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ORIGINAL PAPER

Further on set-valued equilibrium problems in the pseudo-monotone case and applications to Browder variational inclusions

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Abstract This paper deals with set-valued equilibrium problems under conditions of pseudo-monotonicity. Concepts such as strict quasi-convexity, hemicontinuity and pseudo-monotonicity for extended real set-valued mappings are introduced and applied to obtain results on the existence of solutions of set-valued equilibrium problems generalizing those in the literature in the pseudo-monotone case. Applications to Browder variational inclusions under weakened conditions are given. In particular, it is shown that the upper semicontinuity from line segments of the involved pseudo-monotone set-valued operator is not needed in the whole space when solving Browder variational inclusions.

Keywords Equilibrium problem · Set-valued mapping · Convexity · Hemicontinuity · Pseudo-monotonicity · Variational inclusion

Mathematics Subject Classification Primary 26A15; Secondary 26B25 · 26E25 · 47J20 · 49J35

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1 Introduction

Set-valued equilibrium problems are increasingly drawing the attention of many authors since they not only generalize the single-valued equilibrium problems, but also serve as unified models to study multivalued variational inequalities. These inequalities include as a special case Browder variational inclusions which appear in the literature as a generalization of Browder–Hartman–Stampacchia variational inequalities and have many applications, including applications to the surjectivity of set valued-mappings and to nonlinear elliptic boundary value problems, see for example [1,2,6,8,9,16–19,22–25], and the references therein. It is worthwhile recalling that equilibrium problems have also many applications to different areas of mathematics, including optimization problems, fixed point theory and Nash equilibrium problems.

Let C be a nonempty subset of a real topological Hausdorff vector space in the general settings, and $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ a set-valued mapping called *a set-valued bifunction*. The *strong set-valued equilibrium problem* is a problem of the form

$$\text{find } x_0 \in C \text{ such that } \Phi(x_0, y) \subset \overline{\mathbb{R}}_+ \quad \forall y \in C. \tag{Ssvep}$$

The *weak set-valued equilibrium problem* is a problem of the form

$$\text{find } x_0 \in C \text{ such that } \Phi(x_0, y) \cap \overline{\mathbb{R}}_+ \neq \emptyset \quad \forall y \in C. \tag{Wsvep}$$

Due to the importance of the pseudo-monotone case, and motivated by our own recent investigations on the continuity and convexity properties of real set-valued mappings in [8], we continue in this paper developing our ideas in order to apply them to set-valued equilibrium problems in the pseudo-monotone case. Note that although pseudo-monotone single-valued equilibrium problems have been intensively investigated in the literature, there does not seem to be any study of set-valued equilibrium problems in the pseudo-monotone case.

Our objective is twofold. The first is to obtain results on the existence of solutions of set-valued equilibrium problems generalizing those for pseudo-monotone single-valued equilibrium problems. The second objective is to investigate Browder variational inclusions involving pseudo-monotone operators which have already been considered in [24].

In this paper, we extend, in Sect. 2, some notions of convexity introduced recently in [8] to the extended real set-valued mappings case, and introduce different other notions such as strict quasi-convexity, upper hemicontinuity and pseudo-monotonicity. We obtain, in Sect. 3, different results on the existence of solutions of both strong and weak set-valued equilibrium problems generalizing those in the literature for the single-valued equilibrium problems in the pseudo-monotone case. Section 4 of this paper is devoted to the study of Browder variational inclusions involving pseudo-monotone operators. Our results generalize both those in [24] on Browder variational inclusions, and those obtained in [7] on Browder–Hartman–Stampacchia variational inequalities. The reflexivity of the Banach space is omitted, and the improvement concerns also different other conditions including the continuity and the pseudo-monotonicity of the involved set-valued operator. We obtain in particular, that the upper semicontinuity from line segments of the set-valued operator as well as the weak* compactness of the images of compact sets are no longer needed in the whole space when solving Browder variational inclusions involving pseudo-monotone set-valued operators.

2 Notations and preliminary results

In all the paper, $\mathbb{R} =]-\infty, +\infty[$ denotes the set of real numbers and $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$. We also make use of the following notation: $\mathbb{R}_+ = [0, +\infty[$, $\mathbb{R}_+^* =]0, +\infty[$, $\mathbb{R}_- = -\mathbb{R}_+$, $\mathbb{R}_-^* = -\mathbb{R}_+^*$, and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

In the sequel, $\overline{\mathbb{R}}$ will be endowed with the topology extended from the usual topology of $\mathbb{R}$, and with the usual operations involving $+\infty$ and $-\infty$. For a subset A of a Hausdorff topological space X, we denote by cl A, the closure of A.

By a set-valued mapping $F : X \rightrightarrows Y$, we mean a mapping F from a set X to the collection of nonempty subsets of a set Y. In the present paper, a mapping $f : X \to Y$ and the set-valued mapping $F : X \rightrightarrows Y$ defined by $F(x) = \{f(x)\}$ for every $x \in X$, will be identified and both will be called a single-valued mapping. That is, a single-valued mapping is a "classical" mapping or a set-valued mapping with singleton values. By a real set-valued mapping, we mean a set-valued mapping with values in $\mathbb{R}$. A real single-valued mapping is a single-valued mapping with values in $\mathbb{R}$. When $\overline{\mathbb{R}}$ is used instead of $\mathbb{R}$, we talk about extended real single-valued or extended real set-valued mappings.

2.1 Concepts of convexity

As mentioned in [8], the notions of convexity and concavity of set-valued mappings considered in the literature as a generalization of convexity and concavity of real single-valued mappings, are in fact not limited to real set-valued mappings and not really adapted to them. Applied to real single-valued mappings, they are stronger than the convexity and concavity and produce a sort of "linearity on line segments". For more details on the rich field about convexity and related notions of real single-valued mappings, we refer to [10,13].

Let C be a nonempty convex subset of a real topological Hausdorff vector space. Recall that in the literature, a real set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is said to be *convex* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$\sum_{i=1}^{n} \lambda_i F(x_i) \subset F\left(\sum_{i=1}^{n} \lambda_i x_i\right),$$

where the sum denotes here the usual Minkowski sum of sets. The real set-valued mapping $F : C \rightrightarrows \mathbb{R}$ is said to be *concave* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$F\left(\sum_{i=1}^{n}\lambda_i x_i\right)\subset\sum_{i=1}^{n}\lambda_i F(x_i).$$

Let $F : C \rightrightarrows \overline{\mathbb{R}}$ be an extended real set-valued mapping. Following [8], we say that F is *convexly quasi-convex* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n}\lambda_i = 1$, then for every $\{z_1, \ldots, z_n\}$ with $z_i \in F(x_i)$ for every $i = 1, \ldots, n$, there exists $z \in F\left(\sum_{i=1}^{n}\lambda_i x_i\right)$ such that

$$z \le \max\{z_i \mid i = 1, \ldots, n\}.$$

Clearly, the convex quasi-convexity of extended real set-valued mappings generalizes both the convexity of set-valued mappings and the quasi-convexity of extended real single-valued mappings.

For $\lambda \in \overline{\mathbb{R}}$, we set $[F \le \lambda] = \{x \in C \mid F(x) \cap [-\infty, \lambda] \ne \emptyset\}$. With a similar proof to that of [8, Proposition 2.1] but adapted to the extended real set-valued mappings, we obtain the following characterization.

Proposition 2.1 *Let C be a nonempty convex subset of a real topological Hausdorff vector space. An extended real set-valued mapping $F : C \rightrightarrows \overline{\mathbb{R}}$ is convexly quasi-convex on C if and only if the set $[F \le \lambda]$ is convex, for every $\lambda \in \overline{\mathbb{R}}$.*

Following [8], we say that F is *concavely quasi-convex* on C if whenever $\{x_1, \ldots, x_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ such that $\sum_{i=1}^{n}\lambda_i = 1$, then for every $z \in F\left(\sum_{i=1}^{n}\lambda_i x_i\right)$, there exist $\{z_1, \ldots, z_n\}$ with $z_i \in F(x_i)$ for every $i = 1, \ldots, n$ such that

$$z \le \max\{z_i \mid i = 1, \ldots, n\}.$$

Clearly, the concave quasi-convexity of extended real set-valued mappings generalizes both the concavity of set-valued mappings and the quasi-convexity of extended real single-valued mappings.

For $\lambda \in \overline{\mathbb{R}}$, we set $[F \subseteq \lambda] = \{x \in C \mid F(x) \subset [-\infty, \lambda]\}$. With a similar proof to that of [[8], Proposition 2.2] but adapted to the extended real set-valued mappings, we obtain the following characterization.

Proposition 2.2 *Let C be a nonempty convex subset of a real topological Hausdorff vector space. If an extended real set-valued mapping $F : C \rightrightarrows \overline{\mathbb{R}}$ is concavely quasi-convex on C, then the set $[F \subseteq \lambda]$ is convex, for every $\lambda \in \overline{\mathbb{R}}$.*

Note that if F is a real single-valued mapping, then $[F \le \lambda] = [F \subseteq \lambda]$, for every $\lambda \in \overline{\mathbb{R}}$. Furthermore, the quasi-convexity, the convex quasi-convexity and the concave quasi-convexity of F on C are all equivalent.

Now, we introduce the following notion of semistrict quasi-convexity of extended real set-valued mappings. We say that F is *semistrictly convexly quasi-convex* on C if whenever $x_1, x_2 \in C$ such that $F(x_1) \ne F(x_2)$ and $\lambda \in]0, 1[$, then for every $z_1 \in F(x_1)$ and $z_2 \in F(x_2)$, the following holds

(1) there exists $z \in F(\lambda x_1 + (1-\lambda) x_2)$ such that

$$z < \max\{z_1, z_2\},$$

(2) whenever $z' \in F(\lambda x_1 + (1-\lambda) x_2)$, we have

$$\text{if } z' \leq \max\{z_1, z_2\}, \text{ then } z' < \max\{z_1, z_2\}.$$

Every convex extended real set-valued mapping and every semistrictly quasi-convex extended real single-valued mapping is semistrictly convexly quasi-convex extended real set-valued mapping. It is known that there is not any inclusion relationship between the class of semistrictly quasi-convex real single-valued mappings and that of quasi-convex real single-valued mappings, see [13]. It follows that there is not any inclusion relationship between the class of semistrictly convexly quasi-convex extended real set-valued mappings and that of convexly quasi-convex extended real set-valued mappings.

Example 1 Let $C = [-1, 1]$ and $F : C \rightrightarrows \mathbb{R}$ be the set-valued defined by

$$F(x) = \begin{cases} \left[\frac{1}{2}, 1\right] & \text{if } x \neq 0, \\ \left[\frac{3}{2}, 2\right] & \text{if } x = 0. \end{cases}$$

The set-valued F is not convexly quasi-convex since for $x_1 = -1$, $x_2 = 1$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$, we have $\lambda_1 x_1 + \lambda_2 x_2 = 0$. Then clearly, for $z_1 \in F(x_1)$ and $z_2 \in F(x_2)$, we have

$$z > \max\{z_1, z_2\},$$

for every $z \in F(\lambda_1 x_1 + \lambda_2 x_2)$. However, F is semistrictly convexly quasi-convex. Indeed, take $x_1, x_2 \in C$, and in order to apply the definition, we must assume (without loss of generality) that $x_1 = 0$ and $x_2 \neq 0$. Then for $\lambda \in]0, 1[$, we have

$$F(\lambda x_1 + (1-\lambda) x_2) = F((1-\lambda) x_2) = \left[\frac{1}{2}, 1\right] \quad \text{and} \quad F(x_1) = \left[\frac{3}{2}, 2\right].$$

Clearly, for every $z_1 \in F(x_1)$, $z_2 \in F(x_2)$ and $z \in F(\lambda x_1 + (1-\lambda) x_2)$, we have $z < \max\{z_1, z_2\}$.

Remark 1 The notion of semistrictly convexly quasi-convex extended real set-valued mapping will be used in Proposition 3.2 and Proposition 3.4 below where we need only the Condition 2. The Condition 1 has been used in order to make this notion a generalization of the notion of semistrictly quasi-convex extended real single-valued mapping.

In the sequel, an extended real set-valued mappings will be said *explicitly convexly quasi-convex* if it is both convexly quasi-convex and semistrictly convexly quasi-convex. One can consult [11] where the techniques related to explicit quasi-convex single-valued mappings have been first used for solving single-valued equilibrium problems.

2.2 Concepts of continuity

Here, we recall some old and new concepts of continuity of set-valued mappings we need in the paper.

Let X and Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping. Let X and Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping. For a subset B of Y, the lower inverse set of B by F is $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. The upper inverse set of B by F is $F^+(B) = \{x \in X \mid F(x) \subset B\}$.

The set-valued mapping F is said to be lower semicontinuous at a point $x \in X$ if whenever V is an open subset of Y such that $F(x) \cap V \neq \emptyset$, the lower inverse set $F^-(V)$ of V by F is a neighborhood of x. The set-valued mapping F is said to be upper semicontinuous at a point $x \in X$ if whenever V is an open subset of Y such that $F(x) \subset V$, the upper inverse set $F^+(V)$ of V by F is a neighborhood of x. For further details on the rich field about continuity of set-valued mappings with different characterizations, we refer to [20].

Following [6], the set-valued mapping F is lower semicontinuous (*resp.*upper semicontinuous) on a subset S of X if it is lower semicontinuous (*resp.*upper semicontinuous) at every point of S. In particular, it is proved that a set-valued mapping $F : X \rightrightarrows Y$ is upper semicontinuous on a subset S of X if and only if for every closed subset B of Y, $F^-(B) \cap S = \mathrm{cl}\left(F^-(B)\right) \cap S$, see [[6], Proposition 2.4.].

Let X be a Hausdorff topological space and $F : X \rightrightarrows \overline{\mathbb{R}}$ an extended real set-valued mapping. Following [8], the extended real set-valued mapping F is said to be l-lower (*resp.*l-upper, *resp.*u-lower, *resp.*u-upper) semicontinuous at $x \in X$ if for every $\lambda \in \mathbb{R}$ such that $F(x) \cap]\lambda, +\infty] \neq \emptyset$ (*resp.*$F(x) \cap [-\infty, \lambda[\neq \emptyset$, *resp.*$F(x) \subset]\lambda, +\infty]$, *resp.*$F(x) \subset [-\infty, \lambda[$), there exists an open neighborhood U of x such that $F\left(x'\right) \cap]\lambda, +\infty] \neq \emptyset$ (*resp.*$F\left(x'\right) \cap [-\infty, \lambda[\neq \emptyset$, *resp.*$F\left(x'\right) \subset]\lambda, +\infty]$, *resp.*$F\left(x'\right) \subset [-\infty, \lambda[$), for every $x' \in U$.

Obviously, for an extended real single-valued mapping $f : X \to \overline{\mathbb{R}}$, the lower semicontinuity, the l-lower semicontinuity and the u-lower semicontinuity of f at $x \in X$ are all equivalent. Also, the upper semicontinuity, the l-upper semicontinuity and the u-upper semicontinuity of f at $x \in X$ are all equivalent.

The extended real set-valued mapping F is said to be l-lower (*resp.*l-upper, *resp.*u-lower, *resp.*u-upper) semicontinuous on a subset S of X if it is l-lower (*resp.*l-upper, *resp.*u-lower, *resp.*u-upper) semicontinuous at every point of S. One can consult [8] for the proof of the following results and characterizations.

Proposition 2.3 *Let X be a Hausdorff topological space, S a subset of X and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Then, F is*

(1) *l-lower semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^+([-\infty, \lambda]) \cap S = \mathrm{cl}\left(F^+([-\infty, \lambda])\right) \cap S.$$

(2) *l-upper semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{+}([\lambda, +\infty]) \cap S = \mathrm{cl}\left(F^{+}([\lambda, +\infty])\right) \cap S.$$

(3) *u-lower semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{-}([-\infty, \lambda]) \cap S = \mathrm{cl}\left(F^{-}([-\infty, \lambda])\right) \cap S.$$

(4) *u-upper semicontinuous on S if and only if for every $\lambda \in \mathbb{R}$, we have*

$$F^{-}([\lambda, +\infty]) \cap S = \mathrm{cl}\left(F^{-}([\lambda, +\infty])\right) \cap S.$$

2.3 Concepts of continuity on line segments

In the literature, various concepts related to continuity on line segments of single-valued and set-valued mappings defined on real topological Hausdorff vector spaces have been introduced and used in different works. Recently in [3], a weakened notion of hemicontinuity of extended real single-valued mappings have been introduced and employed for the existence of solutions of pseudo-monotone and quasi-monotone single-valued equilibrium problems. Also in [5], the notion of lower quasi-hemicontinuity has been introduced and employed for the existence of solutions of quasi-hemivariational inequalities.

Here, we introduce the notions of upper hemicontinuous and quasi-upper hemicontinuous extended real set-valued mappings which generalize both the upper hemicontinuity of extended real single-valued mapping and the lower semicontinuity of set-valued mappings.

Let X be a real topological Hausdorff vector space. For $x, y \in X$, we put $[x, y] = \{\lambda x + (1-\lambda) y \mid \lambda \in [0, 1]\}$, the line segment starting at x and ending at y. We also put $]x, y[= [x, y] \setminus \{x, y\}$. We say that a set-valued mapping $F : X \rightrightarrows \overline{\mathbb{R}}$ is

(1) *upper hemicontinuous* at a point $x \in X$ if whenever $x' \in X$, there exists a sequence $(t_n)_n$ in $]0, 1[$ converging to 0 such that for every $z \in F(x)$, there exists a sequence $(z_n)_n$ with $z_n \in F\left(t_n x' + (1-t_n) x\right)$ for every n, and such that

$$z \geq \limsup_{n \to +\infty} z_n,$$

where $\limsup_{n\to+\infty} z_n = \inf_n \sup_{k \geq n} z_k$.

(2) *quasi-upper hemicontinuous* at a point $x \in X$ if whenever $x' \in X$ there exist a sequence $(t_n)_n$ in $]0, 1[$ converging to 0, a point $z \in F(x)$, and a sequence $(z_n)_n$ with $z_n \in F\left(t_n x' + (1-t_n) x\right)$ for every n such that

$$z \geq \limsup_{n \to +\infty} z_n.$$

The set-valued mapping F will be said upper hemicontinuous (*resp*.quasi-upper hemicontinuous) on X if it is upper hemicontinuous (*resp*.quasi-upper hemicontinuous) at every point of X. It will be said upper hemicontinuous (*resp*.quasi-upper hemicontinuous) on a subset $S \subset X$, if it is upper hemicontinuous (*resp*.quasi-upper hemicontinuous) at every point of S.

Remark 2 Note that when $x \neq x'$ in the above definition, then we have $t_n x' + (1 - t_n)\, x \in]x', x[$, for every n.

Proposition 2.4 *Let X be a real topological Hausdorff vector space, $x \in X$, and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping.*

1. *If F is lower semicontinuous at x, then F is upper hemicontinuous at x.*
2. *If F has a selection which is upper hemicontinuous at x, then F is quasi-upper hemicontinuous at x.*

Proof The second statement is obvious. The first one comes from the fact that F is lower semicontinuous at $x \in X$ if and only if for every generalized sequence $(x_\lambda)_{\lambda \in \Lambda}$ converging to x, and for every $z \in F(x)$, there exists a generalized sequence $(z_\lambda)_{\lambda \in \Lambda}$ converging to z such that $z_\lambda \in F(x_\lambda)$, for every $\lambda \in \Lambda$, see [[20], Proposition 6.1.4]. □

Even if the existence of continuous selections is subject which is not limited to lower semicontinuous set-valued mapping, Michael's selection theorem remains the pioneering work in this direction which guarantees that every lower semicontinuous set-valued mapping with nonempty, closed and convex values from a paracompact space to a Banach space has a continuous selection.

Proposition 2.5 *Let X be a real topological Hausdorff vector space and $F : X \rightrightarrows \overline{\mathbb{R}}$ a set-valued mapping. Suppose that for every $x \in S$ and $x' \in X$, the restriction of F on $\left[x', x\right]$ has a upper hemicontinuous selection. Then, F is quasi-upper hemicontinuous on S.*

Remark 3 We remark that in Proposition 2.5, we are interested in the restriction of F on the line segment $\left[x', x\right]$ which is a space that enjoys different important properties. In comparaison with Michael's selection theorem, it should be interesting to look for conditions on F in order to obtain such a upper hemicontinuous selection without being necessarily continuous.

In many applications, upper hemicontinuous set-valued mappings will be constructed from upper semicontinuous set-valued operators from line segments as in the results of the last section of this paper. However, by modifying an example from [21], we construct here a quasi-upper hemicontinuous set-valued mapping which is not lower semicontinuous.

Example 2 Let $X = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\} \cup \{(0, 0)\} \subset \mathbb{R}^2$ and define the set-valued mapping $F : X \rightrightarrows \mathbb{R}^2$ by

$$F((x, y)) = \begin{cases} \left[\frac{4x^2}{y}, +\infty\right[\times \left[\frac{x^4}{y^2}, +\infty\right[& \text{if } y > 0, \\ \mathbb{R}^2 & \text{if } y = 0. \end{cases}$$

The function $f : X \rightrightarrows \mathbb{R}^2$ defined by

$$f((x, y)) = \begin{cases} \left(\frac{4x^2}{y}, \frac{x^4}{y^2}\right) & \text{if } y > 0, \\ (0, 0) & \text{if } y = 0, \end{cases}$$

is upper hemicontinuous selection of F which is not continuous. Indeed, the hemicontinuity being obvious, we will just prove that f is not continuous at $(0, 0)$. We have

$$\lim_{\substack{x \to 0 \\ x > 0}} f\left(\left(\sqrt{x}, x\right)\right) = \lim_{\substack{x \to 0 \\ x > 0}} \left(4\sqrt{x}, 1\right) = (0, 1) \neq (0, 0) = f((0, 0)).$$

The set-valued mapping F is not lower semicontinuous at $(0, 0)$. Indeed, let $V = B((0, 0), 1)$ be the open ball around $(0, 0)$ with radius 1. We have $F((0, 0)) \cap V \neq \emptyset$, but for any open neighbourhood U of $(0, 0)$, we can choose a small enough $a > 0$ such that $\left(\sqrt{a}, a\right) \in U$. Now, for every $(x, y) \in F\left(\left(\sqrt{a}, a\right)\right)$, we have $x \geq 4\sqrt{a}$ and $y \geq 1$. Then $\sqrt{x^2 + y2} \geq 1$. It follows that $(x, y) \notin V$, and then $F\left(\left(\sqrt{a}, a\right)\right) \cap V = \emptyset$.

3 Existence of solutions of pseudo-monotone set-valued equilibrium problems

In this section, we deal with the existence of solutions of both strong set-valued equilibrium problems and weak set-valued equilibrium problems in the pseudo-monotone case. Not only these results generalize most of the corresponding results in the literature for single-valued equilibrium problems, including our recent results obtained in [3–5,7] with mild continuity on the set of coerciveness, but also open the way to further investigations on set-valued equilibrium problems in the pseudo-monotone case.

We need in the sequel the notion of KKM mappings and the well-known intersection lemma due to Ky Fan, see [15].

Let X be a real topological Hausdorff vector space and M a subset of X. Recall that a set-valued mapping $F : M \rightrightarrows X$ is said to be a KKM mapping if for every finite subset $\{x_1, \ldots, x_n\}$ of M, we have

$$\text{conv}\{x_1, \ldots x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

It is well known by Ky Fan's lemma [15] that if

(1) F is a KKM mapping,
(2) $F(x)$ is closed for every $x \in M$ and
(3) there exists $x_0 \in M$ such that $F(x_0)$ is compact,

then $\bigcap_{x \in M} F(x) \neq \emptyset$.

We define the following set-valued mappings $\Phi^+, \Phi^{++} : C \rightrightarrows C$ by

$$\Phi^+(y) = \left\{x \in C \mid \Phi(x, y) \cap \overline{\mathbb{R}}_+ \neq \emptyset\right\} \quad \forall y \in C,$$

and

$$\Phi^{++}(y) = \left\{ x \in C \mid \Phi(x, y) \subset \overline{\mathbb{R}}_+ \right\} \quad \forall y \in C.$$

We remark that $\Phi^{++}(y) \subset \Phi^{+}(y)$, for every $y \in C$. We also remark that

(1) $x_0 \in C$ is a solution of the weak set-valued equilibrium problem (Wsvep) if and only if $x_0 \in \bigcap_{y \in C} \Phi^{+}(y)$, and
(2) $x_0 \in C$ is a solution of the strong set-valued equilibrium problem (Ssvep) if and only if $x_0 \in \bigcap_{y \in C} \Phi^{++}(y)$.

In the sequel, we set

$$\operatorname{cl} \Phi^{+}(y) = \operatorname{cl}\left(\Phi^{+}(y)\right) \quad \text{and} \quad \operatorname{cl} \Phi^{++}(y) = \operatorname{cl}\left(\Phi^{++}(y)\right),$$

the closure of $\Phi^{+}(y)$ and $\Phi^{++}(y)$ respectively, for every $y \in C$.

Lemma 3.1 *Let C be a nonempty convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ be a set-valued mapping, and assume that the following conditions hold*:

(1) $\Phi(x, x) \subset \overline{\mathbb{R}}_+$, *for every $x \in C$;*
(2) Φ *is convexly quasi-convex in its second variable on C.*

Then, the set-valued mappings $\operatorname{cl} \Phi^{++} : C \rightrightarrows C$ *and* $\operatorname{cl} \Phi^{+} : C \rightrightarrows C$ *are KKM mappings.*

Proof It suffices to prove that the set-valued mapping $\Phi^{++} : C \rightrightarrows C$ is a KKM mapping. Let $\{y_1, \ldots, y_n\} \subset C$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}_+$ be such that $\sum_{i=1}^{n} \lambda_i = 1$. Put $\tilde{y} = \sum_{i=1}^{n} \lambda_i y_i$. By assumption (2), for $\{z_1, \ldots, z_n\}$ with $z_i \in \Phi(\tilde{y}, y_i)$ for every $i = 1, \ldots, n$, there exists $z \in \Phi(\tilde{y}, \tilde{y})$ such that

$$z \leq \max\{z_i \mid i = 1, \ldots, n\}.$$

We have $z \geq 0$ since $\Phi(\tilde{y}, \tilde{y}) \subset \overline{\mathbb{R}}_+$ by assumption (1). It follows that there exists $i_0 \in \{1, \ldots, n\}$ such that $\Phi\left(\tilde{y}, y_{i_0}\right) \cap \mathbb{R}_-^* = \emptyset$, which implies that $\Phi\left(\tilde{y}, y_{i_0}\right) \subset \overline{\mathbb{R}}_+$. Otherwise, all the z_i can be taken in $\mathbb{R}_-^*$, and therefore $z \in \mathbb{R}_-^*$, which is impossible. We conclude that

$$\sum_{i=1}^{n} \lambda_i y_i = \tilde{y} \in \Phi^{++}\left(y_{i_0}\right) \subset \bigcup_{i=1}^{n} \Phi^{++}(y_i),$$

which proves that Φ^{++} is a KKM mapping. □

First, we deal with strong set-valued equilibrium problems. The following result emphasizes the role of upper hemicontinuity when solving set-valued equilibrium problems. It generalizes [[3],Proposition 1.3] for single-valued mappings.

We define the following set-valued mapping $\Phi^{--} : C \rightrightarrows C$ by

$$\Phi^{--}(y) = \{x \in C \mid \Phi(y, x) \subset \mathbb{R}_-\} \quad \forall y \in C,$$

and we set

$$\operatorname{cl}\Phi^{--}(y) = \operatorname{cl}\left(\Phi^{--}(y)\right),$$

the closure of $\Phi^{--}(y)$, for every $y \in C$.

Proposition 3.2 *Let C be a nonempty convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ be a set-valued mapping and suppose the following assumptions hold*:

(1) $\Phi(x, x) \subset \overline{\mathbb{R}}_+$, *for every* $x \in C$;
(2) Φ *is explicitly convexly quasi-convex in its second variable on* C;
(3) Φ *is upper hemicontinuous in its first variable on a subset* S *of* C.

Then,

$$\bigcap_{y \in C}\left(\Phi^{--}(y) \cap S\right) \subset \bigcap_{y \in C}\Phi^{++}(y).$$

Proof Without loss of generality, we may assume that

$$\bigcap_{y \in C}\left(\Phi^{--}(y) \cap S\right) \neq \emptyset.$$

Take $x \in \bigcap_{y \in C}\left(\Phi^{--}(y) \cap S\right)$ and let $y \in C$ be an arbitrary point. By upper hemicontinuity of Φ in its first variable on S, let $(t_n)_n$ be a sequence in $]0, 1[$ converging to 0, and for $z \in \Phi(x, y)$, let $(z_n)_n$ be a sequence with $z_n \in \Phi(x_n, y)$ for every n, and such that

$$z \geq \limsup_{n \to +\infty} z_n,$$

where $x_n = t_n y + (1 - t_n)x$. We have in particular that $x \in \Phi^{--}(x_n)$ for every n. Thus, $\Phi(x_n, x) \subset \mathbb{R}_-$, for every n. By convex quasi-convexity of Φ in its second variable, for $z_n \in \Phi(x_n, y)$ and $w_x^n \in \Phi(x_n, x)$, there exists $w_n \in \Phi(x_n, x_n)$ such that

$$w_n \leq \max\left\{z_n, w_x^n\right\}.$$

We have $w_n \geq 0$ since $\Phi(x_n, x_n) \subset \overline{\mathbb{R}}_+$. We also have $z_n \geq 0$. Indeed, assume that $z_n < 0$. Then $w_x^n \geq 0$, otherwise $w_n < 0$ which is impossible. This yields that $w_x^n = 0$ and then, $z_n < w_x^n$. Since z_n and w_x^n are arbitrary in $\Phi(x_n, y)$ and $\Phi(x_n, x)$ respectively, then $\Phi(x_n, y) \neq \Phi(x_n, x)$. By semistrict convex quasi-convexity of Φ in its second variable, we obtain $w_n < \max\left\{z_n, w_x^n\right\} = w_x^n = 0$, which is impossible. We conclude that

$$z \geq \limsup_{n \to +\infty} z_n \geq 0.$$

Since z is arbitrary in $\Phi(x, y)$, then $x \in \Phi^{++}(y)$. Since y is arbitrary in C, then $x \in \bigcap_{y \in C} \Phi^{++}(y)$, which completes the proof. □

Now, we obtain a result on the existence of solutions of strong set-valued equilibrium problems generalizing [[3], Theorem 2.1] and [[7], Theorem 3.2] obtained for single-valued equilibrium problems.

We say that a bifunction $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ is *strongly pseudo-monotone on* C if for every $x, y \in C$,

$$\Phi(x, y) \subset \overline{\mathbb{R}}_+ \Longrightarrow \Phi(y, x) \subset \mathbb{R}_-.$$

Example 3 Define the set-valued mapping $\Phi : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$\Phi(x, y) = \begin{cases} \left[y^2 - x^2, +\infty\right[& \text{if } |y| > |x|, \\ \{0\} & \text{if } |y| = |x|, \\ \left]-\infty, y^2 - x^2\right] & \text{if } |y| < |x|. \end{cases}$$

Clearly, $\Phi(x, x) = \Phi(-x, -x) = \Phi(x, -x) = \Phi(-x, x) = \{0\}$, for every $x \in \mathbb{R}$. If $\Phi(x, y) \subset \overline{\mathbb{R}}_+$, then necessarily, we have $|y| \geq |x|$. It follows that $\Phi(y, x)$ is either equal to $\{0\}$ or to $]-\infty, x^2 - y^2]$ which are included in $\mathbb{R}_-$. That is, Φ is strongly pseudo-monotone on $\mathbb{R}$.

Theorem 3.3 *Let C be a nonempty, closed and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ be a set-valued mapping, and assume that the following conditions hold*:

(1) $\Phi(x, x) \subset \overline{\mathbb{R}}_+$, for every $x \in C$;
(2) Φ is strongly pseudo-monotone on C,
(3) Φ is explicitly convexly quasi-convex in its second variable on C;
(4) there exist a compact set K of C and a point $y_0 \in K$ such that $\Phi(x, y_0) \cap \mathbb{R}^*_- \neq \emptyset$, for every $x \in C \setminus K$;
(5) Φ is l-lower semicontinuous in its second variable on K;
(6) Φ is upper hemicontinuous in its first variable on K.

Then, the set of solutions of the set-valued equilibrium problem (Ssvep) *is nonempty compact set. It is also convex whenever Φ is concavely quasi-convex in its second variable on C and K is convex.*

Proof Assumption (1) yields $\Phi^{++}(y)$ is nonempty, for every $y \in C$. Clearly, $\mathrm{cl}\Phi^{++}(y)$ is closed for every $y \in C$, and $\mathrm{cl}\Phi^{++}(y_0)$ is compact since it lies in K by assumption (4). Also, the set-valued mapping $\mathrm{cl}\Phi^{++}$ is a KKM mapping by Lemma 3.1. By using Ky Fan Lemma, we have

$$\bigcap_{y \in C} \mathrm{cl}\Phi^{++}(y) \neq \emptyset.$$

Since the subset $\text{cl}\Phi^{++}(y_0)$ is contained in the compact K, then

$$\bigcap_{y\in C} \text{cl}\Phi^{++}(y) = \bigcap_{y\in C} \left(\text{cl}\Phi^{++}(y) \cap K\right).$$

By strong pseudo-monotonicity, we have $\Phi^{++}(y) \subset \Phi^{--}(y)$, for every $y \in X$. We remark that for every $y \in C$, $\Phi^{--}(y)$ is the upper inverse set $\Phi^{+}(y,]-\infty, 0])$ of $]-\infty, 0]$ by the set-valued mapping $\Phi(y, .)$ which is l-lower semicontinuous on K. Then, by Proposition 2.3, we have $\text{cl}\Phi^{--}(y) \cap K = \Phi^{--}(y) \cap K$. It follows that

$$\bigcap_{y\in C} \left(\text{cl}\Phi^{++}(y) \cap K\right) \subset \bigcap_{y\in C} \left(\text{cl}\Phi^{--}(y) \cap K\right) = \bigcap_{y\in C} \left(\Phi^{--}(y) \cap K\right).$$

By Proposition 3.2, we have

$$\bigcap_{y\in C} \left(\Phi^{--}(y) \cap K\right) \subset \bigcap_{y\in C} \Phi^{++}(y).$$

This yields that

$$\bigcap_{y\in C} \text{cl}\Phi^{++}(y) = \bigcap_{y\in C} \Phi^{++}(y).$$

That is, the set of solutions of the set-valued equilibrium problem (Ssvep) is the nonempty set $\bigcap_{y\in C} \text{cl}\Phi^{++}(y)$ which is compact since it is closed and contained in the compact set K.

By applying Proposition 2.2, the concave quasi-convexity of Φ in its second variable on C yields that the set $\Phi^{--}(y)$ is convex, for every $y \in C$. Since we also have

$$\bigcap_{y\in C} \text{cl}\Phi^{++}(y) = \left(\bigcap_{y\in C} \Phi^{--}(y)\right) \cap K,$$

then the set of solutions of the set-valued equilibrium problem (Ssvep) is convex whenever K is convex. □

Now, we deal with weak set-valued equilibrium problems. The following result emphasizes the role of quasi-upper hemicontinuity when solving set-valued equilibrium problems. It also generalizes [[3],Proposition 1.3] for single-valued mappings.

We define the following set-valued mapping $\Phi^{-} : C \rightrightarrows C$ by

$$\Phi^{-}(y) = \{x \in C \mid \Phi(y, x) \cap \mathbb{R}_{-} \neq \emptyset\} \quad \forall y \in C,$$

and we set

$$\text{cl}\,\Phi^{-}(y) = \text{cl}\left(\Phi^{-}(y)\right),$$

the closure of $\Phi^-(y)$, for every $y \in C$.

Proposition 3.4 *Let C be a nonempty convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ be a set-valued mapping and suppose the following assumptions hold*:

(1) $\Phi(x, x) \subset \overline{\mathbb{R}}_+$, *for every* $x \in C$;
(2) Φ *is explicitly convexly quasi-convex in its second variable on* C;
(3) Φ *is quasi-upper hemicontinuous in its first variable on a subset* S *of* C.

Then,

$$\bigcap_{y \in C} \left(\Phi^-(y) \cap S\right) \subset \bigcap_{y \in C} \Phi^+(y).$$

Proof As in the proof of Proposition 3.2, take $x \in \bigcap_{y \in C}\left(\Phi^-(y) \cap S\right)$ and let $y \in C$ be an arbitrary point. By quasi-upper hemicontinuity of Φ in its first variable on S, let $(t_n)_n$ be a sequence in $]0, 1[$ converging to 0, a point $z \in \Phi(x, y)$, and a sequence $(z_n)_n$ with $z_n \in \Phi(x_n, y)$ for every n, such that

$$z \geq \limsup_{n \to +\infty} z_n,$$

where $x_n = t_n y + (1 - t_n)x$. By convex quasi-convexity of Φ in its second variable, for $z_n \in \Phi(x_n, y)$ and $w_x^n \in \Phi(x_n, x) \cap \mathbb{R}_-$, there exists $w_n \in \Phi(x_n, x_n)$ such that

$$w_n \leq \max\left\{z_n, w_x^n\right\}.$$

By using the semistrict convex quasi-convexity of Φ in its second variable, we obtain that $z_n \geq 0$, and we conclude that

$$z \geq \limsup_{n \to +\infty} z_n \geq 0,$$

which completes the proof. □

Now, we obtain a result on the existence of solutions of weak set-valued equilibrium problems generalizing [[3], Theorem 2.1] and [[7], Theorem 3.2] obtained for single-valued equilibrium problems.

We say that a bifunction $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ is *weakly pseudo-monotone on* C if for every $x, y \in C$,

$$\Phi(x, y) \cap \overline{\mathbb{R}}_+ \neq \emptyset \Longrightarrow \Phi(y, x) \cap \mathbb{R}_- \neq \emptyset.$$

Example 4 Define the set-valued mapping $\Phi : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$ by

$$\Phi(x, y) = \begin{cases} [0, +\infty[& \text{if } y \geq x, \\ ]-\infty, x - y] & \text{if } y < x. \end{cases}$$

We remark that $\Phi(x, y) \cap \mathbb{R}_- \neq \emptyset$, for every $x, y \in \mathbb{R}$. Then, Φ is obviously weakly pseudo-monotone on $\mathbb{R}$. However, Φ can not be strongly pseudo-monotone on $\mathbb{R}$ since $\Phi(1, 2) = [0, +\infty[\subset \overline{\mathbb{R}}_+$, but $\Phi(2, 1) =]-\infty, 1] \not\subset \mathbb{R}_-$.

We note that for real single-valued mappings, the weak pseudo-monotonicity coincides with the strong pseudo-monotonicity, and it is called, in this case, the pseudo-monotonicity.

Theorem 3.5 *Let C be a nonempty, closed and convex subset of a real topological vector space. Let $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ be a set-valued mapping, and assume that the following conditions hold*:

(1) $\Phi(x, x) \subset \overline{\mathbb{R}}_+$, *for every $x \in C$*;
(2) Φ *is weakly pseudo-monotone on C,*
(3) Φ *is explicitly convexly quasi-convex in its second variable on C*;
(4) *there exist a compact set K of C and a point $y_0 \in K$ such that $\Phi(x, y_0) \subset \mathbb{R}_-^*$, for every $x \in C \backslash K$*;
(5) Φ *is u-lower semicontinuous in its second variable on K*;
(6) Φ *is quasi-upper hemicontinuous in its first variable on K.*

Then, the set of solutions of the set-valued equilibrium problem (Wsvep) *is nonempty compact set. It is also convex whenever K is convex.*

Proof As in the proof of Theorem 3.3, $\mathrm{cl}\Phi^+(y)$ is nonempty and closed for every $y \in C$, $\mathrm{cl}\Phi^+(y_0)$ is compact and the set-valued mapping $\mathrm{cl}\Phi^+$ is a KKM mapping. By using Ky Fan Lemma and since the subset $\mathrm{cl}\Phi^+(y_0)$ is contained in the compact K, we have

$$\bigcap_{y \in C} \left(\mathrm{cl}\Phi^+(y) \cap K\right) = \bigcap_{y \in C} \mathrm{cl}\Phi^+(y) \neq \emptyset.$$

By weak pseudo-monotonicity, we have $\Phi^+(y) \subset \Phi^-(y)$, and by Proposition 2.3, since Φ is u-lower semicontinuity in its second variable on K, we have $\mathrm{cl}\Phi^-(y) \cap K = \Phi^-(y) \cap K$, for every $y \in X$. It follows that

$$\bigcap_{y \in C} \left(\mathrm{cl}\Phi^+(y) \cap K\right) \subset \bigcap_{y \in C} \left(\mathrm{cl}\Phi^-(y) \cap K\right) = \bigcap_{y \in C} \left(\Phi^-(y) \cap K\right).$$

By Proposition 3.4, we have

$$\bigcap_{y \in C} \left(\Phi^-(y) \cap K\right) \subset \bigcap_{y \in C} \Phi^+(y).$$

This yields that

$$\bigcap_{y \in C} \mathrm{cl}\Phi^+(y) = \bigcap_{y \in C} \Phi^+(y).$$

Then, the set of solutions of the set-valued equilibrium problem (Wsvep) is the nonempty compact set $\bigcap_{y\in C} \text{cl}\Phi^{+}(y)$. By Proposition 2.1, we have that the set $\Phi^{-}(y)$ is convex, for every $y \in C$. Since we also have

$$\bigcap_{y\in C} \text{cl}\Phi^{+}(y) = \left(\bigcap_{y\in C} \Phi^{-}(y)\right) \cap K,$$

then the set of solutions of the set-valued equilibrium problem (Wsvep) is convex whenever K is convex. □

4 Browder variational inclusions

In this section, we deal with Browder variational inclusions involving pseudo-monotone set-valued operators. Browder variational inclusions which generalize Browder–Hartman–Stampacchia variational inequalities, have many applications, including applications to nonlinear elliptic boundary value problems and the surjectivity of set-valued mappings, see for example [12,24] and the references therein.

In the sequel, for a real normed vector space X, we denote by X^*, the dual space of X, and by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X.

Let C be a nonempty, closed and convex subset of a real normed vector space X. In the literature, some notions of coerciveness for set-valued operators have been introduced a generalizations of those for linear operators and bilinear forms on Hilbert spaces. A set-valued operator $F : C \rightrightarrows X^*$ is said to be coercive on C if there exists $y_0 \in C$ such that

$$\lim_{\substack{\|x\|\to+\infty \\ x\in C}} \inf_{x^*\in F(x)} \langle x^*, x - y_0\rangle > 0,$$

or if the stronger condition

$$\lim_{\substack{\|x\|\to+\infty \\ x\in C}} \frac{\inf_{x^*\in F(x)} \langle x^*, x - y_0\rangle}{\|x\|} = +\infty.$$

is satisfied. It is not hard to see that under both the two notions of coerciveness of F, there exists $R > 0$ such that $y_0 \in K_R$ and $\inf_{x^*\in F(x)} \langle x^*, y_0 - x\rangle < 0$, for every $x \in C\backslash K_R$, where $K_R = \{x \in C \mid \|x\| \le R\}$. Clearly, K_R is weakly compact whenever X is reflexive. The set K_R is called a set of coerciveness, and the couple (y_0, K_R) may not be unique. We will never need such a set of coerciveness K_R, but a weakly compact set of coerciveness without requiring the space X to be reflexive.

Recall that a set-valued operator $F : C \rightrightarrows E^*$ is called pseudo-monotone on C if for every $x, y \in C$

$$\langle x^*, y - x\rangle \ge 0 \Longrightarrow \langle y^*, x - y\rangle \le 0 \quad \forall x^* \in F(x), \forall y^* \in F(y).$$

In the sequel, for $x \in X$ and a subset A of X^*, we set

$$\langle A, x\rangle = \left\{\langle x^*, x\rangle \mid x^* \in A\right\}.$$

Problems of the form: "find $x_0 \in C$ such that $\langle A, x_0\rangle \subset \mathbb{R}_+$" or "find $x_0 \in C$ such that $\langle A, x_0\rangle \cap \mathbb{R}_+ \neq \emptyset$" are called Browder variational inclusions.

Let X and Y be two real Hausdorff topological vector spaces and C is a nonempty convex subset of X. Following [24], recall that $F : C \rightrightarrows Y$ is said to be *upper semicontinuous from line segments in C at $x \in C$* if for every $x' \in C$, the restriction of F on the line segment $\left[x', x\right]$ is upper semicontinuous at x. That is, for every $x' \in C$, there exists an open neighborhood U of x such that $F(z) \subset V$, for every $z \in U \cap \left[x', x\right]$. We say that F is upper semicontinuous from line segments in C on a subset S of C if it is upper semicontinuous from line segments in C at every point of S.

The following result generalizes [[24], Theorem 1] about the existence of solutions of Browder variational inclusions in the pseudo-monotone case. It also generalizes [[7], Proposition 3.1] for the single-valued case corresponding to Browder–Hartman–Stampacchia variational inequality. We note that a weak* compact set S of X^* is norm bounded whenever X is a Banach space or S is convex, see for example [14]. In our result, we do not need the weak* compactness of the images of compact sets by the set-valued operator, but only the norm boundedness of the images of some line segments.

Theorem 4.1 *Let C be a nonempty, closed and convex subset of a real Banach space X, and $F : C \rightrightarrows X^*$ a set-valued operator. Suppose that the following conditions hold*:

(1) *F is pseudo-monotone on C*;
(2) *there exist a weakly compact subset K of C and $y_0 \in K$ such that $\sup_{z^* \in F(x)} \langle z^*, y_0 - x\rangle < 0$, for every $x \in C \backslash K$*;
(3) *F upper semicontinuous from line segments in C on K to X^* endowed with the weak* topology*;
(4) *F has weak* compact values on C,*
(5) *For every $x \in K$ and $x' \in C$, $F\left(\left[x', x\right]\right)$ is norm bounded.*

Then, there exists $x_0 \in K$ such that $\langle F(x_0), y - x_0\rangle \cap \mathbb{R}_+ \neq \emptyset$, for every $y \in C$.

Proof First, we define the following extended real single-valued mapping $\Phi : C \times C \rightrightarrows \mathbb{R} \cup \{+\infty\}$ by

$$\Phi(x, y) = \sup_{z^* \in F(x)} \langle z^*, y - x\rangle,$$

which also, it can be seen as an extended real set-valued mapping. Now, we will verify for Φ and X endowed with the weak topology the assumptions of Theorem 3.3 or Theorem 3.5, where the five first conditions are the same in this case.

Clearly, $\Phi(x, x) = 0 \in \mathbb{R}_+$, for every $x \in C$. For $z \in C$ fixed, the function $z^* \mapsto \langle z^*, y - z\rangle$ is weak* continuous on X^* and therefore, by Weierstrass theorem, it attains its maximum on weak* compact sets. Thus, for every $x \in C$, there exists $x^* \in F(x)$ such that

$$\Phi(x, y) = \langle x^*, y - x \rangle,$$

which provides easily that Φ is pseudo-monotone on C.

For every $x \in C$ and $x^* \in F(x)$, the function $y \to \langle x^*, y - x \rangle$ is linear and weakly continuous. Then Φ being the superior envelope of a family of convex and weakly lower semicontinuous functions, it is then convex and weakly lower semicontinuous in its second variable on C.

It remains just to prove that Φ is upper hemicontinuous in its first variable on K. We note that the strong topology and the weak topology coincide on line segments of X. Let $y \in C$ be fixed, $x \in K$ and $x' \in C$. Take a sequence $(x_n)_n$ in $\left[x', x\right]$ converging to x. Take $x^* \in F(x)$ such that $\Phi(x, y) = \langle x^*, y - x \rangle$, and $x_n^* \in F(x_n)$ such that $\Phi(x_n, y) = \langle x_n^*, y - x_n \rangle$, for every n.

Suppose first that there exists $a \in \mathbb{R}$ such that $\Phi(x_n, y) \geq a$, for every n. We claim that the sequence $\left(x_n^*\right)_n$ has a weak* cluster point $\tilde{x}^* \in F(x)$. Indeed, suppose the contrary holds. Then the weak* compactness of $F(x)$ yields the existence of a weak* open set V containing $F(x)$ and $n_0 \in \mathbb{N}$ such that $x_n^* \notin V$, for every $n \geq n_0$. The upper semicontinuity of F from line segments in C at x yields the existence of an open neighborhood U of x such that $F(z) \subset V$, for every $z \in U \cap \left[x', x\right]$. Since $(x_n)_n$ is converging to x, let $n_1 \in \mathbb{N}$ be such that $x_n \in U$, for every $n \geq n_1$. Then, $x_n^* \in F(x_n) \subset V$, for every $n \geq n_1$. A contradiction.

Let now $\left(x_{n_\lambda}^*\right)_{\lambda \in \Lambda}$ be a subnet of $\left(x_n^*\right)_n$ converging to $\tilde{x}^*$ in the weak* topology of X^*. The subnet $\left(x_{n_\lambda}\right)_{\lambda \in \Lambda}$ also converges to x, and therefore, for $\varepsilon > 0$, let $\lambda_0 \in \Lambda$ be such that for every $\lambda \geq \lambda_0$, we have

$$\|x - x_{n_\lambda}\| \leq \frac{\varepsilon}{2\left(\|F([x', x])\| + 1\right)},$$

where $\|F\left(\left[x', x\right]\right)\| = \sup_{z^* \in F([x', x])} \|z^*\|$. Let also $\lambda_1 \in \Lambda$ be such that

$$\|\tilde{x}^* - x_{n_\lambda}^*\| \leq \frac{\varepsilon}{2\left(\|y - x\| + 1\right)}.$$

Let $\tilde{\lambda} \in \Lambda$ be such that $\tilde{\lambda} \geq \lambda_0$ and $\tilde{\lambda} \geq \lambda_1$. It result that for every $\lambda \geq \tilde{\lambda}$, we have

$$\begin{aligned} \left|\langle \tilde{x}^*, y - x \rangle - \langle x_{n_\lambda}^*, y - x_{n_\lambda} \rangle\right| &= \left|\langle \tilde{x}^* - x_{n_\lambda}^*, y - x \rangle + \langle x_{n_\lambda}^*, x_{n_\lambda} - x \rangle\right| \\ &\leq \|\tilde{x}^* - x_{n_\lambda}^*\| \|y - x\| + \|x_{n_\lambda}^*\| \|x - x_{n_\lambda}\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We conclude that

$$\Phi(x, y) \geq \langle \tilde{x}^*, y - x \rangle = \lim_{\lambda} \langle x_{n_\lambda}^*, y - x_{n_\lambda} \rangle = \lim_{\lambda} \Phi(x_\lambda, y) \geq a.$$

Now, we claim that $\Phi(x, y) \geq \limsup_{n\to+\infty} \Phi(x_n, y)$. Suppose the contrary holds and let $\lambda > 0$ be such that

$$\Phi(x, y) + \lambda < \limsup_{n\to+\infty} \Phi(x_n, y).$$

Put $a = \Phi(x, y) + \lambda$ which is then in $\mathbb{R}$ (but this also holds from the fact that $F(x)$ is weak* compact). Now if a subsequence $\left(x_{n_k}\right)_k$ of $(x_n)_n$ is such that $\Phi\left(x_{n_k}, y\right) \geq a$ for every k, then by the above statement, we obtain

$$\Phi(x, y) \geq a > \Phi(x, y),$$

which is impossible. Then, there exists $n_0 \in \mathbb{N}$ such that $\Phi(x_n, y) < a$, for every $n \geq n_0$. It results that

$$\limsup_{\substack{n\geq n_0\\ n\to+\infty}} \Phi(x_n, y) \leq a < \limsup_{n\to+\infty} \Phi(x_n, y),$$

which yields a contradiction since $\limsup_{\substack{n\geq n_0\\ n\to+\infty}} \Phi(x_n, y) = \limsup_{n\to+\infty} \Phi(x_n, y)$. We conclude that $\Phi(x, y) \geq \limsup_{n\to+\infty} \Phi(x_n, y)$, which completes the proof. □

Remark 4 We remark that in the proof of Theorem 4.1 above, if we assume the mild condition of F *has weak" compact values only on* K instead of all C, we can still prove that

$$\Phi(x, y) \geq a.$$

Indeed, for $\delta > 0$, let $x_n^* \in F(x_n)$ be such that $\langle x_n^*, y - x_n\rangle > a - \delta$, for every n. Using similar arguments, we state that the sequence $\left(x_n^*\right)_n$ has a subnet $\left(x_{n_\lambda}^*\right)_{\lambda\in\Lambda}$ converging to some $\tilde{x}^* \in F(x)$ in the weak* topology of X^*. We also obtain that

$$\Phi(x, y) \geq \langle\tilde{x}^*, y - x\rangle = \lim_{\lambda}\langle x_{n_\lambda}^*, y - x_{n_\lambda}\rangle \geq a - \delta.$$

By letting δ go to zero, we conclude that $\Phi(x, y) \geq a$.

We note that all the other statements of Theorem 4.1 remain true under this mild condition except the pseudo-monotonicity of F. In this case, we can assume the following condition:

$$\sup_{z^*\in F(x)} \langle z^*, y - x\rangle\rangle \geq 0 \Longrightarrow \sup_{z^*\in F(y)} \langle z^*, x - y\rangle \leq 0 \quad \forall x, y \in C,$$

instead of the pseudo-monotonicity of F.

In order to make further discussion in this subject about the existence of solutions of Browder variational inclusions, recall that an open half-space in a real Hausdorff topological vector space E is a subset of the form

$$\{u \in E \mid \varphi(u) < r\}$$

for some continuous linear functional φ on E, not identically zero, and for some real number r.

Let X be a Hausdorff topological space and E a real Hausdorff topological vector space. Following [18], a set-valued operator $F : C \rightrightarrows Y$ is said to be *upper demi-continuous at* $x \in X$ if for every open half-space H containing $F(x)$, there exists a neighborhood U of x such that $F(z) \subset H$ for every $z \in U$. It said to be upper demicontinuous on X if it is upper demicontinuous at every point of X.

We say that a set-valued operator $A : X \rightrightarrows Y$ is *upper demicontinuous from line segments in* X *at* $x \in C$ if for every $x' \in C$ and every open half-space H containing $F(x)$, there exists a neighborhood U of x such that $F(z) \subset H$ for every $z \in U \cap [x', x]$. We say that F is upper demicontinuous from line segments in X on a subset S of C if it is upper demicontinuous from line segments in X at every point of S.

Proposition 4.2 *Let X be a real normed vector space, C a nonempty convex subset of X and $S \subset C$. If $F : C \rightrightarrows X^*$ is upper semicontinuous from line segments in X on S to X^* endowed with the weak* topology, then F is upper demicontinuous from line segments in X on S to X^* endowed with the weak* topology.*

Proof Let $x \in K$ and consider an open half-space H in X^* of the form

$$\left\{u \in X^* \mid \varphi(u) < r\right\}$$

such that $F(x) \subset H$, where φ is a weak* continuous linear functional on X^*, not identically zero, and $r \in \mathbb{R}$. Then, $\varphi(F(x)) \subset]-\infty, r[$. Put $O = \varphi^{-1}(]-\infty, r[)$, which is a weak* open subset containing $F(x)$. By the upper semicontinuity of F from line segments in X on S to X^* endowed with the weak* topology, for every $x' \in C$, there exists a neighborhood U of x such that $F(z) \subset O$, for every $z \in U \cap \left[x', x\right]$. That is, $F(z) \subset H$, for every $z \in U \cap \left[x', x\right]$. □

It is not clear at the stage of development whether upper semicontinuity from line segments in C in Theorem 4.1 can be weakened to upper demicontinuity or to upper demicontinuity from line segments in C.

5 Conclusion

In order to better understand the optimal conditions for solving Browder variational inclusions, we have been led to consider and study set-valued equilibrium problems. Our investigations have brought to light various concepts of convexity, semicontinuity and hemicontinuity involving half intervals rather than open sets. We have obtained results on the existence of solutions of set-valued equilibrium problems in the pseudo-monotone case generalizing the corresponding ones for single-valued equilibrium problems. Browder variational inclusions in the pseudo-monotone case rather than the monotone one have been investigated without the reflexivity of the real normed vector spaces and under weakened conditions on the involved set-valued operator. More advancements and applications to nonlinear elliptic boundary value problems and to the surjectivity of set-valued mappings as well as to other important real world problems, constitute a challenge for further investigations.

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