NoDEA
Nonlinear differ. equ. appl. 8 (2001) 481–497
1021–9722/01/040481–17 \$ 1.50+0.20/0

Nonlinear Differential Equations
and Applications NoDEA

Nonlinear eigenvalue problems for quasilinear operators on unbounded domains

Eugenio MONTEFUSCO
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
via Vanvitelli 1
60123 Perugia, Italy
e-mail: montefus@dipmat.unipg.it

Vicenţiu RĂDULESCU
Department of Mathematics
University of Craiova
1100 Craiova, Romania
e-mail: varadulescu@hotmail.com

Abstract. We prove several existence results for eigenvalue problems involving the p-Laplacian and a nonlinear boundary condition on unbounded domains. We treat the non-degenerate subcritical case and the solutions are found in an appropriate weighted Sobolev space.

2000 Mathematics Subject Classification: 35J20 35J60 35J70.
Key words: Eigenvalue problems, quasilinear operators, unbounded domains.

1 Introduction and preliminary results

The growing attention for the study of the p-Laplacian operator Δ_p in the last few decades is motivated by the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number and the case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudoplastic, dilatant) fluid. The resulting equations of motion then involve $\operatorname{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a \operatorname{div}(\nabla_p u)$, provided that a is constant. The p-Laplacian appears in the study of flow through porous media ($p = 3/2$, see Showalter-Walkington [24]) or glacial sliding ($p \in (1, 4/3]$, see Pélissier-Reynaud [20]). We also refer to Aronsson-Janfalk [4] for the mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep (elastic for $p = 2$, plastic as $p \to \infty$, see Bhattacharya-DiBenedetto-Manfredi [5] and Kawohl [18]). This study is based on the observation that a prismatic material rod subject to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called *creep*. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the *creep-law* (see Kachanov [16, Chapters IV, VIII], Kachanov [17], and Findley-Lai-Onaran [13]). A nonlinear field equation in Quantum Mechanics involving the p-Laplacian, for $p = 6$, has been proposed in Benci-Fortunato-Pisani [6]. Eigenvalue problems involving the p-Laplacian have been the subject of much recent interest (we refer only to Allegretto-Huang [1], Anane [3], Drábek [9], Drábek-Pohozaev [11], Drábek-Simader [12], García-Peral [15], García-Montefusco-Peral [14]).

Let $\Omega \subset \mathbf{R}^N$ be an unbounded domain with (possible noncompact) smooth boundary $\partial\Omega$. We assume throughout this paper that p, q and m are real numbers satisfying $1 < p < q < p^* = \frac{Np}{N-p}$, if $p < N$ ($p^* = +\infty$ if $p \geq N$), $q \leq m < \frac{p(N-1)}{N-p}$ if $p < N$ ($q \leq m < +\infty$ when $p \geq N$).

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbf{R}^N)$-functions restricted on Ω.

We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E = \left(\int_\Omega \left(|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p \right) dx \right)^{1/p}.$$

Denote by $L^p(\Omega;\, w_1), L^q(\Omega;\, w_2)$ and $L^m(\partial\Omega;\, w_3)$ the weighted Lebesgue spaces with weight functions $w_i(x) = (1+|x|)^{\alpha_i}$ ($i = 1, 2, 3$), and the norms defined by

$$\|u\|_{p,w_1}^p = \int_\Omega w_1 |u(x)|^p \, dx, \quad \|u\|_{q,w_2}^q = \int_\Omega w_2 |u(x)|^q \, dx$$

and

$$\|u\|_{m,w_3}^m = \int_{\partial\Omega} w_3 |u(x)|^m \, dS,$$

where $-N < \alpha_1 < -p$ if $p < N$ ($\alpha_1 < -p$ when $p \geq N$), $-N < \alpha_2 < q\frac{N-p}{p} - N$ if $p < N$ ($-N < \alpha_2 < 0$ when $p \geq N$), and $-N < \alpha_3 < m\frac{N-p}{p} - N + 1$ if $p < N$ ($-N < \alpha_3 < 0$ when $p \geq N$).

We shall use in our paper the following embedding result.

Theorem A *Under the above assumptions on p, q and m, the space E is compactly embedded in $L^q(\Omega;\, w_2)$ and also in $L^m(\partial\Omega;\, w_3)$.*

This theorem is a consequence of Theorem 2 and Corollary 6 of Pflüger [22]. Furthermore, with the same proof as in Pflüger [21, Lemma 2], one can show

Lemma 1 *The quantity*

$$\|u\|_b^p = \int_\Omega a(x)|\nabla u|^p\,dx + \int_{\partial\Omega} b(x)|u|^p\,dS$$

defines an equivalent norm on E.

2 The main results

Consider the problem

$$\begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u\cdot n + b(x)|u|^{p-2}u = h(x,u) & \text{on } \partial\Omega, \end{cases} \tag{A}$$

where n denotes the unit outward normal on $\partial\Omega$, $0 < a_0 \le a \in L^\infty(\Omega)$, while $b : \partial\Omega \to \mathbf{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}} \le b(x) \le \frac{C}{(1+|x|)^{p-1}},$$

for some constants $0 < c \le C$.

Problems of this type arise in the study of physical phenomena related to equilibrium of anisotropic continuous media which possibly are somewhere "perfect" insulators, cf. Dautray-Lions [7].

We assume that f and g are nontrivial measurable functions satisfying

$$0 \le f(x) \le C(1+|x|)^{\alpha_1} \quad \text{and} \quad 0 \le g(x) \le C(1+|x|)^{\alpha_2}, \quad \text{for a.e. } x\in\Omega.$$

The mapping $h : \partial\Omega\times\mathbf{R}\to\mathbf{R}$ is a Carathéodory function which fulfills the assumption

(A1) $|h(x,s)| \le h_0(x) + h_1(x)|s|^{m-1}$,

where $h_i : \partial\Omega\to\mathbf{R}$ $(i = 0,1)$ are measurable functions satisfying

$$h_0 \in L^{m/(m-1)}(\partial\Omega;\, w_3^{1/(1-m)}) \quad \text{and} \quad 0 \le h_i \le C_h w_3 \quad \text{a.e. on } \partial\Omega.$$

We also assume

(A2) $\lim_{s\to 0} \dfrac{h(x,s)}{b(x)|s|^{p-1}} = 0$ uniformly in x.

(A3) There exists $\mu \in (p, q]$ such that

$$\mu H(x,t) \leq t h(x,t) \text{ for a.e. } x \in \partial\Omega \text{ and every } t \in \mathbf{R}.$$

(A4) There is a nonempty open set $U \subset \partial\Omega$ with $H(x,t) > 0$ for $(x,t) \in U \times (0,\infty)$, where $H(x,t) = \int_0^t h(x,s)\, ds$.

Our first result asserts that under the above hypotheses, problem (A) has at least a solution.

By weak solution of problem (A) we mean a function $u \in E$ such that, for any $v \in E$,

$$\begin{aligned}&\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\ dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\ dS \\ &\quad = \lambda \int_\Omega f(x)|u|^{p-2}uvdx + \int_\Omega g(x)|u|^{q-2}uvdx + \int_{\partial\Omega} h(x,u)vdS.\end{aligned}$$

Define

$$\tilde{\lambda} := \inf_{u\in E;\ u\neq 0} \left(\frac{\int_\Omega a(x)|\nabla u|^p\, dx + \int_{\partial\Omega} b(x)|u|^p\, dS}{\int_\Omega f(x)|u|^p\, dx} \right).$$

Our first result is

Theorem 1 *Assume that the conditions* (A1)–(A4) *hold. Then, for every* $\lambda < \tilde{\lambda}$, *problem* (A) *has a nontrivial weak solution.*

In the special case $h(x,s) \equiv 0$ we are able to show also a multiplicity result for problem (A). The statement is the following

Theorem 2 *Assume* $h(x,s) \equiv 0$. *Then, for every* $\lambda < \tilde{\lambda}$, *problem* (A) *possesses infinitely many solutions.*

Next we prove the existence of an eigensolution to the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda \left(f(x)|u|^{p-2}u + g(x)|u|^{q-2}u\right) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot n + b(x)|u|^{p-2}u = \lambda h(x,u) & \text{on } \partial\Omega. \end{cases} \tag{B}$$

We stress that for the next existence result of the paper we drop the assumptions (A2) and (A4). By weak solution of problem (B) we mean a function $u \in E$ such that, for any $v \in E$,

$$\begin{aligned}&\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v\ dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\ dS \\ &\quad = \lambda \left[\int_\Omega f(x)|u|^{p-2}uvdx + \int_\Omega g(x)|u|^{q-2}uvdx + \int_{\partial\Omega} h(x,u)vdS\right].\end{aligned}$$

We prove

Theorem 3 *Assume that the hypotheses* (A1) *and* (A3) *hold. Let* d *be an arbitrary real number such that* $1/d$ *is not an eigenvalue* λ *in problem* (B), *and satisfying*

$$d > \frac{1}{\tilde{\lambda}}. \tag{2.1}$$

Then there exists $\overline{\rho} > 0$ *such that for all* $r > \rho \geq \overline{\rho}$, *the eigenvalue problem* (B) *has an eigensolution* $(u, \lambda) = (u_d, \lambda_d) \in E \times \mathbf{R}$ *for which one has*

$$\lambda_d \in \left[\frac{1}{d + r^2 \|u_d\|_b^{m-p}}, \frac{1}{d + \rho^2 \|u_d\|_b^{m-p}} \right].$$

3 Problem (A)

Throughout this section we use the same notations as was previously done in the case of problem (A).

The energy functional corresponding to (A) is defined as $F : E \to \mathbf{R}$

$$F(u) = \frac{1}{p} \int_\Omega a(x) |\nabla u|^p \, dx + \frac{1}{p} \int_{\partial\Omega} b(x) |u|^p \, dS - \frac{\lambda}{p} \int_\Omega f(x) |u|^p \, dx \\ - \int_{\partial\Omega} H(x, u) \, dS - \frac{1}{q} \int_\Omega g(x) |u|^q \, dx$$

where H denotes the primitive function of h with respect to the second variable.

By Lemma 1 we have $\| \cdot \|_b \simeq \| \cdot \|_E$. We may write

$$F(u) = \frac{1}{p} \|u\|_b^p - \frac{\lambda}{p} \int_\Omega f(x) |u|^p \, dx - \int_{\partial\Omega} H(x, u) \, dS - \frac{1}{q} \int_\Omega g(x) |u|^q \, dx.$$

Since $p < q < p^*, -N < \alpha_1 < -p$ and $-N < \alpha_2 < q\frac{N-p}{p} - N$ we can apply Theorem A and we obtain that the embeddings $E \subset L^p(\Omega; w_1)$ and $E \subset L^q$ $(\Omega; w_2)$ are compact. So the functional F is well defined.

We denote by $N_h = h(x, u(x))$, $N_H = H(x, u(x))$ the corresponding Nemytskii operators.

Lemma 2 *The operators*

$$N_h : L^m(\partial\Omega; w_3) \to L^{m/(m-1)}(\partial\Omega; w_3^{1/(1-m)}), \quad N_H : L^m(\partial\Omega; w_3) \to L^1(\partial\Omega)$$

are bounded and continuous.

Proof. The proof follows from Theorem 1.1 in [10]. □

Our hypothesis $\lambda < \tilde{\lambda}$ implies the existence of some $C_0 > 0$ such that, for every $v \in E$

$$\|v\|_b^p - \lambda \int_\Omega f(x)|v|^p dx \geq C_0 \|v\|_b^p.$$

Lemma 3 *Under assumptions* (A1)–(A4)*, the functional F is Fréchet differentiable on E and satisfies the Palais-Smale condition.*

Proof. Denote $I(u) = \frac{1}{p}\|u\|_b^p$, $K_H(u) = \int_{\partial\Omega} H(x,u)\,dS$, $K_\Psi(u) = \int_\Omega \Psi(x,u)\,dx$ and $K_\Phi(u) = \int_\Omega \Phi(x,u)\,dx$, where $\Phi(x,u) = \frac{1}{p}f(x)|u|^p$ and $\Psi(x,u) = \frac{1}{q}g(x)|u|^q$.

Then the directional derivative of F in the direction $v \in E$ is

$$\langle F'(u), v\rangle = \langle I'(u), v\rangle - \lambda\langle K'_\Phi(u), v\rangle - \langle K'_\Psi(u), v\rangle - \langle K'_H(u), v\rangle,$$

where

$$\langle I'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v\,dx + \int_{\partial\Omega} b(x)|u|^{p-2}uv\,dS,$$

$$\begin{aligned} \langle K'_H(u), v\rangle &= \int_{\partial\Omega} h(x,u)v\,dS, \\ \langle K'_\Psi(u), v\rangle &= \int_\Omega g(x)|u|^{q-2}uv\,dx, \\ \langle K'_\Phi(u), v\rangle &= \int_\Omega f(x)|u|^{p-2}uv\,dx. \end{aligned}$$

Clearly, $I' : E \to E^\star$ is continuous. The operator $K_H^{'}$ is a composition of the operators

$$K_H^{'} : E \to L^m(\partial\Omega;\, w_3) \overset{N_h}{\to} L^{m/(m-1)}(\partial\Omega;\, w_3^{1/(1-m)}) \overset{l}{\to} E^\star$$

where $\langle l(u), v\rangle = \int_{\partial\Omega} uv\,dS$. Since

$$\int_{\partial\Omega} |uv|\,dS \leq \left(\int_{\partial\Omega} |u|^{m'} w_3^{1/(1-m)}\,dS\right)^{1/m'} \left(\int_{\partial\Omega} |v|^m w_3\,dS\right)^{1/m},$$

then l is continuous, by Theorem A. As a composition of continuous operators, $K_H^{'}$ is continuous, too. Moreover, by our assumptions on w_3, the trace operator $E \to L^m(\partial\Omega;\, w_3)$ is compact and therefore, $K_H^{'}$ is also compact.

Set $\varphi(u) = f(x)\,|u|^{p-2}u$. By the proof of Lemma 2 we deduce that the Nemytskii operator corresponding to any function which satisfies (A1) is bounded and continuous. Hence N_h and N_φ are bounded and continuous. We note that

$$K_\Phi^{'} : E \subset L^p(\Omega;\, w_1) \overset{N_\varphi}{\to} L^{p/(p-1)}(\Omega;\, w_1^{1/(1-p)}) \overset{\eta}{\to} E^\star$$

where $\langle \eta(u), v\rangle = \int_\Omega uv\,dx$. Since

$$\int_\Omega |uv|\,dx \le \left(\int_\Omega |u|^{p/(p-1)}\,w_1^{1/(1-p)}\,dx\right)^{(p-1)/p}\left(\int_\Omega |v|^p\,w_1\,dx\right)^{1/p},$$

it follows that η is continuous. But $K_\Phi^{'}$ is the composition of three continuous operators and by the assumptions on w_1, the embedding $E \subset L^p(\Omega;\, w_1)$ is compact. This implies that $K_\Phi^{'}$ is compact. In a similar way we obtain that $K_\Psi^{'}$ is compact and the continuous Fréchet differentiability of F follows.

Now, let $u_n \in E$ be a Palais-Smale sequence, i.e.,

$$|F(u_n)| \le C \text{ for all } n \tag{3.1}$$

and

$$\|F'(u_n)\|_{E^\star} \to 0 \text{ as } n \to \infty. \tag{3.2}$$

We first prove that $\{u_n\}$ is bounded in E. Remark that (3.2) implies that

$$|\langle F'(u_n), u_n\rangle| \le \mu \cdot \|u_n\|_b \text{ for } n \text{ large enough.}$$

This and (3.1) imply

$$C + \|u_n\|_b \ge F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle. \tag{3.3}$$

But

$$\langle F'(u_n), u_n\rangle = \int_\Omega a(x)|\nabla u_n|^p\,dx + \int_{\partial\Omega} b(x)|u_n|^p\,dS - \lambda$$
$$\int_\Omega f(x)|u_n|^p\,dx - \int_\Omega g(x)|u_n|^q\,dx - \int_{\partial\Omega} h(x,u_n)u_n\,dS.$$

We have

$$F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(\|u_n\|_b^p - \lambda\int_\Omega f(x)|u|^p dx\right)$$
$$-\left(\int_{\partial\Omega} H(x,u_n)\,dS - \frac{1}{\mu}\int_{\partial\Omega} h(x,u_n)u_n\,dS\right) - \left(\frac{1}{q} - \frac{1}{\mu}\right)\int_\Omega g(x)|u_n|^q\,dx).$$

By (A3) we deduce that

$$\int_{\partial\Omega} H(x,u_n)\,dS \le \frac{1}{\mu}\int_{\partial\Omega} h(x,u_n)u_n\,dS. \tag{3.4}$$

Therefore

$$F(u_n) - \frac{1}{\mu}\langle F'(u_n), u_n\rangle \ge \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p. \tag{3.5}$$

Relations (3.3) and (3.5) yield

$$C + \|u_n\|_b \geq \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0 \|u_n\|_b^p.$$

This shows that $\{u_n\}$ is bounded in E.

To prove that $\{u_n\}$ contains a Cauchy sequence we use the following inequalities for $\xi, \zeta \in \mathbf{R}^N$ (see Diaz [8], Lemma 4.10):

$$|\xi - \zeta|^p \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{3.6}$$

$$|\xi - \zeta|^2 \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } 1 < p < 2. \tag{3.7}$$

Then we obtain in the case $p \geq 2$:

$$\begin{aligned}
\|u_n - u_k\|_b^p &= \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\, dx + \int_{\partial\Omega} b(x)|u_n - u_k|^p\, dS \\
&\leq C(\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle) \\
&= C(\langle F'(u_n), u_n - u_k\rangle - \langle F'(u_k), u_n - u_k\rangle \\
&\quad + \lambda\langle K_\Phi^{'}(u_n), u_n - u_k\rangle - \lambda\langle K_\Phi^{'}(u_k), u_n - u_k\rangle \\
&\quad + \langle K_H^{'}(u_n), u_n - u_k\rangle - \langle K_H^{'}(u_k), u_n - u_k\rangle \\
&\quad + \langle K_\Psi^{'}(u_n), u_n - u_k\rangle - \langle K'_\Psi(u_k), u_n - u_k\rangle) \\
&\leq C(\|F'(u_n)\|_E^\star + \|F'(u_k)\|_{E^\star} + |\lambda|\, \|K_\Phi^{'}(u_n) - K_\Phi^{'}(u_k)\|_{E^\star} \\
&\quad + \|K_H^{'}(u_n) - K_H^{'}(u_k)\|_{E^\star} + \|K_\Psi^{'}(u_n) - K_\Psi^{'}(u_k)\|_{E^\star})\|u_n - u_k\|_b.
\end{aligned}$$

Since $F'(u_n) \to 0$ and $K_\Phi^{'}$, K'_Ψ, K'_H are compact, we can assume, passing eventually to a subsequence, that $\{u_n\}$ converges in E.

If $1 < p < 2$, then we use the estimate

$$\begin{aligned}
\|u_n - u_k\|_b^2 &\leq C'|\langle I'(u_n), u_n - u_k\rangle \\
&\quad - \langle I'(u_k), u_n - u_k\rangle|(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}).
\end{aligned} \tag{3.8}$$

Since $\|u_n\|_b$ is bounded, the same arguments lead to a convergent subsequence. In order to prove the estimate (3.8) we recall the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s) \qquad \text{for any } x, y \in (0, \infty). \tag{3.9}$$

Then we obtain

$$\begin{aligned}
\|u_n - u_k\|_b^2 &= \left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\, dx + \int_{\partial\Omega} b(x)|u_n - u_k|^p\, dS\right)^{\frac{2}{p}} \\
&\leq C_p\left[\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\, dx\right)^{\frac{2}{p}} + \left(\int_{\partial\Omega} b(x)|u_n - u_k|^p\, dS\right)^{\frac{2}{p}}\right].
\end{aligned} \tag{3.10}$$

Using (3.7), (3.9) and the Hölder inequality we find

$$\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx = \int_\Omega a(x)(|\nabla u_n - \nabla u_k|^2)^{\frac{p}{2}}\,dx$$
$$\le C\int_\Omega a(x)((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k))^{\frac{p}{2}}$$
$$(|\nabla u_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}}\,dx$$
$$= C\int_\Omega (a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k))^{\frac{p}{2}}$$
$$(a(x)(|\nabla u_n| + |\nabla u_k|)^p)^{\frac{2-p}{2}}\,dx$$
$$\le C\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}}$$
$$\left(\int_\Omega a(x)(|\nabla u_n| + |\nabla u_k|)^p\,dx\right)^{\frac{2-p}{2}}$$
$$\le \tilde{C}_p\left(\int_\Omega a(x)|\nabla u_n|^p\,dx + \int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}}$$
$$\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}}$$
$$\le \overline{C}_p\left[\left(\int_\Omega a(x)|\nabla u_n|^p\,dx\right)^{\frac{2-p}{2}} + \left(\int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}}\right]$$
$$\times\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}}$$
$$\le \overline{C}_p\left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right]^{\frac{p}{2}}$$
$$(\|u_n\|_b^{\frac{(2-p)p}{2}} + \|u_k\|_b^{\frac{(2-p)p}{2}}).$$

Using the last inequality and (3.9) we have the estimate

$$\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx\right)^{\frac{2}{p}}$$
$$\le C'_p\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)$$
$$(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{3.11}$$

In a similar way we can obtain the estimate

$$\begin{aligned}&\left(\int_{\partial\Omega} b(x)|u_n-u_k|^p\,dS\right)^{\frac{2}{p}}\\&\quad\le C_p'\left(\int_{\partial\Omega} b(x)(|u_n|^{p-2}u_n-|u_k|^{p-2}u_k)(u_n-u_k)\,dx\right)\\&\quad(\|u_n\|_b^{2-p}+\|u_k\|_b^{2-p}).\end{aligned}\tag{3.12}$$

It is now easy to observe that inequalities (3.10), (3.11) and (3.12) imply the estimate (3.8). The proof of Lemma 3 is complete. □

Proof of Theorem 1. We have to verify the geometric assumptions of the Mountain-Pass Theorem. We first show that there exist positive constants R and c_0 such that

$$F(u)\ge c_0,\qquad \text{for any } u\in E \text{ with } \|u\|=R.\tag{3.13}$$

By Theorem A we obtain some $A>0$ such that

$$\|u\|_{q,w_2}^q\le A\|u\|_b^q\quad\text{for all } u\in E.$$

This fact implies that

$$\begin{aligned}F(u)&=\frac{1}{p}(\|u\|_b^p-\lambda\|u\|_{p,w_1}^p)-\frac{1}{q}\int_\Omega g(x)|u|^q\,dx\\&-\int_{\partial\Omega}H(x,u)\,dS\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\int_{\partial\Omega}H(x,u)\,dS.\end{aligned}$$

By (A1) and (A2) we deduce that for every $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\frac{1}{q}|g(x)||u|^q\le\varepsilon b(x)|u|^p+C_\varepsilon w_3(x)|u|^m.$$

Consequently

$$\begin{aligned}F(u)&\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\int_{\partial\Omega}(\varepsilon b(x)|u|^p+C_\varepsilon w_3(x)|u|^m)\,ds\\&\ge\frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\varepsilon c_1\|u\|_b^p-C_\varepsilon C_2\|u\|_b^m.\end{aligned}$$

For $\varepsilon>0$ and $R>0$ small enough, we deduce that for every $u\in E$ with $\|u\|_b=R$, $F(u)\ge c_0>0$, which yields (3.13).

We verify in what follows the second geometric assumption of the Mountain-Pass Theorem, namely

$$\exists v\in E \text{ with } \|v\|>R \text{ such that } F(v)<c_0.\tag{3.14}$$

Choose $\psi \in C_\delta^\infty(\Omega)$, $\psi \geq 0$, such that $\emptyset \neq \mathrm{supp}\psi \cap \partial\Omega \subset U$. From $\frac{1}{q} g(x)|u|^q \geq c_3 s^\mu - c_4$ on $U \times (0,\infty)$ and (A1) we claim that

$$\begin{aligned} F(t\psi) &= \frac{t^p}{p}(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p) - \frac{1}{q}\int_\Omega g(x)|t\psi|^q\,dx - \int_{\partial\Omega} H(x,t\psi)\,dS \\ &\leq \frac{t^p}{p}\left(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p\right) - c_3 t^\mu \int_U \psi^\mu\,dS + c_4|U| - \frac{t^q}{q}\int_\Omega w_2\psi^q\,dx. \end{aligned}$$

Since $q \geq \mu > p$, we obtain $F(t\psi) \to -\infty$ as $t \to \infty$. It follows that if $t > 0$ is large enough, $F(t\psi) < 0$, so $v = t\psi$ satisfies (3.14).

By the Ambrosetti-Rabinowitz Theorem, problem (A) has a nontrivial weak solution.

Next we prove the second existence result about problem (A).

Proof of Theorem 2. In order to show the claim we want to apply a classical tool in critical point theory, precisely we will use the Ljusternik-Schnirelmann theory (see [23]). Consider the even functional

$$J(v) = \frac{1}{p}\int_\Omega a(x)|\nabla v|^p\,dx + \frac{1}{p}\int_{\partial\Omega} b(x)|v|^p\,dS - \frac{\lambda}{p}\int_\Omega f(x)|v|^p\,dx,$$

on the closed symmetric manifold

$$M = \left\{ v \in E : \int_\Omega g(x)|v|^q = 1 \right\}.$$

Note that M is only a C^1-manifold, since we have assumed $1 < p < q$. By our hypotheses on f, g, b and h (note that (A1)–(A4) are easily satisfied), Lemma 3 and Theorem 5.3 in [25], we have that $J|_M$ possesses at least $\gamma(M)$ pairs of critical points (where $\gamma(M)$ stands for the genus of M).

Now we have to estimate $\gamma(M)$. Since $g \not\equiv 0$ there exists an open set $\omega \subset \Omega$ such that $g(x) \geq \delta > 0$ on ω. By the properties of the genus it follows that $\gamma(\omega) \geq \gamma(B)$, where B is the unit ball of $W_0^{1,p}(\omega) \subset E$, but it is well known that the genus of the unit ball of a infinite dimensional Banach space is infinity, so $\gamma(M) = \infty$. Hence there exists a sequence $\{v_n\} \subset E$, such that any v_n (and also $-v_n$) is a constrained critical point of J on M.

By the Lagrange multipliers rule we obtain that there exists a sequence $\{\lambda_n\} \subset \mathbf{R}$ such that

$$\int_\Omega a(x)|\nabla v_n|^p\,dx + \int_{\partial\Omega} b(x)|v_n|^p\,dS - \lambda\int_\Omega f(x)|v_n|^p\,dx = \lambda_n \int_\Omega g(x)|v_n|^q\,dx.$$

Since $v_n \in M$, using our assumption $\lambda < \tilde{\lambda}$ we find

$$\lambda_n = \|v_n\|_b^p - \lambda \int_\Omega f(x)|v_n|^p \, dx > 0,$$

so we can apply the usual scaling. Setting $u_n = \lambda_n^{1/(q-p)} v_n$, we have that u_n satisfies for any n the equation

$$\int_\Omega a(x)|\nabla u_n|^p \, dx + \int_{\partial\Omega} b(x)|u_n|^p \, dS = \lambda \int_\Omega f(x)|u_n|^p \, dx + \int_\Omega g(x)|u_n|^q \, dx,$$

so the claim is proved.

4 Problem (B)

We start with the following auxiliary result.

Lemma 4 *Under assumption* (A1)*, if $q \leq m$, there exists a number $\overline{\rho} > 0$ such that for each $\rho \geq \overline{\rho}$ the function*

$$v \mapsto \frac{\rho^2}{m}\|v\|_b^m - \frac{1}{p}\|v\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v|^q \; dx - \int_{\partial\Omega} H(x,v) \; dS, \quad v \in E,$$

is bounded from below on E.

Proof. The growth condition for h implies

$$\begin{aligned}\left|\int_{\partial\Omega} H(x,v) \; dS\right| &\leq \int_{\partial\Omega}\left(h_0(x)|v| + \frac{1}{m}h_1(x)|v|^m\right) dS \\ &\leq \left(\int_{\partial\Omega} h_0^{\frac{m}{m-1}} w_3^{\frac{1}{1-m}} dS\right)^{\frac{m-1}{m}} \|v\|_{L^m(\partial\Omega;w_3)} + C_h\|v\|_{L^m(\partial\Omega;w_3)}^m \\ &\leq C_0 + C\|v\|_b^m, \quad v \in E,\end{aligned}$$

with constants $C_0, C > 0$. One obtains also that

$$\frac{1}{q}\left|\int_\Omega g(x)|u|^q \; dx\right| \leq C_2\|v\|_b^q \leq \overline{C}_0 + \overline{C}\|v\|_b^m, \quad v \in E,$$

with constants $\overline{C}_0, \overline{C} > 0$. Clearly, we can choose now the positive number $\overline{\rho}$ as desired. □

In view of Lemma 4 one can find numbers $b_0 > 0$ and $\alpha > 0$ such that

$$\frac{\overline{\rho}^2}{m}\|v\|_b^m + \frac{2}{m}b_0 - \frac{1}{p}\|v\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v|^q\,dx \\ -\int_{\partial\Omega} H(x,v)\,dS \geq \alpha > 0, \quad v \in E. \tag{4.1}$$

With $b_0 > 0$ and $\overline{\rho} > 0$ as above we consider numbers $r > \rho \geq \overline{\rho}$ and a function $\beta \in C^1(\mathbf{R})$ such that

$$\beta(0) = \beta(r) = 0,\ \beta(\rho) = b_0, \tag{4.2}$$

$$\beta'(t) < 0 \iff t < 0 \text{ or } \rho < t < r, \tag{4.3}$$

$$\lim_{|t|\to+\infty} \beta(t) = +\infty. \tag{4.4}$$

Lemma 5 *Assume that conditions* (A1) *and* (A3) *are fulfilled. Then, for any* $d > 0$ *satisfying* (3)*, the functional* $J : E \times \mathbf{R} \to \mathbf{R}$ *defined by*

$$J(v,t) = \frac{t^2}{m}\|v\|_b^m + \frac{2}{m}\beta(t) - \frac{1}{p}\int_\Omega f(x)|v|^p \\ -\frac{1}{q}\int_\Omega g(x)|v|^q\,dx - \int_{\partial\Omega} H(x,v)\,dx + \frac{d}{p}\|v\|_b^p \tag{4.5}$$

is of class C^1 *and satisfies the Palais-Smale condition.*

Proof. The property of J to be continuously differentiable has been already justified in the proof of Theorem 1.

In order to check the Palais-Smale condition let the sequences $\{v_n\} \subset E$ and $\{t_n\} \subset \mathbf{R}$ satisfy

$$|J(v_n,t_n)| \leq M,\ \forall n \geq 1 \tag{4.6}$$

$$J_v'(v_n,t_n) = t_n^2\,\|v_n\|_b^{m-p} I'(v_n) - K_\Phi'(v_n) - K_H'(v_n) - K_\Psi'(v_n) + dI'(v_n) \to 0, \tag{4.7}$$

$$J_t'(v_n,t_n) = \frac{2}{m}\,(t_n\|v_n\|_b^m + \beta'(t_n)) \to 0 \tag{4.8}$$

where I, K_Φ, K_H, K_Ψ have been introduced in the proof of Lemma 3.

From (4.1), (4.2), (4.5) and (4.6) we infer that

$$\begin{aligned} M &\geq \frac{t_n^2}{m}\,\|v_n\|_b^m + \frac{2}{m}\beta(t_n) - \frac{1}{p}\|v_n\|_{p,w_1}^p - \frac{1}{q}\int_\Omega g(x)|v_n|^q\,dx \\ &\quad -\int_{\partial\Omega} H(x,v_n)\,dx + \frac{d}{p}\|v_n\|_b^p \\ &\geq \frac{t_n^2-\rho^2}{m}\,\|v_n\|_b^m + \frac{2}{m}(\beta(t_n)-\beta(\rho)) + \frac{d}{p}\,\|v_n\|_b^p. \end{aligned}$$

Condition (4.4) in conjunction with the inequality above yields the boundedness of $\{t_n\}$.

Let us check the boundedness of $\{v_n\}$ along a subsequence. Without loss of generality we may admit that $\{v_n\}$ is bounded away from 0. From (22) we deduce that the sequence $\{t_n \|v_n\|_b^m\}$ is bounded. Therefore it is sufficient to argue in the case where $t_n \to 0$. From (4.6) it turns out that

$$\frac{1}{p}\|v_n\|_{p,w_1}^p + \int_\Omega H(x,v_n)dx + \frac{1}{q}\int_{\partial\Omega} g(x)|v_n|^q dx - \frac{d}{p}\|v_n\|_b^p$$

is bounded. By (4.7) we deduce that

$$\frac{1}{\|v_n\|_b}(-\langle K'_\Phi(v_n), v_n\rangle - \langle K'_H(v_n), v_n\rangle - \langle K'_\Psi(v_n), v_n\rangle + d\|v_n\|_b^p) \to 0.$$

Then, for n sufficiently large, assumption (A3) allows us to write

$$\begin{aligned} M + 1 + \|v_n\|_b &\geq d\left(\frac{1}{p} - \frac{1}{\mu}\right)\|v_n\|_b^p + \left(\frac{1}{\mu} - \frac{1}{q}\right)\|v_n\|_{L^q(\Omega,w_2)}^q \\ &\quad + \int_{\partial\Omega}\left(\frac{1}{\mu}h(x,v_n)v_n - H(x,v_n)\right) dS + \left(\frac{1}{\mu} - \frac{1}{p}\right)\|v_n\|_{p,w_1}^p \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(d\|v_n\|_b^p - \|v_n\|_{p,w_1}^p\right) \geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(d - \frac{1}{\tilde{\lambda}}\right)\|v_n\|_b^p. \end{aligned}$$

By (3), this establishes the boundedness of $\{v_n\}$ in E.

In view of the compactness of the mappings K'_Φ, K'_H, K'_Ψ (see the proof of Lemma 3), by (4.7) we get that

$$(d + t_n^2\, \|v_n\|_b^{m-p})\, I'(v_n)$$

converges in E^* as $n \to \infty$. The boundedness of $\{t_n\}$ and $\{v_n\}$ ensures that $\{I'(v_n)\}$ is convergent in E^* along a subsequence. Assume that $p \geq 2$. Inequality (3.6) shows that

$$\begin{aligned} \|u_n - u_k\|_b^p &\leq C\left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)\cdot(\nabla u_n - \nabla u_k)\, dx \right. \\ &\quad \left. + \int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_k|^{p-2}u_k)(u_n - u_k)\, d\Gamma\right] \\ &= C\langle I'(u_n) - I'(u_k), u_n - u_k\rangle \leq C\|I'(u_n) - I'(u_k)\|_b^*\|u_n - u_k\|_b \qquad \text{if } p \geq 2. \end{aligned}$$

Consequently, if $p \geq 2$, $\{v_n\}$ possesses a convergent subsequence. Proceeding in the same way with inequality (3.7) in place of (3.6) we obtain the result for $1 < p < 2$. □

In the proof of Theorem 3 we shall make use of the following variant of the Mountain Pass Theorem (see Motreanu [19]).

Lemma 6 *Let E be a Banach space and let $J : E \times \mathbf{R} \to \mathbf{R}$ be a C^1 functional verifying the hypotheses*

(a) *there exist constants $\rho > 0$ and $\alpha > 0$ such that $J(v, \rho) \geq \alpha$, for every $v \in E$;*

(b) *there is some $r > \rho$ with $J(0, 0) = J(0, r) = 0$.*
Then the number

$$c := \inf_{g \in \mathcal{P}} \max_{0 \leq \tau \leq 1} J(h(\tau))$$

is a critical value of J, where

$$\mathcal{P} := \{g \in C([0, 1]; E \times \mathbf{R});\ g(0) = (0, 0),\ g(1) = (0, r)\}\,.$$

Proof of Theorem 3. We apply Lemma 6 to the function J defined in (4.5). It is clear that assertion (a) is verified with $\rho > 0$ and $\alpha > 0$ described in Lemma 4 and (4.1). Due to relation (4.2), condition (b) in Lemma 6 holds. Lemma 5 ensures that the functional J satisfies the Palais-Smale condition. Therefore Lemma 6 yields a nonzero element $(u, t) \in E \times \mathbf{R}$ such that

$$J'_v(u, t) = (d + t^2 \|u\|_b^{m-p})\, I'(u) - K'_\Phi(u) - K'_H(u) - K'_\Psi(u) = 0, \tag{4.9}$$

$$J'_t(u, t) = \frac{2}{m}\,(t\|u\|_b^m + \beta'(t)) = 0. \tag{4.10}$$

From (4.10) it follows that

$$t\beta'(t) \leq 0. \tag{4.11}$$

Combining (4.11) and (4.3) we derive that if $t \neq 0$, then $u \neq 0$ and

$$\rho \leq t \leq r. \tag{4.12}$$

Therefore for each d in (3) such that $1/d$ is not an eigenvalue in (B) and each $r > \rho \geq \overline{\rho}$ we deduce that there exists a critical point $(u, t) = (u_d, t_d) \in E \times \mathbf{R}_+$ of J, where $t = t_d$ verifies (4.12). Consequently, relation (4.9) establishes that $u_d \in E$ is an eigenfunction in problem (B) where the corresponding eigenvalue is

$$\lambda_d = \frac{1}{d + t_d^2\, \|u_d\|_b^{m-p}},$$

with $t = t_d$ satisfying (4.12). This completes the proof.

Acknowledgements This work has been performed while V.R. was visiting the Università degli Studi di Perugia with a CNR-GNAFA grant. He would like to thank Professor Patrizia Pucci for the invitation, warm hospitality, and for many stimulating discussions.

References

[1] W. ALLEGRETTO, Y.X. HUANG, Eigenvalues of the indefinite weight p-Laplacian in weighted $\mathbf{R}^N$ spaces, *Funkc. Ekvac.* **38** (1995), 233–242.

[2] A. AMBROSETTI, P.H. RABINOWITZ, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.

[3] A. ANANE, Simplicité et isolation de la première valeur propre du p-laplacien, *C.R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 725–728.

[4] G. ARONSSON, U. JANFALK, On Hele-Shaw flow of power-law fluids, *European J. Appl. Math.* **3** (1992), 343–366.

[5] T. BHATTACHARYA, E. DIBENEDETTO, J. MANFREDI, Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, *Rend. Sem. Mat. Univ. Pol. Torino*, Fascicolo Speciale, 1989, 15–68.

[6] V. BENCI, D. FORTUNATO, L. PISANI, Solitons like solutions of a Lorentz invariant equation in dimension 3, *Rev. Math. Phys.* **10** (1998), 315–344.

[7] R. DAUTRAY, J.-L. LIONS, Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 1: *Physical Origins and Classical Methods*, Springer-Verlag, Berlin, 1985.

[8] J.I. DIAZ, Nonlinear Partial Differential Equations and Free Boundaries. Elliptic Equations, *Research Notes in Mathematics*, 106, Pitman, Boston-London-Melbourne 1986.

[9] P. DRÁBEK, Nonlinear eigenvalue problems for the p-Laplacian in $\mathbf{R}^N$, *Math. Nachr.* **173** (1995), 131–139.

[10] P. DRÁBEK, A. KUFNER, F. NICOLOSI, Quasilinear Elliptic Equations with Degenerations and Singularities, *de Gruyter Series in Nonlinear Analysis and Applications* **5**, W. de Gruyter, Berlin-New York, 1997.

[11] P. DRÁBEK, S. POHOZAEV, Positive solutions for the p-Laplacian: application of the fibrering method, *Proc. Roy. Soc. Edinburgh* **127A** (1997), 703–726.

[12] P. DRÁBEK, C.G. SIMADER, Nonlinear eigenvalue problems for quasilinear equations in unbounded domains, *Math. Nachr.* **203** (1999), 5–30.

[13] W.L. FINDLEY, J.S. LAI, K. ONARAN, Creep and Relaxation of Nonlinear Viscoelastic Materials, *North Holland Publ. House*, Amsterdam - New York - Oxford, 1976.

[14] J. GARCÍA-AZORERO, E. MONTEFUSCO, I. PERAL, Bifurcation for the p-Laplacian in $\mathbf{R}^N$, *Adv. Diff. Equations* **5** (2000), 435–464.

[15] J. GARCÍA-AZORERO, I. PERAL, Existence and uniqueness for the p-Laplacian: nonlinear eigenvalues, *Comm. Partial Differential Equations* **12** (1987), 1389–1430.

[16] L.M. KACHANOV, The Theory of Creep, *National Lending Library for Science and Technology*, Boston Spa, Yorkshire, England, 1967.

[17] L.M. KACHANOV, Foundations of the Theory of Plasticity, North Holland Publ. House, Amsterdam - London, 1971.

[18] B. KAWOHL, On a family of torsional creep problems, *J. Reine Angew. Math.* **410** (1990), 1–22.

[19] D. MOTREANU, A saddle point approach to nonlinear eigenvalue problems, *Math. Slovaca* **47** (1997), 463–477.

[20] M.C. PÉLLISSIER, M.L. REYNAUD, Étude d'un modèle mathématique d'écoulement de glacier, *C.R. Acad. Sci. Paris, Sér. I Math.* **279** (1974), 531–534.

[21] K. PFLÜGER, Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition, *Electronic Journal of Differential Equations* **10** (1998), 1–13.

[22] K. PFLÜGER, Compact traces in weighted Sobolev spaces, *Analysis* **18** (1998), 65–83.

[23] P.H. RABINOWITZ, Minimax Methods in Critical Point Theory with Applications to Differential Equations, C.B.M.S. Regional Conference Series in Mathematics **65**, *Amer. Math. Soc., Providence, R.I.,* 1986.

[24] R.E. SHOWALTER, N.J. WALKINGTON, Diffusion of fluid in a fissured medium with microstructure *SIAM J. Math. Anal.* **22** (1991), 1702–1722.

[25] A. SZULKIN, Ljusternik-Schnirelmann theory on C^1-manifold, *Ann. Ist. H. Poicaré, Analyse non linéaire* **5** (1988), 119–139.

Received May 2000