac{1}{d(x, \partial\Omega)}$$
$$\le \int \mathbf{1}_{\{\varphi_1 > \varepsilon\}} \frac{1}{\varphi_1} \le \int \mathbf{1}_{\{\varphi_1 \ge \varepsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_3} \int \mathbf{1}_{\{\varepsilon/C_4 < d(x,\partial\Omega) < \varepsilon_1/C_3\}} \frac{1}{d(x, \partial\Omega)}.$$

It remains to find, for example, $C_5, C_6 > 0$ such that

$$C_5|\ln\varepsilon| \le I = \int \mathbf{1}_{\{\varepsilon/C_4 < d(x,\partial\Omega) < \varepsilon_1/C_3\}} \frac{1}{d(x, \partial\Omega)} \le C_6(|\ln\varepsilon| + 1).$$

Now with the changement of coordinates, $x = \Psi(x_0, \delta)$, we get

$$I = \int_{\partial\Omega \times (\varepsilon/C_4, \varepsilon_1/C_3)} \frac{1}{\delta} |J(\Psi)| \, ds(x_0) \, d\delta,$$

so that

$$C_1|\partial\Omega|/\ln\frac{C_4\varepsilon_1}{C_3\varepsilon}\le I\le C_2|\partial\Omega|\ln\frac{C_4\varepsilon_1}{C_3\varepsilon}$$

and the desired estimation follows easily. The proof of the lemma is completed. ∎

Now in order to prove (i) of the example 2 it is enough to show that

$$\lim_{\lambda\to\lambda_1}(\lambda_1-\lambda)^{1/3}\|u(\lambda)\|_{L^2(\Omega)}=\infty.$$

Suppose that there exist $\mu_n\to\lambda_1$ and $c<\infty$ such that

$$(\lambda_1-\mu_n)^{1/3}k_n\to c,\qquad \text{where } k_n=\|u(\mu_n)\|_{L^2(\Omega)}.$$

If we divide (10) written with $\lambda=\mu_n$ by $(\lambda_1-\mu_n)^{2/3}$ we get

$$\int\varphi_1(\lambda_1-\mu_n)^{1/3}k_nw_n=\lambda\int\frac{\varphi_1}{(\lambda_1-\mu_n)^{2/3}(k_nw_n+1)^2},\tag{32}$$

where $w_n=(1/k_n)u(\mu_n)$.

If $c=0$ then the left-hand side in (32) tends to 0, while the second one to ∞. Hence $c\in(0,\infty)$. The fact that $k_n\to\infty$ implies that for each $\varepsilon>0$, $2k_nw_n+1<\varepsilon k_n^2$, for large n, so that the right-hand side of (32) is larger that

$$\frac{\lambda}{2c^2}\int\frac{\varphi_1}{\varphi_1^2+\varepsilon}$$

for n large enough to have $(\lambda_1-\mu_1)^{2/3}k_n^2<2c^2$. Since the limit of the left-hand side is c, we get that

$$c\ge\frac{\lambda_1}{2c^2}\int\frac{\varphi_1}{\varphi_1^2+\varepsilon}$$

for all $\varepsilon>0$. Letting $\varepsilon\to0$ we obtain $c=\infty$, the desired contradiction.

(ii) Suppose the contrary. Then there exist $\alpha>\frac{1}{3}$, $\mu_n\to\lambda_1$, $c\in(0,+\infty]$ such that $(\lambda_1-\lambda)^\alpha k_n\to c$, where $k_n=\|u(\mu_n)\|_{L^2(\Omega)}$.

Let $\beta=3\alpha-1>0$. Then (10) with $\lambda=u_n$ divided by $(\lambda_1-\lambda)^{1-\alpha}$ gives

$$\int\varphi_1(\lambda_1-\mu_n)^\alpha k_nw_n=\lambda\int\frac{\varphi_1}{(\lambda_1-\mu_n)^{2\alpha-\beta}(k_nw_n+1)^2}\qquad(=I_n).\tag{33}$$

The limit of the left-hand side is $c\in(0,+\infty]$. I_n can be estimated as follows

$$I_n=\int\cdots=\int\mathbf{1}_{\{\varphi_1<\lambda_1-\mu_n\}}\cdots+\int\mathbf{1}_{\{\varphi_1\ge\lambda_1-\mu_n\}}\cdots=J_n+K_n.$$

Now

$$0<J_n\le\int\frac{\lambda_1-\mu_n}{(\lambda_1-\mu_n)^{2\alpha-\beta}}=(\lambda_1-\mu_n)^\alpha|\Omega|\to0$$

while

$$0 < K_n \leq \frac{M(\lambda_1 - \mu_n)^\beta}{c^2} \int \mathbf{1}_{\{\varphi_1 \geq \lambda_1 - \mu_n\}} \frac{1}{\varphi_1},$$

where $M = \sup_n \max u_n^2/\varphi_1^2 < \infty$ (as shows the proof of proposition 2).

Lemma 13 shows that the last expression is $O(\lambda_1 - \mu_n)^\beta |\ln(\lambda_1 - \mu_n)|)$, that is it tends to zero with n.

In the nonmonotone case $\|v(\lambda)\|_{L^2(\Omega)}$ grows faster to ∞. We have the following proposition.

PROPOSITION 4. Let f obey the nonmonotone case and let

$$\lim_{t\to\infty}[f(t) - at] = l \in [-\infty, 0).$$

Then

$$\lim_{\lambda\to\lambda_1/a} (\lambda_1 - a\lambda)\|v(\lambda)\|_{L^2(\Omega)} = l.$$

The proof is identical to that of the preceding proposition.

The result is good only when $l \in \mathbb{R}$. When $l = -\infty$, we give an example.

Example 3. If $f(t) = t + 2 - \sqrt{t+1}$, then

$$\lim_{\lambda\to\lambda_1} (\lambda - \lambda_1)^2 \|v(\lambda)\|_{L^2(\Omega)} = \left(\int \varphi_1 \sqrt{\varphi_1}\right)^2.$$

Proof. If we multiply (10) by $\lambda - \lambda_1$ we get

$$\int \varphi_1(\lambda - \lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}[\lambda - (\lambda - \lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}]$$

$$= 2\lambda(\lambda - \lambda_1)\int \varphi_1 - \lambda \int \varphi_1[\sqrt{(\lambda - \lambda_1)^2 k(\lambda) w(\lambda) + (\lambda - \lambda_1)^2} - \sqrt{(\lambda - \lambda_1)^2 k(\lambda) w(\lambda)}], \tag{34}$$

where $k(\lambda)$, $w(\lambda)$ are as usual. We prove first that $\limsup_{\lambda\to\lambda_1} (\lambda - \lambda_1)^2 k(\lambda) < \infty$. Suppose there exist $\mu_n \to \lambda_1$ such that $(\mu_n - \lambda_1)^2 k(\mu_n) \to \infty$. Then the right-hand side of (34) tends to 0, while the left-hand side is, for a suitable choice of $C_1, C_2 > 0$, less than

$$C_1(\lambda - \lambda_1)\sqrt{k(\lambda)} - C_2(\lambda - \lambda_1)^2 k(\lambda)$$

so it tends to $-\infty$.

Suppose now that

$$\liminf_{\lambda\to\lambda_1} (\lambda - \lambda_1)^2 k(\lambda) = 0. \tag{35}$$

The last integral in (34) is positive, so that (34) gives

$$\int \varphi_1 \sqrt{k(\lambda)}\sqrt{w(\lambda)}[\lambda - (\lambda - \lambda_1)\sqrt{k(\lambda)}\sqrt{w(\lambda)}] \leq 2\lambda \int \varphi_1. \tag{36}$$

However, the assumption (35) makes the left-hand side of (36) to tend to ∞ for a suitable λ. The contradiction shows that (35) is false.

Now let $c \in (0, +\infty)$ be any limit point of $(\lambda - \lambda_1)^2 k(\lambda)$ when $\lambda \to \lambda_1$. Then (34) shows that $c = (\int \varphi_1 \sqrt{\varphi_1})^2$.

All other functions we have tested behaved well in the sense that $\|v(\lambda)\|_{L^2(\Omega)} \sim Cg(1/(\lambda - \lambda_1))$, where g is the inverse of the antiderivative of

$$[0, +\infty) \ni t \mapsto \frac{1}{at + f(0) + 1 - f(t)}.$$

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