THE LANE-EMDEN EQUATION WITH VARIABLE DOUBLE-PHASE AND MULTIPLE REGIME

CLAUDIANOR O. ALVES AND VICENŢIU D. RĂDULESCU

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ABSTRACT. We are concerned with the study of the Lane-Emden equation with variable exponent and Dirichlet boundary condition. The feature of this paper is that the analysis that we develop does not assume any subcritical hypotheses and the reaction can fulfill a mixed regime (subcritical, critical, and supercritical). We consider the radial and the nonradial cases, as well as a singular setting. The proofs combine variational and analytic methods with a version of the Palais principle of symmetric criticality.

1. INTRODUCTION AND ABSTRACT SETTING

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded regular connected open set and assume that $p, q \in (1, \infty)$. The Lane-Emden problem

$$(1) \qquad \begin{cases} -\Delta_p u = |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega, \end{cases}$$

and its numerous versions play a central role in the qualitative analysis of nonlinear elliptic equations. Usually, this analysis is developed in relationship with the values of q with respect to the Sobolev critical exponent p^* of p, which is defined by

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } 1 < p < N, \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

The following three basic situations can occur:

(i) $q < p^*$ (*subcritical case*). Then the associated energy functional is either coercive (if $q < p$) or has a mountain pass geometry and satisfies the Palais-Smale condition (if $q > p$), hence problem (1) has at least one solution. The case $p = q$ corresponds to an eigenvalue problem, so we cannot exclude a nonexistence property.

(ii) $q = p^*$, provided that $1 < p < N$ (*critical case*). In this case, the topology of Ω plays a crucial role. We recall the following basic result of Bahri and Coron [2]. Let $H_d(\Omega; \mathbb{Z}_2)$ denote the homology of dimension d of Ω with $\mathbb{Z}_2$-coefficients. The Bahri-Coron theorem states that if there exists a positive integer d such that $H_d(\Omega; \mathbb{Z}_2) \neq 0$, then problem (1) has at least one positive solution. In particular, if $p = 2$, $N = 3$, $q = 6$, and Ω is not contractible, then problem (1) has at least one positive solution.

(iii) $q > p^*$, provided that $1 < p < N$ (*supercritical case*). This situation is delicate and a major role is played by the geometry of Ω. For instance, if Ω is starshaped then problem (1) does not have any solution (by Pohozaev's identity). Also, if Ω is an annulus, problem (1) always has at least one solution.

In the case of variable exponents, the Lane-Emden problem (1) becomes

$$
(2) \qquad \begin{cases} -\Delta_{p(x)}u = |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega, \end{cases}
$$

where $\Delta_{p(x)}u := \mathrm{div}\,(|\nabla u|^{p(x)-2}\nabla u)$. In this case, the critical exponent of $p(x)$ depends on the point and it is defined by

$$
p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } 1 < p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}
$$

Problem (2) and its versions have been intensively studied in several recent works, in relationship with numerous applications to non-Newtonian fluids, image reconstruction, etc. We refer to Rădulescu and Repovš [25] for a comprehensive study of several classes of nonlinear problems with variable exponent. For instance, in [25] the problem

$$
(3) \qquad \begin{cases} -\Delta_{p(x)}u = \lambda\,|u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega, \end{cases}
$$

is studied under the following hypotheses:

(h1) $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$;

(h2) $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

Theorem 3 in [25, p. 43] studies the case of low perturbations of the reaction and establishes that there exists $\lambda^* > 0$ such that problem (3) has at least one solution for all $\lambda \in (0, \lambda^*)$.

Hypothesis (h2) corresponds to the *subcritical* case. However, the presence of *variable* exponents allows the following *almost critical* case: there is $z \in \Omega$ such that $q(x) < p^*(x)$ for all $x \in \Omega \setminus \{z\}$ and $q(z) = p(z)$. In this direction, we refer to Alves, Ercole, and Huamán Bolaños [1], where the problem (3) has been considered with $p(x) = 2$ and $q(x)$ a functional that has a critical growth in a subset Ω. By using variational methods combined with the Lions concentration-compactness principle [15], the authors proved the existence of ground state solutions.

Furthermore, problem (2) can fulfill even a "subcritical-critical-supercritical" triple regime, in the sense that $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and

$$
q(x) < p^*(x) \quad \text{if } x \in \Omega_1;
$$

$$
q(x) = p^*(x) \quad \text{if } x \in \Omega_2;
$$

$$
q(x) > p^*(x) \quad \text{if } x \in \Omega_3.
$$

To the best of our knowledge, nonlinear PDEs with variable exponent and multiple regime (subcritical, critical, and supercritical) have not been considered in the literature. The present paper is the first study dedicated to the qualitative analysis of the Lane-Emden equation with variable exponent and multiple regime. We consider the radial and the nonradial cases, as well as a singular setting. More precisely, in Section 2 we study the case where Ω is a ball centered at origin. In

this abstract framework, a key role in the proofs is played by a recent version of the Palais principle of symmetric criticality developed by Kobayashi and Ôtani [14, Theorem 2.2]. Let E be a Banach space on which a symmetric group G acts and let J be a G-invariant functional defined on E. We recall that Palais [21] gave some sufficient conditions to guarantee the principle of symmetric criticality, in the sense that every critical point of J restricted to the subspace of G-symmetric points becomes also a critical point of J on the whole space E. This principle has been generalized by Kobayashi and Ôtani [14] to the case where J is not differentiable within the setting that does not require the full variational structure, under the hypothesis that the action of G is isometry or G is compact. In Sections 3 and 4, we study the case where Ω is not a ball centered at origin, which is more delicate because we cannot use the principle of symmetric criticality mentioned above. In both sections we are concerned with the existence of nontrivial solutions in the nonradial case. We overcome the lack of symmetry properties by developing a new strategy; see the proof of Theorem 5 for more details. In the last section of the present paper we are concerned with a singular setting, which corresponds to a vanishing potential.

1.1. **Variable exponent Lebesgue and Sobolev spaces.** In this subsection, we recall some results on variable exponent Lebesgue and Sobolev spaces. For more details we refer to $[10, 11, 25]$ and their references.

Let $p \in L^\infty(\mathbb{R}^N)$ with $p_- := \operatorname{essinf}_{x \in \mathbb{R}^N} p(x) > 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)} \, dx < \infty \right\}$$

endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_{\mathbb{R}^N} \left| \frac{u}{\lambda} \right|^{p(x)} \, dx \leqslant 1 \right\}.$$

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \in L^{p(x)}(\mathbb{R}^N) \,\middle|\, |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

With these norms, the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are reflexive and separable Banach spaces.

Proposition 1. *The functional $\rho : W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by*

$$(4) \qquad \rho(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx$$

has the following properties:

    (i) *If $\|u\|_{1,p(x)} \geqslant 1$, then $\|u\|_{1,p(x)}^{p_-} \leqslant \rho(u) \leqslant \|u\|_{1,p(x)}^{p_+}$.*
    (ii) *If $\|u\|_{1,p(x)} \leqslant 1$, then $\|u\|_{1,p(x)}^{p_+} \leqslant \rho(u) \leqslant \|u\|_{1,p(x)}^{p_-}$.*

In particular, $\rho(u) = 1$ if and only if $\|u\|_{1,p(x)} = 1$ and if $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$, then $\|u_n\|_{1,p(x)} \to 0$ if and only if $\rho(u_n) \to 0$.

Remark 1. For the functional $\xi : L^{p(x)}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\xi(u) = \int_{\mathbb{R}^N} |u|^{p(x)} \, dx,$$

the conclusion of Proposition 1 also holds, for example, if $(u_n) \subset L^{p(x)}(\mathbb{R}^N)$, then $|u_n|_{p(x)} \to 0$ if and only if $\xi(u_n) \to 0$. Moreover, from (i) and (ii),

$$(5) \qquad |u|_{p(x)} \leqslant \max \left\{ \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, dx \right)^{1/p_-}, \left(\int_{\mathbb{R}^N} |u|^{p(x)} \, dx \right)^{1/p_+} \right\}.$$

Related to the Lebesgue space $L^{h(x)}(\mathbb{R}^N)$, we have the following generalized Hölder-type inequality.

Proposition 2 ([20, p. 9]). *For $p \in L^\infty(\mathbb{R}^N)$ with $p_- > 1$, let $p' : \mathbb{R}^N \to \mathbb{R}$ be such that*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad a.e. \ x \in \mathbb{R}^N.$$

Then, for any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$,

$$(6) \qquad \left| \int_{\mathbb{R}^N} uv \, dx \right| \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) |u|_{p(x)} \, |v|_{p'(x)} .$$

Proposition 3 ([11, Theorems 1.1, 1.3]). *Let $p : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz continuous satisfying $1 < p_- \leqslant p_+ < N$ and let $t : \mathbb{R}^N \to \mathbb{R}$ be a measurable function.*

(i) *If $p \leqslant t \leqslant p^*$, the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{t(x)}(\mathbb{R}^N)$ is continuous.*
(ii) *If $p \leqslant t \ll p^*$, the embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{t(x)}_{\mathrm{loc}}(\mathbb{R}^N)$ is compact.*

The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces provided that p is constant. According to [25, pp. 8-9], these function spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ have some nonusual properties, such as:

(i) Assuming that $1 < p^- \leqslant p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is a smooth function, then the following co-area formula:

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \, |\{x \in \Omega; \ |u(x)| > t\}| \, dt$$

has no analogue in the framework of variable exponents.

(ii) Spaces $L^{p(x)}(\mathbb{R}^N)$ do *not* satisfy the *mean continuity property*. More exactly, if p is nonconstant and continuous in an open ball B, then there is some $u \in L^{p(x)}(B)$ such that $u(x + h) \notin L^{p(x)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponent are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(x)} \leqslant C \, |f|_{p(x)} \, \|g\|_{L^1}$$

remains true if and only if p is constant.

1.2. **Double-phase problems and their historical traces.** Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with a smooth boundary. If $u : \Omega \to \mathbb{R}^N$ is the displacement and Du is the $N \times N$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$(7) \qquad I(u) = \int_{\Omega} f(x, Du(x))dx,$$

where the energy function $f = f(x, \xi) : \Omega \times \mathbb{R}^{N \times N} \to \mathbb{R}$ is quasiconvex with respect to ξ; see Morrey [19]. One of the simplest examples considered by Ball is given by functions f of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geqslant c_1 |\xi|^p; \quad \lim_{t \to +\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < N$. The condition $p \leqslant N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the integral (7) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus it is a continuous function if $p > N$. In accordance with these problems arising in nonlinear elasticity, Marcellini [16–18] considered continuous functions $f = f(x, u)$ with *unbalanced growth* that satisfy

$$c_1 |u|^p \leqslant |f(x, u)| \leqslant c_2 \left(1 + |u|^q\right) \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \leqslant p \leqslant q$. Regularity and existence of solutions of elliptic equations with p, q–growth conditions were studied in [17].

The study of nonautonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been continued by Mingione et al. [4–6, 8, 9], Rădulescu et al. [3, 7, 22–24, 27], etc. These contributions are in relationship with the work of Zhikov [28, 29], which describe the behavior of phenomena arising in nonlinear elasticity. In fact, variational problems with nonstandard integrands were introduced at the beginning of the 1980s and were studied in the context of averaging and the Lavrent'ev phenomenon. Zhikov provided models for strongly anisotropic materials in the context of homogenization. In particular, he considered the following model functional:

$$(8) \qquad \mathcal{P}_{p,q}(u) := \int_{\Omega} (|Du|^p + a(x)|Du|^q)dx, \quad 0 \leqslant a(x) \leqslant L, \ 1 < p < q,$$

where the modulating coefficient $a(x)$ dictates the geometry of the composite made of two differential materials, with hardening exponents p and q, respectively.

Another significant model example of a functional with (p, q)–growth studied by Mingione et al. is given by

$$u \mapsto \int_{\Omega} |Du|^p \log(1 + |Du|)dx, \quad p \geqslant 1,$$

which is a logarithmic perturbation of the p-Dirichlet energy.

2. Problem 1: The radial case

In this section we consider $p, q, m, a : \overline{B}_R(0) \to \mathbb{R}$ four continuous functions satisfying:

(H1)
$$\begin{cases} 1 < p_- = \min_{x \in \overline{B}_R(0)} p(x) \leqslant \max_{x \in \overline{B}_R(0)} p(x) = p_+ < N, \\ 1 < m_- = \min_{x \in \overline{B}_R(0)} m(x) \leqslant \max_{x \in \overline{B}_R(0)} m(x) = m_+ < N, \end{cases}$$

(H2)
$$0 \leqslant a(x) \leqslant L \quad \forall x \in \overline{B}_R(0),$$

(H3) $p(x) = p(|x|), a(x) = a(|x|)$ and $q(x) = q(|x|) \quad \forall x \in \overline{B}_R(0).$

We assume that there exists $0 < r < R$ such that
(H4)
$$q(x) \geqslant 0 \quad \forall x \in \overline{\Omega} \quad \text{and} \quad p_+ < q_-^r = \min_{x \in \overline{B}_r(0)} q(x) \leqslant \max_{x \in \overline{B}_r(0)} q(x) = q_+^r < \min_{x \in \overline{\Omega}} p^*(x).$$

Note that q is subcritical in $\overline{B}_r(0)$, but there is no hypotheses on the function q in the annulus $A_{R,r} = \overline{B}_R(0) \setminus B_r(0)$, hence q can have a supercritical growth close to the boundary. However, note that for any $t \in (0, R)$ we have the continuous embedding
$$W^{1,p(x)}(B_R(0)) \hookrightarrow W^{1,p_-}(A_{R,t})$$

and the compact embedding
$$W_{rad}^{1,p_-}(A_{R,t}) \hookrightarrow C(\overline{A}_{R,t}),$$

which is due to Strauss [26].

Therefore the embedding
$$(9) \qquad\qquad W_{rad}^{1,p(x)}(B_R(0)) \hookrightarrow C(\overline{A}_{R,t})$$

is compact, where
$$W_{rad}^{1,p(x)}(B_R(0)) = \{u \in W^{1,p(x)}(B_R(0)) \ \ u(x) = u(|x|) \quad \text{a.e} \quad x \in B_R(0)\}.$$

Hence, it follows that the embedding
$$(10) \qquad\qquad W_{rad}^{1,p(x)}(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0))$$

is also compact, which is crucial in our approach.

In what follows, $\Delta_{p(x)}$ denotes the well-known $p(x)$-Laplacian operator and $\Delta_{m(x),a(x)}$ is the differential operator defined by
$$\Delta_{m(x),a(x)} u = \operatorname{div}\left(a(x)|\nabla u|^{m(x)-2}\nabla u\right).$$

Moreover, when $a \neq 0$, we set
$$E = W_0^{1,p(x)}(B_R(0)) \cap W_{a(x),0}^{1,m(x)}(B_R(0)),$$

where $W_{a(x),0}^{1,m(x)}(B_R(0))$ is the usual space $W_0^{1,m(x)}(B_R(0))$, endowed with the norm
$$\|\nabla u\|_{m(x),a(x)} = \inf\left\{\lambda > 0 \ \middle| \ \int_{\mathbb{R}^N} a(x)\left|\frac{|\nabla u|}{\lambda}\right|^{m(x)} dx \leqslant 1\right\}.$$

Hereafter, we endow E with the norm
$$\|u\| = \|\nabla u\|_{p(x)} + \|\nabla u\|_{m(x),a(x)}.$$

We observe that if $a = 0$, then $E = W_0^{1,p(x)}(B_R(0))$ and $\| \ \|$ is exactly the usual norm in $W_0^{1,p(x)}(B_R(0))$.

From the definition of E, we have the continuous embedding

$$E \hookrightarrow W_0^{1,p(x)}(B_R(0)).$$

This fact combined with (10) implies that the embedding

(11) $$E_{rad}(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0))$$

is also compact, where

$$E_{rad} = W_{rad,0}^{1,p(x)}(B_R(0)) \cap W_{rad,0}^{1,m(x)}(B_R(0)).$$

In what follows, B_R denotes the ball $B_R(0)$.

Theorem 4. *Assume that conditions* (H1)-(H4) *are fulfilled. Then the following nonhomogeneous boundary value problem:*

(P_1) $$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u & in \quad B_R, \\ u = 0 & on \quad \partial B_R, \end{cases}$$

has a nontrivial solution in E.

Proof. The natural candidate to be the energy associated to problem (P_1) is the following double-phase functional with variable exponents:

$$I(u) = \int_{B_R} \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{a(x)}{m(x)}|\nabla u|^{m(x)} \right) dx - \int_{B_R} \frac{1}{q(x)}|u|^{q(x)} \, dx.$$

However, this functional is not well defined on the whole space E because we do not assume any growth condition on q in the annulus $A_{R,r}$. In what follows we will restrict I to E_{rad}, because $I \in C^1(E_{rad}, \mathbb{R})$ and for all $u, v \in E_{rad}$

$$I'(u)v = \int_{B_R} (|\nabla u|^{p(x)-2}\nabla u\nabla v + a(x)|\nabla u|^{m(x)-2}\nabla u\nabla v) \, dx - \int_{B_R} |u|^{q(x)-2}uv \, dx.$$

Now, it is easy to prove that I satisfies the mountain pass geometry and also the (PS) condition, because we have the compact embedding (10). From this, we can apply the mountain pass theorem to find a nontrivial critical point $u \in E_{rad}$.

Our goal is to prove that u is in fact a critical point of I in the whole space E. However, we cannot apply directly the Palais principle of symmetric criticality, because I is not well defined in whole E. In order to overcome this difficulty, we will use the following trick: consider the function

$$g(x,t) = \xi(|x|)|t|^{q(x)} + (1 - \xi(|x|))|u(x)|^{q(x)} \quad \forall x \in B_R,$$

where $\xi \in C^\infty([0,R],\mathbb{R})$ satisfies

$$\xi(x) = \begin{cases} 1, & x \in x \in \overline{B}_{\frac{r}{2}}(0), \\ 0, & x \in x \in \overline{B}_R(0) \setminus \overline{B}_{\frac{3r}{5}}(0). \end{cases}$$

Since $u \in C(\overline{A}_{R,\frac{r}{2}})$ (see (9)), it follows from (H4) that

$$|g(x,t)| \leqslant C(|t|^{q_+^r} + 1) \quad \forall(x,t) \in B_R \times \mathbb{R}.$$

This fact implies that g has a subcritical growth. Consider the nonlinear problem

(P_g) $$\begin{cases} -\Delta_{p(x)}w - \Delta_{m(x),a(x)}w = g(x,w) & in \quad B_R, \\ w = 0 & on \quad \partial B_R, \end{cases}$$

whose associated energy is given by

$$J(w) = \int_{B_R} \left(\frac{1}{p(x)} |\nabla w|^{p(x)} + \frac{a(x)}{m(x)} |\nabla w|^{m(x)} \right) dx - \int_{B_R} G(x, w) \, dx,$$

where $G(x, t) = \int_0^t g(x, s) \, ds$.

Since g is subcritical, it follows that J is well defined in the whole space E, $J \in C^1(E, \mathbb{R})$ and

$$J'(u)v = \int_{B_R} (|\nabla w|^{p(x)-2} \nabla w \nabla v + a(x) |\nabla w|^{m(x)-2} \nabla w \nabla v) \, dx$$

$$- \int_{B_R} g(x, w)v \, dx \quad \forall u, v \in E.$$

Since

$$g(x, u(x)) = |u|^{q(x)-2} u \quad \forall x \in B_R,$$

we see that u is a critical point of J restricted to E_{rad}. Now we can apply the Palais principle of symmetric criticality developed by Kobayashi and Ôtani [14, Theorem 2.2] to conclude that u is a nontrivial critical point of J in the whole E. □

3. Problem 2: The nonradial case

In this section, we study the existence of nontrivial solutions for the following problem:

$$(\text{P}_2) \qquad \begin{cases} -\Delta_{p(x)} u - \Delta_{m(x),a(x)} u = |u|^{q(x)-2} u & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary. We assume that there exist positive numbers $r < R$ such that $B_R \subset \Omega$ and

$$a(x) = a_0 \qquad \text{for all } x \in A_{R,r}.$$

Related to the functions $p, q, m, a : \overline{\Omega} \to \mathbb{R}$, we assume that they are continuous and satisfy the following conditions:

(H5) $\qquad \begin{cases} 1 < p_- = \min_{x \in \overline{\Omega}} p(x) \leqslant \max_{x \in \overline{\Omega}} p(x) = p_+ < N, \\ 1 < m_- = \min_{x \in \overline{\Omega}} m(x) \leqslant \max_{x \in \overline{\Omega}} m(x) = m_+ < N, \end{cases}$

(H6) $\qquad\qquad\qquad 0 \leqslant a(x) \leqslant L \quad \forall x \in \overline{\Omega},$

(H7) $\qquad\qquad p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|) \quad \forall x \in \overline{A}_{R,r},$

and
(H8)
$$q(x) \geqslant 0 \quad \forall x \in \overline{\Omega} \quad \text{and} \quad p_+! < q_-^A = \min_{x \in \overline{\Omega} \setminus A_{R,r}} q(x) = q_+^A \leqslant \max_{x \in \overline{\Omega} \setminus A_{R,r}} q(x) < \min_{x \in \overline{\Omega}} p^*(x).$$

We point out that we are not assuming any growth condition on q in the annulus $A_{R,r}$, hence q can have a supercritical growth in that region.

Theorem 5. *Assume that hypotheses* (H5)-(H8) *are fulfilled. Then problem* (P_2) *has a nontrivial solution in* E.

Proof. The energy associated to problem (P_2) is the following double-phase varia-
tional integral with variable exponents:

$$I(u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx.$$

However, since we do not assume any growth condition on q in the annulus $A_{R,r}$
I is not well defined on the whole E. Keeping this in mind, we will restrict the
function I to the closed subspace $X \subset E$ given by

$$X = \{ u \in E \, : \, u(x) = u(|x|) \quad \text{a.e.} \quad x \in \overline{A}_{R,r} \}.$$

Arguing as in Section 2, we observe that the compact embedding (10) still holds in
the present case, hence $I \in C^1(X, \mathbb{R})$ and

$$I'(u)v = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|\nabla u|^{m(x)-2} \nabla u \nabla v) \, dx$$

$$- \int_\Omega |u|^{q(x)-2} uv \, dx \quad \forall u, v \in X.$$

Moreover, I also satisfies the (PS) condition in X. Therefore, we can use the
mountain pass theorem to get a nontrivial critical point $u_0 \in X$ of I, that is,
(12)
$$\int_\Omega (|\nabla u_0|^{p(x)-2} \nabla u \nabla v + a(x)|\nabla u_0|^{m(x)-2} \nabla u \nabla v) \, dx = \int_\Omega |u_0|^{q(x)-2} uv \, dx \quad \forall v \in X.$$

Now, we are going to show that u_0 is, in fact, a critical point of I. For this purpose
we cannot use the Palais principle used in Section 2, because Ω is not a ball. Here,
the trick is the following: for all $\varphi \in X_0(A_{R,r}) = \{ u \in X \, : \, u = 0 \quad \text{on} \quad \partial(A_{R,r}) \}$
we have

$$\int_{A_{R,r}} (|\nabla u_0|^{p(x)-2} \nabla u \nabla v + a(x)|\nabla u_0|^{m(x)-2} \nabla u \nabla v) \, dx - \int_{A_{R,r}} |u_0|^{q(x)-2} u_0 \varphi \, dx = 0.$$

Since $p(x) = p(|x|)$, $q(x) = q(|x|)$, and $a(x) = a_0$, the regularity theory ensures
that the function

$$f(s) = s^{N-1} (|u'(s)|^{p(s)-2} u(s) + a_0 |u_0'(s)|^{p(s)-2} u(s))$$

is of class C^1 in the interval (r, R) and satisfies the following equality, in the classical
sense:

$$(s^{N-1}(|u_0'(s)|^{p(s)-2} u(s) + a_0 |u_0'(s)|^{p(s)-2} u(s)))'$$

$$= s^{N-1} |u_0(s)|^{q(s)-2} u_0(s), \quad s \in (r, R).$$

It follows that for all $\psi \in E_0(A_{R,r})$
(13)
$$\int_{A_{R,r}} (|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla \psi + a(x)|\nabla u_0|^{m(x)-2} \nabla u \nabla \psi) \, dx - \int_{A_{R,r}} |u_0|^{q(x)-2} u_0 \psi \, dx = 0,$$

where $E_0(A_{R,r}) = \{ u \in E \, : \, u = 0 \quad \text{on} \quad \partial(A_{R,r}) \}$.
Using the above information, we are ready to prove that

$$\int_\Omega (|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla v + a(x)|\nabla u_0|^{m(x)-2} \nabla u \nabla v) \, dx$$

$$- \int_\Omega |u_0|^{q(x)-2} u_0 v \, dx = 0 \quad \forall v \in E.$$

In what follows, we consider an even function $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$0 \leqslant \phi(s) \leqslant 1 \quad \forall s \in \mathbb{R},$$

$$\phi(s) = 0 \quad \forall s \in [-1, 1],$$

and

$$\phi(s) = 1 \quad \forall s \in (-\infty, -2] \cup [2, +\infty).$$

For $\epsilon > 0$ small enough and $v \in C_0^\infty(B_R) \subset E_0(B_R) = \{u \in E : u = 0 \text{ on } \partial(B_R)\}$, we set the function

$$v_\epsilon(x) = \phi((|x| - r)/\epsilon)v(x) \quad \forall x \in \Omega.$$

From the definition of v_ϵ, we have that $v_\epsilon \in E_0(B_r(0)) \subset X$ and $v_\epsilon \in E_0(A_{R,r})$. Thus, by (12) and (13),

$$\int_{B_r(0)} (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v_\epsilon + a(x)|\nabla u_0|^{m(x)-2}\nabla u \nabla v_\epsilon)\, dx - \int_{B_r(0)} |u_0|^{q(x)-2}u_0 v_\epsilon dx = 0$$

and

$$\int_{A_{R,r}} (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v_\epsilon + a(x)|\nabla u_0|^{m(x)-2}\nabla u \nabla v_\epsilon)\, dx - \int_{A_{R,r}} |u_0|^{q(x)-2}u_0 v_\epsilon dx = 0,$$

leading to

$$\int_\Omega (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v_\epsilon + + a(x)|\nabla u_0|^{m(x)-2}\nabla u \nabla v_\epsilon)\, dx - \int_\Omega |u_0|^{q(x)-2}u_0 v_\epsilon\, dx = 0$$

or equivalently

$$\int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v \phi_\epsilon\, dx + \int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla \phi_\epsilon v\, dx$$

(14)
$$+ \int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v \phi_\epsilon\, dx + \int_\Omega a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla \phi_\epsilon v\, dx$$

$$- \int_{B_R} |u_0|^{q(x)-2}u_0 v \phi_\epsilon\, dx = 0,$$

where

$$\phi_\epsilon(x) = \phi((|x| - r))/\epsilon) \quad \forall x \in \mathbb{R}^N.$$

The Lebesgue dominated convergence theorem ensures that

(15)
$$\lim_{\epsilon \to 0} \int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v \phi_\epsilon\, dx = \int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla v\, dx,$$

(16)
$$\lim_{\epsilon \to 0} \int_\Omega a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla v \phi_\epsilon\, dx = \int_\Omega a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla v\, dx,$$

and

(17)
$$\lim_{\epsilon \to 0} \int_\Omega |u_0|^{q(x)-2}u_0 v \phi_\epsilon\, dx = \int_\Omega |u_0|^{q(x)-2}u_0 v\, dx.$$

On the other hand, it is very important to observe that

$$\int_\Omega |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla \phi_\epsilon v\, dx = \int_{A_\epsilon} |\nabla u_0|^{p(x)-2}\nabla u_0 \nabla \phi_\epsilon v\, dx,$$

where

$$A_\epsilon = \{x \in \Omega : ||x| - r| \leqslant 2\epsilon\}.$$

Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{\epsilon \to 0} \int_{A_\epsilon} |\nabla u_0|^{p(x)}\, dx = 0,$$

and so,

$$\lim_{\epsilon \to 0} \|\nabla u_0\|_{p(x), A_\epsilon} = 0.$$

On the other hand, since $v \in C_0^\infty(B_R)$, we have that

$$\int_{\mathbb{R}^N} |\nabla \phi_\epsilon(x)|^{p+} |v(x)|^{p+}\, dx \leqslant |v|_\infty^{p+} \int_{\mathbb{R}^N} |\nabla \phi_\epsilon(x)|^{p+}\, dx$$

$$\leqslant |v|_\infty^{p+} \epsilon^{N-p+} \int_{A_\epsilon} \left| \phi'\left(|z| - \frac{r}{\epsilon}\right) \right|^{p+}\, dz;$$

then

$$\int_{\mathbb{R}^N} |\nabla \phi_\epsilon(x)|^{p+} |v(x)|^{p+}\, dx$$

$$\leqslant C_N |\phi'|_\infty^{p+} |v|_\infty^{p+} \epsilon^{N-p+} \left((r + 2\epsilon)^N - (r - 2\epsilon)^N\right) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

A similar argument works to prove that

$$\int_{\mathbb{R}^N} |\nabla \phi_\epsilon(x)|^{p-} |v(x)|^{p-}\, dx \to 0 \quad \text{as} \quad \epsilon \to 0.$$

From this,

$$\||\nabla \phi_\epsilon||v|\|_{p(x)} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Hence, by using again Hölder's inequality, we get

$$(18) \qquad \lim_{\epsilon \to 0} \int_\Omega |\nabla u_0|^{p(x)-2} \nabla u_0 \nabla \phi_\epsilon v\, dx = 0.$$

A similar argument gives

$$(19) \qquad \lim_{\epsilon \to 0} \int_\Omega a(x)|\nabla u_0|^{m(x)-2} \nabla u_0 \nabla \phi_\epsilon v\, dx = 0.$$

Taking the limit of $\epsilon \to 0$ in (14) and using (15)-(19), we obtain for all $v \in C_0^\infty(B_R)$

$$\int_\Omega (|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla v + a(x)|\nabla u_0|^{m(x)-2} \nabla u_0 \nabla v)\, dx - \int_\Omega |u_0|^{q(x)-2} u_0 v\, dx = 0;$$

then by density

$$(20) \qquad \int_\Omega (|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla v + a(x)|\nabla u_0|^{m(x)-2} \nabla u_0 \nabla v)\, dx$$

$$- \int_\Omega |u_0|^{q(x)-2} u_0 v\, dx = 0 \quad \forall v \in E_0(B_R).$$

Now, we show that the above equality holds for any $w \in C_0^\infty(\Omega) \subset E$. The idea is as above; we set the function

$$w_\epsilon(x) = \phi((|x| - R)/\epsilon)w(x) \quad \forall x \in \Omega,$$

which belongs to $E_0(B_R)$ and $E_0(\Omega \setminus \overline{B}_R) \subset E$. Since $w_\epsilon|_{\Omega \setminus \overline{B}_R} \in E$, it follows that

$$\int_{\Omega \setminus \overline{B}_R} (|\nabla u_0|^{p(x)-2} \nabla u_0 \nabla w_\epsilon + |\nabla u_0|^{m(x)-2} \nabla u_0 \nabla w_\epsilon)\, dx - \int_{\Omega \setminus \overline{B}_R} |u_0|^{q(x)-2} u_0 w_\epsilon\, dx = 0.$$

On the other hand, as $w_\epsilon|_{B_R} \in E_0(B_R)$, by (20),

$$\int_{B_R} (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla w_\epsilon + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla w_\epsilon)\, dx - \int_{B_R} |u_0|^{q(x)-2} u_0 w_\epsilon\, dx = 0.$$

Combining the last two equalities we obtain

(21)
$$\int_\Omega (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla w_\epsilon + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla w_\epsilon)\, dx - \int_\Omega |u_0|^{q(x)-2} u_0 w_\epsilon\, dx = 0.$$

Now, the same argument used for function v_ϵ works to conclude that taking the limit $\epsilon \to 0$ in (21) we obtain

$$\int_\Omega (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla w + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla w)\, dx$$
$$- \int_\Omega |u_0|^{q(x)-2} u_0 w\, dx = 0 \quad \forall w \in C_0^\infty(\Omega).$$

Again by density, we have that

$$\int_\Omega (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla w + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla w)\, dx$$
$$- \int_\Omega |u_0|^{q(x)-2} u_0 w\, dx = 0 \quad \forall w \in E,$$

showing that u_0 is a nontrivial solution of (P_2). $\qquad\square$

4. Problem 3: The case where q vanishes close to the boundary

In this section, we study the existence of nontrivial solutions for the following class of problems:

$$(P_3) \qquad \begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = \lambda|u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and Ω is a bounded domain with smooth boundary. We assume that there exist positive numbers $r < R$ such that $B_R(0) \subset \Omega$,

$$A_{R,r} \subset \Omega_\delta \quad \text{and} \quad a(x) = a_0 \quad \forall x \in A_{R,r},$$

where

$$\Omega_\delta = \{x \in \Omega \,:\, \text{dist}\,(x, \partial\Omega) > \delta\}.$$

Related to the functions $p, q, m, a : \overline{\Omega} \to \mathbb{R}$, we assume that they are continuous and satisfy the following conditions:

(H9) $\qquad \begin{cases} 1 < p_- = \min_{x \in \overline{\Omega}} p(x) \leqslant \max_{x \in \overline{\Omega}} p(x) = p_+ < N, \\ 1 < m_- = \min_{x \in \overline{\Omega}} m(x) \leqslant \max_{x \in \overline{\Omega}} m(x) = m_+ < N. \end{cases}$

(H10) $\quad \max\{p_+, m_+\} < q_-^A = \min_{x \in \overline{\Omega}_\delta \backslash A_{R,r}} q(x) \leqslant q_+^A = \max_{x \in \overline{\Omega}_\delta \backslash A_{R,r}} q(x) < \min_{x \in \overline{\Omega}} p^*(x),$

(H11) $\qquad\qquad\qquad 0 \leqslant a(x) \leqslant L \quad \forall x \in \overline{\Omega},$

and

(H12) $\qquad\qquad q(x) \geqslant 0 \quad \forall x \in \overline{\Omega} \quad \text{and} \quad \lim_{\text{dist}\,(x, \partial\Omega) \to 0} q(x) = 0.$

Theorem 6. *Assume that hypotheses* (H9)-(H12) *are fulfilled. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (P$_3$) *has at least two nontrivial solutions in* E.

As in the previous section, the natural candidate to be the energy associated to problem (P$_3$) is the following double-phase functional with variable exponents:

$$I(u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_\Omega \frac{\lambda}{q(x)} |u|^{q(x)} dx.$$

Since we do not assume any growth condition on q in the annulus $A_{R,r}$, this functional is not well defined in the whole space E. That is why we restrict the functional I to the closed subspace $X \subset E$ given by

$$X = \{ u \in E \, : \, u(x) = u(|x|) \quad \text{a.e.} \quad x \in \overline{A}_{R,r} \}.$$

Arguing as in Section 2, we deduce that the compact embedding (10) still holds in the present case, hence $I \in C^1(X, \mathbb{R})$ and

$$I'(u)v = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|\nabla u|^{m(x)-2} \nabla u \nabla v) \, dx$$

$$- \lambda \int_\Omega |u|^{q(x)-2} uv \, dx \quad \forall u, v \in X.$$

Moreover, I also satisfies the (PS) condition in X.

Lemma 7. *Given* $\tau > 0$, *there are* $\rho = \rho(\tau) > 0$ *and* $\lambda^* = \lambda^*(\tau)$ *such that*

$$I_\lambda(u) \geqslant \rho \quad \text{for} \quad \|u\| = \tau \quad \text{and} \quad \lambda \in (0, \lambda^*).$$

Proof. By our assumptions we know that there exists $C > 0$ such that

$$\int_\Omega |u|^{q(x)} \, dx \leqslant C \max\{\|u\|^{q+}, \|u\|^{q-}\}.$$

From this, for $\|u\| = \tau$,

$$\int_\Omega |u|^{q(x)} \, dx \leqslant C \max\{\tau^{q+}, \tau^{q-}\} = C_\tau,$$

and so,

$$I_\lambda(u) \geqslant \min\{\tau^{p+}, \tau^{p-}\} - \lambda C_\tau, \quad \|u\| = \tau.$$

Setting $\lambda^* = \frac{\min\{\tau^{p+}, \tau^{p-}\}}{2C_\tau}$ and $\lambda \in (0, \lambda^*)$, we get

$$I_\lambda(u) \geqslant \frac{\min\{\tau^{p+}, \tau^{p-}\}}{2} = \rho(\tau) > 0, \quad \|u\| = \tau.$$

The proof of this auxiliary result is now concluded. □

Lemma 8. *Setting* $A_\lambda = \inf\{I_\lambda(u) \, : \, \|u\| \leqslant \tau\}$, *we have that* $A_\lambda < 0$ *for all* $\lambda \in (0, \lambda^*)$.

Proof. Let $\phi \in C_0^\infty(B_\alpha(x_0))$ such that

$$q_1 = \max_{x \in \overline{B}_\alpha(x_0)} q(x) < \min\{p_-, m_-\}.$$

Here, we are using the fact that $\lim_{dist(x,\partial\Omega) \to 0} q(x) = 0$. If $t > 0$ is small enough, a simple computation gives

$$I_\lambda(t\phi) \leqslant \int_\Omega \left(\frac{t^{p-}}{p(x)} |\nabla \phi|^{p(x)} + \frac{t^{m-} a(x)}{m(x)} |\nabla \phi|^{m(x)} \right) dx - t^{q_1} \int_\Omega \frac{\lambda}{q(x)} |\phi|^{q(x)} \, dx.$$

It follows that

$$I_\lambda(t\phi) < 0 \quad \text{for} \quad t \approx 0^+,$$

showing the desired result. □

The last two lemmas permit one to apply the Ekeland variational principle to conclude that there exists $u_\lambda \in X$ such that

$$I'_\lambda(u_\lambda)v = 0, \quad \forall v \in X \quad \text{and} \quad I_\lambda(u_\lambda) = A_\lambda < 0.$$

Repeating the same arguments of the last section, it follows that u_λ is a critical point of I_λ in E for all $\lambda \in (0, \lambda^*)$.

Lemma 9. *Fix $\phi \in C_0^\infty(\Omega_\delta \setminus \overline{A}_{R,r})$; we have that*

$$I_\lambda(t\phi) \to -\infty \quad as \quad t \to +\infty.$$

Proof. If $\phi \in C_0^\infty(\Omega_\delta \setminus \overline{A}_{R,r})$ and $t > 0$ is large enough, by (H10) we have that

$$I_\lambda(t\phi) \leqslant \int_\Omega \left(\frac{t^{p_+}}{p(x)}|\nabla\phi|^{p(x)} + \frac{t^{m_+}a(x)}{m(x)}|\nabla\phi|^{m(x)} \right) dx - t^{q_-^A} \int_\Omega \frac{\lambda}{q(x)}|\phi|^{q(x)}\, dx.$$

Hence

$$\lim_{t\to+\infty} I_\lambda(t\phi) = -\infty,$$

showing the desired result. □

Proof of Theorem 6. From Lemmas 7 and 9, we derive that I_λ satisfies the mountain pass geometry. Applying [12, Theorem 1.1] (see also [13]), we obtain that for almost every $\lambda \in (0, \lambda^*)$ there is a bounded $(PS)_{c_\lambda}$ sequence for I_λ, where c_λ is the mountain level of I_λ. Since I_λ verifies the (PS) condition, it follows that for almost every $\lambda \in (0, \lambda^*)$ the level c_λ is a critical level, that is, there is $u^\lambda \in X$ such that

$$I'_\lambda(u^\lambda) = 0 \quad \text{and} \quad I_\lambda(u^\lambda) = c_\lambda > 0.$$

We conclude that problem (P_3) has at least two solutions u_λ and u^λ for almost every $\lambda \in (0, \lambda^*)$ with

$$I_\lambda(u_\lambda) = A_\lambda < 0 \quad \text{and} \quad I_\lambda(u^\lambda) = c_\lambda > 0.$$

Finally, we can repeat the same arguments developed in Section 3 to conclude that u_λ and u^λ are, in fact, critical points of I_λ in E, hence two nontrivial solutions of problem (P_3). □

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Unidade Acadêmica de Matemática, Universidade Federal de Campina Grande, 58429-970, Campina Grande - PB, Brazil
Email address: `coalves@mat.ufcg.edu.br`

Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland; and Department of Mathematics, University of Craiova, 200585 Craiova, Romania; and Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80257, Jeddah 21589, Saudi Arabia
Email address: `radulescu@inf.ucv.ro`