

**MULTI-VALUED BOUNDARY VALUE PROBLEMS
INVOLVING LERAY-LIONS OPERATORS
AND DISCONTINUOUS NONLINEARITIES**

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We prove an existence result for a class of Dirichlet boundary value problems with discontinuous nonlinearity and involving a Leray-Lions operator. The proof combines monotonicity methods for elliptic problems, variational inequality techniques and basic tools related to monotone operators. Our work generalizes a result obtained in Carl [4].

Key words: sub- and super-solution, Leray-Lions operator, maximal monotone graph, pseudo-monotone operator, variational inequality.

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1. Introduction and the main result.

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary. Consider the boundary value problem

$$(P) \begin{cases} -\operatorname{div}(a(x, \nabla u(x))) = f(u(x)), & \text{if } x \in \Omega \\ u = 0, & \text{on } \partial \Omega, \end{cases}$$

where $a : \Omega \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ is a Carathéodory function having the properties

- (a₁) there exist $p > 1$ and $\lambda > 0$ such that $a(x, \xi) \cdot \xi \geq \lambda \cdot \|\xi\|^p$, for a.e. $x \in \Omega$ and for any $\xi \in \mathbf{R}^N$;
- (a₂) $(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0$, for any $\xi, \eta \in \mathbf{R}^N$, $\xi \neq \eta$;
- (a₃) there exist $\alpha \in \mathbf{R}^+$ and $k \in L^{p'}(\Omega)$ such that $|a(x, \xi)| \leq \alpha(k(x) + |\xi|^{p-1})$, for a.e. $x \in \Omega$ and for any $\xi \in \mathbf{R}^N$.

Assume that the nonlinearity $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies the hypothesis

- (H₁) there exist nondecreasing functions $f, g : \mathbf{R} \rightarrow \mathbf{R}$ such that $f = g - h$.

Let $\beta : \mathbf{R} \rightarrow 2^{\mathbf{R}}$ be the maximal monotone graph associated with the nondecreasing function h (see Brezis [3]). More exactly,

$$\beta(s) := [h^-(s), h^+(s)], \quad \text{for all } s \in \mathbf{R},$$

where

$$h^-(s) = \lim_{\varepsilon \rightarrow 0^+} h(s - \varepsilon), \quad h^+(s) = \lim_{\varepsilon \rightarrow 0^+} h(s + \varepsilon).$$

Under this assumption we reformulate the problem (P) as follows

$$(P') \begin{cases} -\operatorname{div}(a(x, \nabla(x))) + \beta(u(x)) \ni g(u(x)), & \text{if } x \in \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Denote by G the Nemitskii operator associated with g , that is, $G(u)(x) = g(u(x))$.

DEFINITION 1. A function $u \in W_0^{1,p}(\Omega)$ is called a solution of the problem (P') if there exists $v \in L^{p'}(\Omega)$ such that

- i) $v(x) \in \beta(u(x))$ a.e. in Ω ,
- ii) $\int_{\Omega} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\Omega} v \cdot w \, dx = \int_{\Omega} G(u) \cdot w \, dx$, for any $w \in W_0^{1,p}(\Omega)$.

Let L_+^p be the set of nonnegative elements of $L^p(\Omega)$. For any $v, w \in \Omega$ such that $v \leq w$, we set

$$[v, w] = \{u \in L^p(\Omega) / v \leq u \leq w\}.$$

DEFINITION 2. A function $\bar{u} \in W^{1,p}(\Omega)$ is called an upper solution of the problem (P') if there exists a function $\bar{v} \in L^{p'}(\Omega)$ such that

- i) $\bar{v}(x) \in \beta(\bar{u}(x))$ a.e. in Ω ,
- ii) $\bar{u} \geq 0$ on $\partial\Omega$,
- iii) $\int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla w \, dx + \int_{\Omega} \bar{v} \cdot w \, dx \geq \int_{\Omega} G(\bar{u}) \cdot w \, dx$ for all $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$.

DEFINITION 3. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a lower solution of the problem (P') if there exists a function $\underline{v} \in L^{p'}(\Omega)$ such that

- i) $\underline{v}(x) \in \beta(\underline{u}(x))$ a.e. in Ω ,

- ii) $\underline{u} \leq 0$ on $\partial\Omega$,
- iii) $\int_{\Omega} a(x, \nabla \underline{u}) \cdot \nabla w \, dx + \int_{\Omega} \underline{v} \cdot w \, dx \leq \int_{\Omega} G(\underline{u}) \cdot w \, dx$ for any $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$.

In the sequel the following hypothesis will be needed:

- (H_2) There exist an upper solution \bar{u} and a lower solution \underline{u} of the problem (P') such that $\underline{u} \leq \bar{u}$, and $G(\underline{u}), G(\bar{u}), H^+(\bar{u}), H^-(\underline{u}) \in L^{p'}(\Omega)$.

The following is a generalization of the main result in Carl [4].

THEOREM 1. *Assume hypothesis (H_1) and (H_2) hold and that g is right (resp. left) continuous. Then there exists a maximal (resp. minimal) solution $u \in [\underline{u}, \bar{u}]$ of the problem (P').*

2. Proof of Theorem 1.

We first reformulate the problem (P') in terms of variational inequalities using the subdifferential theory in the sense of convex analysis.

Let $j : \mathbf{R} \rightarrow (-\infty, \infty]$ be a convex, proper and lower semicontinuous function. Let ∂j be the subdifferential of j , that is

$$(1) \quad \partial j(r) = \{\hat{r} \in \mathbf{R} : j(s) \geq j(r) + \hat{r}(s - r) \quad \text{for all } s \in \mathbf{R}\}.$$

We recall the following result concerning maximal monotone graphs in \mathbf{R}^2 (see Brezis [3] [Corollary 2.10], p. 43)

LEMMA 1. *Let $\beta : \mathbf{R} \rightarrow 2^{\mathbf{R}}$ be a maximal monotone graph in \mathbf{R}^2 . Then there exists a convex, proper and lower semicontinuous function $j : \mathbf{R} \rightarrow (-\infty, +\infty]$ such that $\beta = \partial j$. Moreover, the function j is uniquely determined up to an additive constant.*

We observe that the function h appearing in (H_1) can always be chosen so that $h(0) = 0$. Then the maximal monotone graph β has the properties

$$(2) \quad D(\beta) = \mathbf{R} \quad \text{and} \quad 0 \in \beta(0).$$

Since the function j related to β according to Lemma 1 is uniquely determined up to an additive constant we can assume that

$$(3) \quad j(0) = 0.$$

So, by (1), (2) and (3) it follows that

$$(4) \quad j(s) \geq 0 \quad \text{for all } s \in \mathbf{R}.$$

Define $J : L^p(\Omega) \rightarrow (-\infty, +\infty]$ by

$$J(v) = \begin{cases} \int_{\Omega} j(v(x)) dx, & \text{for } j(v(\cdot)) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then J is convex, proper and lower semicontinuous (see Barbu [1]).

Under the above assertions we can reformulate the problem (P') in terms of variational inequalities as follows: find $u \in W_0^{1,p}(\Omega)$ such that

$$(5) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx + J(w) - J(u) \geq \int_{\Omega} G(u)(w - u) dx$$

for all $w \in W_0^{1,p}(\Omega)$.

LEMMA 2. *Let hypotheses (H_1) and (H_2) be fulfilled. Then $u \in [\underline{u}, \bar{u}]$ is a solution of (5) if and only if u is a solution of the problem (P') .*

Proof. Let $u \in [\underline{u}, \bar{u}]$ satisfy the variational inequality (5). Then

$$J(w) \geq J(u) + \int_{\Omega} G(u) \cdot (w - u) dx - \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx.$$

It follows that

$$(6) \quad \operatorname{div}(a(x, \nabla u)) + G(u) \in \partial J(u) \quad \text{in } W^{-1,p'}(\Omega).$$

It follows by Brezis [2] [Corollaire 1] that any subgradient $v \in \partial J(u)$ of the functional $J : W_0^{1,p}(\Omega) \rightarrow (-\infty, +\infty]$ at $u \in W_0^{1,p}(\Omega)$ belongs to $L^1(\Omega)$ and satisfies

$$(7) \quad v(x) \in \partial j(u(x)) = \beta(u(x)) \quad \text{a.e. in } \Omega.$$

Furthermore

$$h^-(\underline{u}(x)) \leq h^-(u(x)) \leq \beta(u(x)) \leq h^+(\bar{u}(x)) \leq h^+(\bar{u}(x)) \quad \text{a.e. in } \Omega.$$

Thus

$$(8) \quad |v| \leq |H^+(\bar{u})| + |H^-(\underline{u})|.$$

By (H_2) , the right-hand side of (8) belongs to $L^{p'}(\Omega)$. It follows that $v \in L^{p'}(\Omega)$. Thus there exists $v \in L^{p'}(\Omega)$ such that

$$\operatorname{div}(a(x, \nabla u)) + G(u) = v \quad \text{in } W^{-1,p'}(\Omega)$$

or, equivalently,

$$(9) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla w \, dx + \int_{\Omega} v \cdot w \, dx = \int_{\Omega} G(u)w \, dx$$

for all $w \in W_0^{1,p}(\Omega)$.

Relations (7) and (9) imply that $u \in W_0^{1,p}(\Omega)$ is a solution of the problem (P') .

Conversely, let $u \in [\underline{u}, \bar{u}]$ be a solution of the problem (P') . Then there exists $v \in L^p(\Omega)$ such that $v \in \beta(u) = \partial j(u(x))$ and the relation (9) is fulfilled. Since $v(x) \in \partial j(u(x))$ we have

$$(10) \quad j(s) \geq j(u(x)) + v(x)(s - u(x)).$$

Taking $s = 0$ in (10) we obtain, by means of (3) and (4) that $0 \leq j(u(x)) \leq v(x)u(x)$. Thus

$$(11) \quad j(u(\cdot)) \in L^1(\Omega) \quad \text{and} \quad J(u) = \int_{\Omega} j(u(x)) \, dx.$$

Let $w \in W_0^{1,p}(\Omega)$. Taking $s = w(x)$ in (10) we obtain

$$(12) \quad \int_{\Omega} j(w(x)) \, dx - \int_{\Omega} j(u(x)) \, dx \geq \int_{\Omega} v(x)(w(x) - u(x)) \, dx.$$

From (9), substituting w by $w - u \in W_0^{1,p}(\Omega)$ we get, by means of (12)

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) \, dx + J(w) - J(u) \geq \int_{\Omega} G(u) \cdot (w - u) \, dx$$

for all $w \in W_0^{1,p}(\Omega)$.

This means that u is a solution of the variational inequality (5). \square

Remark 1. If u is a solution of (P') then, by (11), $J(u) < +\infty$. The result also holds also if we replace u by a super-solution \bar{u} or by a sub-solution \underline{u} .

Set $v^+ = \max\{v, 0\}$.

LEMMA 3. Let $u, v \in L^p(\Omega)$ such that $J(u)$ and $J(v)$ are finite. Then

$$(13) \quad J(u - (u - v)^+) - J(u) + J(v + (u - v)^+) - J(v) = 0.$$

Proof. Let $\Omega_+ := \{x \in \Omega \mid u > v\}$ and $\Omega_- := \{x \in \Omega \mid u \leq v\}$. Since $(u - v)^+ = 0$ in Ω_- and $(u - v)^+ = u - v$ in Ω_+ we obtain

$$(14) \quad J(u - (u - v)^+) = \int_{\Omega_+} j(v) dx + \int_{\Omega_-} j(u) dx \leq \infty$$

$$(15) \quad J(v + (u - v)^+) = \int_{\Omega_+} j(u) dx + \int_{\Omega_-} j(v) dx \leq \infty$$

By (14) and (15) we obtain (13).

Consider now the following variational inequality: given $z \in L^p(\Omega)$, find $u \in W_0^{1,p}(\Omega)$ such that

$$(16) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) + J(w) - J(u) \geq \int_{\Omega} G(z)(w - u) dx$$

for all $w \in W_0^{1,p}(\Omega)$.

The variational inequality (16) defines a mapping $T : z \rightarrow u$ and each fixed point of T yields a solution of (5) and conversely.

LEMMA 4. *Let hypotheses (H_1) and (H_2) be satisfied. Then for each $z \in [\underline{u}, \bar{u}]$ the variational inequality (16) has a unique solution $u = Tz \in [\underline{u}, \bar{u}]$. Moreover, there is a constant $C > 0$ such that $\|Tz\|_{W_0^{1,p}(\Omega)} \leq C$, for any $z \in [\underline{u}, \bar{u}]$.*

Proof. Existence. Let $z \in [\underline{u}, \bar{u}]$ be arbitrarily given. Then $G(z)$ is measurable and $G(z) \in L^{p'}(\Omega)$, due to the estimate

$$|G(z)| \leq |G(\bar{u})| + |G(\underline{u})|$$

and after observing that the right-hand side of the above inequality is in $L^{p'}(\Omega)$, by (H_2) .

We now apply Theorem II.8.5 in Lions [5]. We first observe that the above assertions show that the mapping $W_0^{1,p}(\Omega) \ni u \rightarrow \int_{\Omega} G(z)u$ is in $W^{-1,p'}(\Omega)$.

Consider the Leray-Lions operator $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ defined by

$$\langle Au, w \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla w dx .$$

We show that A is a pseudo-monotone operator. For this aim it is enough to prove that A is bounded, monotone and hemi-continuous (see Lions [5] [Prop. II.2.5]).

Condition (a_3) yields the boundedness of A . Indeed

$$\|Au\|_{W^{-1,p'}(\Omega)} \leq C(\|k\|_{L^{p'}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}^{p-1}).$$

We also observe that (a_2) implies that A is a monotone operator.

In order to justify the hemi-continuity of A , let us consider a sequence $(\lambda_n)_{n \geq 1}$ converging to λ . Then, for given $u, v, w \in W_0^{1,p}(\Omega)$, we have

$$a(x, \nabla(u + \lambda_n v)) \cdot \nabla w \rightarrow a(x, \nabla(u + \lambda v)) \cdot \nabla w \quad \text{a.e. in } \Omega.$$

From the boundedness of $\{\lambda_n\}$ and condition (a_3) we obtain that the sequence $\{|a(x, \nabla(u + \lambda_n v)) \nabla w|\}$ is bounded by a function which belongs to $L^1(\Omega)$. Using the Lebesgue dominated convergence theorem it follows that

$$\langle A(u + \lambda_n v), w \rangle \rightarrow \langle A(u + \lambda v), w \rangle \quad \text{as } n \rightarrow \infty.$$

Hence the application $\lambda \rightarrow \langle A(u + \lambda v), w \rangle$ is continuous.

It follows that all assumptions of Theorem II.8.5 in [5] are fulfilled, so the problem (16) has at least a solution.

Uniqueness. Let u_1 and u_2 be two solutions of (16). Then taking $w = u_2$ as a test function for the solution u_1 , we obtain

$$\int_{\Omega} a(x, \nabla u_1) \cdot \nabla(u_2 - u_1) dx + J(u_2) - J(u_1) \geq \int_{\Omega} G(z)(u_2 - u_1) dx.$$

Similarly we find

$$\int_{\Omega} a(x, \nabla u_2) \cdot \nabla(u_1 - u_2) dx + J(u_1) - J(u_2) \geq \int_{\Omega} G(z)(u_1 - u_2) dx.$$

Therefore

$$\int_{\Omega} (a(x, \nabla u_1) - a(x, \nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) dx \leq 0.$$

So, by (a_2) , it follows that $\nabla u_1 = \nabla u_2$, so $u_1 = u_2 + C$ in Ω . Since $u_1 = u_2 = 0$ on $\partial\Omega$, it follows that $u_1 = u_2$ in Ω .

From (3) and (4) we deduce that $J(0) = 0$ and $J(u) \geq 0$. Moreover, the variational inequality (16) implies

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(-u) dx + J(0) - J(u) \geq - \int_{\Omega} G(z)u dx.$$

Thus

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla u \, dx \leq \int_{\Omega} G(z)u \, dx .$$

This last inequality, assumption (a_1) and Hölder's inequality yield

$$\begin{aligned} \lambda \cdot \|u\|_{W_0^{1,p}(\Omega)}^p &\leq \int_{\Omega} G(z)u \, dx \leq \|G(z)\|_{L^{p'}(\Omega)} \cdot \|u\|_{L^p} \\ &\leq C_1 (\|G(\bar{u})\|_{L^{p'}(\Omega)} + \|G(\underline{u})\|_{L^{p'}(\Omega)}) \|u\|_{W_0^{1,p}(\Omega)} . \end{aligned}$$

Thus $u = Tz$ verifies

$$\|u\|_{W_0^{1,p}(\Omega)}^{p-1} \leq C_1 (\|G(\bar{u})\|_{L^{p'}(\Omega)} + \|G(\underline{u})\|_{L^{p'}(\Omega)}) = C_2 .$$

This implies that there exists a universal constant $C > 0$ such that

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C .$$

So, in order to conclude our proof, it is enough to show that $u \in [\underline{u}, \bar{u}]$. But, by the definition of an upper solution, there exists $\bar{v} \in L^{p'}(\Omega)$ such that $\bar{v} \in \beta(\bar{u}(x))$ and

$$(17) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla w \, dx + \int_{\Omega} \bar{v} \cdot w \, dx \geq \int_{\Omega} G(\bar{u})w \, dx ,$$

for all $w \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega)$.

The solution $u = Tz$ of the variational inequality (16) satisfies

$$(18) \quad \int_{\Omega} a(x, \nabla u) \cdot \nabla (w - u) \, dx + J(w) - J(u) \geq \int_{\Omega} G(z)(w - u) \, dx$$

for all $w \in W_0^{1,p}(\Omega)$.

Setting $\bar{v} \in \beta(\bar{u}) = \partial j(\bar{u})$, we have

$$(19) \quad j(s) \geq j(\bar{u}(x)) + \bar{v}(x)(s - \bar{u}(x)) \quad \text{for all } s \in \mathbf{R} .$$

Taking $s := \bar{u}(x) + (u(x) - \bar{u}(x))^+$ in (19) we find by integration

$$(20) \quad J(\bar{u} + (u - \bar{u})^+) \geq J(\bar{u}) + \int_{\Omega} \bar{v}(u - \bar{u})^+ \, dx .$$

Choosing now $w = (u - \bar{u})^+$ in (17) we obtain

$$(21) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+ dx + \int_{\Omega} \bar{v} \cdot (u - \bar{u})^+ dx \geq \int_{\Omega} G(\bar{u}) \cdot (u - \bar{u})^+ dx.$$

Relations (20) and (21) yield

$$(22) \quad \int_{\Omega} a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+ dx + J(\bar{u} + (u - \bar{u})^+) - J(\bar{u}) \geq \int_{\Omega} G(\bar{u}) \cdot (u - \bar{u})^+ dx.$$

Taking $w = u - (u - \bar{u})^+$ in (18), we obtain

$$\int_{\Omega} a(x, \nabla u) \cdot (-\nabla (u - \bar{u})^+) dx + J(u - (u - \bar{u})^+) - J(u) \geq - \int_{\Omega} G(z)(u - \bar{u})^+ dx.$$

Since $z \in [\underline{u}, \bar{u}]$ and $G : L^p(\Omega) \rightarrow L^p(\Omega)$ is nondecreasing, it follows that

$$(23) \quad \begin{aligned} & \int_{\Omega} a(x, \nabla u) \cdot \nabla (u - \bar{u})^+ dx + J(u - (u - \bar{u})^+) - J(u) \\ & \geq - \int_{\Omega} G(\bar{u})(u - \bar{u})^+ dx. \end{aligned}$$

From (22), (23) and Lemma 3 we have

$$(24) \quad \int_{\Omega} (a(x, \nabla u) - a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+) dx \leq 0.$$

Let $\Omega_+ = \{x \in \Omega \mid u \leq \bar{u}\}$ and $\Omega_- = \{x \in \Omega \mid u > \bar{u}\}$. Since $(u - \bar{u})^+ = 0$ in Ω_+ and $(u - \bar{u})^+ = u - \bar{u}$ in Ω_- , it follows by (24) that

$$\int_{\Omega_-} (a(x, \nabla u) - a(x, \nabla \bar{u}) \cdot \nabla (u - \bar{u})^+) dx \leq 0.$$

So, by (a_2) and the definition of Ω_- , we obtain $\text{meas}(\Omega_-) = 0$, hence $u \leq \bar{u}$ a.e. in Ω . Proceeding in the same way we prove that $\underline{u} \leq u$. \square

LEMMA 5. *The operator T defines a monotone nondecreasing mapping from $[\underline{u}, \bar{u}]$ to $[\underline{u}, \bar{u}]$.*

Proof. Let $z_1, z_2 \in [\underline{u}, \bar{u}]$ be such that $z_1 \leq z_2$. By Lemma 4, we obtain that $Tz_1, Tz_2 \in [\underline{u}, \bar{u}]$ and

$$(25) \quad \int_{\Omega} a(x, \nabla Tz_1) \cdot \nabla(w - Tz_1) dx + J(w) - J(Tz_1) \geq \int_{\Omega} G(z_1)(w - Tz_1) dx$$

$$(26) \quad \int_{\Omega} a(x, \nabla Tz_2) \cdot \nabla(w - Tz_2) dx + J(w) - J(Tz_2) \geq \int_{\Omega} G(z_2)(w - Tz_2) dx.$$

Taking $w = Tz_1 - (Tz_1 - Tz_2)^+$ in (25) and $w = Tz_2 + (Tz_1 - Tz_2)^+$ in (26), we get

$$- \int_{\Omega} a(x, \nabla Tz_1) \cdot \nabla(Tz_1 - Tz_2)^+ dx + J(Tz_1 - (Tz_1 - Tz_2)^+) - J(Tz_1) \geq \int_{\Omega} G(z_1)(-(Tz_1 - Tz_2)^+) dx$$

$$\int_{\Omega} a(x, \nabla Tz_2) \cdot \nabla(Tz_1 - Tz_2)^+ dx + J(Tz_2 + (Tz_1 - Tz_2)^+) - J(Tz_2) \geq \int_{\Omega} G(z_2)(Tz_1 - Tz_2)^+ dx.$$

Summing up these inequalities we get, by means of (13),

$$\int_{\Omega} (a(x, \nabla Tz_1) - a(x, \nabla Tz_2)) \cdot \nabla(Tz_1 - Tz_2)^+ dx \leq \int_{\Omega} (G(z_1) - G(z_2))(Tz_1 - Tz_2)^+ dx.$$

But $G(z_1) \leq G(z_2)$, since G is a nondecreasing operator. Therefore, by the

above inequality we obtain

$$\int_{\Omega} a(x, (\nabla T z_1) - a(x, \nabla T z_2)) \cdot \nabla (T z_1 - T z_2)^+ dx \leq 0.$$

With the same argument as for proving (24) we obtain $T z_1 \leq T z_2$. \square

Proof of Theorem 1 completed. Assume that g is right continuous. Define

$$(27) \quad u^{n+1} = T u^n,$$

where $u^0 = \bar{u}$. Then, by Lemma 4, $\{u^n\}$ is nondecreasing, $u^n \in [\underline{u}, \bar{u}]$, and there is a constant C such that

$$(28) \quad \|u^n\|_{W_0^{1,p}(\Omega)} \leq C.$$

The compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and (28) ensure that there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} u^n &\rightarrow u \quad \text{strongly in } L^p(\Omega) \\ u^n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega) \\ u_n &\rightarrow u \quad \text{a.e. in } \Omega. \end{aligned}$$

By Lemma 4, there exists $u' \in W_0^{1,p}(\Omega)$, $u' \in [\underline{u}, \bar{u}]$ such that $u' = T u$. We prove in what follows that u is a fixed point of T i.e. $u' = u$.

From (27) and by the definition of T we obtain

$$(29) \quad \int_{\Omega} a(x, \nabla u^{n+1}) \nabla (w - u^{n+1}) dx + J(w) - J(u^{n+1}) \geq \int_{\Omega} G(u^n) (w - u^{n+1}) dx$$

for all $w \in W_0^{1,p}(\Omega)$.

Also, from $T u = u'$, we have

$$(30) \quad \int_{\Omega} a(x, \nabla u') \nabla (w - u') dx + J(w) - J(u') \geq \int_{\Omega} G(u) \cdot (w - u') dx$$

for all $w \in W_0^{1,p}(\Omega)$.

Taking $w = u'$ in (29) and $w = u^{n+1}$ in (30), we get

$$\int_{\Omega} a(x, \nabla u^{n+1}) \nabla (u' - u^{n+1}) dx + J(u') - J(u^{n+1}) \geq \int_{\Omega} G(u^n) \cdot (u' - u^{n+1}) dx$$

$$\int_{\Omega} a(x, \nabla u') \nabla(u^{n+1} - u') dx + J(u^{n+1}) - J(u') \geq \int_{\Omega} G(u) \cdot (u^{n+1} - u') dx.$$

So, by (29) and (30), $J(u') < \infty$ and $J(u^{n+1}) < \infty$. Summing up the last two inequalities we obtain

$$(31) \quad \begin{aligned} & \int_{\Omega} (a(x, \nabla u') - a(x, \nabla u^{n+1})) \cdot \nabla(u' - u^{n+1}) dx \\ & \leq \int_{\Omega} (G(u) - G(u^{n+1})) (u' - u^{n+1}) dx. \end{aligned}$$

Since G is right continuous we have $G(u^n) \rightarrow G(u)$ in Ω . We also have

$$|G(u) - G(u^n)| (u - u^{n+1}) \leq 2 (|G(\underline{u})| + |G(\bar{u})|) (|\underline{u}| + |\bar{u}|) \in L^1(\Omega).$$

By (a_2) and the Lebesgue dominated convergence theorem, we deduce from (31) that

$$(32) \quad \int_{\Omega} (a(x, \nabla u') - a(x, \nabla u^n)) \cdot \nabla(u' - u^n) dx \rightarrow 0.$$

This implies that $\nabla u^n \rightarrow \nabla u'$ a.e. in Ω .

Relation (32) implies that (up to a subsequence)

$$(33) \quad (a(x, \nabla u') - a(x, \nabla u^n)) \cdot \nabla(u' - u^n) \rightarrow 0 \quad \text{a.e. } x \in \Omega.$$

This leads to $\nabla u^n \rightarrow \nabla u'$ a.e. in Ω . Indeed, if not, there exists $x \in \Omega$ such that (up to a subsequence), $\nabla u^n(x) \rightarrow \xi \in \overline{\mathbf{R}^N}$ for $\xi \neq \nabla u'$. Passing to the limit in (33) we obtain

$$(a(x, \nabla u') - a(x, \xi)) \cdot (\nabla u' - \xi) = 0,$$

which contradicts (a_2) . So, we have proved that $\nabla u^n \rightarrow \nabla u$. Using the fact that $u^n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we conclude that $\nabla u' = \nabla u$, thus $u' = u$. Replacing u' by u in (30) we get

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla(w - u) dx + J(w) - J(u) \geq \int_{\Omega} G(u)(w - u) dx$$

for all $w \in W_0^{1,p}(\Omega)$.

Hence u is a fixed point of T and a solution for the problem (P') .

In order to prove that u is a maximal solution of (3) with respect to the order interval $[\underline{u}, \bar{u}]$, take any other solution $\hat{u} \in [\underline{u}, \bar{u}]$ of the problem (P') .

Then \hat{u} is in particular a sub-solution satisfying $\hat{u} \leq \bar{u}$. Starting again the iteration (27) with $u^0 = \bar{u}$ we obtain

$$\hat{u} \leq \dots \leq u^{n+1} \leq u^n \leq \dots \leq u^0 = \bar{u}.$$

It follows that $\hat{u} \leq u$, which concludes our proof. \square

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