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ORIGINAL PAPER

Strongly singular nonhomogeneous eigenvalue problems

Nikolaos S. Papageorgiou[1] · **Vicenţiu D. Rădulescu**[2,3,4] · **Lixi Wen**[3,4,5]

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Abstract

We consider a nonlinear Dirichlet problem driven by the sum of a p-Laplacian and of a q-Laplacian, $1 < p < q$, (a (p, q)-equation). The reaction is parametric (eigenvalue problem) and exhibits the competing effects of a strongly singular term and of $(p - 1)$-superlinear Carathéodory perturbation. We show that when the parameter (eigenvalue) is small, then the problem has at least two positive bounded solutions which are bounded away from zero on compact sets.

Keywords Purely singular problem · Regularization · Nonlinear maximum principle · Multiple positive solutions · Superlinear perturbation

Mathematics Subject Classification 35J20 · 35J75

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following singular eigenvalue problem

✉ Lixi Wen
wlx942246762@163.com

Nikolaos S. Papageorgiou
npapg@math.ntua.gr

Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

1 Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece

2 Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

3 Department of Mathematics, University of Craiova, 200585 Craiova, Romania

4 China-Romania Research Center in Applied Mathematics, Craiova, Romania

5 School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, People's Republic of China

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) = \lambda \left(u(z)^{-\eta} + f(z, u(z)) \right) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N, \ 2 \le p, \ \lambda > 0, \ u > 0. \end{array} \right\} \tag{P_λ}$$

If $r \in (1, \infty)$, by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div} \left(|Du|^{r-2} Du \right) \text{ for all } u \in W_0^{1,r}(\Omega).$$

The equation in (P_λ) is driven by the sum of two such operators and so the differential operator in (P_λ) (left-hand side) is not homogeneous. In the parametric reaction (right-hand side with $\lambda > 0$ being the parameter (eigenvalue)), we have the competing effects of a singular term $u \to u^{-\eta}$ with $\eta > 1$ and of a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, the mapping $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, the function $x \to f(z, x)$ is continuous), which is $(p-1)$-superlinear as $x \to +\infty$, but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short).

Since the exponent of the singular term is $\eta > 1$, we have what is called in the literature "a strong singularity" and so the problem is more difficult. When the singularity is "weak" (that is, $0 < \eta < 1$), then we can have global existence and multiplicity results. We refer to the recent works of Bai, Papageorgiou & Zeng [1], Papageorgiou, Rădulescu & Repovš [14] (isotropic problems), Liu-Motreanu-Zeng [11] Papageorgiou, Rădulescu & Zhang [16] (anisotropic problems) and the references therein. Finally, we mention the recent works on double phase obstacle problems by Zeng-Bai-Gasinski-Winkert [22], Zeng-Rădulescu-Winkert [23].

Strongly singular equations are more complicated and of course have not been examined so systematically. Their study was initiated with the seminal paper of Lazer & McKenna [9], who considered semilinear equations driven by the Dirichlet Laplacian and proved that the solution is not $C^1(\bar{\Omega})$ if $\eta > 1$ and it belongs to the Sobolev apace $H_0^1(\Omega)$ if and only if $\eta < 3$. So, when dealing with strongly singular problems, we can not expect good regularity properties for the solutions and this then eliminates from consideration important analytical tools which are available for weakly singular equations (see [14] and [16]). After the work of Lazer and McKenna, further contributions on strongly singular equations were made by Boccardo & Orsina [2], Diaz, Hernandez & Rakotoson [5], Sun [21] (semilinear equations), Chu & Gao [3], Cong & Han [4] (equations driven by the p-Laplacian) and Papageorgiou, Rădulescu & Zhang [15] (double phase equations). These works prove existence but not multiplicity theorems. Here under a compatibility condition relating the exponents η and p, we prove the existence of at least two bounded weak solutions when $\lambda > 0$ is small (continuous spectrum).

2 Mathematical background and hypotheses

The main space in the study of problem (P_λ) is the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, the norm $\|\cdot\|$ of $W_0^{1,p}(\Omega)$ is given by

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

At some point we will also use the space $C_0^1(\bar{\Omega}) = \left\{ u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0 \right\}$. This is an ordered Banach space with positive (order) cone

$$C_+ = \left\{ u \in C_0^1(\bar{\Omega}) : u(z) \ge 0 \text{ for all } z \in \bar{\Omega} \right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0 \right\}$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ and $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

Define the nonlinear operator

$$V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1 \right)$$

by

$$\langle V(u), h \rangle = \int_\Omega \left[|Du|^{p-2} + |Du|^{q-2} \right] (Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

This operator has the following properties (see Gasinski & Papageorgiou [6], Problem 2.192).

Proposition 1 *The operator $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and has the $(S)_+$-property, that is*

"$u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle V(u_n), u_n - u \rangle \le 0$ imply that $u_n \to u$ in $W_0^{1,p}(\Omega)$."

If $u : \Omega \to \mathbb{R}$ is a measurable function, then for every $z \in \Omega$ we define $u^+(z) = \max\{u(z), 0\}$ and $u^-(z) = \max\{-u(z), 0\}$. We know that $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$.

If $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then $g(\cdot, \cdot)$ is jointly measurable (see Papageorgiou & Winkert [17, p.106]). In particular then $g(\cdot, \cdot)$ is superpositionally measurable, that is, if $u : \Omega \to \mathbb{R}$ is measurable, then so is $z \to g(z, u(z))$. By $N_g(\cdot)$ we denote the corresponding Nemytski (superposition) map defined by $N_g(u)(\cdot) = g(\cdot, u(\cdot))$ which maps measurable functions to measurable ones.

Finally, by p^* we denote the critical Sobolev exponent corresponding to p. Since $p < N$, we have $p^* = \frac{Np}{N-p}$.

Now we introduce the hypotheses on the data of problem (P_λ).

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \le f(z, x) \le a(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \ge 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exists $\mu \in \left((r - p)\frac{N}{p}, p^* \right)$ such that

$$0 < \beta_0 \le \liminf_{x\to+\infty} \frac{f(z, x)x - pF(z, x)}{x^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 1 Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, $x \le 0$. Hypothesis H(ii) implies that

$$\lim_{x\to+\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, the perturbation of the singular term, is $(p-1)$-superlinear but need not satisfy the AR-condition (see Rădulescu [19, p. 80]), which is common in the literature when dealing with superlinear problems. The following function $f(x)$ satisfies hypotheses H but fails to satisfy the AR-condition. For the sake of simplicity we drop the z-dependence

$$f(x) = \begin{cases} (x^+)^{\tau-1} & \text{if } x \le 1 \\ x^{p-1}\ln x + x^{\theta-1} & \text{if } 1 < x \end{cases}, \; q < \tau, \; \theta \le p.$$

As we already mentioned in the Introduction, to deal with the strongly singularity, we will need a compatibility condition between the exponents η and p.

$\widehat{H}$: $\eta < \frac{3p^*+1}{2p^*+1} = \frac{3Np+N-p}{2Np+N-p}$.

Remark 2 Note that $\frac{3N+N-p}{2N+N-p} < \frac{3}{2} \le 2 - \frac{1}{p}$ (recall that $2 \le p$).

We mention that, as usual, by a "(weak) solution" of (P_λ), we mean a function $u \in W_0^{1,p}(\Omega)$ such that

$$u^{-1}h \in L^1(\Omega) \text{ for all } h \in W_0^{1,p}(\Omega),$$
$$\langle V(u), h\rangle = \int_\Omega [\lambda u^{-\eta} + f(z,u)]h\,\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega).$$

3 A purely singular problem

In this section, we deal with the following purely singular problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) = \lambda u(z)^{-\eta} \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \lambda > 0, u > 0. \end{array} \right\} \tag{1}$$

We want to produce a weak solution of (1), that is, we want to find $u \in W_0^{1,p}(\Omega)$ such that

$$u^{-\eta}h \in L^1(\Omega) \text{ for all } h \in W_0^{1,p}(\Omega),$$
$$\langle V(u), h\rangle = \int_\Omega \lambda u^{-\eta}h\,\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega).$$

To solve (1), first we consider a regularization of it. So, given $\varepsilon > 0$, we consider the following Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) - \Delta_q u(z) = \lambda\,[u(z)+\varepsilon]^{-\eta} \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \lambda > 0, u > 0. \end{array} \right\} \tag{2}$$

Proposition 2 *For every $\lambda > 0$ and $\varepsilon > 0$, problem (2) has a unique positive solution $\bar{u}_\varepsilon \in \operatorname{int} C_+$ and the map $\varepsilon \to \bar{u}_\varepsilon$ is nonincreasing from $\mathring{\mathbb{R}}_+ = (0,+\infty)$ into $C_0^1(\bar{\Omega})$.*

Proof Let $g \in L^p(\Omega)$ and consider the following Dirichlet problem

$$-\Delta_p u(z) - \Delta_q u(z) = \frac{\lambda}{[|g(z)| + \varepsilon]^\eta} \text{ in } \Omega, \; u\big|_{\partial\Omega} = 0.$$

Note that $\frac{\lambda}{[|g(\cdot)|+\varepsilon]^{\eta}} \in L^{\infty}(\Omega)$ and recall that the operator $V(\cdot)$ is maximal monotone (see Proposition 1). Also, we have

$$\begin{aligned} &\langle V(u), u\rangle \geq \|u\|^{p} \text{ for all } u \in W_0^{1,p}(\Omega), \\ &\Rightarrow V(\cdot) \text{ is coercive.} \end{aligned}$$

A maximal monotone and coercive operator is surjective (see Papageorgiou, Rădulescu & Repovš [13, p.135]). So, we can find $u_g \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that

$$V(u_g) = \frac{\lambda}{[|g| + \varepsilon]^{\eta}}. \tag{3}$$

On account of the strict monotonicity of $V(\cdot)$ (see Proposition 1), this solution u_g is unique. On (3) we act with $-u_g^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} &\|Du_g^-\|_p^p \leq 0, \\ &\Rightarrow u_g \geq 0, u_g \neq 0. \end{aligned}$$

From Theorem 7.1 of Ladyzhenskaya & Uraltseva [8, p.286]), we have that $u_g \in L^{\infty}(\Omega)$ and then using the nonlinear regularity theorem of Lieberman [10], we infer that $u_g \in C_+\backslash\{0\}$. We have

$$\begin{aligned} &-\Delta_p u_g - \Delta_q u_g = \frac{\lambda}{[|g| + \varepsilon]^{\eta}} \text{ in } \Omega, \\ &\Rightarrow \Delta_p u_g + \Delta_q u_g \leq 0 \text{ in } \Omega. \end{aligned}$$

Invoking the nonlinear Hopf maximum principle of Pucci & Serrin [18, p.120], we conclude that

$$u_g \in \operatorname{int} C_+.$$

Let $s : L^p(\Omega) \to W_0^{1,p}(\Omega)$ be the solution map for problem (1) defined by $s(g) = u_g$. Evidently $s(\cdot)$ is continuous. Also acting on (3) with $s(g) = u_g \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} &\|Du_g\|_p^p + \|Du_g\|_q^q = \int_{\Omega} \frac{\lambda u_g}{[|g| + \varepsilon]^{\eta}} dz, \\ &\Rightarrow \|u_g\|^p \leq \frac{\lambda}{\varepsilon^{\eta}} c_1 \|u_g\| \text{ for some } c_1 > 0, \\ &\Rightarrow \|u_g\|^{p-1} \leq \frac{\lambda}{\varepsilon^{\eta}} c_1 \text{ for all } g \in L^p(\Omega). \end{aligned}$$

Hence we have that

$$s(L^p(\Omega)) \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

The compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ (Sobolev embedding theorem), implies that

$$\overline{s(L^p(\Omega))} \subseteq L^p(\Omega) \text{ is compact.}$$

Invoking the Schauder-Tychonov fixed point theorem (see Theorem 4.8.3 of [13, p.357]), we can find $\bar{u}_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$s(\bar{u}_\varepsilon) = \bar{u}_\varepsilon,$$
$$\Rightarrow -\Delta_p \bar{u}_\varepsilon - \Delta_q \bar{u}_\varepsilon = \frac{\lambda}{[\bar{u}_\varepsilon + \varepsilon]^\eta} \text{ in } \Omega.$$

As before the nonlinear regularity theory and the nonlinear maximum principle, imply that

$$\bar{u}_\varepsilon \in \operatorname{int} C_+.$$

We show that this solution of (2) is in fact unique. Indeed, suppose that $\bar{v}_\varepsilon \in W_0^{1,p}(\Omega)$ is another positive solution of (2). Again we show that $\bar{v}_\varepsilon \in \operatorname{int} C_+$. We have

$$\langle V(\bar{u}_\varepsilon), (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \rangle = \int_\Omega \frac{\lambda}{[\bar{u}_\varepsilon + \varepsilon]^\eta} (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \mathrm{d}z, \tag{4}$$
$$\langle V(\bar{v}_\varepsilon), (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \rangle = \int_\Omega \frac{\lambda}{[\bar{v}_\varepsilon + \varepsilon]^\eta} (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \mathrm{d}z. \tag{5}$$

We subtract (5) from (4) and obtain

$$0 \le \langle V(\bar{u}_\varepsilon) - V(\bar{v}_\varepsilon), (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \rangle = \lambda \int_\Omega \left[\frac{1}{[\bar{u}_\varepsilon + \varepsilon]^\eta} - \frac{1}{[\bar{v}_\varepsilon + \varepsilon]^\eta} \right] (\bar{u}_\varepsilon - \bar{v}_\varepsilon)^+ \mathrm{d}z \le 0$$
$$\Rightarrow \bar{u}_\varepsilon \le \bar{v}_\varepsilon \text{ (see Proposition 1).}$$

Interchanging the roles of $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$ in the above argument we also have $\bar{v}_\varepsilon \le \bar{u}_\varepsilon$, to conclude that $\bar{u}_\varepsilon = \bar{v}_\varepsilon$.

This proves the uniqueness of the solution $\bar{u}_\varepsilon \in \operatorname{int} C_+$ of problem (2).

Next we show that the map $\varepsilon \to \bar{u}_\varepsilon$ is nonincreasing from $\mathring{\mathbb{R}}_+ = (0, \infty)$ into $C_+ \backslash \{0\}$. So, let $0 < \varepsilon' < \varepsilon$. We have

$$-\Delta_p \bar{u}_{\varepsilon'} - \Delta_q \bar{u}_{\varepsilon'} = \lambda \left[\bar{u}_{\varepsilon'} + \varepsilon'\right]^{-\eta} \ge \lambda \left[\bar{u}_{\varepsilon'} + \varepsilon\right]^{-\eta} \text{ in } \Omega. \tag{6}$$

We introduce the Carathéodory function $k_\varepsilon(z, x)$ defined by

$$k_\varepsilon(z, x) = \begin{cases} \lambda[x^+ + \varepsilon]^{-\eta} & \text{if } x \le \bar{u}_{\varepsilon'}(z) \\ \lambda[\bar{u}_{\varepsilon'}(z) + \varepsilon]^{-\eta} & \text{if } \bar{u}_{\varepsilon'}(z) < x. \end{cases} \tag{7}$$

We set $K_\varepsilon(z, x) = \int_0^x k_\varepsilon(z, s) ds$ and consider the C^1-functional $\psi_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\varepsilon(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \int_\Omega k_\varepsilon(z, u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (7) that $\psi_\varepsilon(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\psi_\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. Then the Weierstrass-Tonelli theorem implies the existence of $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\psi_\varepsilon(\tilde{u}_\varepsilon) = \inf \left[\psi_\varepsilon(u) : u \in W_0^{1,p}(\Omega) \right],$$
$$\Rightarrow \langle \psi_\varepsilon'(\tilde{u}_\varepsilon), h \rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega). \tag{8}$$

In (8) first we use the test function $h = -\tilde{u}_\varepsilon^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\|D\tilde{u}_\varepsilon^-\|_p \le 0,\\ &\Rightarrow \tilde{u}_\varepsilon \ge 0.\end{aligned}$$

Also, in (8) we choose $h = \left[\tilde{u}_\varepsilon - \bar{u}_{\varepsilon'}\right]^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(\tilde{u}_\varepsilon), (\tilde{u}_\varepsilon - \bar{u}_{\varepsilon'})^+\rangle &= \int_\Omega \frac{\lambda}{[\bar{u}_{\varepsilon'} + \varepsilon]^\eta}(\tilde{u}_\varepsilon - \bar{u}_{\varepsilon'})^+ \,\mathrm{d}z \text{ (see (7))}\\ &\le \langle V(\bar{u}_{\varepsilon'}), (\tilde{u}_\varepsilon - \bar{u}_{\varepsilon'})^+\rangle \text{ (see (6))},\\ &\Rightarrow \tilde{u}_\varepsilon \le \bar{u}_{\varepsilon'}, \text{ (see Proposition 1)}.\end{aligned}$$

So, we have proved that

$$\tilde{u}_\varepsilon \in [0, \bar{u}_{\varepsilon'}]. \tag{9}$$

From (9), (7) and (8) it follows that

$$\begin{aligned}&\tilde{u}_\varepsilon = \bar{u}_\varepsilon,\\ &\Rightarrow \bar{u}_\varepsilon \le \bar{u}_\varepsilon, \text{ (see (9))},\\ &\Rightarrow \varepsilon \to \bar{u}_\varepsilon \text{ is nonincreasing from } \mathring{\mathbb{R}}_+ = (0,\infty) \text{ into } C_+\backslash\{0\}.\end{aligned}$$

The proof is now complete. □

Now we will pass to the limit as $\varepsilon \to 0^+$ in order to produce a solution for problem (1).

Proposition 3 *If* $1 < \eta < 2 - \frac{1}{p}$, *then problem* (1) *admits a unique solution* $\bar{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, *and for every* $K \subseteq \Omega$ *compact we have* $0 < c_K \le \bar{u}(z)$ *for a.a.* $z \in K$.

Proof Let $\varepsilon_n \to 0^+$ and let $\bar{u}_n = \bar{u}_{\varepsilon_n} \in \operatorname{int} C_+$ be the unique positive solution of the corresponding regularized problem (2) with $\varepsilon = \varepsilon_n$, $n \in \mathbb{N}$. (see Proposition 2). We know that

$$\{\bar{u}_n\}_{n\in\mathbb{N}} \subseteq \operatorname{int} C_+ \text{ is nondecreasing (recall that } \bar{u}_n = \bar{u}_{\varepsilon_n} \text{ and } \varepsilon_n \searrow 0). \tag{10}$$

For every $n \in \mathbb{N}$ we have

$$\langle V(\bar{u}_n), h\rangle = \int_\Omega \frac{\lambda h}{[\bar{u}_n + \varepsilon_n]^\eta}\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega). \tag{11}$$

Using the test function $h = \bar{u}_n \in W_0^{1,p}(\Omega)$, we obtain

$$\|D\bar{u}_n\|_p^p \le \int_\Omega \frac{\lambda}{\bar{u}_n^{\eta-1}}\mathrm{d}z \le \int_\Omega \frac{\lambda}{\bar{u}_1^{\eta-1}}\mathrm{d}z \text{ for all } n \in \mathbb{N} \text{ (see (10))}. \tag{12}$$

By hypotheses $\eta < 2$ and so $\eta - 1 < 1$. Since $\bar{u}_1 \in \operatorname{int} C_+$ from the Lemma (and its proof) in Lazer & McKenna [9], we have

$$\frac{1}{\bar{u}_1^{\eta-1}} \in L^1(\Omega).$$

Then from (12) we have

$$\|\bar{u}_n\|^p \le \lambda c_2 \text{ for some } c_2 > 0, \text{ all } n \in \mathbb{N}.$$

Therefore $\{\bar{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded, and so we may assume that

$$\bar{u}_n \xrightarrow{w} \bar{u} \text{ in } W_0^{1,p}(\Omega),\ \bar{u}_n \to \bar{u} \text{ in } L^p(\Omega),\ \bar{u}_n(z) \to \bar{u}(z) \text{ for a.a. } z \in \Omega. \tag{13}$$

Note that

$$\left|\int_\Omega \frac{\lambda(\bar{u}_n-\bar{u})}{[\bar{u}_n+\varepsilon_n]^\eta}\,\mathrm{d}z\right| \le \int_\Omega \frac{\lambda_n|\bar{u}_n-\bar{u}|}{\bar{u}_1^\eta}\,\mathrm{d}z$$
$$\le \int_\Omega \lambda \bar{u}_1^{1-\eta}\frac{|\bar{u}_n-\bar{u}|}{\bar{u}_1}\,\mathrm{d}z. \tag{14}$$

From Lemmas 14 and 16 of Gilbarg & Trudinger [7, p.355], we know that there exists $\delta > 0$ such that if $\widehat{d}(\cdot) = d(\cdot, \partial\Omega)$ on $\bar{\Omega}$, then $\widehat{d} \in C^2(\Omega_\delta)$, where $\Omega_\delta = \{z \in \bar{\Omega} : \widehat{d}(z) < \delta\}$. It follows that $\widehat{d} \in C_+\backslash\{0\}$. Since $\bar{u}_1 \in \operatorname{int} C_+$, using Proposition 4.1.22 of Papageorgiou, Rădulescu & Repovš [13, p.274], we can find $c_3 > 0$ such that $c_3\hat{d} \le \bar{u}_1$. We have

$$\lambda \int_\Omega \bar{u}_1^{1-\eta}\frac{|\bar{u}_n-\bar{u}|}{\bar{u}_1}\mathrm{d}z$$
$$\le \frac{\lambda}{c_3}\int_\Omega \bar{u}_1^{1-\eta}\frac{|\bar{u}_n-\bar{u}|}{\widehat{d}}\mathrm{d}z. \tag{15}$$

Using Hardy's inequality, we have that

$$\frac{|\bar{u}_n-\bar{u}|}{\widehat{d}} \in L^p(\Omega). \tag{16}$$

Since by hypothesis $\eta < 2 - \frac{1}{p}$, we have $(\eta-1)p' < 1$ and so using once again the Lemma of Lazer & McKenna [9], we have

$$\bar{u}_1^{1-\eta} \in L^{p'}(\Omega). \tag{17}$$

From (15), (16), (17) and Hölder's inequality, we have

$$\lambda \int_\Omega \bar{u}_1^{1-\eta}\frac{|\bar{u}_n-\bar{u}|}{\bar{u}_1}\mathrm{d}z \le \frac{\lambda}{c_3}\|\bar{u}^{-1}\|^{\eta-1}_{(\eta-1)p'}\|\frac{\bar{u}_n-\bar{u}}{\widehat{d}}\|_p. \tag{18}$$

Note that

$$\left(\frac{|\bar{u}_n-\bar{u}|}{\widehat{d}}\right)^p = \left(\frac{\bar{u}_n-\bar{u}}{\widehat{d}}\right)^p \le \left(\frac{2\bar{u}}{\widehat{d}}\right)^p \in L^1(\Omega). \tag{19}$$

We have used (12) and Hardy's inequality which says that $\frac{\bar{u}}{\widehat{d}} \in L^p(\Omega)$.
From (13) we have

$$\frac{|(\bar{u}_n-\bar{u})(z)|}{\widehat{d}(z)} \to 0 \text{ for a.a. } z \in \Omega,\ \text{ as } n \to \infty. \tag{20}$$

Then (19), (20) and the Lebesgue dominated convergence theorem imply that

$$\left\|\frac{\bar{u}_n-\bar{u}}{\widehat{d}}\right\|_p \to 0 \text{ as } n \to \infty,$$
$$\Rightarrow \int_\Omega \frac{\lambda|\bar{u}_n-\bar{u}|}{[\bar{u}_n+\varepsilon_n]^\eta}\mathrm{d}z \to 0 \text{ as } n \to \infty \text{ (see (18), (14))}. \tag{21}$$

Therefore, if in (11) we use the test function $h = \bar{u}_n - \bar{u} \in W_0^{1,p}(\Omega)$, passing to the limit as $n \to \infty$ and using (21), we obtain

$$\lim_{n\to\infty}\langle V(\bar{u}_n), \bar{u}_n - \bar{u}\rangle = 0,$$
$$\Rightarrow \bar{u}_n \to \bar{u} \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1)},\ \bar{u}_1 \le \bar{u}. \tag{22}$$

We know that

$$\langle V(\bar{u}_n), h\rangle = \int_\Omega \frac{\lambda h}{[\bar{u}_n + \varepsilon_n]^\eta} dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{23}$$

For every $n \in \mathbb{N}$, we have

$$\frac{|h|}{[\bar{u}_n + \varepsilon_n]^\eta} \le \frac{|h|}{\bar{u}_1^\eta} \text{ (see (10))}. \tag{24}$$

As above, via Hardy's and Hölder's inequalities, we have

$$\frac{|h|}{\bar{u}_1^\eta} \in L^1(\Omega). \tag{25}$$

Moreover, from (13) we have

$$\frac{h(z)}{(\bar{u}_n + \varepsilon_n)(z)^\eta} \to \frac{h(z)}{\bar{u}(z)^\eta} \text{ for a.a. } z \in \Omega. \tag{26}$$

Then (24), (25), (26) and the Lebesgue dominated convergence theorem imply that

$$\int_\Omega \frac{\lambda h}{[\bar{u}_n + \varepsilon_n]^\eta} dz \to \int_\Omega \frac{\lambda h}{\bar{u}_1^\eta} dz. \tag{27}$$

If in (23) we pass to the limit as $n \to \infty$ and use (22) and (27) we obtain

$$\langle V(\bar{u}), h\rangle = \int_\Omega \frac{\lambda h}{\bar{u}^\eta} dz \text{ for all } h \in W_0^{1,p}(\Omega), \; \bar{u}_1 \le u \text{ (see (22))},$$
$$\frac{h}{\bar{u}^\eta} \in L^1(\Omega) \text{ for all } h \in W_0^{1,p}(\Omega) \text{ (see (25))}.$$

We conclude that $\bar{u} \in W_0^{1,p}(\Omega)$ is a positive solution of problem (1). As before, exploiting the strict monotonicity of the operator $V(\cdot)$ (see Proposition 1) and the fact that the map $x \to x^{-\eta}$, $x > 0$, is strictly decreasing, we infer that $\bar{u} \in W_0^{1,p}(\Omega)$ is unique. Moreover, on account of the fact that $\bar{u}_1 \le \bar{u}$ (see (22)), since $\bar{u}_1 \in \operatorname{int} C_+$, we have that for all $K \subseteq \Omega$ compact

$$0 < c_K \le \bar{u}(z) \text{ for a.a. } z \in K.$$

Finally let $k > 1$ and set $\xi_k(t) = [t - k]^+$. This is a Lipschitz function and so $\xi_k(u_n) \in W_0^{1,p}(\Omega)$ for all $n \in \mathbb{N}$ (see [13, p.22]). In (23) we choose as test function $h = \xi_k(\bar{u}_n) \in W_0^{1,p}(\Omega)$. We obtain

$$\|D\xi_k(\bar{u}_n)\|_p^p \le \int_\Omega \frac{\lambda \xi_k(\bar{u}_n)}{[\bar{u}_n + \varepsilon_n]^\eta} dz,$$
$$\Rightarrow \|D\xi_k(\bar{u}_n)\|_p^p \le \lambda \int_\Omega \xi_k(\bar{u}_n) dz \text{ (recall the definition of } \xi_k(\cdot)).$$

From the estimate as in the proof of Proposition 2.10 of Papageorgiou & Rădulescu [12] (see also Stampacchia [20], Theorems 4.1, 4.2), we obtain

$$\|\bar{u}_n\|_\infty \le c_4 \text{ for some } c_4 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \bar{u} \in L^\infty(\Omega) \text{ (see (22))}.$$

The proof is now complete. □

4 Multiplicity theorem

In this section, using the results of Section 3, we prove a multiplicity result for the positive solutions of problem (P_λ) when $\lambda > 0$ is small. To the best of our knowledge, this is the first multiplicity theorem for strongly singular (p, q)- equations.

To this end, we introduce the Carathéodory function $\widehat{k}_\lambda(z, x)$ defined by

$$\widehat{k}_\lambda(z,x) = \begin{cases} \lambda \bar{u}(z)^{-\eta} & \text{if } x \le \bar{u}(z) \\ \lambda x^{-\eta} & \text{if } \bar{u}(z) < x. \end{cases} \tag{28}$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s)ds$ and consider the functional $\widehat{\gamma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\gamma}_\lambda(u) = \int_\Omega \widehat{K}_\lambda(z, u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

In what follows by $C_w^1(W_0^{1,p}(\Omega))$ we denote the space of all functions $\gamma : W_0^{1,p}(\Omega) \to \mathbb{R}$ which are differentiable and the derivative $u \to \gamma'(u)$ is continuous from $W_0^{1,p}(\Omega)$ with the norm topology into $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ with the weak topology.

Lemma 4 *If* $1 < \eta < 2 - \frac{1}{p}$ *and* $\lambda > 0$, *then* $\widehat{\gamma}_\lambda \in C_w^1(W_0^{1,p}(\Omega))$ *and* $\widehat{\gamma}_\lambda'(u) = N_{\widehat{k}_\lambda}(u)$ *for all* $u \in W_0^{1,p}(\Omega)$.

Proof Let $t \in \mathbb{R}\backslash\{0\}$ and $h \in C_c^\infty(\Omega)$. We have

$$\begin{aligned} &\frac{1}{t}\left[\widehat{\gamma}_\lambda(u + th) - \widehat{\gamma}_\lambda(u)\right] \\ &\quad = \frac{1}{t}\int_\Omega \left[\widehat{K}_\lambda(z, u + th) - \widehat{K}_\lambda(z, u)\right]\mathrm{d}z \\ &\quad = \int_\Omega \left[\int_0^1 \widehat{k}_\lambda(z, u + sth)\mathrm{d}s\right] h\,\mathrm{d}z. \end{aligned} \tag{29}$$

Note that

$$\int_0^1 \widehat{k}_\lambda(z, u + sth)\mathrm{d}s \to \widehat{k}_\lambda(z, u) \text{ for a.a. } z \in \Omega, \text{ as } t \to 0. \tag{30}$$

Also we have:

- on $\{u < \bar{u}\}$, for $|t| < 1$ small we have

$$|\widehat{k}_\lambda(z, u + sth)| = \lambda\bar{u}^{-\eta} \le \lambda\bar{u}_1^{-\eta} \text{ (see (28) and recall } h \in C_c^\infty(\Omega),\ \bar{u}_1 \le \bar{u});$$

- on $\{\bar{u} < u\}$, for $|t| < 1$ small we have

$$|\widehat{k}_\lambda(z, u + sth)| = \lambda(u + sth)^{-\eta} \le \lambda\bar{u}_1^{-\eta} \text{ (see (28) and recall } h \in C_c^\infty(\Omega),\ \bar{u}_1 \le \bar{u}).$$

By continuity we also have that

$$|\widehat{k}_\lambda(z, \bar{u} + sth)| \le \lambda\bar{u}_1^{-\eta}.$$

We already know that $\frac{h}{\bar{u}_1^{\eta}} \in L^1(\Omega)$ (see (25)). So, using the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\left[\widehat{\gamma}_\lambda(u+th)-\widehat{\gamma}_\lambda(u)\right] \\
&\quad = \lambda \int_\Omega [\max\{u,\bar{u}\}]^{-\eta} h \, dz \\
&\quad = \lambda \int_\Omega \widehat{k}_\lambda(z,u) h \, dz \text{ for all } h \in C_c^\infty(\Omega) \text{ (see (29), (30))}, \\
&\quad \Rightarrow \widehat{\gamma}_\lambda'(u)(h) = \int_\Omega \widehat{k}_\lambda(z,u) h \, dz \text{ for all } h \in C_c^\infty(\Omega).
\end{aligned}
$$

The density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$ implies that

$$
\widehat{\gamma}_\lambda'(u)(h) = \int_\Omega \widehat{k}_\lambda(z,u) h \, dz \text{ for all } h \in W_0^{1,p}(\Omega).
$$

Let $u_n \to u$ in $W_0^{1,p}(\Omega)$. We have

$$
|\langle \widehat{\gamma}_\lambda'(u_n) - \widehat{\gamma}_\lambda'(u), h\rangle| = \left| \int_\Omega [\widehat{k}_\lambda(z,u_n) - \widehat{k}_\lambda(z,u_n)] h \, dz \right|. \tag{31}
$$

Note that

$$
|\widehat{k}_\lambda(z,u_n) - \widehat{k}_\lambda(z,u)| \le 2\bar{u}_1^{-\eta}.
$$

Moreover, at least for a subsequence, we have

$$
\widehat{k}_\lambda(z,u_n) \to \widehat{k}_\lambda(z,u) \text{ for a.a. } z \in \Omega, \text{ as } n \to \infty.
$$

Then from (31) and using the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\langle \widehat{\gamma}_\lambda'(u_n) - \widehat{\gamma}_\lambda'(u), h\rangle \to 0 \qquad \text{for all } h \in W_0^{1,p}(\Omega), \\
&\Rightarrow \widehat{\gamma}_\lambda'(u_n) \xrightarrow{w} \widehat{\gamma}_\lambda'(u) \quad \text{in} \quad W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*, \\
&\Rightarrow \widehat{\gamma}_\lambda \in C_w^1(W_0^{1,p}(\Omega)) \quad \text{and} \quad \gamma_\lambda'(u) = N_{\widehat{k}_\lambda}(u).
\end{aligned}
$$

The proof is complete. □

Now we are ready to state and prove the multiplicity theorem.

Theorem 5 *If hypotheses H, $\widehat{H}$ hold, then for $\lambda > 0$ small, problem (P_λ) has at least two positive solutions $u_0, \hat{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for all $K \subseteq \Omega$ compact we have*

$$
0 < c_K \le u_0(z),\ \hat{u}(z) \text{ for a.a. } z \in K.
$$

Proof With $\bar{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ being the unique positive solution of problem (1) (see Proposition 3), we introduce the Carathéodory function $k(z,x)$ defined by

$$
k(z,x) = \begin{cases} \bar{u}(z)^{-\eta} + f(z,\bar{u}(z)) & \text{if } x \le \bar{u}(z) \\ x^{-\eta} + f(z,x) & \text{if } \bar{u}(z) < x. \end{cases} \tag{32}
$$

We set $K(z,x) = \int_0^x k(z,s) ds$ and consider the functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\varphi_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \int_\Omega \lambda K(z,u) dx \quad \text{for all } u \in W_0^{1,p}(\Omega).
$$

Using Lemma 4, we have that $\varphi_\lambda \in C^1_w(W^{1,p}_0(\Omega))$. For every $u \in W^{1,p}_0(\Omega)$, we have

$$\int_\Omega K(z,u)\mathrm{d}z = \int_{\{u\le\bar u\}} [\bar u^{-\eta} + f(z,\bar u)]u\,\mathrm{d}z + \int_{\bar u<u} [\bar u^{1-\eta} + f(z,\bar u)\bar u]dz \\ + \frac{1}{1-\eta}\int_{\{u<\bar u\}} [u^{1-\eta} - \bar u^{1-\eta}]\,\mathrm{d}z + \int_{\{\bar u<u\}} [F(z,u) - F(z,\bar u)]\,\mathrm{d}z. \quad (33)$$

We estimate the terms in the right-hand side of (33). First we deal with the first two summands. We have

$$\left|\int_{\{u\le\bar u\}} \bar u^{-\eta}u\,\mathrm{d}z + \int_{\bar u<u} \bar u^{1-\eta}dz\right| \\ \le \int_\Omega \bar u^{-\eta}|u|dz \\ \le \int_\Omega \bar u^{1-\eta}\frac{|u|}{\bar u_1}dz \qquad \text{(see (22))}. \quad (34)$$

As before (see the proof of Proposition 3), using Hardy's and Hölder's inequalities, we have

$$\left|\int_\Omega \bar u_1^{1-\eta}\frac{u}{\bar u_1}\,\mathrm{d}z\right| \le c_5\|\frac{u}{\hat d}\|_p \quad \text{for some } c_5>0 \\ \le c_6\|Du\|_p \quad \text{for some } c_6>0. \quad (35)$$

We use (35) in (34) and obtain

$$\left|\int_{\{u\le\bar u\}} \bar u^{-\eta}u\,\mathrm{d}z + \int_{\{\bar u<u\}} \bar u^{1-\eta}\,\mathrm{d}z\right| \le c_6\|u\|. \quad (36)$$

On account of hypothesis H(i), we have

$$\left|\int_{\{u\le\bar u\}} f(z,\bar u)u\,\mathrm{d}z + \int_{\{\bar u<u\}} f(z,\bar u)\bar u\,\mathrm{d}z\right| \\ \le \int_\Omega f(z,\bar u)|u|\,\mathrm{d}z \le c_7\|u\| \quad \text{for some } c_7>0. \quad (37)$$

From (36) and (37) it follows that

$$\left|\int_{\{u\le\bar u\}} [\lambda\bar u^{-\eta} + f(z,\bar u)]u\,\mathrm{d}z + \int_{\{\bar u<u\}} [\lambda\bar u^{1-\eta} + f(z,\bar u)\bar u]\,\mathrm{d}z\right| \le c_8\|u\| \\ \text{for some } c_8>0. \quad (38)$$

Next we estimate the third summand in the right-hand side of (33). We have

$$\left|\frac{1}{1-\eta}\int_{\{\bar u<u\}} [u^{1-\eta} - \bar u^{1-\eta}]\,\mathrm{d}z\right| \\ = \frac{1}{\eta-1}\int_{\{\bar u<u\}} \left[\frac{1}{\bar u^{\eta-1}} - \frac{1}{u^{\eta-1}}\right]\mathrm{d}z \quad (\text{recall that } \eta>1) \\ = \frac{1}{\eta-1}\int_{\{\bar u<u\}} \frac{u^{\eta-1-\bar u^{\eta-1}}}{(\bar u u)^{\eta-1}}\,\mathrm{d}z \\ \le \frac{1}{\eta-1}\int_{\{\bar u<u\}} \left(\frac{u}{\bar u_1^2}\right)^{\eta-1}\mathrm{d}z \quad \text{(see (22))}. \quad (39)$$

From hypothesis $\widehat{H}$ we have $\eta < \frac{3}{2} \Rightarrow 2(\eta-1) < 1$. Therefore, if $\xi \in \left(1, \frac{1}{2(\eta-1)}\right)$, then using the Lemma of Lazer & McKenna [9] (recall that $\bar{u}_1 \in \operatorname{int} C_+$), we have

$$\bar{u}_1^{-2(\eta-1)} \in L^{\xi}(\Omega). \tag{40}$$

Since by hypothesis $\widehat{H}$, $\eta < \frac{3p^*+1}{2p^*+1}$, we have $\frac{p^*}{p^*+1-\eta} < \frac{1}{2(\eta-1)}$ and so if we restrict further $\xi \in \left[\frac{p^*}{p^*+1-\eta}, \frac{1}{2(\eta-1)}\right)$, then we have

$$(\eta-1)\xi' = (\eta-1)\frac{\xi}{\xi-1} \leq p^*.$$

Therefore by the Sobolev embedding theorem, we have

$$u^{\eta-1} \in L^{\xi'}(\Omega). \tag{41}$$

We return to (39) and use (40), (41) and Hölder's inequality. We have

$$\begin{aligned}
&\frac{1}{\eta-1}\int_{\{\bar{u}<u\}}\left(\frac{u}{\bar{u}_1^2}\right)^{\eta-1}\mathrm{d}z \\
&\leq \frac{1}{\eta-1}\int_{\Omega}\left(\frac{|u|}{\bar{u}_1^2}\right)^{\eta-1}\mathrm{d}z \\
&\leq \left[\int_{\Omega}|u|^{(\eta-1)\xi'}\,\mathrm{d}z\right]^{\frac{1}{\xi'}}\left[\int_{\Omega}\bar{u}_1^{-2(\eta-1)\xi}\,\mathrm{d}z\right]^{\frac{1}{\xi}} \\
&\leq c_9\|u\|_{(\eta-1)\xi'}^{\eta-1} \quad \text{for some } c_9 > 0 \text{ (recall } \bar{u}_1 \in \operatorname{int} C_+ \text{ and } 2(\eta-1)\xi < 1) \\
&\leq c_{10}\|u\|^{\eta-1} \quad \text{for some } c_{10} > 0 \text{ (recall that } (\eta-1)\xi' \leq p^*).
\end{aligned}$$

Thus we have the following estimate for the third summand of the right-hand side of (33)

$$\left|\frac{1}{1-\eta}\int_{\{\bar{u}<u\}}\left[u^{1-\eta}-\bar{u}^{1-\eta}\right]\mathrm{d}z\right| \leq c_{10}\|u\|^{\eta-1} \quad \text{(see (39))}. \tag{42}$$

Finally we examine the fourth summand in the right-hand side of (33). On account of hypotheses H(i), (iii), given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$0 \leq F(z,x) \leq \frac{\varepsilon}{q}|x|^q + c_{11}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{43}$$

So, we have

$$\begin{aligned}
0 &\leq \int_{\{\bar{u}<u\}}[F(z,u)-F(z,\bar{u})]\,\mathrm{d}z \\
&\leq \int_{\{\bar{u}<u\}}F(z,u)\,\mathrm{d}z \quad (\text{since } F \geq 0, \text{ see hypotheses } H(\text{i})) \\
&\leq \frac{\varepsilon}{q}\|u\|_q^q + c_{12}\|u\|^r \quad \text{for some } c_{12} > 0 \\
&\leq \frac{\varepsilon}{\widehat{\lambda}_1(q)}\|Du\|_q^q + c_{12}\|u\|^r
\end{aligned} \tag{44}$$

with $\widehat{\lambda}_1(q) > 0$ being the principle eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$.

Using (38), (42) and (44), we have

$$\varphi_\lambda(u) \geq \frac{1}{p}\|u\|^p + \frac{1}{q}\left[1 - \frac{\lambda\varepsilon}{\widehat{\lambda}_1(q)}\right]\|Du\|_q^q$$
$$- c_{14}\lambda\left[\|u\|^{\eta-1} + \|u\| + \|u\|^{r-1}\right] \quad \text{for some } c_{14} > 0.$$

Note that $\eta - 1 < 1 < r$. So, we have

$$\|u\| \leq \|u\|^{1-\eta} + \|u\|^r. \tag{45}$$

Choose $\varepsilon \in \left(0, \frac{\widehat{\lambda}_1(q)}{\lambda}\right)$. We have $1 - \frac{\lambda\varepsilon}{\widehat{\lambda}_1(q)} > 0$ and so

$$\begin{aligned}\varphi_\lambda(u) &\geq \frac{1}{p}\|u\|^p - 2c_{14}\lambda\left[\|u\|^{\eta-1} + \|u\|^r\right] \quad (\text{see } (45)) \\ &\geq \left[\frac{1}{p} - 2c_{14}\lambda\left(\|u\|^{\eta-1-p} + \|u\|^{r-p}\right)\right]\|u\|^p.\end{aligned} \tag{46}$$

Let $\xi(t) = t^{\eta-1-p} + t^{r-p}$, $t > 0$. Evidently $\xi \in C^1(0,\infty)$ and since $\eta - 1 < 1 < p < r$, we have

$$\xi(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 > 0$ such that

$$\begin{aligned}\xi(t_0) &= \min\{\xi(t) : t > 0\}, \\ &\Rightarrow \xi'(t_0) = 0, \\ &\Rightarrow (p + 1 - \eta)t_0^{\eta-2-p} = (r - p)t_0^{r-p-1}, \\ &\Rightarrow t_0 = \left(\frac{p+1-\eta}{r-p}\right)^{\frac{1}{r+1-\eta}}.\end{aligned}$$

Then for $u \in W_0^{1,p}(\Omega)$ with $\|u\| = t_0$ from (46)

$$\varphi_\lambda(u) \geq \left[\frac{1}{p} - 2c_{14}\lambda\xi(t_0)\right]t_0^p.$$

We see that we can find $\lambda^* > 0$ such that

$$\varphi_\lambda(u) \geq c_\lambda > 0 \quad \text{for all } \|u\| = t_0, \text{ all } \lambda \in (0, \lambda^*). \tag{47}$$

We introduce the closed ball $\bar{B}_0 = \{u \in W_0^{1,p} : \|u\| \leq t_0\}$ and consider the following minimization problem

$$\inf\left[\varphi_\lambda(u) : u \in \bar{B}_0\right] = m_\lambda. \tag{48}$$

The Eberlein-Smulian theorem says that $\bar{B}_0$ is sequentially weakly compact. Also, using the Sobolev embedding theorem we see that $\varphi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = m_\lambda \quad (\text{see } (48)). \tag{49}$$

For $t \in (0, 1)$, we have

$$\varphi_\lambda(t\bar{u}) \leq \frac{t^p}{p}\|D\bar{u}\|_p^p + \frac{t^q}{q}\|D\bar{u}\|_q^q - \lambda t\int_\Omega \bar{u}^{1-\eta}\,dz \ (\text{see } (32) \text{ and recall } f \geq 0, \text{ see } H(\mathrm{i})).$$

We know that $0 \leq \bar{u}^{1-\eta} \leq \bar{u}_1^{1-\eta} \in L^1(\Omega)$ (see (22)). Moreover, since $t \in (0,1)$ and $q < p$, we obtain

$$\varphi_\lambda(t\bar{u}) \leq c_{15}t^q - \lambda c_{16}t \quad \text{for some } c_{15},\ c_{16} > 0.$$

Recalling that $q > 1$, if we choose $t \in (0,1)$, we have

$$\begin{aligned} &\varphi_\lambda(t\bar{u}) < 0, \qquad t\bar{u} \in \bar{B}_0, \\ &\quad \Rightarrow \varphi_\lambda(u_0) < 0 = \varphi_\lambda(0) \qquad (\text{see } (48)), \\ &\quad \Rightarrow u_0 \neq 0. \end{aligned} \tag{50}$$

Then (47) and (50) imply that

$$\begin{aligned} &0 < \|u_0\| < t_0,\ \text{that is},\ u_0 \in B_0 \setminus \{0\}, \\ &\quad \Rightarrow \varphi_\lambda'(u_0) = 0 \quad (\text{see } (48), (49) \text{ and recall } \varphi_\lambda \in C_w^1(W_0^{1,p}(\Omega))), \\ &\quad \Rightarrow \langle V(u_0), h\rangle = \lambda \int_\Omega k(z,u_0)h \,\mathrm{d}z \end{aligned} \tag{51}$$

for all $h \in W_0^{1,p}(\Omega)$ (see Lemma 4). In (51) we choose $h = [\bar{u} - u_0]^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} \langle V(u_0), (\bar{u}-u_0)^+\rangle &= \int_\Omega [\lambda \bar{u}^{-\eta} + f(z,\bar{u})](\bar{u}-u_0)^+ \,\mathrm{d}z \quad (\text{see } (32)) \\ &\geq \int_\Omega \lambda \bar{u}^{-\eta}(\bar{u}-u_0)^+ \,\mathrm{d}z \quad (\text{since } f \geq 0,\ \text{see } H(\mathrm{i})) \\ &= \langle V(\bar{u}), (\bar{u}-u_0)^+\rangle \quad (\text{see Proposition } 3) \\ &\Rightarrow \bar{u} \leq u_0 \ (\text{see Proposition } 1). \end{aligned} \tag{52}$$

From (52), (32) and (51), we see that $u_0 \in W_0^{1,p}(\Omega)$ is a positive solution of (P_λ) (for $\lambda \in (0,\lambda^*)$) and $\bar{u}_1 \leq \bar{u} \leq u_0$. Then as before we have

$$\begin{aligned} &u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \\ &0 < c_K \leq u_0(z) \quad \text{for a.a. } z \in \Omega,\ \text{all } K \subseteq \Omega \text{ compact}. \end{aligned}$$

From the previous arguments we know that

$$\varphi_\lambda(0) = 0 < c_\lambda \leq \varphi_\lambda(u) \qquad \text{for all } \|u\| = t_0. \tag{53}$$

On account of hypotheses $H(\mathrm{ii})$, if $u \in \operatorname{int} C_+$, then

$$\varphi_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{54}$$

Also, from the claim in the proof of Proposition 4 of Papageorgiou, Rădulescu & Zhang [16], we know that

$$\varphi_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{55}$$

Then (53), (54) and (55) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widehat{u} \in K_{\varphi_\lambda} = \{u \in W_0^{1,p}(\Omega) : \varphi_\lambda'(u) = 0\} \\ &\qquad\qquad \subseteq [\bar{u}) \cap L^\infty(\Omega) = \{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \bar{u}(z) \leq u(z), \text{a.e.}\}. \\ &\varphi_\lambda(u_0) < 0 = \varphi_\lambda(0) < c_\lambda \leq \varphi_\lambda(\widehat{u}). \end{aligned}$$

Therefore $\widehat{u} \notin \{0, u_0\}$ is a positive solution of (P_λ) (for $\lambda \in (0, \lambda^*)$) and, moreover, $0 < c_K \leq \widehat{u}(z)$ for a.a. $z \in K$, all $K \subseteq \Omega$ compact. □

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