ON A NONHOMOGENEOUS QUASILINEAR EIGENVALUE PROBLEM IN SOBOLEV SPACES WITH VARIABLE EXPONENT

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Abstract. We consider the nonlinear eigenvalue problem
\[-\text{div} \left( |\nabla u|^{p(x)} - 2 \nabla u \right) = \lambda |u|^{q(x)} - 2 u\]
in \(\Omega\), \(u = 0\) on \(\partial \Omega\), where \(\Omega\) is a bounded open set in \(\mathbb{R}^N\) with smooth boundary
and \(p, q\) are continuous functions on \(\Omega\) such that \(1 < \inf_{\Omega} q < \inf_{\Omega} p < \sup_{\Omega} q\),
\(\sup_{\Omega} p < N\), and \(q(x) < Np(x)/(N - p(x))\) for all \(x \in \Omega\). The main result
of this paper establishes that any \(\lambda > 0\) sufficiently small is an eigenvalue of
the above nonhomogeneous quasilinear problem. The proof relies on simple
variational arguments based on Ekeland’s variational principle.

1. Introduction and preliminary results

A basic result in the elementary theory of linear partial differential equations
asserts that the spectrum of the Laplace operator in \(H^1_0(\Omega)\) is discrete, where \(\Omega\) is
a bounded open set in \(\mathbb{R}^N\) with smooth boundary. More precisely, the problem
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has an unbounded sequence of eigenvalues \(0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots\). This
celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact
operators on Hilbert spaces. The anisotropic case
\[
\begin{cases}
-\Delta u = \lambda a(x) u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
was considered by Bocher [5], Hess and Kato [16], Minakshisundaram and Pleijel [20, 22]. For instance, Minakshisundaram and Pleijel proved that the above eigen-
value problem has an unbounded sequence of positive eigenvalues if \(a \in L^\infty(\Omega)\),
\(a \geq 0\) in \(\Omega\), and \(a > 0\) in \(\Omega_0 \subset \Omega\), where \(|\Omega_0| > 0\). Eigenvalue problems for ho-
mogeneous quasilinear problems have been intensively studied in the last decades
(see, e.g., Anane [4] and Lindqvist [18]).

This paper is motivated by recent advances in elastic mechanics and electrorhe-
ological fluids (sometimes referred to as “smart fluids”), where some processes are
modeled by nonhomogeneous quasilinear operators (see Acerbi and Mingione [1],
Diening [7], Halsey [15], Ruzicka [24], Zhikov [26, 27], and the references therein). We refer mainly to the \( p(x) \)-Laplace operator \( \Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2} \nabla u) \), where \( p \) is a continuous nonconstant function. This differential operator is a natural generalization of the \( p \)-Laplace operator \( \Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) \), where \( p > 1 \) is a real constant. However, the \( p(x) \)-Laplace operator possesses more complicated nonlinearities than the \( p \)-Laplace operator, due to the fact that \( \Delta_{p(x)} \) is not homogeneous. Recent qualitative properties of solutions to quasilinear problems in Sobolev spaces with variable exponent have been obtained by Alves and Souto [2], Chabrowski and Fu [6], and Mihăilescu and Rădulescu [19]. We also refer to El Hamidi [12], where it is proved a multiplicity result for a class of symmetric systems involving \( (p(x), q(x)) \)-Laplace operators.

In this paper we are concerned with the nonhomogeneous eigenvalue problem

\[
\begin{align*}
\left\{ 
-\text{div}(|\nabla u|^{p(x)-2} \nabla u) &= \lambda |u|^{q(x)-2} u, & x \in \Omega, \\
u &= 0, & x \in \partial\Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded domain with smooth boundary, \( \lambda > 0 \) is a real number, and \( p, q \) are continuous on \( \bar{\Omega} \).

The case \( p(x) = q(x) \) was considered by Fan, Zhang and Zhao in [14] who, using the Ljusternik-Schnirelmann critical point theory, established the existence of a sequence of eigenvalues. Denoting by \( \Lambda \) the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that \( \sup \Lambda = +\infty \), and they pointed out that only under additional assumptions we have \( \inf \Lambda > 0 \). We remark that for the \( p \)-Laplace operator (corresponding to \( p(x) \equiv p \)) we always have \( \inf \Lambda > 0 \).

In this paper we study problem (1.1) under the basic assumption

\[
1 < \min_{x \in \Omega} q(x) < \min_{x \in \Omega} p(x) < \max_{x \in \Omega} q(x).
\]

Our main result establishes the existence of a continuous family of eigenvalues for problem (1.1) in a neighborhood of the origin. More precisely, we show that there exists \( \lambda^* > 0 \) such that \( \arg\max \lambda \in (0, \lambda^*) \) is an eigenvalue for problem (1.1).

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by Musielak [21] and the papers by Edmunds et al. [8, 9, 10], Kovacik and Rákosník [17], and Samko and Vakulov [24].

We assume that \( p \in C(\bar{\Omega}) \) and \( p(x) > 1 \), for all \( x \in \bar{\Omega} \).

Set

\[
C_+(\bar{\Omega}) = \{ h; \ h \in C(\bar{\Omega}), \ h(x) > 1 \text{ for all } x \in \bar{\Omega} \}.
\]

For any \( h \in C_+(\bar{\Omega}) \) we define

\[
h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).
\]

For any \( p(x) \in C_+(\bar{\Omega}) \), we define the variable exponent Lebesgue space

\[
L^{p(x)}(\Omega) = \{ u; \ u \text{ is a measurable real-valued function and } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \}.
\]

We define a norm, the so-called Luxemburg norm, on this space by the formula

\[
|u|_{p(x)} = \inf \left\{ \mu > 0; \ \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} \, dx \leq 1 \right\}.
\]
We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < \|\Omega\| < \infty$ and $p_1, p_2$ are variable exponent, so that $p_1(x) \leq p_2(x)$ almost everywhere in $\Omega$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^p(x)(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. For any $u \in L^p(x)(\Omega)$ and $v \in L^{p'}(x)(\Omega)$ the Hölder type inequality

$$
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x)} |v|_{p'(x)} \tag{1.3}
$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$ 

If $(u_n), u \in L^{p(x)}(\Omega)$, then the following relations hold true:

$$
|u|_{p(x)} > 1 \Rightarrow |u|^p_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^p_{p(x)}, \tag{1.4}
$$

$$
|u|_{p(x)} < 1 \Rightarrow |u|^p_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^p_{p(x)}, \tag{1.5}
$$

$$
|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0. \tag{1.6}
$$

Next, we define $W^{1,p(x)}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$ 

The space $(W^{1,p(x)}_0(\Omega), \| \cdot \|)$ is a separable and reflexive Banach space. We note that if $s(x) \in C_+ (\overline{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}_0(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{n p(x)}{N - p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$.

We refer to Kovářík and Rákosník [17] for more properties of Lebesgue and Sobolev spaces with variable exponent.

2. The main result

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u \in W^{1,p(x)}_0(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} |u|^{q(x)-2} u v \, dx = 0,$$

for all $v \in W^{1,p(x)}_0(\Omega)$. We point out that if $\lambda$ is an eigenvalue of problem (1.1), then the corresponding $u \in W^{1,p(x)}_0(\Omega) \setminus \{0\}$ is a weak solution of (1.1).

Our main result is given by the following theorem.

**Theorem 2.1.** Assume that condition (1.2) is fulfilled, $\max_{x \in \overline{\Omega}} p(x) < N$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. Then there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem (1.1).
The above result implies
\[
\inf_{u \in W^{1,p(x)}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^{p(x)} \, dx}{\int_{\Omega} |u|^{q(x)} \, dx} = 0.
\]

Thus, for any positive constant \(C\), there exists \(u_0 \in W^{1,p(x)}_0(\Omega)\) such that
\[C \int_{\Omega} |u_0|^{q(x)} \, dx \geq \int_{\Omega} |\nabla u_0|^{p(x)} \, dx.\]

Let \(E\) denote the generalized Sobolev space \(W^{1,p(x)}_0(\Omega)\).

For any \(\lambda > 0\) the energy functional corresponding to problem (1.1) is defined as
\[J_\lambda : E \to \mathbb{R}, \quad J_\lambda(u) = \int_{\Omega} \frac{1}{p(x)}|\nabla u|^{p(x)} \, dx - \lambda \int_{\Omega} \frac{1}{q(x)}|u|^{q(x)} \, dx.\]

Standard arguments imply that \(J_\lambda \in C^1(E, \mathbb{R})\) and
\[\langle J'_\lambda(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} |u|^{q(x)-2} uv \, dx,\]
for all \(u, v \in E\). Thus the weak solutions of (1.1) coincide with the critical points of \(J_\lambda\). If such a weak solution exists and is nontrivial, then the corresponding \(\lambda\) is an eigenvalue of problem (1.1).

**Lemma 2.2.** There exists \(\lambda^* > 0\) such that for any \(\lambda \in (0, \lambda^*)\) there exist \(\rho, a > 0\) such that \(J_\lambda(u) \geq a > 0\) for any \(u \in E\) with \(\|u\| = \rho\).

**Proof.** Since \(q(x) < p^*(x)\) for all \(x \in \Omega\) it follows that \(E\) is continuously embedded in \(L^{q(x)}(\Omega)\). So, there exists a positive constant \(c_1\) such that
\[(2.1) \quad |u|^{q(x)} \leq c_1|u|, \quad \forall u \in E.\]

We fix \(\rho \in (0, 1)\) such that \(\rho < 1/c_1\). Then relation (2.1) implies
\[|u|^{q(x)} < 1, \quad \forall u \in E, \text{ with } \|u\| = \rho.\]

Furthermore, relation (1.5) yields
\[(2.2) \quad \int_{\Omega} |u|^{q(x)} \, dx \leq |u|^{q^-}_{q(x)}, \quad \forall u \in E, \text{ with } \|u\| = \rho.\]

Relations (2.1) and (2.2) imply
\[(2.3) \quad \int_{\Omega} |u|^{q(x)} \, dx \leq c_1^{q^-} \|u\|^{q^-}, \quad \forall u \in E, \text{ with } \|u\| = \rho.\]
Taking into account relations (1.5) and (2.3) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true:

$$
J_\lambda(u) \geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)} \, dx - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)} \, dx
$$

$$
\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} \|u\|^{q^-}
$$

$$
= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} \rho^{q^-}
$$

$$
= \rho^{q^-} \left( \frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} \rho^{q^-} \right).
$$

By the above inequality we remark that if we define

(2.4)

$$
\lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \cdot \frac{q^-}{c_1^q},
$$

then for any $\lambda \in (0, \lambda^*)$ and any $u \in E$ with $\|u\| = \rho$ there exists $a = \frac{p^+}{2p^+} > 0$ such that

$$
J_\lambda(u) \geq a > 0.
$$

The proof of Lemma 2.2 is complete. \(\square\)

**Lemma 2.3.** There exists $\phi \in E$ such that $\phi \geq 0$, $\phi \neq 0$ and $J_\lambda(t\phi) < 0$, for $t > 0$ small enough.

**Proof.** Assumption (1.2) implies that $q^- < p^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < p^-$ for all $x \in \Omega_0$.

Let $\phi \in C^\infty_0(\Omega)$ be such that $\text{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \phi \leq 1$ in $\Omega$. Then using the above information for any $t \in (0, 1)$ we have

$$
J_\lambda(t\phi) = \int_\Omega \left( \frac{t^{p(x)}}{p(x)} |\nabla \phi|^{p(x)} \right) \, dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |\phi|^{q(x)} \, dx
$$

$$
\leq \frac{t^{p^+}}{p^+} \int_\Omega |\nabla \phi|^{p(x)} \, dx - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |\phi|^{q(x)} \, dx
$$

$$
\leq \frac{t^{p^-}}{p^-} \int_\Omega |\nabla \phi|^{p(x)} \, dx - \lambda \int_\Omega \frac{t^{q(x)}}{q^-} |\phi|^{q(x)} \, dx
$$

$$
\leq \frac{t^{p^-}}{p^-} \int_\Omega |\nabla \phi|^{p(x)} \, dx - \lambda \cdot \frac{t^{q^+ - \epsilon_0}}{q^+} \int_\Omega_0 |\phi|^{q(x)} \, dx.
$$

Therefore

$$
J_\lambda(t\phi) < 0
$$

for $t < \delta^{1/(p^- - q^- - \epsilon_0)}$ with

$$
0 < \delta < \min \left\{ 1, \frac{\lambda p^-}{q^-} \int_\Omega_0 |\phi|^{q(x)} \, dx \right\}.
$$
Finally, we point out that \( \int_{\Omega} |\nabla \phi|^{p(x)} \, dx > 0 \). Indeed, it is clear that
\[
0 \leq \int_{\Omega} |\phi|^{q(x)} \, dx \leq \int_{\Omega} |\phi|^{q^-} \, dx.
\]
On the other hand, \( W^{1,p(x)}_0(\Omega) \) is continuously embedded in \( L^{q^-}(\Omega) \), and thus there exists a positive constant \( c_2 \) such that
\[
|\phi|_{q^-} \leq c_2 \|\phi\|.
\]
The last two inequalities imply that
\[
\|\phi\| > 0,
\]
and combining that fact with relations (1.4) or (1.5) we deduce that
\[
\int_{\Omega} |\nabla \phi|^{p(x)} \, dx > 0.
\]
The proof of Lemma 2.3 is complete. \( \square \)

**Proof of Theorem 2.1.** Let \( \lambda^* > 0 \) be defined as in (2.4) and \( \lambda \in (0, \lambda^*) \). By Lemma 2.2 it follows that on the boundary of the ball centered at the origin and of radius \( \rho \) in \( E \), denoted by \( B_{\rho}(0) \), we have
\[
\inf_{\partial B_{\rho}(0)} J_{\lambda} > 0.
\]
On the other hand, by Lemma 2.3, there exists \( \phi \in E \) such that \( J_{\lambda}(t\phi) < 0 \) for all \( t > 0 \) small enough. Moreover, relations (2.3) and (1.5) imply that for any \( u \in B_{\rho}(0) \) we have
\[
J_{\lambda}(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} c_1 \|u\|^{q^-}.
\]
It follows that
\[
-\infty < \epsilon := \inf_{B_{\rho}(0)} J_{\lambda} < 0.
\]
We now let \( 0 < \epsilon < \inf_{\partial B_{\rho}(0)} J_{\lambda} - \inf_{B_{\rho}(0)} J_{\lambda} \). Applying Ekeland’s variational principle to the functional \( J_{\lambda} : B_{\rho}(0) \to \mathbb{R} \), we find \( u_\epsilon \in B_{\rho}(0) \) such that
\[
J_{\lambda}(u_\epsilon) < \inf_{B_{\rho}(0)} J_{\lambda} + \epsilon,
\]
\[
J_{\lambda}(u_\epsilon) < J_{\lambda}(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon.
\]
Since
\[
J_{\lambda}(u_\epsilon) \leq \inf_{B_{\rho}(0)} J_{\lambda} + \epsilon \leq \inf_{B_{\rho}(0)} J_{\lambda} + \epsilon < \inf_{\partial B_{\rho}(0)} J_{\lambda},
\]
we deduce that \( u_\epsilon \in B_{\rho}(0) \). Now, we define \( I_{\lambda} : B_{\rho}(0) \to \mathbb{R} \) by \( I_{\lambda}(u) = J_{\lambda}(u) + \epsilon \cdot \|u - u_\epsilon\| \). It is clear that \( u_\epsilon \) is a minimum point of \( I_{\lambda} \) and thus
\[
\frac{I_{\lambda}(u_\epsilon + tv) - I_{\lambda}(u_\epsilon)}{t} \geq 0
\]
for small \( t > 0 \) and any \( v \in B_{\epsilon}(0) \). The above relation yields
\[
\frac{J_{\lambda}(u_\epsilon + tv) - J_{\lambda}(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.
\]
Letting \( t \to 0 \) it follows that \( \langle J'_{\lambda}(u_\epsilon), v \rangle + \epsilon \cdot \|v\| > 0 \), and we infer that \( \|J'_{\lambda}(u_\epsilon)\| \leq \epsilon \).
We deduce that there exists a sequence \( \{w_n\} \subset B_{\rho}(0) \) such that
\[
J_\lambda(w_n) \to c \quad \text{and} \quad J'_\lambda(w_n) \to 0.
\]
It is clear that \( \{w_n\} \) is bounded in \( E \). Thus, there exists \( w \in E \) such that, up to a subsequence, \( \{w_n\} \) converges weakly to \( w \) in \( E \). Since \( q(x) < p^*(x) \) for all \( x \in \Omega \) we deduce that \( E \) is compactly embedded in \( L^{q(x)}(\Omega) \), hence \( \{w_n\} \) converges strongly to \( w \) in \( L^{p(x)}(\Omega) \). So, by relations (1.6) and (1.3),
\[
\lim_{n \to \infty} \int_\Omega |w_n|^{q(x)-2} w_n (w_n - w) \, dx = 0.
\]
On the other hand, relation (2.6) yields
\[
\lim_{n \to \infty} \langle J'_\lambda(w_n), w_n - w \rangle = 0.
\]
Using the above information we find
\[
\lim_{n \to \infty} \int_\Omega |\nabla w_n|^{p(x)-2} \nabla w_n \nabla (w_n - w) \, dx = 0.
\]
Relation (2.7) and the fact that \( \{w_n\} \) converges weakly to \( w \) in \( E \) enable us to apply Theorem 3.1 in Fan and Zhang [13] in order to obtain that \( \{w_n\} \) converges strongly to \( w \) in \( E \). So, by (2.6),
\[
J_\lambda(w) = c < 0 \quad \text{and} \quad J'_\lambda(w) = 0.
\]
We conclude that \( w \) is a nontrivial weak solution for problem (1.1) and thus any \( \lambda \in (0, \lambda^*) \) is an eigenvalue of problem (1.1).

The proof of Theorem (2.1) is complete. \( \square \)

Let us now assume that the hypotheses of Theorem (2.1) are fulfilled and, furthermore,
\[
\max_{\Omega} p(x) < \max_{\Omega} q(x).
\]
Then, using similar arguments as in the proof of Lemma 2.3 we find some \( \psi \in E \) such that
\[
\lim_{t \to \infty} J_\lambda(t\psi) = -\infty.
\]
That fact combined with Lemma 2.2 and the mountain pass theorem (see [3]) implies that there exists a sequence \( \{u_n\} \) in \( E \) such that
\[
J_\lambda(u_n) \to \tau > 0 \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{in} \quad E^*.
\]
However, relation (2.9) is not useful because we cannot show that the sequence \( \{u_n\} \) is bounded in \( E \) since the functional \( J_\lambda \) does not satisfy a relation of the Ambrosetti-Rabinowitz type. This enables us to affirm that we cannot obtain a critical point for \( J_\lambda \) by using this method.

On the other hand, we point out that we will fail in trying to show that the functional \( J_\lambda \) is coercive since by relation (1.2) we have \( q^+ > p^- \). Thus, we cannot apply (as in the homogeneous case) a result as Theorem 1.2 in Struwe [25] in order to obtain a critical point of the functional \( J_\lambda \).

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