Opuscula Math. 40, no. 1 (2020), 71–92
https://doi.org/10.7494/OpMath.2020.40.1.71 Opuscula Mathematica

Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

NONHOMOGENEOUS EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH AND LACK OF COMPACTNESS

Giovany M. Figueiredo and Vicenţiu D. Rădulescu

Communicated by Marius Ghergu

Abstract. We study the existence and multiplicity of positive solutions for the following class of quasilinear problems

$$-\mathrm{div}\,(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(\epsilon x)b(|u|^p)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

where ϵ is a positive parameter. We assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential and $f : \mathbb{R} \to \mathbb{R}$ is a smooth reaction term with critical exponential growth.

Keywords: exponential critical growth, quasilinear equation, Trudinger–Moser inequality, Moser iteration.

Mathematics Subject Classification: 35J62, 35A15, 35B30, 35B33, 58E05.

1. INTRODUCTION AND MAIN RESULT

In this paper, we are concerned with the existence of positive solutions for the following class of quasilinear problems

$$\begin{cases} -\mathrm{div}\,(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) + V(\epsilon x)b(|u|^p)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{P_ϵ}$$

where $\epsilon > 0$ and $1 < p < N$. The hypotheses on the functions a, b, V and f are the following:

(a_1) the function a is of class C^1 and there exist constants $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^N \leq a(t^p)t^p \leq k_2 t^p + t^N \quad \text{for all } t > 0;$$

(a_2) the mapping $t \mapsto A(t^p)$ is convex on $(0, \infty)$, where $A(t) = \int_0^t a(s)ds$;

(a_3) the mapping $t \mapsto \frac{a(t^p)}{t^{N-p}}$ is nonincreasing for $t > 0$.

As a direct consequence of (a_3) we obtain that the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(N-p)}{p} a(t) \quad \text{for all } t > 0. \tag{1.1}$$

Now if we define the function $h(t) = a(t)t - \frac{N}{p}A(t)$, using (1.1) we can prove that the function h is decreasing. Then, there exists a positive real constant $\gamma \geq \frac{N}{p}$ such that

$$\frac{1}{\gamma} a(t)t \leq A(t) \quad \text{for all } t \geq 0. \tag{1.2}$$

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^N \leq b(t^p)t^p \leq k_4 t^p + t^N \quad \text{for all } t > 0;$$

(b_2) the mapping $t \mapsto B(t^p)$ is convex on $(0,\infty)$, where $B(t) = \int_0^t b(s)ds$;
(b_3) the mapping $t \mapsto \frac{b(t^p)}{t^{N-p}}$ is nonincreasing for $t > 0$.

Using the hypothesis (b_3) and arguing as (1.1) and (1.2), we also can prove that there exists $\gamma \geq \frac{N}{p}$ such that

$$\frac{1}{\gamma} b(t)t \leq B(t) \quad \text{for all } t \geq 0. \tag{1.3}$$

We assume that V is a continuous potential such that

$$V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{\mathbb{R}^N} V(x) > 0. \tag{V}$$

This kind of hypothesis was introduced by Rabinowitz in [22]. The nonsmooth setting was considered by Gazzola and Rădulescu [11].

In this paper, an important role is played by the existence of solutions to the problem

$$\begin{cases} -k_2\Delta_p u - \Delta_N u + V^* k_4 |u|^{p-2}u + V^*|u|^{N-2}u = |u|^{r-2}u \text{ in } \mathbb{R}^N, \\ u(z) > 0 \text{ for all } z \in \mathbb{R}^N. \end{cases} \tag{P_r}$$

where $r > N$, k_2 appears in hypothesis (a_1), k_4 appears in hypothesis (b_1), and V^* is a real number. The energy functional associated to this problem is $\Phi \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ and is defined by

$$\begin{aligned} \Phi(u) = \frac{k_2}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^N dx \\ + \frac{k_4}{p} V^* \int_{\mathbb{R}^N} |u|^p dx + \frac{1}{N} V^* \int_{\mathbb{R}^N} |u|^N dx - \frac{1}{r} \int_{\mathbb{R}^N} |u|^r dx. \end{aligned}$$

Since $W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$, we can repeat the computation developed in Lemmas 2.1–2.3, Proposition 2.1 and Proposition 2.2 of [4] in order to obtain that $w_r \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ is a ground state solution of (P_r), that is,

$$\Phi(w_r) = c_r \quad \text{and} \quad \Phi'(w_r) = 0.$$

Note that

$$c_r = \Phi(w_r) = \Phi(w_r) - \frac{1}{N}\Phi'(w_r)w_r \geq \frac{r-N}{pN} \int_{\mathbb{R}^N} |u|^r dx. \tag{1.4}$$

The nonlinearity f has critical exponential growth at $+\infty$, that is, f behaves $\exp(\alpha_0 |t|^{N/N-1})$ for some α_0. More precisely, there exists $\alpha_0 > 0$ such that the function f satisfies

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = 0 \quad \text{for } \alpha > \alpha_0$$

and

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha |t|^{N/N-1}) - S_{N-2}(\alpha, t)} = \infty \quad \text{for } \alpha < \alpha_0,$$

where

$$S_{N-2}(\alpha, t) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |t|^{N/(N-1)k}.$$

Moreover, we assume the following growth conditions in the origin and at infinity for the continuous function $f : \mathbb{R} \to \mathbb{R}$:

(f_1) the following limit holds:

$$\lim_{t\to 0^+} \frac{f(t)}{t^{N-1}} = 0;$$

(f_2) there exists $C > 0$ such that

$$|f(t)| \leq \exp(\alpha_N |t|^{N/N-1}) - S_{N-2}(\alpha_N, t),$$

for all $t > 0$, where $\alpha_N := N w_{N-1}^{1/N-1}$ and w_{N-1} is the $(N-1)$-dimensional measure of $(N-1)$ sphere;

(f_3) there exist $\theta > p\gamma$ (with the same γ as in (1.3)) such that

$$0 < \theta F(t) \leq f(t)t$$

for all $t > 0$;

(f_4) the function $t \mapsto \frac{f(t)}{t^{N-1}}$ is increasing in $(0, +\infty)$;

(f_5) there exist $r > N$, $\tau > \tau^*$, $s > 1$ and $\delta > 0$ such that

$$f(t) \geq \tau t^{r-1},$$

for all $t \geq 0$, where

$$\tau > \tau^* := \Big[\frac{2^N}{\min\{k_1, k_3, 1\}} \frac{\theta p \gamma c_r N r (r-p)}{(\theta - p\gamma)(r-N)pr}\Big]^{(r-p)/p}.$$

We denote by M and M_δ the following sets:

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : \text{dist}\,(x, M) \leq \delta\} \quad \text{for } \delta > 0.$$

Our main theorem is the following.

Theorem 1.1. *Suppose that hypotheses* (a_1)–(a_3), (b_1)–(b_3), (f_1)–(f_5) *and* (V) *are fulfilled. Then, for any* $\delta > 0$, *there exists* $\epsilon_\delta > 0$ *such that problem* (P_ϵ) *has at least* $\text{cat}_{M_\delta}(M)$ *positive solutions, for any* $0 < \epsilon < \epsilon_\delta$. *Moreover, if* u_ϵ *denotes one of these positive solutions and* $\eta_\epsilon \in \mathbb{R}^N$ *is its global maximum point, then*

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$

In this work, we use the following version of the Trudinger–Moser inequality in the whole Euclidean space $\mathbb{R}^N$, which is due do Ó [7] (for the case $N = 2$, see Cao [9]).

Proposition 1.2. *If* $N \geq 2$, $\alpha > 0$ *and* $u \in W^{1,N}(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{(N/N-1)}\right) - S_{N-2}(\alpha, u)\right] < \infty.$$

Moreover, if $|\nabla u|_{L^N}^N \leq 1$, $|u|_{L^N} \leq K < \infty$ *and* $\alpha < \alpha_N$, *then there exists a constant* $C = C(N, K, \alpha)$, *which depends only on* N, K *and* α, *such that*

$$\int_{\mathbb{R}^N} \left[\exp\left(\alpha |u|^{(N/N-1)}\right) - S_{N-2}(\alpha, u)\right] \leq C.$$

We give in what follows some examples of functions a in order to illustrate the degree of generality of the kind of problems studied here.

Considering $a(t) = t^{\frac{N-p}{p}}$, $b(t) = t^{\frac{N-p}{p}}$ we deduce that a and b satisfy the hypotheses (a_1)–(a_3), (b_1)–(b_3) with $k_1 = k_3 = k_5 = k_7 = 0$ and $k_2 = k_4 = k_6 = k_8 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_N u + V(\epsilon x)|u|^{N-2}u = f(u) \text{ in } \mathbb{R}^N. \tag{nL}$$

Considering $a(t) = 1 + t^{\frac{N-p}{p}}$, $b(t) = 1 + t^{\frac{N-p}{p}}$ then the functions a, b satisfy the hypotheses (a_1)–(a_3), (b_1)–(b_3) with $k_1 = k_2 = k_3 = k_4 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u - \Delta_N u + V(\epsilon x)(|u|^{p-2}u + |u|^{N-2}u) = f(u) \text{ in } \Omega. \tag{pnL}$$

This class of equations comes, for example, from a general reaction-diffusion system:

$$u_t = \text{div}\,[D(u)\nabla u] + c(x, u), \tag{1.5}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{N-2})$.

This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.5) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ is a polynomial of u with variable coefficients (see [10, 13, 18–20]).

Other examples that are also interesting from mathematical point of view are now presented. Considering

$$a(t) = 1 + t^{\frac{N-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}, \quad b(t) = 1 + t^{\frac{N-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}},$$

it follows that the functions a, b satisfy the hypotheses (a_1)–(a_3), (b_1)–(b_3) with $k_1 = k_3 = 1$, and $k_2 = k_4 = 2$. Hence, Theorem 1.1 is valid for the problem

$$\begin{aligned} &-\Delta_p u - \Delta_N u - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) \\ &+ V(\epsilon x)\left(|u|^{p-2}u + |u|^{N-2}u + \left(\frac{|u|^{p-2}u}{(1+|u|^p)^{\frac{p-2}{p}}}\right)\right) = f(u) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Other combinations can be made with the functions presented in the examples above, generating very interesting elliptic problems from the mathematical point of view. For the abstract methods used in this paper we refer to the recent monograph by Papageorgiou, Rădulescu and Repovš [21].

2. A PERIODIC PROBLEM

The main objective of this section is to study the existence of solutions for the problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) + W(x)b(|u|^p)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N, \\ u(z) > 0 \text{ for all } z \in \mathbb{R}^N, \end{cases} \tag{P_W}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a continuous 1-periodic function. More precisely, we have $W(x+y) = W(x)$ for all $y \in Z^N$, and $V^* \geq W(x) \geq W_0 > 0$ for all $x \in \mathbb{R}^N$, where $V^* = V_\infty$ if $V_\infty < \infty$ or $V^* = V_0$ if $V_\infty = \infty$. Without loss of generality, we can assume that $W_0 = 1$.

Let us consider the space

$$X_W = W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$$

endowed with the norm $\|u\|_W = \|u\|_{p,W} + \|u\|_{N,W}$, where

$$\|u\|_{m,W} = \Big(\int\limits_{\mathbb{R}^N} |\nabla u|^m dx + \int\limits_{\mathbb{R}^N} W(x)|u|^m dx\Big)^{1/m}.$$

Note that, by hypotheses (a_1) and (b_1), we have $A(t^p) \leq k_2 t^p + t^N$ and $B(t^p) \leq k_4 t^p + t^N$ for $t > 0$. Thus, the functional $J_W : X_W \to \mathbb{R}$ given by

$$J_W(u) = \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla u|^p)dx + \frac{1}{p}\int\limits_{\mathbb{R}^N} W(x)B(|u|^p)dx - \int\limits_{\mathbb{R}^N} F(u)dx$$

is well-defined. Moreover, note that

$$\begin{aligned} J'_W(u)\phi = &\int\limits_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \phi \, dx \\ &+ \int\limits_{\mathbb{R}^N} W(x)b(|u|^p)|u|^{p-2}u\phi \, dx - \int\limits_{\mathbb{R}^N} f(u)\phi \, dx \end{aligned}$$

for all $\phi \in X_W$. We conclude that J_W is a C^1 functional on X_W and its critical points are weak solution of (P_W). The Nehari manifold associated to the functional J_W is given by

$$\mathcal{M} = \left\{ u \in X\backslash\{0\} : \int\limits_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p dx + \int\limits_{\mathbb{R}^N} W(x)b(|u|^p)|u|^p dx = \int\limits_{\mathbb{R}^N} f(u)u \, dx \right\}.$$

Since we are looking for positive solutions, we consider $f(t) = 0$, for all $t \leq 0$.

Note that from (f_1) and (f_2), given $\Upsilon > 0$, $q \geq 0$ and $\alpha \geq 1$, there exists $C_\Upsilon > 0$ such that for all $u \in X_W$

$$\int\limits_{\mathbb{R}^N} f(u)udx \leq \Upsilon \int\limits_{\mathbb{R}^N} |u|^p dx + C_\Upsilon \int\limits_{\mathbb{R}^N} |u|^q[\exp(\alpha\alpha_N |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha\alpha_N, u)]dx \quad (2.1)$$

and

$$\int\limits_{\mathbb{R}^N} F(u)dx \leq \frac{\Upsilon}{p}\int\limits_{\mathbb{R}^N} |u|^p dx + \tilde{C_\Upsilon} \int\limits_{\mathbb{R}^N} |u|^q[\exp(\alpha\alpha_N |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha\alpha_N, u)]dx. \quad (2.2)$$

The next result concerns with the mountain pass geometry of J.

Lemma 2.1. *The functional J_W satisfies the following properties:*

(i) *there exist ρ, $\eta > 0$, such that $J_W(u) \geq \eta$, if $\|u\|_W = \rho$;*
(ii) *for any $u \in X_W\backslash\{0\}$ with $u \geq 0$, $J_W(tu) \to -\infty$ as $t \to +\infty$.*

Proof. (i) Note that from (a_1), (b_1) and (2.2) we get

$$\begin{aligned} J_W(u) \geq &\frac{k_1}{p}\int\limits_{\mathbb{R}^N} |\nabla u|^p dx + \frac{k_3}{p}\int\limits_{\mathbb{R}^N} W(x)|u|^p dx \\ &+ \frac{1}{N}\int\limits_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{N}\int\limits_{\mathbb{R}^N} W(x)|u|^N dx \\ &- \frac{\Upsilon}{p}\int\limits_{\mathbb{R}^N} |u|^p dx - \tilde{C_\Upsilon}\int\limits_{\mathbb{R}^N} |u|^q[\exp(\alpha\alpha_N |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha\alpha_N, u)]dx. \end{aligned}$$

Using Hölder's inequality with $\frac{1}{s}+\frac{1}{s'}=1$ we obtain

$$\begin{aligned}J_W(u) &\geq \frac{k_1}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx+\frac{k_3}{p}\int_{\mathbb{R}^N}W(x)|u|^p dx\\ &+\frac{1}{N}\int_{\mathbb{R}^N}|\nabla u|^N dx+\frac{1}{N}\int_{\mathbb{R}^N}W(x)|u|^N dx-\frac{\Upsilon}{p}\int_{\mathbb{R}^N}|u|^p dx\\ &-\tilde{C_\Upsilon}\Big(\int_{\mathbb{R}^N}|u|^{qs'}dx\Big)^{1/s'}\\ &\cdot C\int_{\mathbb{R}^N}\Big[\exp\Big(s\alpha\alpha_N\|u\|_W^{\frac{N}{N-1}}\Big(\frac{|u|}{\|u\|_W}\Big)^{\frac{N}{N-1}}\Big)-S_{N-2}(s\alpha\alpha_N,u)\Big]dx.\end{aligned}$$

By the Sobolev embedding theorem, there are positive constants C_1, C_2, C_3 such that

$$\begin{aligned}J_W(u) &\geq C_1\|u\|_{p,W}^p+C_2\|u\|_{N,W}^N\\ &-C_3\|u\|_W^q\int_{\mathbb{R}^N}\Big[\exp\Big(\alpha s\|u\|_W^{\frac{N}{N-1}}\Big(\frac{|u|}{\|u\|_W}\Big)^{\frac{N}{N-1}}\Big)-S_{N-2}(s\alpha,u)\Big]dx.\end{aligned}$$

Considering $\|u\|_W=\rho$, where $0<\rho<\min\{1,\frac{\alpha_N}{\alpha s}\}$, then

$$\|u\|_{p,W}^N\leq\|u\|_{p,W}^p,\quad |\nabla u|_{L^N}^N\leq 1, |u|_L^N\leq K\quad\text{and}\quad \rho^{N/N-1}s\alpha<1.$$

By Proposition 1.2, there are positive constants C_4, C_5 such that

$$J_W(u)\geq C_4\rho^N-C_5\rho^q.$$

Since $q>N$ the proof of (i) is over.

(ii) For any $u\in X_W\backslash\{0\}$, from (a_1), (b_1), (f_5) we conclude that

$$\begin{aligned}J_W(tu) \leq \frac{t^pK_2}{p}\int_{\mathbb{R}^N}|\nabla u|^p dx+\frac{t^N}{N}\int_{\mathbb{R}^N}|\nabla u|^N dx+\frac{t^pk_4}{p}\int_{\mathbb{R}^N}W(x)|u|^p dx\\ +\frac{t^N}{N}\int_{\mathbb{R}^N}W(x)|u|^N dx-\frac{t^r}{r}\int_{\mathbb{R}^N}|u|^r dx.\end{aligned}$$

Since $r>N>p$, the proof is finished. □

Now, in view of lemma above, we can apply the Ambrosetti–Rabinowitz mountain pass theorem without the Palais–Smale condition [25, Theorem 1.15] in order to get a sequence $(u_n)\subset X_W$ verifying

$$J_W(u_n)\to c_W\quad\text{and}\quad J'_W(u_n)\to 0\quad\text{as}\quad n\to\infty,$$

where the level c_W is characterized by

$$c_W = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_W(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], X_W) : J_W(0) = 0, J_W(\gamma(1)) < 0\}.$$

Lemma 2.2. *Let (u_n) be a $(PS)_{c_W}$ sequence for J_W. Then*

(i) $c_W \in [\eta, \frac{(r-p)}{pr\tau^{p/(r-p)}} \frac{c_r N r}{(r-N)}]$,
(ii) $u_n \rightharpoonup u_0$ *in* X_W,
(iii) $J'_W(u_0) = 0$,
(iv) $u_n \geq 0$ *for* $n \in \mathbb{N}$.

Proof. (i) In view of Lemma 2.1, $c_W \geq \eta$. Note that, by the hypotheses (a_1), (b_1) and (f_5), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} a(|\nabla w_r|^p)|\nabla w_r|^p dx + \int_{\mathbb{R}^N} W(x) b(|w_r|^p)|w_r|^p dx\\ &\leq k_2 \int_{\mathbb{R}^N} |\nabla w_r|^p dx + \int_{\mathbb{R}^N} |\nabla w_r|^N dx + k_4 V^* \int_{\mathbb{R}^N} |w_r|^p dx + V^* \int_{\mathbb{R}^N} |w_r|^N dx\\ &= \int_{\mathbb{R}^N} |w_r|^r \leq \int_{\mathbb{R}^N} f(w_r) w_r dx.\end{aligned}$$

This inequality implies that $J'_W(w_r)w_r \leq 0$. Thus, there exists $\beta \in (0,1)$ such that $\beta w_r \in \mathcal{M}$. Using (a_1), (b_1) and (f_5) again, we obtain

$$\begin{aligned}c_W \leq J(\beta w_r) \leq \frac{k_3}{p}\beta^p \int_{\mathbb{R}^N} |\nabla w_r|^p dx + \frac{k_4}{N}\beta^N \int_{\mathbb{R}^N} |\nabla w_r|^N dx + \frac{k_7}{p}\beta^p V^* \int_{\mathbb{R}^N} |w_r|^p dx\\ + \frac{k_8}{N}\beta^N V^* \int_{\mathbb{R}^N} |w_r|^N dx - \frac{\tau}{r}\beta^r \int_{\mathbb{R}^N} |w_r|^r dx.\end{aligned}$$

Since $\beta \in (0,1)$, we get

$$\begin{aligned}c_W \leq J(\beta w_r) \leq \frac{\beta^p}{p}\Bigg[k_2 \int_{\mathbb{R}^N} |\nabla w_r|^p dx + \int_{\mathbb{R}^N} |\nabla w_r|^N dx + k_4 V^* \int_{\mathbb{R}^N} |w_r|^p dx\\ + V^* \int_{\mathbb{R}^N} |w_r|^N dx\Bigg] - \frac{\tau}{r}\beta^r \int_{\mathbb{R}^N} |w_r|^r dx.\end{aligned}$$

Since $\Phi'_r(w_r) = 0$, we conclude that

$$c_W \leq \Big[\frac{\beta^p}{p} - \tau\frac{\beta^r}{r}\Big] \int_{\mathbb{R}^N} |w_r|^r dx.$$

Using (1.4), we have

$$c_W \leq \Big[\frac{\beta^p}{p} - \tau\frac{\beta^r}{r}\Big]\frac{c_r N r}{(r-N)} \leq \max_{s\geq 0}\Big[\frac{s^p}{p} - \tau\frac{s^r}{r}\Big]\frac{c_r N r}{(r-N)}.$$

By standard algebraic manipulations, we get

$$c_W \leq \Big[\frac{(r-p)}{pr\tau^{p/(r-p)}}\Big]\frac{c_r N r}{(r-N)}$$

and the result follows.

(ii) Note that from (f_3) we obtain

$$\begin{aligned} c_W \geq\ & \frac{1}{p}\int\limits_{\mathbb{R}^N} A(|\nabla u_n|^p)dx - \frac{1}{\theta}\int\limits_{\mathbb{R}^N} a(|\nabla u_n|^p)|\nabla u_n|^p dx \\ & + \frac{1}{p}\int\limits_{\mathbb{R}^N} W(x)B(|u_n|^p)dx - \frac{1}{\theta}\int\limits_{\mathbb{R}^N} W(x)b(|u_n|^p)|u_n|^p dx. \end{aligned}$$

Using (1.3) we get

$$c_W \geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\left[\int\limits_{\mathbb{R}^N} a(|\nabla u_n|^p)|\nabla u_n|^p dx + \int\limits_{\mathbb{R}^N} W(x)b(|u_n|^p)|u_n|^p dx\right].$$

Using now (a_1), (b_1), we obtain

$$\begin{aligned} c_W \geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\Bigg[& k_1\int\limits_{\mathbb{R}^N} |\nabla u_n|^p dx + \int\limits_{\mathbb{R}^N} |\nabla u_n|^N dx \\ & + K_3\int\limits_{\mathbb{R}^N} W(x)|u_n|^p dx + \int\limits_{\mathbb{R}^N} W(x)|u_n|^N dx\Bigg]. \end{aligned} \tag{2.3}$$

Then there are positive constants C_1, C_2 such that

$$c_W \geq C_1\|u\|_{p,W}^p + C_2\|u\|_{N,W}^N.$$

Arguing by contradiction, we assume that, up to a subsequence, $\|u_n\|_W \to +\infty$. It occurs one of the following situations:

(a) $\|u_n\|_{p,W} \to +\infty$ and $\|u_n\|_{N,W} \to +\infty$;
(b) $\|u_n\|_{p,N} \to +\infty$ and $\|u_n\|_{N,W}$ is bounded;
(c) $\|u_n\|_{p,W}$ is bounded and $\|u_n\|_{N,W} \to +\infty$.

But in all the cases, we obtain a contradiction with (i). Thus, for a subsequence still denoted by (u_n), there is $u_0 \in X_W$ such that $u_n \rightharpoonup u_0$ in X_W.

(iii) We have $u_n \to u_0$ in $L^s_{loc}(\mathbb{R}^N)$ for $s \geq N$ and $u_n(x) \to u_0(x)$ a.e in $\mathbb{R}^N$. Now considering the inequality (2.3) we obtain

$$\|u_n\|^p_{p,W} + \|u_n\|^N_{N,W} \leq \frac{1}{\min\{k_1, k_3, 1\}} \frac{\theta p\gamma}{(\theta - p\gamma)} C_W.$$

Using (i), we get

$$\|u_n\|^p_{p,W} + \|u_n\|^N_{N,W} \leq \frac{1}{\min\{k_1, k_3, 1\}} \frac{\theta p\gamma}{(\theta - p\gamma)} \frac{(r-p)}{pr\tau^{p/(r-p)}} \frac{c_r N r}{(r-N)}.$$

Using now (f_5), we obtain

$$\|u_n\|^p_{p,W} + \|u_n\|^N_{N,W} \leq \frac{1}{2^N}$$

and

$$\int\limits_{\mathbb{R}^N} |\nabla u_n|^N dx \leq \frac{1}{2^N}.$$

Then, $\|u\|_W \leq 1$ with $\int_{\mathbb{R}^N} |\nabla u_n|^N dx < 1$ and $\int_{\mathbb{R}^N} |u_n|^N dx \leq C$, for some $C > 0$.

By Proposition 1.2, there exist $C > 0$, $\alpha \geq 1$ sufficiently close to 1, such that the sequence (h_n) given by

$$h_n(x) = \exp\Big(\frac{\alpha}{2^N}\alpha_N\Big|\frac{u_n}{\|u_n\|}\Big|^{N/N-1}\Big) - S_{N-2}\Big(\frac{\alpha}{2^N}\alpha_N, u_n\Big)$$

belongs to $L^r(\mathbb{R}^N)$ and $|h_n|_r \leq C$. Combining this fact with the Brezis–Lieb lemma [8] (see also [14, Lemma 4.6]) we deduce that

$$\int\limits_{\mathbb{R}^N} f(u_n)\phi \to \int\limits_{\mathbb{R}^N} f(u_0)\phi \quad \text{for all } \phi \in X_W. \tag{2.4}$$

From now on, the proof of (iii) and (iv) follows by [4, Lemma 2.3]. □

The next proposition is a version of a Lions-type result in the framework of the critical exponential growth in $\mathbb{R}^N$ (for the case $N = 2$, see [5]).

Proposition 2.3. *Assume that* (f_1)–(f_3) *hold and let* $(u_n) \subset X_W$ *be a sequence with* $u_n \rightharpoonup 0$ *and* $\limsup_{n\to+\infty} \|u_n\|^N < 1$. *Assume that there exists* $R > 0$ *such that*

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^N = 0.$$

Then

$$\int\limits_{\mathbb{R}^N} f(u_n)u_n \to 0 \quad \textit{and} \quad \int\limits_{\mathbb{R}^N} F(u_n) \to 0.$$

Proof. By hypothesis

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^N = 0,$$

combined with Lemma 8.4 in [15], we deduce that

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^N) \text{ for all } t \in (N, +\infty).$$

From (2.1), for each $\Upsilon > 0$, $q = 1$ and $\alpha > 1$ closed to 1, it follows that

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \leq \Upsilon \int_{\mathbb{R}^N} |u_n|^p dx + C_\Upsilon \int_{\mathbb{R}^N} |u_n|[\exp(\alpha\alpha_N |u_n|^{\frac{N}{N-1}}) - S_{N-2}(\alpha\alpha_N, u_n)]dx.$$

By Proposition 1.2, there exists γ enough large such that

$$\int_{\mathbb{R}^N} f(u_n)u_n \leq \Upsilon C + C\Big(\int_{\mathbb{R}^N} |u_n|^\gamma\Big)^{1/\gamma} \to 0.$$

Thus, in view of assumption (f_3),

$$\int_{\mathbb{R}^N} F(u_n) \to 0.$$

This concludes the proof. □

Proposition 2.4. *Suppose that* $0 \leq W_0 \leq W(x) \leq V_0$ *and that* (f_1)*–*(f_6) *occur. Then problem* (P_W) *has a solution in* $C^{1,\alpha}(\mathbb{R}^N)$*, with* $0 < \alpha < 1$.

Proof. By Lemmas 2.1 and 2.2, there exists $u_0 \in X_W$ such that $J'(u_0) = 0$ and $u_0 \geq 0$. Suppose that $u_0 \not\equiv 0$. Adapting arguments found in [12, Theorem 1.11], we deduce that $u \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$, and therefore, from Harnack's inequality [23] it follows that $u_0(x) > 0$ for all $x \in \mathbb{R}^N$. If $u_0 \equiv 0$, we have the following claim:

Claim. There is a sequence $(y_n) \in \mathbb{R}^N$, and $R, \alpha > 0$ such that

$$\liminf_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^N > \alpha. \tag{2.5}$$

This claim is true, because for the contrary case, using the Proposition 2.3, we have

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0,$$

which implies that u_n converges strongly to zero, and consequently, $c_W = 0$. Thus, the last limit does not hold, and the claim is proved. It is clear that we may assume, without loss of generality, that $(y_n) \in Z^N$. Now, letting $\widetilde{u}_n(x) = u_n(x - y_n)$, since W is 1-periodic function, by a routine calculus we obtain $\|\widetilde{u}_n\| = \|u_n\|$, $J_W(\widetilde{u}_n) = J_W(u_n)$ and $J'_W(\widetilde{u}_n) = 0$. Thus, there exists $\widetilde{u}_0$ such that $\widetilde{u}_n \rightharpoonup \widetilde{u}_0$ weakly in X_W and as before it follows that $J'(\widetilde{u}_0) = 0$. Now, by (2.5), taking a subsequence and bigger R, we conclude that $\widetilde{u}_0$ is nontrivial and the proposition is proved. □

3. THE NONPERIODIC PROBLEM

The main tool employed to prove Theorem 1.1 is the variational method, namely to find critical points of the functional

$$I_\epsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(|\nabla u|^p) + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)B(|u|^p) - \int_{\mathbb{R}^N} F(u).$$

It is obvious that I_ϵ is well defined on the Banach space W_ϵ given by

$$W_\epsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)(|u|^p + |u|^N) < \infty \right\},$$

endowed with the norm

$$\|u\|_\epsilon = \|u\|_{V,p} + \|u\|_{V,N},$$

where

$$\|u\|^m_{V,m} = \int_{\mathbb{R}^N} |\nabla u|^m + \int_{\mathbb{R}^N} V(\epsilon x)|u|^m.$$

Let $\mathcal{N}_\epsilon$ denote the Nehari manifold related to I_ϵ given by

$$\mathcal{N}_\epsilon = \{u \in W_\epsilon \setminus \{0\} : I'_\epsilon(u)u = 0\}.$$

In the next result we prove that if (u_n) is a sequence in $\mathcal{N}$, then (u_n) cannot converge to 0.

Lemma 3.1. *There exists a constant $C > 0$ such that $0 < C \le \|u\|_\epsilon$ for every $u \in \mathcal{N}_\epsilon$.*

Proof. Suppose, by contradiction, that there is $(u_n) \subset \mathcal{N}_\epsilon$ such that

$$u_n \to 0 \quad \text{in} \quad W_\epsilon. \tag{3.1}$$

From $(a_1), (b_1)$ and (2.1), for each $\Upsilon > 0$, $q > N$ and $\alpha > 1$ closed to 1, it follows that

$$\begin{aligned} &k_1\|u_n\|^p_{V,p} + \|u_n\|^N_{V,N} \\ &\le \Upsilon \int_{\mathbb{R}^N} |u_n|^p dx + C_\Upsilon \int_{\mathbb{R}^N} |u_n|^q[\exp(\alpha\alpha_N|u_n|^{\frac{N}{N-1}}) - S_{N-2}(\alpha\alpha_N, u_n)]dx. \end{aligned}$$

Using Sobolev embeddings and since $\|u\|^{N-p}_{V,p} \le 1$, there exists $C_1 > 0$ such that

$$C_1(\|u_n\|^N_{V,p} + \|u_n\|^N_{V,N}) \le C_\Upsilon \int_{\mathbb{R}^N} |u_n|^q \exp\left(\alpha\alpha_N\|u_n\|^{N/N-1}_\epsilon\Big(\frac{|u_n|}{\|u_n\|_\epsilon}\Big)^{N/N-1}\right)dx.$$

By Hölder's inequality with $s', s > 1$ we find $C_2 > 0$ such that

$$C_2\|u_n\|^N_\epsilon \le C_\Upsilon\left(\int_{\mathbb{R}^N} |u_n|^{qs'}dx\right)^{1/s'}\left(\int_{\mathbb{R}^N}\exp\left(\alpha s\alpha_N\|u_n\|^{N/N-1}_\epsilon\Big(\frac{u_n}{\|u_n\|_\epsilon}\Big)^{N/N-1}\right)dx\right)^{1/s}.$$

Note that by (3.1), there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n\| \leq \left[\frac{1}{\alpha s}\right]^{N-1/N}$$

for all $n \geq n_0$. Thus, by Proposition 1.2 and Sobolev embeddings, we have

$$C_2\|u_n\|_\epsilon^N \leq MC_\Upsilon\Big(\int\limits_{\mathbb{R}^N} |u_n|^{qs'}dx\Big)^{1/s'} \leq MC_\Upsilon C\|u_n\|_\epsilon^q.$$

This inequality implies

$$\frac{C_2}{MC_\Upsilon C} \leq \|u_n\|_\epsilon^{q-N}.$$

Since $q > N$, the above inequality contradicts (3.1) and the lemma is proved. □

3.1. TECHNICAL RESULTS

In this subsection, we establish some properties concerning the functional I_ϵ. Arguing as in Lemma 2.1, we can show that I_ϵ satisfies the mountain pass geometry. This fact is stated in the below lemma.

Lemma 3.2. *The functional I_ϵ satisfies the following conditions:*

(i) *there exists α, $\rho > 0$ such that*

$$I_\epsilon(u) \geq \eta \quad \textit{for all } u \in W_\epsilon \textit{ with } \|u\|_\epsilon = \rho;$$

(ii) *there exists $e \in B_\rho^c(0)$ with $I_\epsilon(e) < 0$.*

By the mountain pass theorem of Ambrosetti and Rabinowitz without (PS) condition [25, Theorem 1.15], it follows that there exists a $(PS)_{c_\epsilon}$ sequence $(u_n) \subset W_\epsilon$, that is,

$$I_\epsilon(u_n) \to c_\epsilon \quad \text{and} \quad I_\epsilon'(u_n) \to 0,$$

where c_ϵ is the minimax level of mountain pass theorem applied to I_ϵ. Arguing as in the proof of Lemma 2.2, we can show that for each n, u_n is nonnegative and that (u_n) is bounded. Thus, there exist a subsequence, still denoted by (u_n), and $u \in W_\epsilon$ such that

$$u_n \rightharpoonup u \text{ in } W_\epsilon \quad \text{and} \quad u_n(x) \to u(x) \text{ a.e in } \mathbb{R}^N.$$

As in [4, Lemma 2.2], we obtain the following equivalent characterization of c_ϵ, which is more adequate to our purpose:

$$c_\epsilon = \inf_{u \in W_\epsilon \setminus \{0\}} \sup_{t \geq 0} I_\epsilon(tu) = \inf_{u \in \mathcal{N}_\epsilon} I_\epsilon(u).$$

Remark 3.3. It is easy to check that for each nonzero nonnegative $u \in W_\epsilon$, there exists a unique $t_0 = t_0(u)$ such that

$$I_\epsilon(t_0 u) = \max_{t \geq 0} I_\epsilon(tu).$$

3.2. A COMPACTNESS CONDITION

In this section, we prove some important lemmas to establish a compactness condition.

Lemma 3.4. *Let* (u_n) *be a sequence in* $W^{1,N}(\mathbb{R}^N)$ *with* $\sup_{n\in\mathbb{N}} \|u_n\|^N \le 1/2^{2N+3}$ *and* u *its weak limit in* $W^{1,N}(\mathbb{R}^N)$. *If* $v_n = u_n - u$ *and* (f_1)–(f_5) *hold, then the following properties hold:*

(I) *we have*

$$\int_{\mathbb{R}^N} |F(v_n+u) - F(v_n) - F(u)| = o_n(1)$$

where F *is the primite of* f*;*

(II) *there exists* $r \ge 1$, r *closed to 1, such that*

$$\int_{\mathbb{R}^N} |f(v_n+u) - f(v_n) - f(u)|^r = o_n(1).$$

Proof. The proof follows by similar arguments as in [3, Lemma 7]. □

Lemma 3.5. *Assume that* $V_\infty < +\infty$ *and let* (v_n) *be a* $(PS)_d$ *sequence for* I_ϵ *in* W_ϵ *with* $\limsup_{n\to+\infty} \|\nabla v_n\|^N_{L^N} \le m < 1$ *and* $v_n \rightharpoonup 0$ *in* $W^{1,p}(\mathbb{R}^N)$. *If* $v_n \not\to 0$ *in* $W^{1,N}(\mathbb{R}^N)$, *then* $d \ge c_{V_\infty}$, *where* c_{V_∞} *is the minimax level of* J_{V_∞}.

Proof. The proof follows by using similar arguments as in [4, Lemma 3.3]. □

3.3. PALAIS–SMALE CONDITION

In order to apply the Lusternik–Schnirelman category theory, we need to prove that I_ϵ satisfies the Palais–Smale condition on $\mathcal{N}_\epsilon$. Since the Sobolev embedding $W^{1,N}(\mathbb{R}^N) \subset L^s(\mathbb{R}^N)$ $(s \ge N)$ is continuous but is not compact, it is well known that, in general, such a condition is not fulfilled. Nevertheless, we still can prove (Proposition 3.7) that the Palais–Smale condition holds at a suitable sublevel, related to the ground energy "at infinity".

Proposition 3.6. *Let* (u_n) *be a sequence* $(PS)_c$ *for* I_ϵ, *with* $\|\nabla u_n\|^N_{L^N} \le 1/2^N$ *for all* $n \in \mathbb{N}$, *and assume that* $c < c_{V_\infty}$ *when* $V_\infty < \infty$, *or* $c \in \mathbb{R}$ *if* $V_\infty = \infty$. *Then* (u_n) *has a convergent subsequence in* W_ϵ.

Proof. It follows from [4, Lemmas 3.5 and 3.6] □

Proposition 3.7. *Let* (u_n) *be a sequence* $(PS)_c$ *for* I_ϵ *restricted to* $\mathcal{N}_\epsilon$, *with* $\|\nabla u_n\|^N_{L^N} \le 1/2^N$ *for all* $n \in \mathbb{N}$, *and assume that* $c < c_{V_\infty}$ *when* $V_\infty < \infty$, *or* $c \in \mathbb{R}$ *if* $V_\infty = \infty$. *Then* (u_n) *has a convergent subsequence in* W_ϵ.

Proof. Let $(u_n) \subset \mathcal{N}_\epsilon$ be such that $I_\epsilon(u_n) \to c$ and $\|I'_\epsilon(u_n)\|_* = o_n(1)$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\epsilon(u_n) = \lambda_n J'_\epsilon(u_n) + o_n(1), \tag{3.2}$$

where $J_\epsilon : X_\epsilon \to \mathbb{R}$ is given by

$$J_\epsilon(u) = \int\limits_{\mathbb{R}^N} a(|\nabla u|^p)|\nabla u|^p + \int\limits_{\mathbb{R}^N} V(\epsilon x)b(|u|^p)|u|^p - \int\limits_{\mathbb{R}^N} f(u)u.$$

Note that by (a_3), (b_3) and (f_4),

$$\begin{aligned} J'_\epsilon(u_n)u_n &= p\int\limits_{\mathbb{R}^N} a'(|\nabla u_n|^p)|\nabla u_n|^{2p} + p\int\limits_{\mathbb{R}^N} a(|\nabla u_n|^p)|\nabla u_n|^p \\ &\quad + p\int\limits_{\mathbb{R}^N} V(\epsilon x)b'(|u_n|^p)|u_n|^{2p} + p\int\limits_{\mathbb{R}^N} V(\epsilon x)b(|u_n|^p)|u_n|^p \\ &\quad - \int\limits_{\mathbb{R}^N} f(u_n)u_n - \int\limits_{\mathbb{R}^N} f'(u_n)(u_n)^2 \\ &\leq \int\limits_{\mathbb{R}^N} (N-1)f(u_n)u_n - \int\limits_{\mathbb{R}^N} f'(u_n)(u_n)^2 \leq 0, \end{aligned}$$

which implies that $\limsup_{n\to+\infty} J'_\epsilon(u_n)u_n = l \leq 0$.

Arguing as in the proof of Proposition 2.4, we can consider that $u_n \rightharpoonup u \neq 0$ in W_ϵ. Then, $l \neq 0$, leading to $\lambda_n = o_n(1)$. From (3.2), $I'_\epsilon(u_n) = o_n(1)$, hence (u_n) is a $(PS)_c$ sequence for I_ϵ and the result follows from Proposition 3.6. □

Corollary 3.8. *The critical points of I_ϵ on $\mathcal{N}_\epsilon$ are critical points of I_ϵ in W_ϵ.*

Proof. The proof follows by using similar arguments as in the previous proof. □

4. EXISTENCE OF A GROUND STATE SOLUTION

In this section, we prove the existence of a nonnegative ground state solution to problem (P_ϵ), that is, a nonnegative solution u_ϵ of (P_ϵ) satisfying $I_\epsilon(u_\epsilon) = c_\epsilon$ and $I'_\epsilon(u_\epsilon) = 0$. To this end, we adapt some ideas developed in [6].

Theorem 4.1. *Suppose that a, b and f verify (a_1)–(a_3), (b_1)–(b_3), (V_0) and (f_1)–(f_5), respectively. Then, there exists $\bar{\epsilon} > 0$ such that problem (P_ϵ) has a nonnegative ground state solution u_ϵ for all $0 < \epsilon < \bar{\epsilon}$.*

Proof. From Lemma 3.2, I_ϵ verifies the mountain pass geometry. Then, there exists a bounded sequence $(u_n) \subset W_\epsilon$ satisfying

$$I_\epsilon(u_n) \to c_\epsilon \quad \text{and} \quad I'_\epsilon(u_n) \to 0.$$

We show that there exists $\bar{\epsilon} > 0$ such that $c_\epsilon < c_{V_0}$, for all $\epsilon \in (0, \bar{\epsilon})$. Since $c_{V_0} < c_{V_\infty}$ when $V_\infty < \infty$, from Proposition 3.6, we conclude that I_ϵ satisfies the $(PS)_{c_\epsilon}$ condition. Thus, there exists $u \in W_\epsilon$ such that

$$I_\epsilon(u) = c_\epsilon \quad \text{and} \quad I'_\epsilon(u) = 0.$$

If $V_\infty < \infty$, let us consider without loss of generality that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x).$$

Let $\mu \in \mathbb{R}$ such that $V_0 < \mu < V_\infty$. Since $c_{V_0} < c_\mu < c_\infty$, there exists a nonnegative function $w \in X_\mu$ with compact support such that $J_\mu(w) = \max_{t \geq 0} J_\mu(tw)$ and $J_\mu(w) < c_\infty$. The condition (V_0) implies that for some $\overline{\epsilon} > 0$

$$V(\epsilon x) \leq \mu \quad \text{for all } x \in \operatorname{supp} w \text{ and } \epsilon \leq \overline{\epsilon},$$

so

$$\int_{\mathbb{R}^N} V(\epsilon x) B(|tw|^p) \leq \int_{\mathbb{R}^N} \mu B(|tw|^p) \quad \text{for all } \epsilon \leq \overline{\epsilon} \text{ and } t > 0.$$

Consequently

$$I_\epsilon(tw) \leq J_\mu(tw) \leq J_\mu(w) \quad \text{for all } t > 0$$

from where it follows that

$$\max_{t > 0} I_\epsilon(tw) \leq J_\mu(w),$$

showing that $c_\epsilon < c_\infty$. Therefore, the theorem follows from Proposition 3.6. □

5. MULTIPLICITY OF SOLUTIONS TO (P_ϵ)

In this section, our main goal is to show the existence of multiple solutions and to study the behavior of its maximum points in relationship with the set M.

5.1. PRELIMINARY RESULTS

Let $\delta > 0$ be fixed and w be a ground state solution of problem (P_{V_0}), that is, $J_{V_0}(w) = c_{V_0}$ and $J'_{V_0}(w) = 0$. Let η be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that $\eta(s) = 1$ if $0 \leq s \leq \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \geq \delta$.

For any $y \in M$, let us define

$$\Psi_{\epsilon,y}(x) = \eta(|\ \epsilon x - y\ |) w\Big(\frac{\epsilon x - y}{\epsilon}\Big).$$

and $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by

$$\Phi_\epsilon(y) = t_\epsilon \Psi_{\epsilon,y}.$$

where $t_\epsilon > 0$ verifies

$$\max_{t \geq 0} I_\epsilon(t\Psi_{\epsilon,y}) = I_\epsilon(t_\epsilon \Psi_{\epsilon,y}).$$

By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$.

Lemma 5.1. *The function Φ_ϵ satisfies*

$$\lim_{\epsilon\to 0} I_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \quad \textit{uniformly in } y \in M.$$

Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - c_{V_0}| \geq \delta_0. \tag{5.1}$$

Repeating the same arguments as in [1], it is possible to check that $t_{\epsilon_n} \to 1$. Next, by Lebesgue's theorem,

$$\lim_{n\to\infty} \| \Psi_{\epsilon_n, y_n} \|_{\epsilon_n} = \Big(\int_{\mathbb{R}^n} (|\nabla w|^p + V_0|w|^p) \Big)^{\frac{1}{p}} + \Big(\int_{\mathbb{R}^n} (|\nabla w|^N + V_0|w|^N) \Big)^{\frac{1}{N}}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(\Psi_{\epsilon_n, y_n}) = \int_{\mathbb{R}^N} F(w).$$

From the above limits, we obtain

$$\lim_{n\to\infty} I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = J_{V_0}(w) = c_{V_0},$$

which contradicts (5.1), and the proof of the lemma is finished. □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| \leq \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider $\beta : \mathcal{N}_\epsilon \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon x)|u(x)|^p}{\int_{\mathbb{R}^N} |u(x)|^p}.$$

The next three lemmas follow by using the same arguments found in [2, Lemma 4.3]. For this reason, we omit their proofs.

Lemma 5.2. *The function Φ_ϵ verifies the following limit*

$$\lim_{\epsilon\to 0} \beta(\Phi_\epsilon(y)) = y, \quad \textit{uniformly in } y \in M.$$

Lemma 5.3 (A compactness principle)**.** *Let $(u_n) \subset \mathcal{M}$ be a sequence satisfying $J_{V_0}(u_n) \to c_{V_0}$. Then the following alternative holds:*

(a) *(u_n) has a subsequence strongly convergent in X_{V_0}*

or

(b) *there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $v_n(x) = u_n(x+\tilde{y}_n)$ converges strongly in X_{V_0}.*

In particular, there exists a minimizer for c_{V_0}.

Proposition 5.4. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to c_0$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in X_{V_0}. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \epsilon_n \tilde{y}_n$.*

Corollary 5.5. *Given $\xi > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|u_n\|_{X_{V_0}(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \quad \text{for all } n \geq n_0.$$

Proof. By Proposition 5.4, there exists $v \in X_{V_0}$ such that $v_n \to v$ in X_{V_0}, that is,

$$\Big(\int_{\mathbb{R}^N} |\nabla(u_n(x + \tilde{y}_n) - v)|^p \Big)^{1/p} + \Big(\int_{\mathbb{R}^N} |\nabla(u_n(x + \tilde{y}_n) - v)|^N \Big)^{1/N}$$
$$+ \Big(\int_{\mathbb{R}^N} |u_n(x + \tilde{y}_n) - v|^p \Big)^{1/p} + \Big(\int_{\mathbb{R}^N} |u_n(x + \tilde{y}_n) - v|^N \Big)^{1/N} \to 0.$$

Consequently

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u_n(x + \tilde{y}_n) - v|^p \to 0$$

and

$$\int_{\mathbb{R}^N \setminus B_R(0)} |u_n(x + \tilde{y}_n) - v|^N \to 0.$$

By the change of variable $z = x + \tilde{y}_n$, we obtain

$$\int_{\mathbb{R}^N \setminus B_R(\tilde{y}_n)} |u_n(z) - v(z - \tilde{y}_n)|^p \to 0 \tag{5.2}$$

and

$$\int_{\mathbb{R}^N \setminus B_R(\tilde{y}_n)} |u_n(z) - v(z - \tilde{y}_n)|^N \to 0. \tag{5.3}$$

On the other hand, given $\xi > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_R(\tilde{y}_n)} |v(x - \tilde{y}_n)|^p = \int_{{}^N \setminus B_R(0)} |v|^p < \frac{\xi}{2}$$

and

$$\int_{\mathbb{R}^N \setminus B_R(\tilde{y}_n)} |v(x - \tilde{y}_n)|^N = \int_{\mathbb{R}^N \setminus B_R(0)} |v|^N < \frac{\xi}{2}$$

for all $n \in \mathbb{N}$. Hence, by (5.2) and (5.3), there exists $n_0 \in \mathbb{N}$ such that

$$|u_n|_{L^p(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \quad \text{for all } n \geq n_0$$

and

$$|u_n|_{L^N(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \quad \text{for all } n \geq n_0.$$

Similar arguments show that

$$|\nabla u_n|_{L^p(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \quad \text{for all } n \geq n_0,$$

and

$$|\nabla u_n|_{L^N(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \quad \text{for all } n \geq n_0,$$

which completes the proof. □

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function tending to 0 such that $h(\epsilon) \to 0$ as $\epsilon \to 0$ and let

$$\widetilde{\mathcal{N}}_\epsilon = \{u \in \mathcal{N}_\epsilon : I_\epsilon(u) \leq d_{V_0} + h(\epsilon)\}.$$

Lemma 5.6. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\}$. Then*

$$\lim_{\epsilon \to 0} \sup_{u \in \widetilde{\mathcal{N}}_\epsilon} \inf_{y \in M_\delta} |\beta(u) - y| = 0.$$

Proof. The proof follows by using the same arguments found in [2]. □

5.2. PROOF OF THEOREM 1.1

We divide the proof in two parts.

Part I: Multiplicity of solutions. In the sequel, $\epsilon > 0$ is small enough. Then, by Lemmas 5.1 and 5.6, we have $\beta \circ \Phi_\epsilon$ is homotopic to the inclusion map $id : M \to M_\delta$. This fact implies

$$\operatorname{cat}_{\widetilde{\mathcal{N}}_\epsilon}(\widetilde{\mathcal{N}}_\epsilon) \geq \operatorname{cat}_{M_\delta}(M).$$

Since that functional I_ϵ satisfies the $(PS)_c$ condition for $c \in (c_0, c_0 + h(\epsilon))$, by the Lusternik–Schnirelman theory of critical points (see [25]), we can conclude that I_ϵ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\epsilon$. Consequently by Corollary 3.8, I_ϵ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in X_ϵ.

Part II: The behavior of maximum points. The next two lemmas play an important role in the study of the behavior of the maximum points of the solutions. In the proof of the next lemma, we adapted some arguments found in [12] and [13], which are related to the Moser iteration method [16].

Lemma 5.7. *Let u_n be a solution of the following problem*

$$\begin{cases} -\operatorname{div}(a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n) + V(\epsilon_n x) b(|u_n|^p)|u_n|^{p-2}u_n = f(u_n) \text{ in } \mathbb{R}^N, \\ u_n \in X_{\epsilon_n}, \quad 1 < p < N, \\ u_n(z) > 0 \text{ for all } z \in \mathbb{R}^N. \end{cases}$$

Then, for each n, $u_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|u_n|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Furthermore, given $\xi > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$|u_n|_{L^\infty(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < \xi \text{ for all } n \geq n_0,$$

where $\{\tilde{y}_n\}$ was given in Proposition 5.4.

Proof. It is sufficient to use (a_1), (b_1) and [3, Lemma 15]. □

Lemma 5.8. *There exists $\delta > 0$ such that $|u_n|_{L^\infty(|x-\tilde{y}_n|<R)} \geq \delta$ for all $n \geq n_0$.*

Proof. If $|u_n|_{L^\infty(|x-\tilde{y}_n|<R)} \to 0$, by Lemma 5.7, we have $|u_n|_{L^\infty(\mathbb{R}^N)} \to 0$. Fixed $\epsilon_0 = \frac{V_0}{2}$, it follows from (f_5) that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{f(u_n)}{|u_n|^{N-1}} < \epsilon_0 \quad \text{for } n \geq n_0.$$

Therefore

$$\int_{\mathbb{R}^N} a(|\nabla u_n|^p)|\nabla u_n|^p + \int_{\mathbb{R}^N} V(\epsilon_n x) b(|u_n|^p)|u_n|^p \leq 0,$$

that is,

$$\|u_n\|_{X_{\epsilon_n}} = 0 \quad \text{for all } n \geq n_0,$$

which is absurd, because $u_n \neq 0$ for all $n \in \mathbb{N}$. Thus, there exists $\delta > 0$ such that $|u_n|_{L^\infty(|x-\tilde{y}_n|<R)} \geq \delta$, for all $n \geq n_0$.

Considering $v_n(x) = u_n(x + \tilde{y}_n)$, by Lemmas 5.7 and 5.8, we have that

$$|v_n|_{L^\infty(|x|\geq R)} < \xi \quad \text{for all } n \geq n_0$$

and

$$|v_n|_{L^\infty(|x|\leq R)} \geq \delta \quad \text{for all } n \geq n_0.$$

Thus, there exists $q_n \in B_R(0)$ such that $v(q_n) = \max_{z \in \mathbb{R}^N} v_n(z)$ and $v(q_n) = u_n(q_n + \tilde{y}_n)$. Hence, $x_n = q_n + \tilde{y}_n$ is a maximum point of $\{u_n\}$ and

$$\lim_{n\to\infty} \epsilon_n x_n = \lim_{n\to\infty} \epsilon_n \tilde{y}_n = y \in M.$$

Since V is a continuous functions, we get

$$\lim_{n\to\infty} V(\epsilon_n x_n) = V(y) = V_0.$$

The proof is now complete. □

Acknowledgments

Giovany M. Figueiredo is supported by CNPq 300959/2005–2.

Vicenţiu D. Rădulescu is supported was supported by the Slovenian Research Agency grants P1–0292, J1–8131, J1–7025, N1–0064, and N1–0083.

REFERENCES

[1] C.O. Alves, *Existence and multiplicity of solutions for a class of quasilinear equations*, Adv. Nonlinear Studies **5** (2005), 73–87.

[2] C.O. Alves, G.M. Figueiredo, *Existence and multiplicity of positive solutions to a p-Laplacian equation in* $\mathbb{R}^N$, Differential and Integral Equations **19** (2006), 143–162.

[3] C.O. Alves, G.M. Figueiredo, *On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in* $\mathbb{R}^N$, J. Differential Equations **246** (2009), 1288–1311.

[4] C.O. Alves, G.M. Figueiredo, *Multiplicity and concentration of positive solutions for a class of quasilinear problems*, Adv. Nonlinear Studies **11** (2011), 265–294.

[5] C.O. Alves, J.M. Bezerra do Ó, O.H. Miyagaki, *On pertubations of a class of a periodic m-Laplacian equation with critical growth*, Nonlinear Anal. **45** (2001), 849–863.

[6] C.O. Alves, M.A.S. Souto, *On existence and concentration behavior of ground state solutions for a class of problems with critical growth*, Comm. Pure Appl. Anal. **1** (2002), 417–431.

[7] J.M. Bezerra do Ó, *N-Laplacian equations in $\mathbb{R}^N$ with critical growth,* Abstract and Applied Analysis **2** (1997), 301–315.

[8] H. Brezis, E.H. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **8** (1983), 486–490.

[9] D.M. Cao, *Nontrivial solutions of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Diff. Equations **17** (1992), 407–435.

[10] L. Cherfils, Y. Il'yasov, *On the stationary solutions of generalized reaction difusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal. **4** (2005), 9–22.

[11] F. Gazzola, V.D. Rădulescu, *A nonsmooth critical point theory approach to some nonlinear elliptic equations in* $\mathbb{R}^n$, Differential Integral Equations **13** (2000), 47–60.

[12] Li Gongbao, *Some properties of weak solutions of nonlinear scalar field equations*, Annales Acad. Sci. Fennicae, Series A **14** (1989), 27–36.

[13] C. He, G. Li, *The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians*, Annales Acad. Sci. Fennicae, Series A **33** (2008), 337–371.

[14] O. Kavian, *Introduction à la théorie des points critiques et applications aux problèmes elliptiques*, Springer, Heidelberg, 1983.

[15] P.L. Lions, *The concentration-compacteness principle in the calculus of variation. The locally compact case, part II*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 223–283.

[16] J. Moser, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations,* Comm. Pure Appl. Math. **13** (1960), 457–468.

[17] J.Y. Oh, *On positive multi-bump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Partial Diff. Equations **131** (1990), 223–253.

[18] N.S. Papageorgiou, V.D. Rădulescu, *Resonant* $(p, 2)$*-equations with asymmetric reaction*, Analysis and Applications **13** (2015), 481–506.

[19] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *On a class of parametric* $(p, 2)$*-equations*, Applied Mathematics and Optimization **75** (2017), 193–228.

[20] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, $(p, 2)$*-equations symmetric at both zero and infinity*, Advances in Nonlinear Analysis **7** (2018), 327–351.

[21] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, Springer, Berlin, 2019.

[22] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 27–42.

[23] N.S. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **XX** (1967), 721–747.

[24] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[25] M. Willem, *Minimax Theorems*, Birkhäuser, Basel, 1996.

Giovany M. Figueiredo
giovany@unb.br

Universidade de Brasília – UnB
Departamento de Matemática
CEP: 70910-900 Brasília–DF, Brazil

Vicenţiu D. Rădulescu (corresponding author)
Vicentiu.Radulescu@imar.ro

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Krakow, Poland

Department of Mathematics
University of Craiova
200585 Craiova, Romania

"Simion Stoilow" Institute of Mathematics of the Romanian Academy
21 Calea Grivitei, 010702 Bucharest, Romania

Received: February 13, 2019.
Accepted: February 27, 2019.