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Normalized ground states for the Sobolev critical Schrödinger equation with at least mass critical growth

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Abstract

In the present paper, we investigate the existence of ground state solutions to the Sobolev critical nonlinear Schrödinger equation

$$\begin{cases} -\Delta u+\lambda u=g(u)+|u|^{2^*-2}u \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx=m^2, \end{cases} \tag{$\mathrm{P_m}$}$$

where $N\geqslant 3$, $m>0$, $2^*:=\frac{2N}{N-2}$, λ is an unknown parameter that will appear as a Lagrange multiplier, g is a mass critical or supercritical but Sobolev subcritical nonlinearity. With the aid of the minimization of the energy functional over a linear combination of the Nehari and Pohozaev constraints intersected with

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the product of the closed balls in $L^2(\mathbb{R}^N)$ of radii m and the profile decomposition, we obtain a couple of the normalized ground state solution to (P_m) that is independent of the sign of the Lagrange multiplier. This result complements and extends the paper by Bieganowski and Mederski (2021 *J. Funct. Anal.* **280** 108989) concerning the above problem from the Sobolev subcritical setting to the Sobolev critical framework. We also answer an open problem that was proposed by Jeanjean and Lu (2020 *Calc. Var. PDE* **59** 174). Furthermore, the asymptotic behavior of the ground state energy map is also studied.

Keywords: normalized ground states, Pohozaev manifold, profile decomposition, Sobolev critical exponent

Mathematics Subject Classification numbers: 35J20, 35J50, 35J70

1. Introduction and main results

In this paper, for given $m > 0$, we shall investigate the existence of $(u,\lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ satisfying

$$-\Delta u + \lambda u = g(u) + |u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

and

$$\int_{\mathbb{R}^N} |u|^2 dx = m^2,$$

where

$$H^1\left(\mathbb{R}^N\right) := \left\{u \in L^2\left(\mathbb{R}^N\right) : |\nabla u| \in L^2\left(\mathbb{R}^N\right)\right\}$$

is endowed with the natural norm

$$\|u\| := \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx\right]^{\frac{1}{2}}.$$

Such (u,λ) is called a *couple of solution* to the system (P_m).

It is well known that solutions of (1.1) are related to the existence of standing waves, which can help us to understand the dynamics property to the following time-dependent nonlinear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = \Delta\Psi + |\Psi|^{2^*-2}\Psi + f(|\Psi|)\,\Psi, \tag{1.2}$$

where i denotes the imaginary unit, $\Psi = \Psi(t,x) \in \mathbb{C}$ is the wave function, and f is an appropriate nonlinearity.

To study the standing wave solutions to problem (1.2), set

$$\Psi(t,x) = e^{-i\lambda t}u(x),$$

where $u \in H^1(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$ is the frequency or the chemical potential. Then equation (1.2) can be transformed to equation (1.1) with $g(u) = f(|u|)u$.

About the frequency λ in problem (1.1), there exist two substantially different points of view. One is to regard λ as a given constant. At this time, we call (1.1) a *fixed frequency problem*. For this class of nonlinear problems, existence, multiplicity and concentration of solutions have already been studied. It seems almost impossible for us to give a complete list of references. We refer the readers to [3, 4, 12, 29] and the references therein. The other point of view is to regard λ as an unknown quantity to problem (1.1). At this moment, it becomes very natural to prescribe the value of the integral $\int_{\mathbb{R}^N}|u|^2dx$ so that λ can be interpreted as the Lagrange multiplier, and $\int_{\mathbb{R}^N}|u|^2dx$ is referred to as the mass. This point of view has often a powerful physical meaning. For example, it represents the power supply in nonlinear optics and the total number of atoms in Bose–Einstein condensation. These are two basic fields of application of nonlinear Schrödinger equations. In this way, a new critical exponent $r = 2+\frac{4}{N}$ named the L^2-critical exponent (also named mass critical exponent) appears in (1.1). Which is the threshold exponent for many dynamical properties, such as global existence, blow-up and the stability or instability of ground states. Alternatively, the L^2-*critical exponent* will substantially affect the geometry structure of the energy functional, and will produce some mathematical difficulties that make such type of study particularly interesting. Usually we say that the region where $r<2+\frac{4}{N}$ is L^2-*subcritical*, while $r>2+\frac{4}{N}$ signifies the L^2-*supercritical* regime.

It is of great interest to consider the solutions of problem (P_m) that admit prescribed L^2-norm, namely, for given $m>0$, to study the solutions of (1.1) under the L^2-norm constrained manifold

$$\mathcal{S} := \left\{u\in H^1\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N}|u|^2dx = m^2\right\}.$$

Physically, such type of solutions are known as *normalized solutions* to (P_m), which are the critical points of the energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2dx - \int_{\mathbb{R}^N}G(u)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx$$

restricted to the manifold $\mathcal{S}$. At this time, the unknown frequency λ is determined as the Lagrange multiplier associated to the constraint S. In addition, the mass is conserved along the trajectories of (1.2), that is,

$$\int_{\mathbb{R}^N}|\Psi(t,x)|^2dx = \int_{\mathbb{R}^N}|u|^2dx$$

for all $t>0$, and it can provide a good insight of the dynamical properties (such as, orbital stability and instability) of solutions to problem (1.2) (see [11, 27]).

In recent years, in consideration of the strong physical background of normalized solutions, more and more researchers began to pay attention to the study of normalized solutions to elliptic PDEs and systems, especially normalized ground state solutions. This is because they share further properties, like stability, positivity and symmetry, which are important for both physical and mathematical point of view. If a nontrivial solution minimizes I among all nontrivial solutions, we call it a *normalized ground state solution* to (P_m).

In [15], Jeanjean considered a semilinear elliptic Equation

$$-\Delta u+\lambda u = g(u),\quad x\in\mathbb{R}^N, \tag{1.3}$$

where $N\geqslant 1$, $\lambda\in\mathbb{R}$, and g satisfies

(g_0) $g\in C(\mathbb{R},\mathbb{R})$ and g is odd;

(g_1) there exist $\alpha,\ \beta\in\mathbb{R}$ with $2+\frac{4}{N}<\alpha\leqslant\beta<2^*$ such that

$$0<\alpha G(t)\leqslant g(t)t\leqslant\beta G(t)$$

for all $t\in\mathbb{R}\backslash\{0\}$, where $2^*=\frac{2N}{N-2}$ for $N\geqslant 3$ and $2^* := +\infty$ for $N=1,\ 2$;

(g_2) $H(t) := g(t)t-2G(t)\in C^1(\mathbb{R},\mathbb{R})$ and

$$h(t)t > \left(2 + \frac{4}{N}\right) H(t)$$

for all $t \in \mathbb{R} \backslash \{0\}$, where $h := H'$.

It is easy to see that the corresponding energy functional is unbounded from below on $\mathcal{S}$. By making use of a minimax procedure, Jeanjean showed that for each $m > 0$, (1.3) possesses at least one couple $(u_m, \lambda_m) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solution with $\|u_m\|_2 = m$ and u_m is radial under (g_0)–(g_1) for $N \geqslant 2$. Furthermore, when (g_2) is also assumed, he obtained the existence of ground states for $N \geqslant 1$. But, afterwards, there was little progress about the study of normalized solutions for a long time. One of the main reasons is that it is very difficult to prove the boundedness of constrained Palais-Smale sequence when the functional is unbounded from below on the constraint manifold. More recently, problems of such type began to receive much attention. Still under (g_0)–(g_1), by virtue of a fountain theorem type argument, Bartsch and de Valeriola [2] got a multiplicity result of (1.3) with $\|u\|_2 = m > 0$. About another proof for this multiplicity result can be seen [14], and [7, 8] but requires the additional assumption (g_2). Soave [30] studied the existence and properties of ground states to the nonlinear Schrödinger equation with combined power nonlinearities

$$-\Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{p-2} u, \ \ x \in \mathbb{R}^N$$

on $\mathcal{S}$, where $N \geqslant 1$, $2 < q \leqslant 2 + \frac{4}{N} \leqslant p < 2^*$. Which is more difficult and substantially different with purely subcritical or supercritical cases, because the interplay between subcritical, critical and supercritical nonlinearities has deep impacts on the geometry of the functional and on the existence and properties of ground states. Recently, the idea of [30] was used to deal with the fractional order case, see [24]. More results, we refer the readers to [6, 7, 13] for normalized solutions to systems in the whole space $\mathbb{R}^N$, [26, 28] for normalized solutions on bounded domains, [1, 17, 18, 20, 21, 23] for normalized solutions to the other equations, such as fractional Schrödinger equations, Hartree equations and Kirchhoff equations.

All the papers mentioned above only deal with the Sobolev subcritical case. As far as we know, it seems that there is few result for the normalized solutions to the Sobolev critical problem (P_m) which usually creates some thorny difficulties in using variational methods due to the double lack of compactness (see remark 1.7). In 2020, Soave [31] considered the equation

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{2^*-2} u, \ \ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = m^2, \end{cases} \tag{1.4}$$

where $N \geqslant 3$. When $2 + \frac{4}{N} < q < 2^*$, he proved that there exists a constant $\alpha = \alpha(N, q) > 0$ such that if $\mu \cdot m^{(1-\delta_q)q} < \alpha$, then (1.4) possesses a couple $(u_m, \lambda_m) \in H^1_{rad}(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solution and u_m is a real valued, positive function, where

$$H^1_{rad}\left(\mathbb{R}^N\right) := \left\{u \in H^1\left(\mathbb{R}^N\right) : u(|x|) = u(x)\right\}.$$

In particular, it is a critical point of mountain pass type, where $\delta_q = N(\frac{1}{2} - \frac{1}{q})$. Noting that Soave [31] only considered the case that $\mu \cdot m^{(1-\delta_q)q} > 0$ small. Recently, Wei and Wu [32] also considered the same problem (1.4). Especially, the first author and co-author in [22] obtained the existence of ground states that does not depend on the range of $\mu \cdot m^{(1-\delta_q)q}$, which improves and extends the result in [31]. Based on the direct minimization of the energy functional on the linear combination of Nehari and Pohozaev constraints, Bieganowski and Mederski [5] proposed a simple minimization method to prove the existence of normalized ground states to (1.3) for the Sobolev subcritical equation. A natural problem produces:

(Q_1) Do the results of [5] concerning Sobolev subcritical equation can be extended to the Sobolev critical equation (P_m)?

In this paper, we will give an affirmative answer for problem (Q_1). Before stating our results, we make the following assumptions:

$(A0)$ $g,h\in C(\mathbb{R},\mathbb{R})$ and there exists a constant $C>0$ such that

$$|h(t)|\leqslant C\left(|t|+|t|^{2^*-1}\right)\quad\text{for } t\in\mathbb{R}.$$

$(A1)$ $\eta:=\limsup_{|t|\to 0} G(t)/|t|^{2+\frac{4}{N}}<+\infty$.

$(A2)$ $\lim_{|t|\to\infty} G(t)/|t|^{2_*}=+\infty$.

$(A3)$ $\lim_{|t|\to\infty} G(t)/|t|^{2^*}=0$.

$(A4)$ $(2+\frac{4}{N})H(t)\leqslant h(t)t$ for any $t\in\mathbb{R}$.

$(A5)$ $(2+\frac{4}{N})G(t)\leqslant g(t)t\leqslant 2^*G(t)$ for any $t\in\mathbb{R}$.

$(A6)$ $H(\zeta_0)>0$ for some $\zeta_0\neq 0$.

We point out that $(A1)$ allows G to admit L^2-critical growth $G(t)\sim|t|^{2+\frac{4}{N}}$ at 0, but $(A2)$ rules out the same behavior at infinity. Furthermore, $(A2)$ allows G to possess L^2-supercritical growth, and $(A3)$ allows G to have Sobolev subcritical growth at infinity. Alternatively, we need the following relation:

Let $f_1,f_2\in C(\mathbb{R}^N,\mathbb{R})$. Then $f_1\preceq f_2$ if and only if $f_1\leqslant f_2$ and for each $\varepsilon>0$ there exists $x\in\mathbb{R}^N$ with $|x|<\varepsilon$ such that $f_1(x)<f_2(x)$.

Hence, the pure L^2-critical case for $|t|$ small is ruled out by $(A4,\preceq)$ or the first part of $(A5,\preceq)$.

Clearly, u solves (P_m), then u satisfies the following Pohozaev identity

$$\begin{aligned}P(u):&=\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{N}{2}\int_{\mathbb{R}^N}[g(u)u-2G(u)]\,dx-\int_{\mathbb{R}^N}|u|^{2^*}dx\\&=\int_{\mathbb{R}^N}|\nabla u|^2dx-\frac{N}{2}\int_{\mathbb{R}^N}H(u)\,dx-\int_{\mathbb{R}^N}|u|^{2^*}dx.\end{aligned}$$

For simplicity, we set

$$\mathcal{P}:=\left\{u\in H^1\left(\mathbb{R}^N\right)\setminus\{0\}:P(u)=0\right\},$$

and

$$\mathcal{D}:=\left\{u\in H^1\left(\mathbb{R}^N\right):\|u\|_2\leqslant m\right\}.$$

The constraint $\mathcal{P}$ contains all the nontrivial solutions to (P_m) and does not depend on λ. And any nontrivial (normalized) solution to (P_m) belongs to $\mathcal{S}\cap\mathcal{P}\subset\mathcal{D}\cap\mathcal{P}$. Hence, if u solves (P_m) and $I(u)=\inf_{\mathcal{S}\cap\mathcal{P}} I$, then u is a normalized ground state solution to (P_m). We have the following theorems.

Theorem 1.1. *Let* $(A0)$–$(A6)$ *hold and*

$$2^*\eta C_{N,2_*}^{2_*}m^{\frac{4}{N}}<1. \tag{1.5}$$

Then there exists $u\in\mathcal{D}\cap\mathcal{P}$ *such that* $I(u)=\inf_{\mathcal{D}\cap\mathcal{P}} I>0$*, and if, in addition, g is odd, then u is radially symmetric.*

(i) Suppose that $g(t)t\preceq 2^*G(t)$ *in* $(A5)$ *holds. Then* $\inf_{\mathcal{D}\cap\mathcal{P}} I=\inf_{\mathcal{S}\cap\mathcal{P}} I$.

(1) *If* $g'(t) = o(1)$ *as* $t \to 0$*, then* $u \in \mathcal{S} \cap \mathcal{P}$ *is a normalized ground state solution to* (P_m).
(2) *If* g *is odd, then* $u \in \mathcal{S} \cap \mathcal{P}$ *is a positive and radially symmetric normalized ground state solution to* (P_m).

(ii) *Suppose that* g *is odd and* $N \in \{3,4\}$*. Then* $\inf_{\mathcal{D} \cap \mathcal{P}} I = \inf_{\mathcal{S} \cap \mathcal{P}} I$*. Alternatively, if* $g'(t) = o(1)$ *as* $t \to 0$*, then* $u \in \mathcal{S} \cap \mathcal{P}$ *is a positive and radially symmetric normalized ground state solution to* (P_m).

To study the asymptotic behavior of the ground state energy map $m \mapsto \inf_{\mathcal{S} \cap \mathcal{P}} I$, for any $m > 0$, let us set

$$\mathcal{D}_m := \left\{u \in H^1\left(\mathbb{R}^N\right) : \|u\|_2 \leqslant m\right\}.$$

and

$$\mathcal{S}_m := \left\{u \in H^1\left(\mathbb{R}^N\right) : \|u\|_2 = m\right\}.$$

Then, the following theorem holds.

Theorem 1.2. *Let* $(A0)$–$(A6)$ *and* (1.5) *hold,*

(i) *the ground state energy map* $m \mapsto \inf_{S_m \cap \mathcal{P}} I$ *is strictly decreasing;*
(ii) *if* $\eta = 0$,
(1) *the map* $m \mapsto \inf_{S_m \cap \mathcal{P}} I$ *is continuous;*
(2) *in addition, if* $\lim_{t \to 0} \frac{G(t)}{|t|^{2^*}} = +\infty$*, then* $\inf_{S_m \cap \mathcal{P}} I \to 0^+$ *as* $m \to +\infty$.

Remark 1.3. It is easy to see that $(A5)$ are weaker than (g_1) that has been widely used, see [2, 7, 8, 15].

Remark 1.4. Suppose that $(A5, \preceq)$ holds or $\frac{4}{N} G(t) \preceq H(t)$ for $t \in R$, Bieganowski and Mederski [5] obtained a corresponding result to Sobolev subcritical normalized problem

$$\begin{cases} -\Delta u + \lambda u = g(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = m^2. \end{cases}$$

In our paper, we do not need the condition $\frac{4}{N} G(t) \preceq H(t)$ for $t \in R$. Furthermore, we consider Sobolev critical case which can cause the lack of compactness.

Remark 1.5. Suppose that g satisfies $(A0)$, $(A1)$ with $\eta = 0$, $(A2)$ and $(A3)$ and the second part of $(A5)$, and the following conditions

$(\widetilde{A}4)$ $t \mapsto \frac{H(t)}{|t|^{2+\frac{4}{N}}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$.
$(A7)$ $\lim_{t \to 0} \frac{g(t)t}{|t|^{2^*}} = +\infty$,

Jeanjean and Lu [16] considered the following Sobolev subcritical problem

$$\begin{cases} -\Delta u + \lambda u = g(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = m^2, \\ u \in H^1\left(\mathbb{R}^N\right) \end{cases}$$

and obtained the existence of ground states and the limit behavior of the ground state energy as $m > 0$ varies. Our results extend their results from Sobolev subcritical case to Sobolev critical case that is much more challenging and less straightforward. Furthermore, Jeanjean and Lu [16] proposed the following open problem:

(Q_2) Does there exist an alternative approach, not relying on the sign of the Lagrange multiplier, to give more general existence results?

In our paper, the existence of normalized ground state solution to (P_m) is independent of the sign of the Lagrange multiplier. Consequently, our result can answer the above open problem proposed by Jeanjean and Lu [16].

Remark 1.6. We point out that $(\widetilde{A}4)$ is weaker than (g_2), and $(A5)$ is weaker than (g_1). By $(A0)$, $(A1)$ with $\eta = 0$, $(A2)$, $(\widetilde{A}4)$ and lemma 2.3 in [16], we can deduce that

$$g(t)t > \left(2 + \frac{4}{N}\right) G(t) > 0, \ \ \forall t \in \mathbb{R}\setminus\{0\},$$

which is weaker than the first part of (g_1) that is a technical and essential condition, see [22].

Remark 1.7. (i) Usually, we can take the space of radial functions $H^1_{rad}(\mathbb{R}^N)$ as the working space to overcome compactness that is caused by the whole space $\mathbb{R}^N$, and we can use the concentration-compactness principle to overcome compactness that is caused by the Sobolev critical exponent. In this paper, we neither work in $H^1_{rad}(\mathbb{R}^N)$ nor use the concentration-compactness principle. Alternatively, we do not need to work with Palais-Smale sequences. Consequently, we avoid the mini-max approach in $\mathcal{P}$ that has been recently intensively used by many authors (see [6–8, 16, 24, 30, 31]). Which are necessary in [8, 16]. A minimizing sequences of I on $\mathcal{D}\cap\mathcal{P}$ is directly considered. Which seen impossible to mass critical and supercritical cases for a long time.

(ii) Profile decomposition is used to overcome the lack of the compactness which is caused by the Sobolev critical exponent, but its calculation is complex.

2. Some lemmas

To begin with, we give some estimates. By $(A0)$, $(A1)$, $(A3)$ and $(A5)$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$H(t) \leqslant (2^*-2)G(t) \leqslant (2^*-2)\left(\varepsilon|t|^{2^*} + (\varepsilon+\eta)|t|^{2+\frac{4}{N}} + C_\varepsilon|t|^q\right) \tag{2.1}$$

for any $t \in \mathbb{R}$, where $q \in [2, 2^*]$. Alternatively, it follows from $(A5)$ that

$$G(t),\ H(t) \geqslant 0, \ \ \forall t \in \mathbb{R}. \tag{2.2}$$

Let S be the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, i.e.

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

For $q \in (2, 2^*]$, the following Gagliardo-Nirenberg inequality holds:

$$\|u\|_q \leqslant C_{N,q}\|u\|_2^{1-\delta_q}\|\nabla u\|_2^{\delta_q}, \quad \forall u \in H^1\left(\mathbb{R}^N\right),$$

where $\delta_q = N(\frac{1}{2} - \frac{1}{q})$, $C_{N,q} > 0$. Especially, when $q = 2^*$, it is easy to see that $C^2_{N,2^*} = S^{-1}$.

By $(A6)$ and arguing as in [3], for any $R > 1$ there exists a radial function $u \in H^1_0(B(0,R)) \cap L^\infty(B(0,R))$ such that $\int_{\mathbb{R}^N} H(u)dx > 0$.

Set

$$r^2(u) := \frac{\frac{N}{2}\int_{\mathbb{R}^N} H(u)\,dx + \int_{\mathbb{R}^N}|u|^{2^*}dx}{\int_{\mathbb{R}^N}|\nabla u|^2 dx}.$$

Then $u(r(u)\cdot) \in \mathcal{P}$, so $\mathcal{P} \neq \emptyset$

Lemma 2.1. *Assume that* $(A0)$, $(A1)$, $(A3)$, $(A5)$, $(A6)$ *and* (1.5) *hold. Then*

$$\inf_{\mathcal{D}\cap\mathcal{P}} \|\nabla u\|_2 > 0.$$

Proof. Taking $2+\frac{4}{N}<p<2^*$. For any $u\in\mathcal{D}\cap\mathcal{P}\subset\mathcal{D}$, by virtue of the Gagliardo-Nirenberg inequality we have

$$\|u\|_t \leqslant C_{N,t}\|\nabla u\|_2^{\delta_t}\|u\|_2^{1-\delta_t} \leqslant C_{N,t}m^{1-\delta_t}\|\nabla u\|_2^{\delta_t}, \tag{2.3}$$

where $\delta_t = N(\frac{1}{2}-\frac{1}{t})$, $t\in(2,2^*]$. Consequently, using (2.1)–(2.3) we derive that

$$\begin{aligned}
\|\nabla u\|_2^2 =& \frac{N}{2}\int_{\mathbb{R}^N}[g(u)u-2G(u)]\,dx+\int_{\mathbb{R}^N}|u|^{2^*}dx\\
=&\frac{N}{2}\int_{\mathbb{R}^N}H(u)\,dx+\int_{\mathbb{R}^N}|u|^{2^*}dx\\
\leqslant&\frac{N}{2}(2^*-2)\int_{\mathbb{R}^N}\left[\varepsilon|u|^{2^*}+(\varepsilon+\eta)|u|^{2+\frac{4}{N}}+C_\varepsilon|u|^p\right]dx+\int_{\mathbb{R}^N}|u|^{2^*}dx\\
=&(2^*\varepsilon+1)\|u\|_{2^*}^{2^*}+2^*(\varepsilon+\eta)\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}}+2^*C_\varepsilon\|u\|_p^p\\
\leqslant&(2^*\varepsilon+1)S^{-\frac{2^*}{2}}\|\nabla u\|_2^{2^*}+2^*(\varepsilon+\eta)C_{N,2_*}^{2_*}\cdot m^{\frac{4}{N}}\|\nabla u\|_2^2\\
&+2^*C_\varepsilon C_{N,p}^p m^{p(1-\delta_p)}\|\nabla u\|_2^{p\delta_p}.
\end{aligned}$$

Taking $\varepsilon<\frac{1-2^*\eta C_{N,2_*}^{2_*}m^{\frac{4}{N}}}{2^*C_{N,2_*}^{2_*}m^{\frac{4}{N}}}$ and using (1.5), we can obtain that there exists a constant $C>0$ such that $\|\nabla u\|_2\geqslant C$, since $p\delta_p = pN(\frac{1}{2}-\frac{1}{p}) = \frac{N(p-2)}{2}>2$. □

Let $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfy

$$2\eta C_{N,2_*}^{2_*}\left(\int_{\mathbb{R}^N}u^2dx\right)^{\frac{2}{N}}<1. \tag{2.4}$$

For any $\lambda>0$, set $\varphi(\lambda):=I(\lambda^{\frac{N}{2}}u(\lambda\cdot))$. We have the following lemma.

Lemma 2.2. *Assume that* $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ *satisfies* (2.4) *and* $(A0)$ *and* $(A1)$ *and* $(A3)$–$(A6)$ *hold. Then there exists an interval* $[a,b]\subset(0,+\infty)$ *such that any* $\lambda\in[a,b]$ *is a global maximizer for* φ *and* φ *is strictly increasing on* $(0,a)$ *and strictly decreasing on* $(b,+\infty)$. *Furthermore,* $P(\lambda^{\frac{N}{2}}u(\lambda\cdot))=0$ *if and only if* $\lambda\in[a,b]$, $P(\lambda^{\frac{N}{2}}u(\lambda\cdot))>0$ *if and only if* $\lambda\in(0,a)$, *and* $P(\lambda^{\frac{N}{2}}u(\lambda\cdot))<0$ *if and only if* $\lambda\in(b,+\infty)$.

Proof. Let $u\in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfy (2.4). By $(A1)$ we have

$$\begin{aligned}
\varphi(\lambda) =& I\left(\lambda^{\frac{N}{2}}u(\lambda x)\right)\\
=&\frac{1}{2}\int_{\mathbb{R}^N}\left|\nabla\left[\lambda^{\frac{N}{2}}u(\lambda x)\right]\right|^2dx-\int_{\mathbb{R}^N}G\left(\lambda^{\frac{N}{2}}u(\lambda x)\right)dx-\frac{1}{2^*}\int_{\mathbb{R}^N}|\lambda^{\frac{N}{2}}u(\lambda x)|^{2^*}dx\\
=&\frac{1}{2}\lambda^2\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}\frac{G\left(\lambda^{\frac{N}{2}}u\right)}{\lambda^N}dx-\frac{1}{2^*}\lambda^{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx\\
\to& 0
\end{aligned}$$

as $\lambda \to 0^+$, and from (2.2) one has $\varphi(\lambda) \to -\infty$ as $\lambda \to +\infty$. Set $R := \|u\|_2^2 = \|\lambda^{\frac{N}{2}} u(\lambda\cdot)\|_2^2 > 0$. By using of (2.1), Sobolev embedding theorem and the Gagliardo–Nirenberg inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} G(u)\,dx &\leqslant (\varepsilon+\eta)\,\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + C_\varepsilon \|u\|_{2^*}^{2^*} \\ &\leqslant (\varepsilon+\eta)\,C_{N,2_*}^{2_*} R^{\frac{2}{N}} \|\nabla u\|_2^2 + S^{-\frac{2^*}{2}} C_\varepsilon \|\nabla u\|_2^{2^*}.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\varphi(\lambda)}{\lambda^2} =& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{\lambda^2}\int_{\mathbb{R}^N} G\left(\lambda^{\frac{N}{2}} u(\lambda x)\right) dx - \frac{1}{2^*}\lambda^{2^*-2}\int_{\mathbb{R}^N} |u|^{2^*} dx \\ \geqslant& \frac{1}{2}\|\nabla u\|_2^2 - (\varepsilon+\eta)\,C_{N,2_*}^{2_*} R^{\frac{2}{N}} \|\nabla u\|_2^2 - S^{-\frac{2^*}{2}} C_\varepsilon \lambda^{2^*-2} \|\nabla u\|_2^{2^*} - \frac{1}{2^*}\lambda^{2^*-2}\|u\|_{2^*}^{2^*} \\ =& \frac{1}{2}\|\nabla u\|_2^2 \left[1 - 2(\varepsilon+\eta)\,C_{N,2_*}^{2_*} R^{\frac{2}{N}}\right] - S^{-\frac{2^*}{2}} C_\varepsilon \lambda^{2^*-2} \|\nabla u\|_2^{2^*} - \frac{1}{2^*}\lambda^{2^*-2}\|u\|_{2^*}^{2^*}.\end{aligned}$$

Combining with (2.4) we get $\varphi(\lambda) > 0$ for sufficiently small $\varepsilon > 0$ and $\lambda > 0$. Therefore, there exists an interval $[a,b] \subset (0,+\infty)$ such that $\varphi|_{[a,b]} = \max \varphi$. It is easy to see that

$$\varphi'(\lambda) = \lambda\left[\|\nabla u\|_2^2 - \frac{N}{2}\lambda^{-N-2}\int_{\mathbb{R}^N} H\left(\lambda^{\frac{N}{2}} u\right) dx - \lambda^{2^*-2}\|u\|_{2^*}^{2^*}\right],$$

and the function

$$\lambda \in (0,+\infty) \mapsto \frac{N}{2}\lambda^{-N-2}\int_{\mathbb{R}^N} H\left(\lambda^{\frac{N}{2}} u\right) dx + \lambda^{2^*-2}\|u\|_{2^*}^{2^*}$$

is strictly increasing by $(A4)$ and tends to $+\infty$ as $\lambda \to +\infty$. It follows that $\varphi'(\lambda) > 0$ if $\lambda \in (0,a)$ and $\varphi'(\lambda) < 0$ if $\lambda \in (b,+\infty)$. Noting that

$$\lambda\varphi'(\lambda) = \lambda^2\|\nabla u\|_2^2 - \frac{N}{2}\lambda^{-N}\int_{\mathbb{R}^N} H\left(\lambda^{\frac{N}{2}} u\right) dx - \lambda^{2^*}\|u\|_{2^*}^{2^*} = P\left(\lambda^{\frac{N}{2}} u(\lambda\cdot)\right),$$

which easily yields that the rest conclusions hold. □

Lemma 2.3. *Assume that* $(A0)$ *and* $(A1)$, $(A3)$–$(A6)$ *and* (1.5) *hold. Then* I *is coercive on* $\mathcal{D} \cap \mathcal{P}$.

Proof. For any $u \in \mathcal{D} \cap \mathcal{P}$, by $(A5)$ we get

$$\begin{aligned}I(u) =& I(u) - \frac{1}{2}P(u) \\ =& \frac{N}{4}\int_{\mathbb{R}^N}\left[g(u)u - \left(2+\frac{4}{N}\right)G(u)\right] dx + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} |u|^{2^*} dx \\ \geqslant& 0.\end{aligned} \tag{2.5}$$

Then I is bounded from below on $\mathcal{D} \cap \mathcal{P}$. Argument by indirection. Suppose that there exist a sequence $\{u_n\} \subset \mathcal{D} \cap \mathcal{P}$ and a positive number $d > 0$ such that $\|u_n\| \to +\infty$ and $I(u_n) \leqslant d$. By the definition of $\mathcal{D}$ we see that $\|\nabla u_n\|_2^2 \to +\infty$ as $n \to \infty$. Set $\lambda_n := \frac{1}{\|\nabla u_n\|_2} \to 0^+$ as $n \to \infty$ and $v_n(\cdot) = \lambda_n^{\frac{N}{2}} u_n(\lambda_n \cdot)$. Then

$$\int_{\mathbb{R}^N} |v_n|^2 dx = \lambda_n^N \int_{\mathbb{R}^N} |u_n(\lambda_n x)|^2 dx = \int_{R^N} |u_n|^2 dx \leqslant m^2$$

and

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lambda_n^2 \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \frac{1}{\|\nabla u_n\|_2^2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1.$$

Consequently, $v_n \in \mathcal{D}$ and $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $v_n \neq 0$, there exists a point $y \in \mathbb{R}^N$ such that

$$\int_{B_1(y)} |v_n|^{2^*} dx := \eta > 0.$$

Set $l(z) := \int_{B_1(z)} |v_n|^{2^*} dx$. It follows from the integral absolute continuity that $l(z)$ is continuous on $\mathbb{R}^N$. Take a large $R > 0$ with $\int_{\mathbb{R}^N \setminus B_R(0)} |v_n|^{2^*} dx < \eta$. Then, for any $z \in \mathbb{R}^N \setminus B_{R+1}(0)$,

$$l(z) = \int_{B_1(z)} |v_n|^{2^*} dx < \eta.$$

As a consequence,

$$\sup_{z\in\mathbb{R}^N} l(z) = \sup_{z\in\overline{B}_{R+1}(0)} l(z).$$

Combining the continuity of $l(\cdot)$ and the compactness of $\overline{B}_{R+1}(0)$ that we can conclude that there exists $y_n \in \overline{B}_{R+1}(0)$ such that $l(y_n) = \sup_{z\in\overline{B}_{R+1}(0)} l(z)$. Consequently,

$$\int_{B_1(y_n)} |v_n|^{2^*} dx = \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |v_n|^{2^*} dx.$$

Then,

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^{2^*} dx > 0.$$

Otherwise,

$$\lim_{n\to\infty} \int_{B_1(y_n)} |v_n|^{2^*} dx = 0.$$

Lemma 3.8 in [9] yields that $v_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$. Making use of the interpolation inequality, we obtain $v_n \to 0$ in $L^{2+\frac{4}{N}}(\mathbb{R}^N)$, which together with (2.1) and (2.2) implies that

$$\int_{\mathbb{R}^N} G\left(\lambda^{\frac{N}{2}} v_n(\lambda x)\right) dx \to 0$$

as $n \to \infty$ for any $\lambda > 0$. Noting that for any $u \in \mathcal{D} \cap \mathcal{P}$, by (1.5), u clearly satisfies the inequality (2.4). Hence, by lemma 2.2 we deduce that

$$\begin{aligned}
d \geqslant & I(u_n) \geqslant I\left(\lambda^{\frac{N}{2}} v_n(\lambda x)\right) \\
= & \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \left[\lambda^{\frac{N}{2}} v_n(\lambda x)\right]|^2 dx - \int_{\mathbb{R}^N} G\left(\lambda^{\frac{N}{2}} v_n(\lambda x)\right) dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |\lambda^{\frac{N}{2}} v_n(\lambda x)|^{2^*} dx \\
= & \frac{1}{2}\lambda^2 \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} G\left(\lambda^{\frac{N}{2}} v_n(\lambda x)\right) dx - \frac{1}{2^*}\lambda^{2^*} \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\
= & \frac{1}{2}\lambda^2 + o(1).
\end{aligned}$$

This is impossible for large λ. As a result, by (2.2) and (2.5) we deduce that

$$\begin{aligned}0 &\leqslant \lambda_n^{2^*} I(u_n) = \lambda_n^{2^*}\left[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 dx - \int_{\mathbb{R}^N} G(u_n)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*} dx\right]\\ &\leqslant \frac{1}{2}\lambda_n^{2^*}\|\nabla u_n\|_2^2 - \frac{1}{2^*}\lambda_n^{2^*}\int_{\mathbb{R}^N}|u_n|^{2^*}dx\\ &= \frac{1}{2}\lambda_n^{2^*-2} - \frac{1}{2^*}\int_{\mathbb{R}^N}|v_n|^{2^*}dx\\ &\leqslant \frac{1}{2}\lambda_n^{2^*-2} - \frac{1}{2^*}\int_{B_1(y_n)}|v_n|^{2^*}dx\\ &< 0\end{aligned}$$

for large n, a contradiction. □

Lemma 2.4. *Assume that* (A0) *and* (A1), (A3)–(A6) *and* (1.5) *hold. Then* $c := \inf_{\mathcal{D}\cap\mathcal{P}} I > 0$.

Proof. For any $u \in \mathcal{D}$, by (1.5) and (2.1) and the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}&\int_{\mathbb{R}^N} G(u)\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx\\ &\quad\leqslant (\varepsilon+\eta)\|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + (C_\varepsilon+1)\|u\|_{2^*}^{2^*}\\ &\quad\leqslant (\varepsilon+\eta)\,C_{N,2_*}^{2_*} m^{\frac{4}{N}}\|\nabla u\|_2^2 + (C_\varepsilon+1)\,S^{-\frac{2^*}{2}}\|\nabla u\|_2^{2^*}\\ &\quad\leqslant \left[\varepsilon C_{N,2_*}^{2_*} m^{\frac{4}{N}} + (C_\varepsilon+1)\,S^{-\frac{2^*}{2}}\|\nabla u\|_2^{\frac{4}{N-2}} + \eta C_{N,2_*}^{2_*} m^{\frac{4}{N}}\right]\|\nabla u\|_2^2\\ &\quad\leqslant \left[\varepsilon C_{N,2_*}^{2_*} m^{\frac{4}{N}} + (C_\varepsilon+1)\,S^{-\frac{2^*}{2}}\|\nabla u\|_2^{\frac{4}{N-2}} + \frac{1}{2^*}\right]\|\nabla u\|_2^2.\end{aligned}$$

Taking

$$\varepsilon = \frac{1}{4NC_{N,2_*}^{2_*} m^{\frac{4}{N}}} > 0 \text{ and } \delta = \left[\frac{1}{4N(C_\varepsilon+1)\,S^{-\frac{2^*}{2}}}\right]^{\frac{N-2}{4}} > 0,$$

then when $\|\nabla u\|_2 \leqslant \delta$ we have

$$\int_{\mathbb{R}^N} G(u)\,dx + \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx \leqslant \left(\frac{1}{4N}+\frac{1}{4N}+\frac{1}{2^*}\right)\|\nabla u\|_2^2 = \left(\frac{1}{2}-\frac{1}{2N}\right)\|\nabla u\|_2^2.$$

Then, when $\|\nabla u\|_2 \leqslant \delta$,

$$\begin{aligned}I(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} G(u)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx\\ &\geqslant \frac{1}{2}\|\nabla u\|_2^2 - \left(\frac{1}{2}-\frac{1}{2N}\right)\|\nabla u\|_2^2\\ &= \frac{1}{2N}\|\nabla u\|_2^2.\end{aligned}$$

Again noting that for any $u \in \mathcal{D}\cap\mathcal{P}$, by (1.5), u clearly satisfies the inequality (2.4). Consequently, by lemma 2.2,

$$I(u) \geqslant I\left(\lambda^{\frac{N}{2}} u(\lambda x)\right) := I(v(\cdot))$$

for any $\lambda > 0$, where $v(\cdot) := \lambda^{\frac{N}{2}} u(\lambda \cdot)$. Especially, taking $\lambda = \frac{\delta}{\|\nabla u\|_2}$. It is easy to see that $\|v\|_2 = \|u\|_2$, $\|\nabla v\|_2 = \delta$. Hence, from the above two inequalities we obtain

$$I(u) \geqslant I(v) \geqslant \frac{1}{2N}\|\nabla v\|_2^2 = \frac{1}{2N}\delta^2 > 0.$$

Therefore, $c = \inf_{u \in \mathcal{D}\cap\mathcal{P}} I(u) \geqslant \frac{1}{2N}\delta^2 > 0.$ □

Lemma 2.5. *Assume that $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ satisfies (2.4) and (A0)–(A6) hold. Then,*

$$c = \inf_{\mathcal{D}\cap\mathcal{P}} I < \frac{1}{N} S^{\frac{N}{2}}.$$

Proof. It is well known to us that the Aubin-Talanti babble

$$U_\varepsilon(x) = [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}$$

is the unique solution to the following equation:

$$\begin{cases} -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(0) = \max_{x\in\mathbb{R}^N} u(x), \\ u(x) > 0, \\ u(x) \to 0 \text{ as } |x| \to +\infty, \end{cases}$$

and satisfies

$$\|\nabla U_\varepsilon\|_2^2 = \|U_\varepsilon\|_{2^*}^{2^*} = S^{\frac{N}{2}}.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\varphi(x) \equiv 1$ for $|x| \leqslant 1$ and $\varphi(x) \equiv 0$ for $|x| \geqslant 2$. Set

$$v_\varepsilon(x) = \varphi(x) U_\varepsilon(x).$$

Then by [10, 19] we get that

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = S^{\frac{N}{2}} + O\left(\varepsilon^{N-2}\right),$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O\left(\varepsilon^N\right),$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^2 dx = \begin{cases} C_1\varepsilon^2 + O\left(\varepsilon^{N-2}\right), & N \geqslant 5, \\ C_1\varepsilon^2|\ln\varepsilon| + O\left(\varepsilon^2\right), & N = 4, \\ C_1\varepsilon, & N = 3, \end{cases}$$

where $C_1 > 0$. Set

$$u_\varepsilon(x) = \left(m^{-1}\|v_\varepsilon\|_2\right)^{\frac{N-2}{2}} v_\varepsilon\left(m^{-1}\|v_\varepsilon\|_2 x\right).$$

Then it is easy to see that $u_\varepsilon \in \mathcal{S}$ and

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx = S^{\frac{N}{2}} + O\left(\varepsilon^{N-2}\right),$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx = \int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O\left(\varepsilon^N\right)$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N}|u_\varepsilon|^{2_*}dx &= \left(m^{-1}\|v_\varepsilon\|_2\right)^{-\frac{4}{N}}\int_{\mathbb{R}^N}|v_\varepsilon|^{2_*}dx\\
&\geqslant \left(m^{-1}\|v_\varepsilon\|_2\right)^{-\frac{4}{N}}[N(N-2)]^{\frac{N^2-4}{2N}}\int_{B_1(0)}\left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{N^2-4}{N}}dx\\
&= \left(m^{-1}\|v_\varepsilon\|_2\right)^{-\frac{4}{N}}[N(N-2)]^{\frac{N^2-4}{2N}}\varepsilon^{\frac{4}{N}}\int_{B_{\frac{1}{\varepsilon}}(0)}\left(\frac{1}{1+|x|^2}\right)^{\frac{N^2-4}{N}}dx\\
&\geqslant C_2\left(m^{-1}\|v_\varepsilon\|_2\right)^{-\frac{4}{N}}\varepsilon^{\frac{4}{N}}\\
&= C_2 m^{\frac{4}{N}}\varepsilon^{\frac{4}{N}}\|v_\varepsilon\|_2^{-\frac{4}{N}}\\
&\geqslant \frac{1}{2}C_2C_1^{-\frac{2}{N}}m^{\frac{4}{N}}\cdot\begin{cases}1, & N\geqslant 5,\\ \left|\ln\varepsilon\right|^{-\frac{2}{N}}, & N=4,\\ \varepsilon^{\frac{2}{N}}, & N=3\end{cases}
\end{aligned}$$

for $\varepsilon>0$ small enough. It follows from lemma 2.2 that there exists $\lambda_\varepsilon>0$ such that $\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\in\mathcal{P}$. Since $u_\varepsilon\in\mathcal{S}$, $\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\in\mathcal{S}\cap\mathcal{P}\subset\mathcal{D}\cap\mathcal{P}$.

We claim that there exist two constants $A_1,A_2\in\mathbb{R}^+$ independent of ε such that $A_1\leqslant\lambda_\varepsilon\leqslant A_2$. Indeed, by lemma 2.4 and (2.2) we know that

$$\begin{aligned}
0<c&\leqslant I\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)\\
&=\frac{1}{2}\lambda_\varepsilon^2\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2dx-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx-\frac{1}{2^*}\lambda_\varepsilon^{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx\\
&\leqslant\frac{1}{2}\lambda_\varepsilon^2\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2dx-\frac{1}{2^*}\lambda_\varepsilon^{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx,
\end{aligned}$$

which yields that the assertion holds.

As a consequence,

$$\begin{aligned}
c&=\inf_{\mathcal{D}\cap\mathcal{P}}I\leqslant I\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)\\
&=\frac{1}{2}\lambda_\varepsilon^2\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2dx-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx-\frac{1}{2^*}\lambda_\varepsilon^{2^*}\int_{\mathbb{R}^N}|u_\varepsilon|^{2^*}dx\\
&=\frac{1}{2}\lambda_\varepsilon^2\left(S^{\frac{N}{2}}+O\left(\varepsilon^{N-2}\right)\right)-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx-\frac{1}{2^*}\lambda_\varepsilon^{2^*}\left(S^{\frac{N}{2}}+O\left(\varepsilon^N\right)\right)\\
&\leqslant\left(\frac{1}{2}\lambda_\varepsilon^2-\frac{1}{2^*}\lambda_\varepsilon^{2^*}\right)S^{\frac{N}{2}}+\frac{1}{2}A_2^2O\left(\varepsilon^{N-2}\right)-\frac{1}{2^*}A_1^{2^*}O\left(\varepsilon^N\right)-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx\\
&\leqslant\sup_{\tau\in R}\left(\frac{1}{2}\tau^2-\frac{1}{2^*}\tau^{2^*}\right)S^{\frac{N}{2}}+\frac{1}{2}A_2^2O\left(\varepsilon^{N-2}\right)-\frac{1}{2^*}A_1^{2^*}O\left(\varepsilon^N\right)-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx\\
&=\frac{1}{N}S^{\frac{N}{2}}+\frac{1}{2}A_2^2O\left(\varepsilon^{N-2}\right)-\frac{1}{2^*}A_1^{2^*}O\left(\varepsilon^N\right)-\int_{\mathbb{R}^N}G\left(\lambda_\varepsilon^{\frac{N}{2}}u_\varepsilon(\lambda_\varepsilon\cdot)\right)dx
\end{aligned}$$

By $(A2)$ and (2.2), for any $M>0$, there exists $C_M>0$ such that

$$G(t)\geqslant M|t|^{2_*}-C_M|t|^2,\quad\forall t\in R.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^N} G\left(\lambda_\varepsilon^{\frac{N}{2}} u_\varepsilon(\lambda_\varepsilon x)\right) dx \geqslant & M\int_{\mathbb{R}^N} |\lambda_\varepsilon^{\frac{N}{2}} u_\varepsilon(\lambda_\varepsilon x)|^{2_*} dx - C_M \int_{\mathbb{R}^N} |\lambda_\varepsilon^{\frac{N}{2}} u_\varepsilon(\lambda_\varepsilon x)|^2 dx \\ = & M\lambda_\varepsilon^2 \int_{\mathbb{R}^N} |u_\varepsilon|^{2_*} dx - C_M \int_{\mathbb{R}^N} |u_\varepsilon|^2 dx \\ \geqslant & MA_1^2 \int_{\mathbb{R}^N} |u_\varepsilon|^{2_*} dx - C_M \int_{\mathbb{R}^N} |u_\varepsilon|^2 dx \\ \geqslant & \frac{1}{2} MA_1^2 C_2 C_1^{-\frac{2}{N}} m^{\frac{4}{N}} \cdot \begin{cases} 1, & N \geqslant 5, \\ \left|\ln\varepsilon\right|^{-\frac{2}{N}}, & N=4, \\ \varepsilon^{\frac{2}{N}}, & N=3 \end{cases} - C_M m^2.\end{aligned}$$

Consequently,

$$\begin{aligned} c \leqslant & \frac{1}{N} S^{\frac{N}{2}} + \frac{1}{2} A_2^2 O\left(\varepsilon^{N-2}\right) - \frac{1}{2^*} A_1^{2^*} O\left(\varepsilon^N\right) \\ & - \frac{1}{2} MA_1^2 C_2 C_1^{-\frac{2}{N}} m^{\frac{4}{N}} \cdot \begin{cases} 1, & N \geqslant 5, \\ \left|\ln\varepsilon\right|^{-\frac{2}{N}}, & N=4, \\ \varepsilon^{\frac{2}{N}}, & N=3 \end{cases} + C_M m^2 \\ < & \frac{1}{N} S^{\frac{N}{2}} \end{aligned}$$

for $\varepsilon > 0$ small enough and M big enough. □

Theorem 2.6 ([25] Profile decomposition). *Assume that* $\{u_n\}$ *is bounded in* $H^1(\mathbb{R}^N)$. *Then, there exist sequences* $\{\tilde{u}_i\}_{i=0}^{\infty} \subset H^1(\mathbb{R}^N)$, $\{y_n^i\}_{i=0}^{\infty} \subset \mathbb{R}^N$ *for any* $n \geqslant 1$, *such that* $y_n^0 = 0$, $|y_n^i - y_n^j| \to +\infty$ *as* $n \to \infty$ *for* $i \neq j$, *and up to a subsequence, the following conclusions hold for any* $i \geqslant 0$:

$$u_n\left(\cdot + y_n^i\right) \rightharpoonup \tilde{u}_i \text{ in } H^1\left(\mathbb{R}^N\right) \text{ as } n \to \infty, \tag{2.6}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 = \sum_{j=0}^{i} \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n^i|^2 dx, \tag{2.7}$$

where $v_n^i(\cdot) := u_n - \sum_{j=0}^{i} \tilde{u}_j(\cdot - y_n^j)$ *and*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} H(u_n)\, dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^N} H(\tilde{u}_j)\, dx. \tag{2.8}$$

Lemma 2.7. *Assume that* (A0)–(A6) *and* (1.5) *hold. Then* $c = \inf_{\mathcal{D}\cap\mathcal{P}} I$ *is attained. If, in addition, g is odd, then c is attained by a nonnegative and radially symmetric function in* $\mathcal{D}\cap\mathcal{P}$.

Proof. Let $\{u_n\} \subset \mathcal{D}\cap\mathcal{P}$ be such that $I(u_n) \to c$ as $n \to \infty$. By lemma 2.3, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. With the aid of theorem 2.6, we can find a profile decomposition of $\{u_n\}$ satisfying (2.6)–(2.8). Set $\mathcal{A} := \{i \geqslant 0 : \tilde{u}_i \neq 0\}$. If $\mathcal{A} = \emptyset$, namely, for any $i \geqslant 0$, $\tilde{u}_i = 0$. By (2.2) and (2.8),

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} H(u_n)\, dx = 0.$$

Since $u_n \in \mathcal{D} \cap \mathcal{P} \subset \mathcal{P}$,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |u_n|^{2^*} dx = o(1).$$

Up to a subsequence, denote by $l \geqslant 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \to l$$

and

$$\int_{\mathbb{R}^N} |u_n|^{2^*} dx \to l$$

as $n \to \infty$. If $l > 0$, then

$$S \leqslant \frac{\|\nabla u_n\|_2^2}{\|u_n\|_{2^*}^2} \to \frac{l}{l^{\frac{2}{2^*}}} = l^{\frac{2}{N}},$$

so $l \geqslant S^{\frac{N}{2}}$. Together with $(A5)$ we see that

$$\begin{aligned} c &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} G(u_n)\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right] \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) l = \frac{1}{N} l \geqslant \frac{1}{N} S^{\frac{N}{2}}, \end{aligned}$$

a contradiction to lemma 2.5. Consequently, $l = 0$. Which yields that $c = \lim_{n\to\infty} I(u_n) = 0$, a contradiction to lemma 2.4. Hence, $\mathcal{A} \neq \emptyset$.

In the sequel, we assert that for every $i \geqslant 0$ there holds that

$$u_n\left(\cdot + y_n^i\right) \to \tilde{u}_i \tag{2.9}$$

in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ or

$$0 < \int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 dx \leqslant \frac{N}{2}\int_{\mathbb{R}^N} H(\tilde{u}_i)\, dx + \int_{\mathbb{R}^N} |\tilde{u}_i|^{2^*} dx. \tag{2.10}$$

Argument by indirection. Suppose that there exists $i \geqslant 0$ such that

$$\nu := \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla \left[u_n\left(x + y_n^i\right) - \tilde{u}_i(x)\right]|^2 dx > 0$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 dx > \frac{N}{2}\int_{\mathbb{R}^N} H(\tilde{u}_i)\, dx + \int_{\mathbb{R}^N} |\tilde{u}_i|^{2^*} dx, \tag{2.11}$$

where $v_n(\cdot) := u_n(\cdot + y_n^i) - \tilde{u}_i(\cdot)$. Set $F(\cdot) = \frac{N}{2}H(\cdot) + |\cdot|^{2^*}$, with the aid of Vitali's convergence theorem, we know

$$\begin{aligned}\int_{\mathbb{R}^N} [F(u_n) - F(v_n)]\, dx &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} F(u_n - s\tilde{u}_i)\, ds dx \\ &= \int_{\mathbb{R}^N} \int_0^1 f(u_n - s\tilde{u}_i)\, \tilde{u}_i ds dx \\ &\to \int_0^1 \int_{\mathbb{R}^N} f(\tilde{u}_i - s\tilde{u}_i)\, \tilde{u}_i dx ds \\ &= \int_{\mathbb{R}^N} \int_0^1 -\frac{d}{ds} F(\tilde{u}_i - s\tilde{u}_i)\, ds dx \\ &= \int_{\mathbb{R}^N} F(\tilde{u}_i)\, dx\end{aligned}$$

as $n \to \infty$. There holds that

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |\nabla \tilde{u}_i|^2 dx \\ &= \frac{N}{2} \int_{\mathbb{R}^N} H(v_n)\, dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx + \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u}_i)\, dx + \int_{\mathbb{R}^N} |\tilde{u}_i|^{2^*} dx + o(1).\end{aligned}$$

Consequently, by (2.11) we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leqslant \frac{N}{2} \int_{\mathbb{R}^N} H(v_n)\, dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx + o(1). \tag{2.12}$$

Taking

$$R_n^2 = \frac{\frac{N}{2} \int_{\mathbb{R}^N} H(v_n)\, dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx}{\int_{\mathbb{R}^N} |\nabla v_n|^2 dx}.$$

Then, $v_n(R_n \cdot) \in \mathcal{P}$. In the following, we prove that $R_n \to 1$ as $n \to \infty$. We divide the proof into two cases.

Case (i): If

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx > \frac{N}{2} \int_{\mathbb{R}^N} H(v_n)\, dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx$$

holds for a.e. n, then by (2.12) and the fact that $\nu > 0$ we have $R_n \to 1$ as $n \to \infty$.

Case (ii): If, up to a subsequence,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \leqslant \frac{N}{2} \int_{\mathbb{R}^N} H(v_n)\, dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx,$$

then $R_n^2 \geqslant 1$. Noticing that

$$\lim_{n\to\infty} \left(\|u_n\|_2^2 - \|v_n\|_2^2\right) = \|\tilde{u}_i\|_2^2 > 0,$$

so $v_n \in \mathcal{D}$ for a.e. on n and hence $v_n(R_n\cdot) \in \mathcal{D}\cap\mathcal{P}$ for a.e. on n. Therefore, by Brézis-Lieb lemma and $(A5)$ we get

$$\begin{aligned}
c = \inf_{\mathcal{D}\cap\mathcal{P}} I \leqslant I(v_n(R_n\cdot)) = I(v_n(R_n\cdot)) - \frac{1}{2}P(v_n(R_n\cdot))\\
=&\frac{N}{4}\int_{\mathbb{R}^N}\left[g(v_n(R_nx))v_n(R_nx) - \left(2+\frac{4}{N}\right)G(v_n(R_nx))\right]dx\\
&+\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|v_n(R_nx)|^{2^*}dx\\
=&\frac{N}{4}R_n^{-N}\int_{\mathbb{R}^N}\left[g(v_n)v_n - \left(2+\frac{4}{N}\right)G(v_n)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)R_n^{-N}\int_{\mathbb{R}^N}|v_n|^{2^*}dx\\
\leqslant&\frac{N}{4}\int_{\mathbb{R}^N}\left[g(v_n)v_n - \left(2+\frac{4}{N}\right)G(v_n)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|v_n|^{2^*}dx\\
=&\frac{N}{4}\int_{\mathbb{R}^N}\left[g(u_n)u_n - \left(2+\frac{4}{N}\right)G(u_n)\right]dx - \frac{N}{4}\int_{\mathbb{R}^N}\left[g(\tilde{u}_i)\tilde{u}_i - \left(2+\frac{4}{N}\right)G(\tilde{u}_i)\right]dx\\
&+\left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|u_n|^{2^*}dx - \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\tilde{u}_i|^{2^*}dx + o(1)\\
\leqslant&\frac{N}{4}\int_{\mathbb{R}^N}\left[g(u_n)u_n - \left(2+\frac{4}{N}\right)G(u_n)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|u_n|^{2^*}dx + o(1)\\
=&I(u_n) - \frac{1}{2}P(u_n) + o(1)\\
=&I(u_n) + o(1) = c + o(1),
\end{aligned}$$

which yields that $R_n \to 1$ as $n \to \infty$.

As a result, by theorem 2.6 one has

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla v_n|^2dx =& \frac{N}{2}\int_{\mathbb{R}^N}H(v_n)dx + \int_{\mathbb{R}^N}|v_n|^{2^*}dx + o(1)\\
=&\int_{\mathbb{R}^N}|v_n|^{2^*}dx + o(1),
\end{aligned}$$

which together with the definition of S implies that $\nu \geqslant S^{\frac{N}{2}}$. Since

$$I(u_n) - I(v_n) = I(\tilde{u}_i) + o(1),$$

and by $(A5)$ we have

$$\begin{aligned}
I(\tilde{u}_i) =& I(\tilde{u}_i) - \frac{1}{2}P(\tilde{u}_i)\\
=&\frac{N}{4}\int_{\mathbb{R}^N}\left[g(\tilde{u}_i)\tilde{u}_i - \left(2+\frac{4}{N}\right)G(\tilde{u}_i)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\tilde{u}_i|^{2^*}dx \geqslant 0.
\end{aligned}$$

Thereby,

$$\begin{aligned}
c &= \lim_{n\to\infty} I(u_n) \\
&= \lim_{n\to\infty} [I(v_n) + I(\tilde{u}_i)] \\
&= I(\tilde{u}_i) + \lim_{n\to\infty} \left[\frac{1}{2}\|\nabla v_n\|_2^2 - \int_{\mathbb{R}^N} G(v_n)\,dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx\right] \\
&= I(\tilde{u}_i) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\nu \\
&\geqslant \frac{1}{N}\nu \geqslant \frac{1}{N} S^{\frac{N}{2}},
\end{aligned}$$

a contradiction to lemma 2.5.

In a word, for every $i \geqslant 0$, (2.9) or (2.10) holds. If (2.10) holds, there exists $R \geqslant 1$ such that $\tilde{u}_i(R\cdot) \in \mathcal{P}$ and $\tilde{u}_i(R\cdot) \in \mathcal{D}$. Hence, by $(A5)$ and Fatou lemma we deduce that

$$\begin{aligned}
c &\leqslant I(\tilde{u}_i(Rx)) = I(\tilde{u}_i(Rx)) - \frac{1}{2}P(\tilde{u}_i(Rx)) \\
&= \frac{N}{4}\int_{\mathbb{R}^N}\left[g(\tilde{u}_i(Rx))\tilde{u}_i(Rx) - \left(2+\frac{4}{N}\right)G(\tilde{u}_i(Rx))\right]dx \\
&\quad + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\tilde{u}_i(Rx)|^{2^*}dx \\
&= \frac{N}{4}R^{-N}\int_{\mathbb{R}^N}\left[g(\tilde{u}_i)\tilde{u}_i - \left(2+\frac{4}{N}\right)G(\tilde{u}_i)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)R^{-N}\int_{\mathbb{R}^N}|\tilde{u}_i|^{2^*}dx \\
&< \frac{N}{4}\int_{\mathbb{R}^N}\left[g(\tilde{u}_i)\tilde{u}_i - \left(2+\frac{4}{N}\right)G(\tilde{u}_i)\right]dx + \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^N}|\tilde{u}_i|^{2^*}dx \\
&\leqslant \frac{N}{4}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[g(u_n)u_n - \left(2+\frac{4}{N}\right)G(u_n)\right]dx + (\frac{1}{2}-\frac{1}{2^*})\liminf_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{2^*}dx \\
&\leqslant \lim_{n\to\infty}\left[I(u_n) - \frac{1}{2}P(u_n)\right] \\
&= \lim_{n\to\infty} I(u_n) = c,
\end{aligned} \tag{2.13}$$

a contradiction. Therefore, $u_n(\cdot + y_n^i) \to \tilde{u}_i$ in $D^{1,2}(\mathbb{R}^N)$. So $u_n(\cdot + y_n^i) \to \tilde{u}_i$ in $L^{2^*}(\mathbb{R}^N)$. Then, it results by (2.1) that

$$\int_{\mathbb{R}^N} H\left(u_n\left(\cdot + y_n^i\right)\right)dx \to \int_{\mathbb{R}^N} H(\tilde{u}_i)\,dx$$

as $n \to \infty$. Thereby, by $u_n \in \mathcal{D}\cap\mathcal{P}$ we know that $\tilde{u}_i \in \mathcal{D}\cap\mathcal{P}$. Arguing as before but with $R = 1$ we can prove that $I(\tilde{u}_i) = c$. If, in addition, g is odd, then $G(|\cdot|) = G(\cdot)$ and $H(|\cdot|) = H(\cdot)$. Set $\tilde{v}_i := |\tilde{u}_i|^*$ as the Schwarz symmetrization of $|\tilde{u}_i|$. Then, $\|\tilde{v}_i\|_2 = \|\tilde{u}_i\|_2$, so $\tilde{v}_i \in \mathcal{D}$. Furthermore,

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla\tilde{v}_i|^2dx \leqslant \int_{\mathbb{R}^N}|\nabla\tilde{u}_i|^2dx &= \frac{N}{2}\int_{\mathbb{R}^N}H(\tilde{u}_i)\,dx + \int_{\mathbb{R}^N}|\tilde{u}_i|^{2^*}dx \\
&= \frac{N}{2}\int_{\mathbb{R}^N}H(\tilde{v}_i)\,dx + \int_{\mathbb{R}^N}|\tilde{v}_i|^{2^*}dx.
\end{aligned}$$

Then

$$r(\tilde{v}_i) := \frac{\frac{N}{2}\int_{\mathbb{R}^N} H(\tilde{v}_i)\,dx + \int_{\mathbb{R}^N} |\tilde{v}_i|^{2^*}dx}{\int_{\mathbb{R}^N} |\nabla \tilde{v}_i|^2 dx} \geqslant 1$$

and $\tilde{v}_i(r(\tilde{v}_i)\cdot) \in \mathcal{P}$. If $r(\tilde{v}_i) > 1$, with a similar argument as the proof in (2.13) we can obtain a contradiction. As a consequence, $r(\tilde{v}_i) = 1$, and so $\tilde{v}_i \in \mathcal{P}$, $I(\tilde{v}_i) = c$. □

Lemma 2.8. *Assume that* (A0)–(A6) *and* (1.5) *hold. Alternative, suppose that*

$$(a)\;\; g(t)t \preceq 2^* G(t) \text{ for } t \in \mathbb{R},$$

or

$$(b)\;\; g \text{ is odd and } N \in \{3,\,4\}\,.$$

For any $u \in (\mathcal{D}\backslash S) \cap \mathcal{P}$, $\inf\limits_{S\cap\mathcal{P}} I < I(u)$.

Proof. We argue by contradiction. Suppose that there exists $\tilde{u} \in (\mathcal{D}\backslash S) \cap \mathcal{P}$ such that

$$\inf_{S\cap\mathcal{P}} I \geqslant I(\tilde{u}) = c = \inf_{\mathcal{D}\cap\mathcal{P}} I.$$

Which indicates that $\tilde{u}$ is a local minimizer for I on $\mathcal{D}\cap\mathcal{P}$. Since

$$\mathcal{D}\backslash S = \left\{u \in H^1\left(\mathbb{R}^N\right) : \|u\|_2 < m\right\}$$

is an open set in $\mathcal{P}$, $\tilde{u}$ is a local minimizer of I on $\mathcal{P}$. Consequently, there exists a Lagrange multiplier $\mu \in R$ such that

$$I'(\tilde{u})\,v + \mu\left(\int_{\mathbb{R}^N} \nabla\tilde{u}\nabla v dx - \frac{N}{4}\int_{\mathbb{R}^N} h(\tilde{u})\,v dx - \frac{2^*}{2}\int_{\mathbb{R}^N} |\tilde{u}|^{2^*-2}\tilde{u} v dx\right) = 0$$

for any $v \in H^1(\mathbb{R}^N)$. Namely, $\tilde{u}$ solves

$$-\Delta\tilde{u} - g(\tilde{u}) - |\tilde{u}|^{2^*-2}\tilde{u} + \mu\left(-\Delta\tilde{u} - \frac{N}{4}h(\tilde{u}) - \frac{2^*}{2}|\tilde{u}|^{2^*-2}\tilde{u}\right) = 0,$$

i.e.

$$-(1+\mu)\,\Delta\tilde{u} = g(\tilde{u}) + \frac{N}{4}\mu h(\tilde{u}) + \left(1 + \frac{2^*}{2}\mu\right)|\tilde{u}|^{2^*-2}\tilde{u}.$$

Especially, $\tilde{u}$ satisfies the following Nehari-type identity

$$(1+\mu)\int_{\mathbb{R}^N} |\nabla\tilde{u}|^2 dx = \int_{\mathbb{R}^N}\left[g(\tilde{u})\,\tilde{u} + \frac{N}{4}\mu h(\tilde{u})\,\tilde{u} + \left(1+\frac{2^*}{2}\mu\right)|\tilde{u}|^{2^*}\right]dx.$$

If $\mu = -1$, then

$$\int_{\mathbb{R}^N}\left[g(\tilde{u})\,\tilde{u} - \frac{N}{4}h(\tilde{u})\,\tilde{u} + \left(1 - \frac{2^*}{2}\right)|\tilde{u}|^{2^*}\right]dx = 0.$$

By $(A4)$ and $(A5)$,

$$0 = \int_{\mathbb{R}^N} \left[g(\tilde{u})\tilde{u} - \frac{N}{4} h(\tilde{u})\tilde{u} + \left(1 - \frac{2^*}{2}\right) |\tilde{u}|^{2^*} \right] dx$$
$$\leqslant \int_{\mathbb{R}^N} \left[g(\tilde{u})\tilde{u} - \frac{N}{4}\left(2 + \frac{4}{N}\right) H(\tilde{u}) + \left(1 - \frac{2^*}{2}\right) |\tilde{u}|^{2^*} \right] dx$$
$$\leqslant \left(1 - \frac{2^*}{2}\right) \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx < 0,$$

a contradiction. Consequently, $\mu \neq -1$. Since $\tilde{u} \in \mathcal{P}$,

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u})\, dx + \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx.$$

On the other hand, $\tilde{u}$ satisfies Pohozaev and Nehari identities. Hence,

$$(1+\mu) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx$$
$$= \frac{N}{2} \int_{\mathbb{R}^N} \left[H(\tilde{u}) + \frac{N}{4}\mu (h(\tilde{u})\tilde{u} - 2H(\tilde{u})) + \left(1 - \frac{2}{2^*}\right)\left(1 + \frac{2^*}{2}\mu\right) |\tilde{u}|^{2^*} \right] dx.$$

Combining there two identities we obtain

$$(1+\mu) \left[\frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u})\, dx + \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx \right]$$
$$= \frac{N}{2} \int_{\mathbb{R}^N} \left[H(\tilde{u}) + \frac{N}{4}\mu (h(\tilde{u})\tilde{u} - 2H(\tilde{u})) + \left(1 - \frac{2}{2^*}\right)\left(1 + \frac{2^*}{2}\mu\right) |\tilde{u}|^{2^*} \right] dx.$$

If $\mu \neq 0$, then

$$\int_{\mathbb{R}^N} \left[h(\tilde{u})\tilde{u} - \left(2 + \frac{4}{N}\right) H(\tilde{u}) \right] dx + \frac{16}{N^2 (N-2)} \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} = 0.$$

It yields by $(A4)$ that $\tilde{u} = 0$, a contradiction to $\tilde{u} \in (\mathcal{D} \setminus S) \cap \mathcal{P}$. Consequently, $\mu = 0$, and then $\tilde{u}$ solves

$$-\Delta \tilde{u} = g(\tilde{u}) + |\tilde{u}|^{2^*-2}\tilde{u}$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx = \int_{\mathbb{R}^N} g(\tilde{u})\tilde{u} dx + \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx.$$

Since $\tilde{u} \in \mathcal{P}$, one has

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u})\, dx + \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx.$$

Therefore,

$$\int_{\mathbb{R}^N} g(\tilde{u})\tilde{u} dx = \frac{N}{2} \int_{\mathbb{R}^N} H(\tilde{u})\, dx.$$

Which yields that

$$\int_{\mathbb{R}^N} [2^* G(\tilde{u}) - g(\tilde{u})\tilde{u}]\, dx = 0.$$

By the elliptic regularity theory, $\tilde{u}$ is continuous on $\mathbb{R}^N$. By $(A5)$,

$$2^*G(\tilde{u}(x)) = g(\tilde{u}(x))\,\tilde{u}(x)$$

for $x \in \mathbb{R}^N$. By $\tilde{u} \in H^1(\mathbb{R}^N)$, there exists an open interval $\Lambda \subset \mathbb{R}$ such that $0 \in \bar{\Lambda}$ and $2^*G(u) = g(u)u$ for $u \in \bar{\Lambda}$, which implies that there exists $C > 0$ such that

$$G(u) = C|u|^{2^*},$$

where $u \in \bar{\Lambda}$. We next continue our arguments by distinguishing two cases.

Case (i): If the inequality $g(t)t \preceq 2^*G(t)$ for $t \in \mathbb{R}$, then we can easily obtain a contradiction.

Case (ii): If g is odd and $N \in \{3,\ 4\}$, then by lemma 2.7, we may assume that $\tilde{u}$ is nonnegative and radially symmetric. Alternatively,

$$-\Delta\tilde{u} = (2^*C+1)\,|\tilde{u}|^{2^*-2}\tilde{u}. \tag{2.14}$$

This is impossible, since the nonnegative and radial solution of (2.14) is a Aubin–Talenti instanton, up to a scaling and a translation, which is not L^2-integrable if $N \in \{3,\ 4\}$. □

3. Proof of theorem 1.1

Proof of theorem 1.1. Taking into account of lemmas 2.7 and 2.8, $c = \inf_{S\cap\mathcal{P}} I$ is attained at $\tilde{u} \in S\cap\mathcal{P}$. Consequently, there exist two Lagrange multipliers $\lambda,\ \mu \in \mathbb{R}$ such that $\tilde{u}$ solves

$$-\Delta\tilde{u} - g(\tilde{u}) - |\tilde{u}|^{2^*-2}\tilde{u} + \lambda\tilde{u} + \mu\left(-\Delta\tilde{u} - \frac{N}{4}h(\tilde{u}) - \frac{2^*}{2}|\tilde{u}|^{2^*-2}\tilde{u}\right) = 0,$$

i.e.

$$-(1+\mu)\,\Delta\tilde{u} + \lambda\tilde{u} = g(\tilde{u}) + \frac{N}{4}\mu h(\tilde{u}) + \left(1 + \frac{2^*}{2}\mu\right)|\tilde{u}|^{2^*-2}\tilde{u}.$$

If $\mu = -1$, then

$$\lambda\tilde{u} = g(\tilde{u}) - \frac{N}{4}h(\tilde{u}) + \left(1 - \frac{2^*}{2}\right)|\tilde{u}|^{2^*-2}\tilde{u}.$$

In the sequel, we divide the following argument into two cases.

(a) Suppose $g'(t) = o(1)$ as $t \to 0$. By $(A4)$ and $(A5)$,

$$\begin{aligned}
\lambda\int_{\mathbb{R}^N}|\tilde{u}|^2dx &= \int_{\mathbb{R}^N}\left[g(\tilde{u})\,\tilde{u} - \frac{N}{4}h(\tilde{u})\,\tilde{u}\right]dx + \left(1 - \frac{2^*}{2}\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx \\
&\leqslant \int_{\mathbb{R}^N}\left[g(\tilde{u})\,\tilde{u} - \left(\frac{N}{2}+1\right)H(\tilde{u})\right]dx + \left(1 - \frac{2^*}{2}\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx \\
&= \frac{N}{2}\int_{\mathbb{R}^N}\left[\left(2 + \frac{4}{N}\right)G(\tilde{u}) - g(\tilde{u})\,\tilde{u}\right]dx + \left(1 - \frac{2^*}{2}\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx \\
&\leqslant \left(1 - \frac{2^*}{2}\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx < 0,
\end{aligned}$$

which implies that $\lambda < 0$. On the other hand, set

$$\Sigma := \left\{x \in \mathbb{R}^N : \lambda\tilde{u}(x) = g(\tilde{u}(x)) - \frac{N}{4}h(\tilde{u}(x)) + \left(1 - \frac{2^*}{2}\right)|\tilde{u}(x)|^{2^*-2}\tilde{u}(x)\right\},$$
$$\Omega := \{x \in \Sigma : \tilde{u}(x) \neq 0\}.$$

Then, $|\Omega| > 0$. Set $\delta := ess \inf_{x\in\Omega} |\tilde{u}(x)| \geqslant 0$. If $\delta > 0$, since $\tilde{u} \in L^2(\mathbb{R}^N)\backslash\{0\}$, then Ω admits finite positive measure and observe that

$$\int_{\mathbb{R}^N} |\tilde{u}(x+h) - \tilde{u}(x)|^2 dx \geqslant \delta^2 \int_{\mathbb{R}^N} |\chi_\Omega(x+h) - \chi_\Omega(x)|^2 dx$$

for any $h \in \mathbb{R}^N$, where χ_Ω is the characteristic function of Ω. Which means by theorem 2.1.6 in [33] that $\chi_\Omega \in H^1(\mathbb{R}^N)$, a contradiction to the fact that $\chi_\Omega \notin H^1(\mathbb{R}^N)$. Therefore, there exists a sequence $\{x_n\} \subset \Omega$ such that $\tilde{u}(x_n) \to 0$ and

$$\lambda = \frac{g(\tilde{u}(x_n))\tilde{u}(x_n) - \frac{N}{4}h(\tilde{u}(x_n))\tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} + \left(1 - \frac{2^*}{2}\right)|\tilde{u}(x_n)|^{2^*-2}$$

for any $n \in \mathbb{N}$. By $(A1)$ and $(A5)$,

$$\frac{g(\tilde{u}(x_n))\tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} = \frac{H(\tilde{u}(x_n))}{|\tilde{u}(x_n)|^2} + \frac{2G(\tilde{u}(x_n))}{|\tilde{u}(x_n)|^2} \to 0$$

as $n \to \infty$, which together with $g'(t) = o(1)$ as $t \to 0$ we see that

$$\lim_{n\to\infty} \frac{h(\tilde{u}(x_n))\tilde{u}(x_n)}{|\tilde{u}(x_n)|^2} = \lim_{n\to\infty}\left[g'(\tilde{u}(x_n)) - \frac{g(\tilde{u}(x_n))\tilde{u}(x_n)}{|\tilde{u}(x_n)|^2}\right] = 0.$$

Then $\lambda = 0$. This contradicts with $\lambda < 0$.

(b) Assume that g is odd. Then, by lemma 2.7 and the strong maximum principle, we may assume that $\tilde{u}$ is positive and radially symmetric. Hence, by Strauss lemma ([29]) we may assume that $\tilde{u}$ is continuous and

$$\lambda\tilde{u}(x) = g(\tilde{u}(x)) - \frac{N}{4}h(\tilde{u}(x)) + \left(1 - \frac{2^*}{2}\right)|\tilde{u}(x)|^{2^*-2}\tilde{u}(x)$$

holds for $x \in \mathbb{R}^N$. Since $\tilde{u}$ is continuous and $\tilde{u} \in H^1(\mathbb{R}^N)$, there exists an open interval Λ such that $0 \in \bar{\Lambda}$ and

$$\lambda u = g(u) - \frac{N}{4}h(u) + \left(1 - \frac{2^*}{2}\right)|u|^{2^*-2}u, \quad u \in \bar{\Lambda}.$$

Namely,

$$\begin{aligned}\lambda u =& g(u) - \frac{N}{4}[g'(u)u - g(u)] + \left(1 - \frac{2^*}{2}\right)|u|^{2^*-2}u \\ =& \left(1 + \frac{N}{4}\right)g(u) - \frac{N}{4}g'(u)u + \left(1 - \frac{2^*}{2}\right)|u|^{2^*-2}u, \quad u \in \bar{\Lambda}.\end{aligned}$$

Hence,

$$g(u) = C_1|u|^{\frac{4}{N}}u + \lambda u - |u|^{\frac{4}{N-2}}u = C_1|u|^{\frac{4}{N}}u + \lambda u - |u|^{2^*-2}u, \quad u \in \bar{\Lambda}$$

for some $C_1 \in \mathbb{R}$. Then,

$$G(u) = \frac{C_1}{2+\frac{4}{N}}|u|^{2+\frac{4}{N}} + \frac{1}{2}\lambda u^2 - \frac{1}{2^*}|u|^{2^*}, \quad u \in \bar{\Lambda}.$$

By $(A1)$, we infer that $\lambda = 0$, then

$$g(u) = C_1|u|^{\frac{4}{N}}u - |u|^{2^*-2}u, \quad u \in \bar{\Lambda}$$

and

$$G(u) = \frac{C_1}{2+\frac{4}{N}}|u|^{2+\frac{4}{N}} - \frac{1}{2^*}|u|^{2^*}, \quad u \in \bar{\Lambda}.$$

This contradicts with $(A5)$.

From the above arguments, $\mu \neq -1$. Since $\tilde{u}$ satisfies Nehari and Pohozaev identities, namely,

$$
\begin{aligned}
&(1+\mu)\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx\\
&\quad=\int_{\mathbb{R}^N}\left[g(\tilde{u})+\frac{N}{4}\mu h(\tilde{u})\right]\tilde{u}dx+\left(1+\frac{2^*}{2}\mu\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx-\lambda\int_{\mathbb{R}^N}|\tilde{u}|^2dx
\end{aligned}
$$

and

$$
\begin{aligned}
&(1+\mu)\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx\\
&\quad=2^*\int_{\mathbb{R}^N}\left[G(\tilde{u})+\frac{N}{4}\mu H(\tilde{u})+\left(1+\frac{2^*}{2}\mu\right)\cdot\frac{1}{2^*}|\tilde{u}|^{2^*}-\frac{1}{2}\lambda|\tilde{u}|^2\right]dx,
\end{aligned}
$$

we can deduce that

$$
\begin{aligned}
&(1+\mu)\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx\\
&\quad=\frac{N}{2}\int_{\mathbb{R}^N}\left[H(\tilde{u})+\frac{N}{4}\mu\left(h(\tilde{u})\tilde{u}-2H(\tilde{u})\right)\right]dx+\left(1+\frac{2^*}{2}\mu\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}.
\end{aligned}
$$

Since $\tilde{u}\in S\cap\mathcal{P}\subset\mathcal{P}$, then

$$
\int_{\mathbb{R}^N}|\nabla\tilde{u}|^2dx=\frac{N}{2}\int_{\mathbb{R}^N}H(\tilde{u})\,dx+\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx.
$$

Consequently,

$$
\begin{aligned}
&(1+\mu)\left[\frac{N}{2}\int_{\mathbb{R}^N}H(\tilde{u})\,dx+\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx\right]\\
&\quad=\frac{N}{2}\int_{\mathbb{R}^N}\left[H(\tilde{u})+\frac{N}{4}\mu\left(h(\tilde{u})\tilde{u}-2H(\tilde{u})\right)\right]dx+\left(1+\frac{2^*}{2}\mu\right)\int_{\mathbb{R}^N}|\tilde{u}|^{2^*},
\end{aligned}
$$

which yields that

$$
\mu\int_{\mathbb{R}^N}\left[h(\tilde{u})\tilde{u}-\left(2+\frac{4}{N}\right)H(\tilde{u})\right]+\frac{16}{N^2(N-2)}\mu\int_{\mathbb{R}^N}|\tilde{u}|^{2^*}dx=0.
$$

Arguing as the proof of lemma 2.8 we can infer that $\mu=0$. As a result,

$$
-\Delta\tilde{u}+\lambda\tilde{u}=g(\tilde{u})+|\tilde{u}|^{2^*-2}\tilde{u}.
$$

Consequently, $\tilde{u}\in S\cap\mathcal{P}$ is a normalized ground state solution to (P_m). Noting that if g is odd, it follow from lemma 2.7 and the maximum principle that $\tilde{u}$ is positive and radially symmetric. □

4. Proof of theorem 1.2

Proof of theorem 1.2. (i) For any $m_1 > m_2 > 0$, by Lemmas 2.7 and 2.8, there exists $u_{m_1} \in S_{m_1} \cap \mathcal{P}$, $u_{m_2} \in S_{m_2} \cap \mathcal{P}$ such that $E_{m_1} = I(u_{m_1})$, $E_{m_2} = I(u_{m_2})$. Since $m_1 > m_2 > 0$, $u_{m_2} \in (\mathcal{D}_{m_1} \setminus \mathcal{S}_{m_1}) \cap \mathcal{P}$. Again by lemma 2.8 we have

$$E_{m_1} = \inf_{\mathcal{S}_{m_1} \cap \mathcal{P}} I < I(u_{m_2}) = E_{m_2}.$$

(ii) Assume that $\eta = 0$.

(1) Firstly, let us assume that $m_n \to m^+$ as $n \to +\infty$. By lemma 2.7, there exists $u_n \in \mathcal{D}_{m_n} \cap \mathcal{P}$ such that $I(u_n) = \inf_{\mathcal{D}_{m_n} \cap \mathcal{P}} I$. Up to a translation and up to a subsequence, we obtain $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^N)$ and $r(\tilde{u}) \geqslant 1$. Clearly, $\tilde{u}(r(\tilde{u})\cdot) \in \mathcal{D}_m \cap \mathcal{P}$. We claim that $r(\tilde{u}) = 1$. Otherwise, if $r(\tilde{u}) > 1$, then by $(A5)$, lemma 2.4, Fatou lemma and $\mathcal{D}_m \cap \mathcal{P} \subset \mathcal{D}_{m_n} \cap \mathcal{P}$ we derive that

$$\begin{aligned}
0 < c = &\inf_{\mathcal{D}_m \cap \mathcal{P}} I \\
&\leqslant I(\tilde{u}(r(\tilde{u})\cdot)) = I(\tilde{u}(r(\tilde{u})\cdot)) - \frac{1}{2} P(\tilde{u}(r(\tilde{u})\cdot)) \\
&= \frac{N}{4} \int_{\mathbb{R}^N} \left[g(\tilde{u}(r(\tilde{u})\cdot)) \tilde{u}(r(\tilde{u})\cdot) - \left(2 + \frac{4}{N}\right) G(\tilde{u}(r(\tilde{u})\cdot)) \right] dx \\
&\quad + \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\tilde{u}(r(\tilde{u})\cdot)|^{2^*} dx \\
&= \frac{N}{4} r^{-N}(\tilde{u}) \int_{\mathbb{R}^N} \left[g(\tilde{u})\tilde{u} - \left(2 + \frac{4}{N}\right) G(\tilde{u}) \right] dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) r^{-N}(\tilde{u}) \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx \\
&< \frac{N}{4} \int_{\mathbb{R}^N} \left[g(\tilde{u})\tilde{u} - (2 + \frac{4}{N}) G(\tilde{u}) \right] dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |\tilde{u}|^{2^*} dx \\
&\leqslant \frac{N}{4} \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[g(u_n)u_n - \left(2 + \frac{4}{N}\right) G(u_n) \right] dx + \left(\frac{1}{2} - \frac{1}{2^*}\right) \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx \\
&\leqslant \lim_{n\to\infty} \left[I(u_n) - \frac{1}{2} P(u_n) \right] \\
&= \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \inf_{\mathcal{D}_{m_n} \cap \mathcal{P}} I \leqslant \lim_{n\to\infty} \inf_{\mathcal{D}_m \cap \mathcal{P}} I = \inf_{\mathcal{D}_m \cap \mathcal{P}} I.
\end{aligned}$$

This is impossible. Hence, $r(\tilde{u}) = 1$. And so the above proof yields that

$$c = \inf_{\mathcal{D}_m \cap \mathcal{P}} I \leqslant I(\tilde{u}) \leqslant \liminf_{n\to\infty} I(u_n) = \liminf_{n\to\infty} \inf_{\mathcal{D}_{m_n} \cap \mathcal{P}} I \leqslant \liminf_{n\to\infty} \inf_{\mathcal{D}_m \cap \mathcal{P}} I = \inf_{\mathcal{D}_m \cap \mathcal{P}} I,$$

which implies that

$$I(\tilde{u}) = \inf_{\mathcal{D}_m \cap \mathcal{P}} I = \lim_{n\to\infty} \inf_{\mathcal{D}_{m_n} \cap \mathcal{P}} I.$$

Alternatively, suppose that $m_n \to m^-$ as $n \to \infty$. Take $u \in \mathcal{D}_m \cap \mathcal{P}$ be such that $I(u) = \inf_{\mathcal{D}_m \cap \mathcal{P}} I$. Set $s_n := \frac{m_n}{m} \to 1^-$ as $n \to \infty$, $v_n = s_n u$. By lemma 2.2, there exists $\lambda_n > 0$ such that $\lambda_n^{\frac{N}{2}} v_n(\lambda_n \cdot) \in \mathcal{P}$. In the sequel, we assert that there exists $\lambda > 0$ such that $\lambda_n \to \lambda$ as $n \to \infty$. Indeed, up to a subsequence, if $\lambda_n \to +\infty$ as $n \to \infty$, by $\lambda_n^{\frac{N}{2}} v_n(\lambda_n \cdot) \in \mathcal{P}$ and $(A5)$ we see that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geqslant \lambda_n^{2^*-2} s_n^{2^*-2} \int_{\mathbb{R}^N} |u|^{2^*} dx \to +\infty$$

as $n \to \infty$, a contradiction. Up to a subsequence, if $\lambda_n \to 0$ as $n \to \infty$, again by $\lambda_n^{\frac{N}{2}} v_n(\lambda_n \cdot) \in \mathcal{P}$ and Fatou lemma we obtain that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{N}{2} s_n^{\frac{4}{N}} \int_{\mathbb{R}^N} \frac{H\left(\lambda_n^{\frac{N}{2}} s_n u(x)\right)}{|\lambda_n^{\frac{N}{2}} s_n u(x)|^{2+\frac{4}{N}}} |u|^{2+\frac{4}{N}} dx + \lambda_n^{2^*-2} s_n^{2^*-2} \int_{\mathbb{R}^N} |u|^{2^*} dx$$
$$\to 0,$$

a contradiction. Consequently, the assertion is true, and so $\lambda^{\frac{N}{2}} u(\lambda \cdot) \in \mathcal{P}$. By lemma 2.2 we have

$$\begin{aligned}
&\limsup_{n\to\infty} \inf_{\mathcal{D}_{m_n}\cap\mathcal{P}} I \\
&\leqslant \lim_{n\to\infty} I\left(\lambda_n^{\frac{N}{2}} v_n(\lambda_n \cdot)\right) \\
&= \lim_{n\to\infty} \left[\frac{1}{2}\lambda_n^2 s_n^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda_n^{-N} \int_{\mathbb{R}^N} G\left(\lambda_n^{\frac{N}{2}} s_n u\right) dx - \frac{1}{2^*}\lambda_n^{2^*} s_n^{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx\right] \\
&= \frac{1}{2}\lambda^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda^{-N} \int_{\mathbb{R}^N} G\left(\lambda^{\frac{N}{2}} u\right) dx - \frac{1}{2^*}\lambda^{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx \\
&= I\left(\lambda^{\frac{N}{2}} u(\lambda \cdot)\right) = I(u) = \inf_{\mathcal{D}_m\cap\mathcal{P}} I.
\end{aligned}$$

Noting that $\mathcal{D}_{m_n} \cap \mathcal{P} \subset \mathcal{D}_m \cap \mathcal{P}$ we derive that

$$\inf_{\mathcal{D}_m\cap\mathcal{P}} I \leqslant \liminf_{n\to\infty} \inf_{\mathcal{D}_{m_n}\cap\mathcal{P}} I \leqslant \limsup_{n\to\infty} \inf_{\mathcal{D}_{m_n}\cap\mathcal{P}} I \leqslant \inf_{\mathcal{D}_m\cap\mathcal{P}} I,$$

i.e. $\lim_{n\to\infty} \inf_{\mathcal{D}_{m_n}\cap\mathcal{P}} I = \inf_{\mathcal{D}_m\cap\mathcal{P}} I$. This ends the proof of the continuity of the ground state energy map.

(2) Let $m_n \to +\infty$ as $n \to \infty$. By lemmas 2.7 and 2.8, we may assume that u is a ground state solution to (P_1), i.e. $I(u) = \inf_{\mathcal{D}_1\cap\mathcal{P}} I = \inf_{\mathcal{S}_1\cap\mathcal{P}} I$. For simplicity, we may assume that $m_n > 1$. Set $u_n = m_n u$. Then, $u_n \in S_{m_n} \subset \mathcal{D}_{m_n}$. With the aid of lemma 2.2, there exists $\lambda_n > 0$ such that $v_n(\cdot) := \lambda_n^{\frac{N}{2}} u_n(\lambda_n \cdot) \in \mathcal{P}$. Then, $\|v_n\|_2 = \|u_n\|_2 = m_n$, i.e. $v_n \in \mathcal{D}_{m_n} \cap \mathcal{P}$. Consequently, by lemma 2.4 one has

$$0 < \inf_{\mathcal{D}_{m_n}\cap\mathcal{P}} I \leqslant I(v_n) \leqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \frac{1}{2}\lambda_n^2 \cdot m_n^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{4.1}$$

Since $v_n \in \mathcal{P}$,

$$\begin{aligned}
\lambda_n^2 \cdot m_n^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx &= \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \\
&= \frac{N}{2}\int_{\mathbb{R}^N} H(v_n) dx + \int_{\mathbb{R}^N} |v_n|^{2^*} dx \\
&= \frac{N}{2}\int_{\mathbb{R}^N} H\left(\lambda_n^{\frac{N}{2}} m_n u(\lambda_n x)\right) dx + \int_{\mathbb{R}^N} |\lambda_n^{\frac{N}{2}} m_n u(\lambda_n x)|^{2^*} dx \\
&= \frac{N}{2}\lambda_n^{-N} \int_{\mathbb{R}^N} H\left(\lambda_n^{\frac{N}{2}} m_n u\right) dx + \lambda_n^{2^*} m_n^{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{N}{2}\lambda_n^{-N-2} m_n^{-2} \int_{\mathbb{R}^N} H\left(\lambda_n^{\frac{N}{2}} m_n u\right) dx + \lambda_n^{2^*-2} m_n^{2^*-2} \int_{\mathbb{R}^N} |u|^{2^*} dx$$
$$= \frac{N}{2} m_n^{\frac{4}{N}} \int_{\mathbb{R}^N} \frac{H\left(\lambda_n^{\frac{N}{2}} m_n u\right)}{|\lambda_n^{\frac{N}{2}} m_n u|^{2+\frac{4}{N}}} |u|^{2+\frac{4}{N}} dx + (\lambda_n m_n)^{2^*-2} \int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{4.2}$$

Consequently, by $m_n \to +\infty$ as $n \to \infty$ we derive that

$$\int_{\mathbb{R}^N} \frac{H\left(\lambda_n^{\frac{N}{2}} m_n u\right)}{|\lambda_n^{\frac{N}{2}} m_n u|^{2+\frac{4}{N}}} |u|^{2+\frac{4}{N}} dx \to 0$$

as $n \to \infty$ and so $\lambda_n^{\frac{N}{2}} m_n \to 0$ as $n \to \infty$. Taking into account $(A5)$ and the condition $\lim_{t\to 0} \frac{G(t)}{|t|^{2^*}} = +\infty$ we derive that for any $\eta > 0$, there exists $\delta > 0$ such that when $t \in (-\delta, 0) \cup (0, \delta)$,

$$H(t) \geqslant \frac{4}{N} G(t) \geqslant \eta |t|^{2^*}.$$

Therefore, by (4.2) one has

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geqslant \frac{N}{2}\lambda_n^{-N-2} m_n^{-2} \eta \int_{\mathbb{R}^N} |\lambda_n^{\frac{N}{2}} m_n u|^{2^*} dx$$
$$= \eta \frac{N}{2} (\lambda_n m_n)^{2^*-2} \int_{\mathbb{R}^N} |u|^{2^*} dx, \tag{4.3}$$

which indicates that $\lambda_n m_n \to 0$ as $n \to \infty$. By (4.1) we see that $\inf_{\mathcal{D}_{m_n} \cap \mathcal{P}} I \to 0$ as $n \to \infty$. Consequently, $\inf_{\mathcal{D}_m \cap \mathcal{P}} I \to 0^+$ as $m \to +\infty$. □

Data availability statement

No new data were created or analysed in this study.

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Conflict of interest

Authors state that there is no conflict of interest.

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