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Results in Mathematics

Variational Approach to Fourth-Order Impulsive Differential Equations with Two Control Parameters

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Abstract. In this paper, we are concerned with the multiplicity of solutions for a fourth-order impulsive differential equation with Dirichlet boundary conditions and two control parameters. Using variational methods and a three critical points theorem, we give some new criteria to guarantee that the impulsive problem has at least three classical solutions. We also provide an example in order to illustrate the main abstract results of this paper.

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1. Introduction

In this paper, we consider the fourth-order boundary value problem with two control parameters and impulsive effects

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = \lambda f(t, u(t)) + \mu g(t, u(t)), \quad t \neq t_j,\ t \in [0,1], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \ldots, n, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \ldots, n, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where A and B are two real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous, $I_{1j}, I_{2j} \in C(\mathbb{R};\mathbb{R})$ for $1 \leq j \leq n$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$, the operator Δ is defined as $\Delta(u(t_j)) := u(t_j^+) - u(t_j^-)$, where $u(t_j^+)$ and $u(t_j^-)$ denote the right and the left limits, respectively, of u at t_j, and $\lambda > 0$ and $\mu \geq 0$ are referred to as control parameters.

Many dynamical systems describing models in applied sciences have an *impulsive* dynamical behaviour due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to *impulsive differential equations*; they describe various processes of the real world described by models that are subject to *sudden changes* in their states. Essentially, impulsive differential equations correspond to a *smooth evolution* that may *change instantaneously* or even *abruptly*, as happens in various applications that describe mechanical or natural phenomena. These changes correspond to impulses in the smooth system, such as for example in the model of a mechanical clock. Impulsive differential equations also study models in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, chaos theory. Associated with this development, a theory of impulsive differential equations has been given extensive attention. For an introduction of the basic theory of impulsive differential equations in $\mathbb{R}^n$, see [2,11,19]. Some classical tools have been used to study such problems in the literature, such as the coincidence degree theory of Mawhin, the method of upper and lower solutions with the monotone iterative technique, and some fixed point theorems in cones (see [8,12,17]). Recently, the existence and multiplicity of solutions for impulsive boundary value problems by using variational methods and critical point theory has been considered and here we cite the papers [14,20–24].

Motivated by the above facts, in this paper, our aim is to study the existence of solutions for fourth-order impulsive boundary value problem (1.1). By employing a three critical point theorem which we recall in the next section (Theorem 2.1), we establish the exact collections of the parameters λ and μ, for which problem (1.1) admits at least three solutions; see Theorem 3.1.

2. Abstract Setting

The original *three critical point* theorem is due to Pucci and Serrin [15,16] and establishes that if X is a real Banach space and a function $f : X \to \mathbb{R}$ is of class C^1, satisfies the Palais–Smale condition, and has two local minima, then f has at least three distinct critical points. This result has been extended in the framework of problems depending on a real parameter by Ricceri [18], who also established a precise range of the parameter that guarantees the existence of at least three critical points.

Our main tool is a three critical point theorem that we recall here in a convenient form. We also refer the reader to the recent papers [1,5–7,13] where an analogous variational approach has been developed on studying different elliptic problems.

Theorem 2.1 ([4, Theorem 3.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a* *continuous inverse on* X^*; $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ *and* $\overline{x} \in X$, *with* $r < \Phi(\overline{x})$, *such that*

(a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)}[$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* I_λ *has at least three distinct critical points in* X.

Here and in the sequel, we suppose that A and B satisfy the following condition:

$$A \le 0 \le B. \tag{2.1}$$

Define

$$H_0^1([0,1]) := \{u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0\},$$
$$H^2([0,1]) := \{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\}.$$

Take $X := H^2([0,1]) \cap H_0^1([0,1])$ and define

$$\|u\|_X := \left(\int_0^1 \left(|u''(t)|^2 - A|u'(t)|^2 + B|u(t)|^2\right) dt \right)^{1/2}, \quad u \in X. \tag{2.2}$$

Since A and B satisfy (2.1), it is straightforward to verify that (2.2) defines a norm for the Sobolev space X and this norm is equivalent to the usual norm defined as follows:

$$\|u\| := \left(\int_0^1 |u''(t)|^2\, dt \right)^{1/2}$$

It follows from (2.1) that $\|u\| \le \|u\|_X$. For the norm in $C^1([0,1])$,

$$\|u\|_\infty := \max\left\{ \max_{t\in[0,1]} |u(t)|, \max_{t\in[0,1]} |u'(t)| \right\},$$

we have the following relation.

Lemma 2.2 ([24, Lemma 2.1]). *Let* $M_1 := 1 + 1/\pi$. *Then* $\|u\|_\infty \le M_1\|u\|_X$ *for all* $u \in X$.

Throughout the sequel, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\lambda > 0$ and $\mu \ge 0$ are real parameters. Put

$$F(t,\xi) := \int_0^{\xi} f(t,x)\,dx \text{ and } G(t,\xi) := \int_0^{\xi} g(t,x)\,dx,$$

for all $(t,\xi) \in [0,1] \times \mathbb{R}$.

Moreover, set $G^c := \int_0^1 \max_{|\xi|\le c} G(t,\xi)\,dt$ for all $c > 0$ and $G_d := \inf_{[0,1]\times[0,d]} G$ for all $d > 0$. Clearly, $G^c \ge 0$ and $G_d \le 0$.

We say that $u \in C([0,1])$ is a *classical solution* of problem (1.1), if it satisfies the equation in (1.1) a.e. on $[0,1] \setminus \{t_1, t_2, \ldots, t_n\}$, the limits $u''(t_j^+)$, $u''(t_j^-), u'''(t_j^+)$ and $u'''(t_j^-)$, $1 \le j \le n$, exist, satisfy two impulsive conditions in (1.1) and the boundary condition $u(0) = u(1) = u''(0) = u''(1) = 0$.

A *weak solution* of problem (1.1) is a function $u \in X$ such that the equality

$$\begin{aligned}\int_0^1 &(u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t))\,dt \\ &= -\sum_{j=1}^{n} I_{2j}(u(t_j))v(t_j) - \sum_{j=1}^{n} I_{1j}(u'(t_j))v'(t_j) \\ &\quad +\lambda \int_0^1 f(t,u(t))v(t)\,dt + \mu \int_0^1 g(t,u(t))v(t)\,dt\end{aligned}$$

holds for all $v \in X$.

We consider the functional $I_\lambda : X \to \mathbb{R}$, defined by

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \quad u \in X, \tag{2.3}$$

where

$$\Phi(u) := \frac{1}{2}\|u\|_X + \sum_{j=1}^{n} \int_0^{u'(t_j)} I_{1j}(s)\,ds + \sum_{j=1}^{n} \int_0^{u(t_j)} I_{2j}(s)\,ds \tag{2.4}$$

and

$$\Psi(u) := \int_0^1 F(t,u(t))\,dt + \frac{\mu}{\lambda}\int_0^1 G(t,u(t))\,dt. \tag{2.5}$$

It is clear that I_λ is differentiable at any $u \in X$ and

$$\begin{aligned}I'_\lambda(u)(v) = \int_0^1 &\big(u''(t)v''(t) - Au'(t)v'(t) + Bu(t)v(t)\big)\,dt + \sum_{j=1}^{n} I_{2j}(u(t_j))v(t_j) \\ &+ \sum_{j=1}^{n} I_{1j}(u'(t_j))v'(t_j) - \lambda\int_0^1 f(t,u(t))v(t)\,dt - \mu\int_0^1 g(t,u(t))v(t)\,dt\end{aligned}$$

for any $v \in X$. Hence, a critical point of I_λ gives a weak solution of problem (1.1).

Lemma 2.3 ([24, Lemma 2.2]). *If $u \in X$ is a weak solution of problem* (1.1)*, then u is also a classical solution of problem* (1.1).

We assume throughout, and without further mention, that the following conditions hold:

(H1) Assume that $\{t_1, t_2, \ldots, t_n\} \subseteq [\frac{1}{4}, \frac{3}{4}]$;
(H2) Assume that there exist two positive constants k_1 and k_2 such that for each $u \in X$,

$$0 \leq \sum_{j=1}^{n} \int_{0}^{u'(t_j)} I_{1j}(s)\, ds \leq k_1 \max_{j\in\{1,2,\ldots,n\}} |u'(t_j)|^2$$

and

$$0 \leq \sum_{j=1}^{n} \int_{0}^{u(t_j)} I_{2j}(s)\, ds \leq k_2 \max_{j\in\{1,2,\ldots,n\}} |u(t_j)|^2.$$

Also put $k_0 := 2{,}048\big(\frac{3}{8} - \frac{9}{10\cdot 4^4}A + \frac{79}{14\cdot 4^8}B\big)$ and $k_3 := k_0 + k_2$. These constants will be used in some of our hypotheses in the next section.

In conclusion, we cite a recent monograph by Kristály et al. [10] as a general reference on variational methods adopted here.

3. Main Result and Proof

In this section, we present our main results on the existence of at least three classical solutions for the problem (1.1).

In order to introduce our first result, we fix $c, d > 0$ such that

$$\frac{k_3 d^2}{\int_{1/4}^{3/4} F(t,d)\, dt} < \frac{c^2}{2M_1^2 \int_0^1 \max_{|\xi|\leq c} F(t,\xi)\, dt}$$

and pick

$$\lambda \in \Lambda := \left(\frac{k_3 d^2}{\int_{1/4}^{3/4} F(t,d)\, dt}, \frac{c^2}{2M_1^2 \int_0^1 \max_{|\xi|\leq c} F(t,\xi)\, dt} \right). \tag{3.1}$$

Set

$$\delta := \min\left\{ \frac{c^2 - 2\lambda M_1^2 \int_0^1 \max_{|\xi|\leq c} F(t,\xi)\, dt}{2M_1^2 G^c}, \frac{k_3 d^2 - \lambda \int_{1/4}^{3/4} F(t,d)\, dt}{G_d} \right\} \tag{3.2}$$

and

$$\overline{\delta} := \min\left\{\delta, \frac{1}{\max\left\{0, 4M_1^2 \limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2}\right\}}\right\}, \tag{3.3}$$

where we read $r/0 = +\infty$. For instance, $\overline{\delta} = +\infty$ when $\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2} \leq 0$ and $G_d = G^c = 0$.

With the above notations we are able to prove the following multiplicity property.

Theorem 3.1. *Assume that there exist two positive constants c, d, with $c < \sqrt{2k_0}M_1 d$, such that*

(A1) *$F(t,\xi) \geq 0$, for each $(t,\xi) \in ([0,\frac{1}{4}] \cup [\frac{3}{4},1]) \times [0,d]$;*

(A2) $\frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2} < \frac{\int_{1/4}^{3/4} F(t,d)\,dt}{2k_3(M_1 d)^2}$;

(A3) $\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} < \frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{2c^2}$.

Then, for every $\lambda \in \Lambda$, where Λ is given by (3.1), and for every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2} < +\infty,$$

there exists $\overline{\delta} > 0$ given by (3.3) such that, for each $\mu \in [0,\overline{\delta})$, problem (1.1) admits at least three classical solutions.

Proof. Fix λ, g and μ as in the conclusion. By Lemma 2.3, it suffices to show that the functional I_λ defined in (2.3) has at least three critical points in X. We prove this by verifying the conditions given in Theorem 2.1. Note that Φ defined in (2.4) is a nonnegative Gâteaux differentiable, coercive, and sequentially weakly lower semicontinuous functional, and its Gâteaux derivative admits a continuous inverse on X^*. Moreover, Ψ defined in (2.5) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. We will verify (a_1) and (a_2) of Theorem 2.1.

Let w be the function defined by

$$w(t) := \begin{cases} 64d(t^3 - \frac{3}{4}t^2 + \frac{3}{16}t), & t \in [0,1/4) \\ d, & t \in [1/4,3/4] \\ 64d(-t^3 + \frac{9}{4}t^2 - \frac{27}{16}t + \frac{7}{16}), & t \in (3/4,1], \end{cases}$$

and put $r := \frac{c^2}{2M_1^2}$. Clearly, $w \in X$ and from the condition (H2) one has

$$\Phi(w) \geq \frac{1}{2}\|w\|_X^2 = 2{,}048\left(\frac{3}{8} - \frac{9}{10\cdot 4^4}A + \frac{79}{14\cdot 4^8}B\right)d^2 = k_0 d^2 > r.$$

Also, by using condition (A1), since $0 \leq w(t) \leq d$ for each $t \in [0,1]$, we infer

$$\begin{aligned}\Psi(w) &= \int_0^1 F(t, w(t))\,dt + \frac{\mu}{\lambda}\int_0^1 G(t, w(t))\,dt \\ &\geq \int_{1/4}^{3/4} F(t, d)\,dt + \frac{\mu}{\lambda}\int_0^1 G(t, w(t))\,dt \\ &\geq \int_{1/4}^{3/4} F(t, d)\,dt + \frac{\mu}{\lambda}G_d.\end{aligned}$$

For all $u \in X$ satisfying $\Phi(u) \leq r$, by Lemma 2.2, we obtain

$$\|u\|_\infty^2 \leq M_1^2\|u\|_X^2 \leq 2M_1^2\Phi(u) \leq 2M_1^2 r = c^2.$$

Therefore

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} &\leq \frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt + \frac{\mu}{\lambda}\int_0^1 \max_{|\xi|\leq c} G(t,\xi)\,dt}{\frac{c^2}{2M_1^2}} \\ &= 2M_1^2\frac{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{c^2} + 2M_1^2\frac{\mu}{\lambda}\frac{G^c}{c^2}.\end{aligned}$$

From this, if $G^c = 0$, we deduce that

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{1}{\lambda}, \tag{3.4}$$

while, if $G^c > 0$, it turns out to be true bearing in mind that

$$\mu < \frac{c^2 - 2\lambda M_1^2\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}{2M_1^2 G^c}.$$

On the other hand, taking into account (H2), we have

$$\begin{aligned}\Phi(w) &= 2{,}048\left(\frac{3}{8} - \frac{9}{10\cdot 4^4}A + \frac{79}{14\cdot 4^8}B\right)d^2 + \sum_{j=1}^m \int_0^d I_{2j}(s)\,ds \\ &\leq k_0 d^2 + k_2 d_2 = k_3 d^2,\end{aligned}$$

and so,

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\geq \frac{\int_{1/4}^{3/4} F(t,d)\,dt + \frac{\mu}{\lambda}G_d}{k_3 d^2} \\ &= \frac{\int_{1/4}^{3/4} F(t,d)\,dt}{k_3 d^2} + \frac{\mu}{\lambda}\frac{G_d}{k_3 d^2}.\end{aligned}$$

Hence, if $G_d = 0$, we find

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda}, \tag{3.5}$$

while, if $G_d < 0$, the same relation holds since

$$\mu < \frac{k_3 d^2 - \lambda \int_{1/4}^{3/4} F(t,d)\,dt}{G_d}.$$

Therefore, from (3.4) and (3.5), condition (a_1) of Theorem 2.1 is verified.

Now, in order to prove the coercivity of the functional I_λ, first we assume that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} > 0.$$

Therefore, fix

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} < \varepsilon < \frac{\int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}{c^2}.$$

From (A3), there is a function $h_\varepsilon \in L^1([0,1])$ such that

$$F(t,\xi) \le \varepsilon\xi^2 + h_\varepsilon(t),$$

for each $t \in [0,1]$ and $\xi \in \mathbb{R}$. Taking (2.3) into account and since $\lambda < \frac{c^2}{2M_1^2 \int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}$, it follows that

$$\begin{aligned}
\lambda \int_0^1 F(t,u(t))\,dt &\le \lambda\left(\varepsilon\int_0^1 (u(t))^2\,dt + \int_0^1 h_\varepsilon(t)\,dt\right)\\
&< \frac{c^2}{2M_1^2 \int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}\left(\varepsilon\int_0^1 (u(t))^2\,dt + \int_0^1 h_\varepsilon(t)\,dt\right)\\
&\le \frac{c^2}{2M_1^2 \int_0^1 \max_{|\xi|\le c} F(t,\xi)\,dt}\left(\varepsilon M_1^2 \|u\|_X^2 + \|h_\varepsilon\|_{L^1([0,1])}\right),
\end{aligned} \tag{3.6}$$

for each $u \in X$. Moreover, since $\mu < \overline{\delta}$, we obtain

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2} < \frac{1}{4\mu M_1^2}.$$

Thus, there is a function $h_\mu \in L^1([0,1])$ such that

$$G(t,\xi) \le \frac{1}{4\mu M_1^2}\xi^2 + h_\mu(t),$$

for each $t \in [0,1]$ and $\xi \in \mathbb{R}$. Thus, taking again Lemma 2.2 into account, it follows that

$$\begin{aligned}\int_0^1 G(t,u(t))\,dt &\leq \frac{1}{4\mu M_1^2}\int_0^1 (u(t))^2\,dt + \int_0^1 h_\mu(t)\,dt \\ &\leq \frac{1}{4\mu}\|u\|_X^2 + \|h_\mu\|_{L^1([0,1])}, \end{aligned} \tag{3.7}$$

for each $u \in X$. Finally, putting together (3.6) and (3.7), we have

$$\begin{aligned} I_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\geq \frac{1}{2}\|u\|_X^2 - \frac{c^2}{2M_1^2\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}\left(\varepsilon M_1^2\|u\|_X^2 + \|h_\varepsilon\|_{L^1([0,1])}\right) \\ &\quad -\frac{1}{4}\|u\|_X^2 - \mu\|h_\mu\|_{L^1([0,1])} \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{c^2}{\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt}\varepsilon\right)\|u\|_X^2 - \frac{c^2\|h_\varepsilon\|_{L^1([0,1])}}{2M_1^2\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt} \\ &\quad -\mu\|h_\mu\|_{L^1([0,1])}. \end{aligned}$$

On the other hand, if

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} F(t,\xi)}{\xi^2} \leq 0,$$

there exists a function $h_\varepsilon \in L^1([0,1])$ such that $F(t,\xi) \leq h_\varepsilon(t)$ for each $t \in [0,1]$ and $\xi \in \mathbb{R}$, and arguing as before we obtain

$$I_\lambda(u) \geq \frac{1}{4}\|u\|_X^2 - \frac{c^2\|h_\varepsilon\|_{L^1([0,1])}}{2M_1^2\int_0^1 \max_{|\xi|\leq c} F(t,\xi)\,dt} - \mu\|h_\mu\|_{L^1([0,1])}.$$

Both cases lead to the coercivity of I_λ and condition (a_2) of Theorem 2.1 is verified.

Since, from (3.4) and (3.5),

$$\lambda \in \Lambda \subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[,$$

Theorem 2.1 ensures the existence of at least three critical points for the functional I_λ and the proof is complete. □

The technical approach used to prove the previous result uses some ideas from [3]. In the cited work, the existence of at least three classical solutions for a perturbed two-point boundary value problem has been investigated under suitable conditions on the potentials F and G; see also [9], where analogous variational approaches have been developed to study a perturbed mixed boundary value problem.

4. Particular Cases and An Example

A particular case of Theorem 3.1 is the following multiplicity property.

Theorem 4.1. *Let $\alpha \in L^1([0,1])$ be a nonnegative and non-zero function. Further, let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $\alpha_0 := \int_{1/4}^{3/4} \alpha(t)\, dt$ and $H(\xi) := \int_0^\xi h(x)\, dx$ for all $\xi \in \mathbb{R}$, and assume that there exist two positive constants c, d, with $c < 16\sqrt{6}M_1 d$, such that*

(A4) $\frac{H(c)}{c^2} < \frac{\alpha_0}{2(768+k_2)M_1^2\|\alpha\|_{L^1([0,1])}} \frac{H(d)}{d^2}$;

(A5) $\limsup_{|\xi|\to+\infty} \frac{H(\xi)}{\xi^2} \leq 0.$

Then, for each parameter λ belonging to

$$\Lambda := \left(\frac{768+k_2}{\alpha_0} \frac{d^2}{H(d)}, \frac{1}{2M_1^2\|\alpha\|_{L^1([0,1])}} \frac{c^2}{H(c)} \right),$$

and for every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2} < +\infty,$$

there exists

$$\delta := \min\left\{ \frac{c^2 - 2\lambda M_1^2\|\alpha\|_{L^1([0,1])}H(c)}{2M_1^2 G^c}, \frac{(768+k_2)d^2 - \lambda\alpha_0 H(d)}{G_d} \right\}$$

such that, for each

$$\mu \in \left[0, \min\left\{\delta, \frac{1}{\max\left\{0, 4M_1^2 \limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,1]} G(t,\xi)}{\xi^2}\right\}}\right\}\right),$$

the problem

$$\begin{cases} u^{(iv)}(t) = \lambda\alpha(t)h(u(t)) + \mu g(t,u(t)), \quad t \neq t_j,\ t \in [0,1], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1,2,\ldots,n, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1,2,\ldots,n, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{4.1}$$

admits at least three classical solutions.

A direct consequence of the previous multiplicity property reads as follows.

Corollary 4.2. *Let $\alpha \in L^1([0,1])$ be a nonnegative and non-zero function. Moreover, let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative (not identically zero) and continuous function such that*

$$\liminf_{\xi\to0^+} \frac{h(\xi)}{\xi} = \limsup_{|\xi|\to+\infty} \frac{h(\xi)}{\xi} = 0. \tag{4.2}$$

Then, for each

$$\lambda > \frac{768 + k_2}{\alpha_0} \inf_{d \in S} \frac{d^2}{H(d)},$$

where $S := \{d > 0 : H(d) > 0\}$*, the problem*

$$\begin{cases} u^{(iv)}(t) = \lambda \alpha(t) h(u(t)), \quad t \neq t_j,\ t \in [0,1], \\ \Delta(u''(t_j)) = I_{1j}(u'(t_j)), \quad j = 1, 2, \dots, n, \\ -\Delta(u'''(t_j)) = I_{2j}(u(t_j)), \quad j = 1, 2, \dots, n, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

admits at least three classical solutions.

Proof. Fix $\lambda > \frac{768+k_2}{\alpha_0} \inf_{d \in S} \frac{d^2}{H(d)}$. Then, there exists $\overline{d} > 0$ such that $H(\overline{d}) > 0$ and $\lambda > \frac{(768+k_2)\overline{d}^2}{\alpha_0 H(\overline{d})}$. By using condition (4.2) we obtain

$$\liminf_{\xi \to 0^+} \frac{H(\xi)}{\xi^2} = 0.$$

Therefore, we can find a positive constant $\overline{c}$ such that $\overline{c} < 16\sqrt{6} M_1 d$ and

$$\frac{H(\overline{c})}{\overline{c}^2} < \min \left\{ \frac{\alpha_0 H(\overline{d})}{(768 + k_2)(M_1 \overline{d})^2}, \frac{1}{2\lambda M_1^2 \|\alpha\|_{L^1([0,1])}} \right\}.$$

Hence

$$\lambda \in \left(\frac{768 + k_2}{\alpha_0} \frac{\overline{d}^2}{H(\overline{d})}, \frac{1}{2M_1^2 \|\alpha\|_{L^1([0,1])}} \frac{\overline{c}^2}{H(\overline{c})} \right).$$

All the hypotheses of Theorem 4.1 are satisfied and problem (4.1) admits at least three distinct classical solutions. The proof is complete. □

In conclusion we present a concrete example of application of Corollary 4.2.

Example 4.3. Consider the following problem:

$$\begin{cases} u^{(iv)}(t) = \lambda e^{16t} h(u(t)), \quad t \in [0,1] \setminus \{\frac{1}{2}\}, \\ \Delta(u''(t_1)) = 2u'(t_1), \quad t_1 = \frac{1}{2}, \\ -\Delta(u'''(t_1)) = 4u(t_1), \quad t_1 = \frac{1}{2}, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{4.3}$$

where $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) := \begin{cases} x^2, & |x| \leq 1, \\ \frac{1}{x^2}, & |x| > 1. \end{cases}$$

Here, $I_{11}(s) = 2s$ and $I_{21}(s) = 4s$ for all $s \in \mathbb{R}$. It is easy to verify that (H2) is satisfied with $k_1 = 1$ and $k_2 = 2$. Direct calculations give

$$\liminf_{\xi \to 0^+} \frac{h(\xi)}{\xi} = \lim_{\xi \to 0^+} \xi = 0,$$
$$\limsup_{|\xi| \to +\infty} \frac{h(\xi)}{\xi} = \lim_{|\xi| \to +\infty} \frac{1}{\xi^3} = 0.$$

Also we have

$$\inf_{d \in S} \frac{d^2}{H(d)} = \inf_{0<d\leq 1} \frac{3}{d} = 3.$$

Put $\alpha(t) = e^{16t}$ for all $t \in \mathbb{R}$. Therefore,

$$\alpha_0 = \int_{1/4}^{3/4} \alpha(t)\, dt = \frac{1}{16}(e^{12} - e^4).$$

From Corollary 4.2, for each parameter $\lambda > \frac{36{,}960}{e^{12}-e^4}$, problem (4.3) admits at least three classical solutions. In particular, the problem

$$\begin{cases} u^{(iv)}(t) = e^{16t} h(u(t)), \quad t \in [0,1] \setminus \{\frac{1}{2}\}, \\ \Delta(u''(t_1)) = 2u'(t_1), \quad t_1 = \frac{1}{2}, \\ -\Delta(u'''(t_1)) = 4u(t_1), \quad t_1 = \frac{1}{2}, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

admits at least three classical solutions.

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