Problem 11073, American Mathematical Monthly, 3/111 (2004)
Proposed by Vicenţiu Rădulescu, University of Craiova, Romania

Let a and b be positive real numbers. Let f and g be functions from $\mathbb{R}$ into $\mathbb{R}$, twice differentiable, with initial conditions $f(0) = a$, $f'(0) = 0$, $g(0) = 0$, $g'(0) = b$, and satisfying the differential equations

$$f'' = -f(1 - f^2 - g^2), \quad g'' = -g(1 - f^2 - g^2).$$

(a) *Show that there is a nontrivial polynomial function $E(X, Y)$ such that for all $a, b > 0$, $E(f^2(t) + g^2(t), f'^2(t) + g'^2(t))$ is independent of t.*

(b) *Show that if f and g are both periodic in t, with period T, and if at $t = 0$, $f^2(t)+g^2(t)$ is not at a local minimum, then $a \le 1$, $b^2 \le a^2(1 - a^2)$, and $T > 2\pi$.*

(c) *Give an example of f and g satisfying the premises of part (b).*

(d) *Prove that there exist choices of a and b such that the resulting (f, g) is periodic, and $\min(f^2 + g^2) < (1/2)\max(f^2 + g^2)$.*

Solution. **(a)** This statement is a kind of energy conservation law.
Let $r = \sqrt{f^2 + g^2}$ and $s = \sqrt{f'^2 + g'^2}$. Then

$$(f^2 + g^2)(1 - (1/2)(f^2 + g^2)) + (f'^2 + g'^2) = r^2(1 - r^2/2) + s^2.$$

Define $E(X, Y) = X - X^2/2 + Y$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(f^2(t) + g^2(t), f'^2(t) + g'^2(t)) = 2\left[(ff' + gg')(1 - f^2 - g^2) + (f'f'' + g'g'')\right](t).$$

Using now the differential equation fulfilled by f and g we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} E(f^2(t) + g^2(t), f'^2(t) + g'^2(t)) = \;& 2(ff' + gg')(1 - f^2 - g^2)(t) - \\ & 2\left[f'f(1 - f^2 - g^2) + g'g(1 - f^2 - g^2)\right](t) = 0,\end{aligned}$$

for any $t \in \mathbb{R}$. Hence

$$E(f^2(t) + g^2(t), f'^2(t) + g'^2(t)) \equiv E(f^2(0) + g^2(0), f'^2(0) + g'^2(0)) = a^2 - a^4/2 + b^2.$$

Alternative proof of (a). We multiply by f' the differential equation $f'' = -f(1 - f^2 - g^2)$ and then we integrate on $[0, t]$. Using the assumptions $f(0) = a$ and $f'(0) = 0$ we obtain

$$f^2(t) - \frac{f^4(t)}{2} + f'^2(t) - \int_0^t f(s)f'(s)g^2(s)ds = a^2 - \frac{a^4}{2}.$$

Similarly, using the differential equation satisfied by g we find

$$g^2(t) - \frac{g^4(t)}{2} + g'^2(t) - \int_0^t f^2(s)g(s)g'(s)ds = b^2.$$

By addition we obtain, for any $t \in \mathbb{R}$,

$$f^2(t) + g^2(t) - \frac{\left(f^2(t) + g^2(t)\right)^2}{2} + f'^2(t) + g'^2(t) = a^2 - \frac{a^4}{2} + b^2.$$

(b) We first prove that $a \leq 1$. Indeed, arguing by contradiction, let us assume the contrary. Set $u := f^2 + g^2$. The assumption $a > 1$ enables us to choose $M > 1$ such that

$$\min\{u(x);\ x \in \mathbb{R}\} < M^2 < a^2.$$

Let $I \subset \mathbb{R}$ be a bounded interval such that $u > M^2$ in I, and $u = M^2$ on ∂I. But

$$u'' = 2u(u-1) + 2(f'^2 + g'^2) \geq 2u(u-1) > 0 \qquad \text{in } I.$$

So, u is convex in I and $u = M^2$ on ∂I. Hence $u \leq M^2$, which contradicts the choice of I.

Applying Taylor's formula we have

$$f(x) = a - \frac{a(1-a^2)}{2}\, x^2 + O(x^3), \qquad \text{as } x \to 0$$

and

$$g(x) = bx + O(x^3), \qquad \text{as } x \to 0.$$

So

$$u(x) = a^2 + \left[b^2 - a^2(1-a^2)\right]\, x^2 + O(x^3), \qquad \text{as } x \to 0.$$

Since $x = 0$ is a local maximum point of u, it follows that $b^2 \leq a^2(1-a^2)$.

An alternative proof of this statement is based on the fact that $u''(0) \leq 0$ (since $x = 0$ is a local maximum point of u) combined with $u''(0) = 2\left[b^2 + a^2(a^2-1)\right]$.

The above arguments also show that $a < 1$. Indeed, if $a = 1$, then $u''(0) = 2b^2 > 0$, a contradiction with the fact that the origin is a local maximum point of u.

Let us now prove that $T > 2\pi$. We first notice that f (or g) cannot have the same sign on an unbounded interval. Indeed, in this case, f'' (or g'') would have the same sign. But, due to the periodicity, this is possible only for constant functions, which is impossible in our case.

Let x_1, x_2 be two consecutive zeros of f. We can assume that $f > 0$ in (x_1, x_2), so that $f'(x_1) > 0$ and $f'(x_2) < 0$. Denote by x_3 the smallest real number greater than x_2 such that $f(x_3) = 0$. Hence $f < 0$ in (x_2, x_3). If we prove that $x_2 - x_1 > \pi$, it will also follow that $x_3 - x_2 > 2\pi$ and there does not exist $x \in (x_1, x_3)$ such that $f(x) = 0$ and $f'(x) > 0$. This implies that the principal period of f must be greater than 2π. For our purpose, we multiply by $\varphi(x) := \sin \frac{\pi(x-x_1)}{x_2-x_1}$ in $f'' + f(1 - f^2 - g^2) = 0$ and then we integrate on $[x_1, x_2]$. Hence

$$\left(\frac{\pi}{x_2 - x_1}\right)^2 \int_{x_1}^{x_2} f(x)\varphi(x)dx = \int_{x_1}^{x_2} f(x)\left(1 - f^2(x) - g^2(x)\right)\varphi(x)dx < \int_{x_1}^{x_2} f(x)\varphi(x)dx.$$

It follows that $x_2 - x_1 > \pi$.

Alternative proof of (b). Define $u : \mathbb{R} \to [0, \infty)$ by $u(x) = f^2(x) + g^2(x)$, $x \in \mathbb{R}$. Clearly, u is a T-periodic function of class $C^2(\mathbb{R})$, and

$$u''(x) = 2(u(x)(u(x)-1) + (f'(x))^2 + (g'(x))^2 \tag{1}$$

for all real x. In particular, $u''(0) = 2\left[a^2(a^2-1)+b^2\right]$. Since u has a local maximum at the origin, it follows that $u''(0) \le 0$, which yields immediately $b^2 \le a^2(1-a^2)$ by the preceding. This establishes the first inequality in (ii).

The proof of the second is a bit more tricky, so let us first outline the strategy. The main idea consists in showing that the distance between two 'consecutive' zeros of f must exceed π. To be more precise, we shall prove:

Claim 1: *If $x_1 < x_2$ are real numbers such that $f(x_1) = f(x_2) = 0$, but $f(x) \ne 0$ for $x_1 < x < x_2$, then $x_2 - x_1 > \pi$.*

Therefore, if we are able to produce two disjoint open subintervals, say I and J, of an interval of length T, with the property that f does not vanish on $I \cup J$, but $f = 0$ on $\partial I \cup \partial J$, then the length of each of these subintervals exceeds π, by Claim 1, and we are done: $T \ge \operatorname{length} I + \operatorname{length} J > \pi + \pi = 2\pi$.

The easiest way to produce such intervals consists in showing that f must take on values of either sign on each interval of length T. We shall prove:

Claim 2: *The function f takes on values of either sign on each interval of length T.*

Assuming this, let us see how it applies to produce the desired subintervals. First, let $I = (\alpha, \beta)$ be the connected component of the open set $\{x : f(x) > 0\}$ which contains the origin (recall that $f(0) = a > 0$). Then $f(\alpha) = f(\beta) = 0$ by continuity of f and maximality of I. Next, by Claim 2, $f(x_0) < 0$ for some $x_0 \in [0, T]$. But since $f(T) = f(0) = a > 0$, x_0 must be an interior point of $[0, T]$. Now let $J = (\gamma, \delta)$ be the connected component of the open set $\{x : 0 < x < T \text{ and } f(x) < 0\}$ which contains x_0. Again, $f(\gamma) = f(\delta) = 0$ by continuity of f and maximality of J. Finally, observe that $\beta \le \gamma$ and $\delta \le \alpha + T$ (here we use the T-periodicity of f), to conclude that I and J are indeed disjoint open subintervals of $[\alpha, \alpha + T]$, satisfying the required conditions.

So all it remains to prove are Claims 1 and 2 above. Both proofs rely upon

Claim 3: *$u(x) \le 1$ for all $x \in \mathbb{R}$.*

Proof of Claim 3. Recall the inequality $b^2 \le a^2(1-a^2)$, proved in the first part. Since $b > 0$, we deduce that $a^2 < 1$, so $u(0) = a^2 < 1$. Now we argue by *reductio ad absurdum*. Suppose, by contradiction, that $u(x_0) > 1$ for some real x_0. Then $U = \{x : u(x) > 1\}$ is an open non-empty set. Let K denote the connected component of U which contains x_0. Since u is periodic and $u(0) = a^2 < 1$, it follows that $\alpha = \inf K > -\infty$ and $\beta = \sup K < \infty$. Observe now that $u(\alpha) = u(\beta) = 1$, by continuity of u and maximality of K, while $u(x) > 1$ for $\alpha < x < \beta$. This leads to the following two contradictory facts: on the one hand, by virtue of (1),

$$u''(x) \ge 2u(x)(u(x)-1) > 0 \tag{2}$$

for $\alpha < x < \beta$; on the other hand, since u is continuous, it must attain a *maximum* value on the compact set $[\alpha, \beta]$ at some *interior* point x_1, so $u''(x_1) \le 0$ thus contradicting (2) at x_1. Consequently, $u(x) \le 1$ for any $x \in \mathbb{R}$.

Proof of Claim 1. To make a choice, let $f(x) > 0$ for $x_1 < x < x_2$; in case when $f(x) < 0$ for $x_1 < x < x_2$, we merely replace f by $-f$ everywhere. Now set $\varphi(x) = \sin(\pi(x-x_1)/(x_2-x_1))$ for $x_1 \le x \le x_2$, and note that $\varphi(x_1) = \varphi(x_2) = 0$ and $\varphi(x) > 0$

for $x_1 < x < x_2$. Then

$$\begin{aligned}
\int_{x_1}^{x_2} f(x)\varphi(x)dx &> \int_{x_1}^{x_2} f(x)(1-u(x))\varphi(x)dx && \text{(by Claim 3)} \\
&= -\int_{x_1}^{x_2} f''(x)\varphi(x)dx && \text{for } f''+f(1-u)=0 \\
&= \tfrac{\pi}{x_2-x_1}\int_{x_1}^{x_2} f'(x)\cos\tfrac{\pi(x-x_1)}{x_2-x_1}dx && \text{for } \varphi(x_1)=\varphi(x_2)=0 \\
&= \left(\tfrac{\pi}{x_2-x_1}\right)^2\int_{x_1}^{x_2} f(x)\varphi(x)dx && \text{for } f(x_1)=f(x_2)=0,
\end{aligned}$$

which shows that $x_2 - x_1$ is indeed strictly greater than π.

Proof of Claim 2. Since f is T-periodic, there is no loss in considering the interval $[0,T]$. Clearly, f takes on positive values around the origin, for $f(0) = a > 0$ by hypothesis. To prove that it also takes on negative values, we argue by *reductio ad absurdum.* Suppose, if possible, that $f(x) \geq 0$ for all $x \in [0,T]$. By Claim 3, it then follows that $f''(x) = -f(x)(1-u(x)) \leq 0$ for all $x \in [0,T]$, that is, f' is decreasing on $[0,T]$. But f' is itself T-periodic, so it must be constant on $[0,T]$, which implies in turn that f must be constant on $[0,T]$, by T-periodicity. Thus, f'' vanishes on $[0,T]$, and since $f''+f(1-f^2-g^2)=0$, we deduce that g is itself constant $[0,T]$; that is, g' vanishes on $[0,T]$ thus contradicting the hypothesis $g'(0) = b > 0$. We conclude that f must take on values of either sign on $[0,T]$.

(c) Choose $a, b > 0$ such that $b^2 = a^2(1-a^2)$. Define $f(x) = a\cos\frac{bx}{a}$ and $g(x) = a\sin\frac{bx}{a}$. It follows that $T = 2\pi a/b$. In particular, we observe that $T > 2\pi$.

(d) We shall prove the following more general result.

Fix arbitrarily $\delta > 0$. Then there exist choices of a and b such that the resulting (f,g) is periodic, and $\min(f^2+g^2) < \delta\ \max(f^2+g^2)$.

Denote

$$\Omega := \{(a,b) \in (0,1] \times [0,1];\ \ b^2 \leq a^2(1-a^2)\}.$$

Let $(a,b) \in \operatorname{Int}\Omega$ and set $v(x) := f(x) + ig(x)$. Since $v(0) = a \neq 0$, it follows that, for small x,

$$v(x) = e^{i\varphi(x)}r(x), \qquad r(x) = \sqrt{f^2(x)+g^2(x)},$$

where $\varphi(0) = 0$ and $r > 0$. Then r satisfies

$$\begin{cases} -r'' &= r(1-r^2) - \dfrac{a^2b^2}{r^3} \\ r(0) &= a,\ r'(0) = 0, \end{cases} \tag{3}$$

while φ is given by

$$\varphi' = \frac{ab}{r^2}, \qquad \varphi(0) = 0.$$

Hence, if the problem (3) has a global positive solution, it follows that v is global. Moreover, if r is periodic of period T_0, then

$$v(nT_0 + x) = e^{in\varphi(T_0)}e^{i\varphi(x)}r(x), \qquad \forall\, 0 \leq x < T_0,\ \forall\, n \in \mathbb{N},$$

so that (3) gives a periodic solution if and only if $\varphi(T_0) \in \pi\mathbb{Q}$.

We prove in what follows the global existence. More precisely, if $(a,b)\in \operatorname{Int}\Omega$, then (3) has a global positive periodic solution. We first observe that the assumption made on (a,b) implies $r''(0)<0$. So, multiplying in (3) by r', we obtain, for small $x>0$,

$$r'^2 = -r^2 + \frac{r^4}{2} - \frac{a^2b^2}{r^2} + a^2 - \frac{a^4}{2} + b^2 \tag{4}$$

and

$$r' = -\sqrt{-r^2 + \frac{r^4}{2} - \frac{a^2b^2}{r^2} + a^2 - \frac{a^4}{2} + b^2}. \tag{5}$$

Now, relation (4) implies that r and r' are bounded as far as the solution exists and, moreover, that

$$\inf\{r(x);\quad r \text{ exists}\} > 0.$$

It follows that r is a global solution.

Set

$$t_0 := \sup\{x>0;\quad r'(y)<0 \quad \text{for all}\;\; 0<y<x\}.$$

Note that (5) is valid if $0<x<t_0$.

Let $0<c<a$ be the **unique** root of

$$\psi(x) := \frac{x^4}{2} - x^2 - \frac{a^2b^2}{x^2} + a^2 - \frac{a^4}{2} + b^2 = 0.$$

Since $\psi(x)<0$ if $x\in(0,c)$ or if $x>a$, x close to a, it follows from relation (4) that

$$c \le r(x) \le a \qquad \text{for all}\;\; x\in\mathbb{R}. \tag{6}$$

Claim 4. *We have* $\lim_{x\nearrow t_0} r(x) = c$.

Proof of Claim 4. If $t_0<\infty$, it follows that $r'(t_0)=0$. Now, relation (4) in conjunction with the definitions of t_0 and c shows that $r(t_0)=c$. If $t_0=+\infty$, then we have $\lim_{x\to\infty} r(x)\ge c$. If $\lim_{x\to\infty} r(x) > c$, then there exists a constant $M>0$ such that $r'(x)\le -M$ for each $x>0$. The latest inequality contradicts (6) for large x.

For any $0<x<t_0$, relation (5) yields

$$x = \int_{r(x)}^{a} \frac{dt}{\sqrt{-t^2 + \frac{t^4}{2} - \frac{a^2b^2}{t^2} + a^2 - \frac{a^4}{2} + b^2}}.$$

Therefore

$$t_0 = \int_{c}^{a} \frac{dt}{\sqrt{-t^2 + \frac{t^4}{2} - \frac{a^2b^2}{t^2} + a^2 - \frac{a^4}{2} + b^2}} < \infty.$$

It follows by a reflection argument that $r(2t_0)=r(0)=a$, $r'(2t_0)=r'(0)=0$, so that r is $(2t_0)$–periodic. This concludes the proof of Claim 4.

Denote $\Psi(x) := 2x^2\psi(x) = x^6 - 2x^4 - 2a^2b^2 + (2a^2 - a^4 + 2b^2)x^2$. Taking into account Claim 4 and observing that $\Psi(0)<0$, it follows that it is enough to show that $\Psi(\varepsilon)>0$, for some $\varepsilon>0$ and for a convenient choice of $(a,b)\in\operatorname{Int}\Omega$. But

$$\Psi(\varepsilon) > \varepsilon^2[\varepsilon^4 - 2\varepsilon^2 + a^2(2-a^2)] - 2a^2b^2.$$

For $a = 2^{-1/2}$ we obtain

$$\Psi(\varepsilon) > \varepsilon^2[\varepsilon^4 - 2\varepsilon^2 + 3 \cdot 4^{-1}] - b^2 > \frac{\varepsilon^2}{2} - b^2 > 0,$$

provided that $\varepsilon > 0$ is sufficiently small and $b = \varepsilon/2$. It is obvious that this choice of a and b guarantees $(a, b) \in \operatorname{Int} \Omega$.