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Nikolaos S. Papageorgiou & Vicenţiu D. Rădulescu

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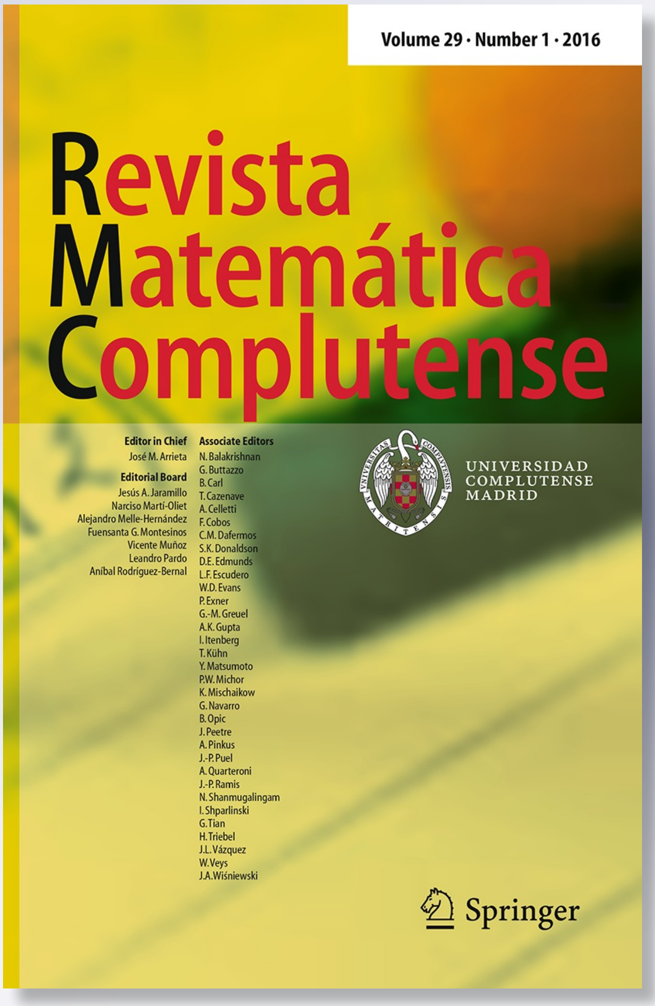

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Robin problems with indefinite, unbounded potential and reaction of arbitrary growth

Nikolaos S. Papageorgiou[1] ·
Vicenţiu D. Rădulescu[2,3]

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Abstract We study an elliptic Robin problem driven by the negative Laplacian plus an indefinite and unbounded potential and with a reaction of arbitrary growth which exhibits z-dependent zeros of constant sign. We prove multiplicity theorems producing three or four nontrivial solutions, all with precise sign information. As a particular case we consider a generalized equidiffusive logistic equation with potential.

Keywords Indefinite and unbounded potential · Robin boundary condition · Constant sign and nodal solutions · Multiplicity theorem · Critical groups

Mathematical Subject Classification 35J20 · 35J60 · 58E05

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded with a C^2-boundary $\partial\Omega$. In this paper, we study the following Robin problem

✉ Vicenţiu D. Rădulescu
vicentiu.radulescu@imar.ro

Nikolaos S. Papageorgiou
npapg@math.ntua.gr

1 Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

2 Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

3 Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

$$-\Delta u(z)+\xi(z)u(z)=f(z,u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}+\beta(z)u=0 \quad \text{on } \partial\Omega. \qquad (1)$$

Here $\xi \in L^s(\Omega)$ with $s > N$ and in general it is sign changing. Also, $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geqslant 0$. When $\beta \equiv 0$, then we have the Neumann problem. The reaction $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous). The interesting feature of our work here, is that we do not impose any global growth condition on $x \longmapsto f(z,x)$. Instead, we assume that $f(z,\cdot)$ admits z-dependent zeros of constant sign. Our aim is to prove a multiplicity theorem providing precise sign information for all the solutions. Using variational methods based on the critical point theory, together with suitable truncation and perturbation techniques and Morse theory (critical groups), we prove two multiplicity theorems producing three nontrivial solutions (two of constant sign and the third nodal (sign changing)). The two multiplicity results differ on the behavior of the reaction $f(z,\cdot)$ near zero. Subsequently, by improving the regularity condition on $x \longmapsto f(z,x)$ and using tools from Morse theory, we prove a third multiplicity theorem, producing four nontrivial solutions, two of constant sign and two nodal. Our work here extends the semilinear part of the recent work of Papageorgiou and Rădulescu [23].

Semilinear equations with indefinite and bounded potential, were studied recently under different conditions on the reaction and under different boundary conditions. We mention the works of Kyritsi and Papageorgiou [13], Papageorgiou and Papalini [21] (Dirichlet problems), and Papageorgiou and Rădulescu [22,24], Papageorgiou and Smyrlis [25] (Neumann problems).

None of the aforementioned works addresses the general boundary condition used in this paper (which incorporates as a special case the Neumann problem for $\beta \equiv 0$) and all assumed that the reaction term $f(z,\cdot)$ has subcritical polynomial growth. In contrast here, the behavior of $f(z,\cdot)$ near $\pm\infty$ is irrelevant and instead we assume a kind of oscillatory behavior near zero for the nonlinearity $x \longmapsto f(z,x)-\xi(z)x$, by requiring the presence of z-dependent zeros for the function. In this way we can focus our analysis on an interval $[-\rho,\rho]$ ignoring the structure of the reaction term outside it.

2 Mathematical background

In this section, we briefly review the main mathematical tools which we will use in this paper.

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Palais–Smale condition" (the PS-condition for short), if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ, which leads to a deformation theorem, from which one can derive the minimax theory for the critical values of φ. A basic result in that theory, is the so-called "mountain pass theorem".

Theorem 1 *Assume that $\varphi \in C^1(X)$ satisfies the PS-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| > r > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = r] = m_r$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geqslant m_\rho$ and c is critical value of φ.

In the study of problem (1) we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$. By $||\cdot||$ we denote the norm of the Sobolev space $H^1(\Omega)$, defined by

$$||u| = [||u||_2^2 + ||Du||_2^2]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

The Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$ are defined as follows. On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma_0(\cdot)$. Then using $\sigma_0(\cdot)$ we can introduce the spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$ in the usual way. From the trace theorem, we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map", such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. This map is compact into $L^\eta(\partial\Omega)$ for $1 \leqslant \eta < \frac{2(N-1)}{N-2}$. Moreover, we know that

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = H_0^1(\Omega)$$

(for details see, for example, Gasinski and Papageorgiou [10]). In what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. Every Sobolev function defined on $\partial\Omega$ is understood in the sense of traces.

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in H^1(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in H^1(\Omega), \quad |u| = u^+ + u^- \quad \text{and} \quad u = u^+ - u^-.$$

We will also use some aspects of the spectrum of $-\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following eigenvalue problem

$$-\Delta u(z) + \xi(z)u(z) = \hat{\lambda}u(z) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

This eigenvalue problem was studied for the Neumann boundary condition (that is, $\beta \equiv 0$), in Papageorgiou and Rădulescu [22] and Papageorgiou and Smyrlis [25]. For the p-Laplacian and Neumann boundary condition, it was investigated by Mugnai and Papageorgiou [18] and for the p-Laplacian with Robin boundary condition and $\xi \equiv 0$ by Papageorgiou and Rădulescu [23]. An analogous study can be conducted for problem (2) and leads to similar results. More, precisely, assume that $\xi \in L^{\frac{N}{2}}(\Omega)$ if $N \geqslant 3$, $\xi \in L^r(\Omega)$ for $r > 1$ when $N = 2$ and $\xi \in L^1(\Omega)$ when $N = 1$, $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geqslant 0$ and let $\sigma : H^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$\sigma(u) = ||Du||_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma_0 \quad \text{for all } u \in H^1(\Omega).$$

Then the eigenvalue problem (2) admits a smallest eigenvalue $\hat{\lambda}_1(\beta) > -\infty$ given by

$$\hat{\lambda}_1(\beta) = \inf\left[\frac{\sigma(u)}{||u||_2^2} : u \in H^1(\Omega),\ u \neq 0\right]. \tag{3}$$

So, we can find $\mu > \max\{-\hat{\lambda}_1(\beta), 1\}$ such that

$$\sigma(u) + \mu||u||_2^2 \geqslant c_0||u||^2 \quad \text{for all } u \in H^1(\Omega),\ \text{with } c_0 > 0. \tag{4}$$

Using (4) and the spectral theorem for compact self-adjoint operators, exactly as in [22,25], we have a sequence $\{\hat{\lambda}_k(\beta)\}_{k\geqslant 1}$ such that $\hat{\lambda}_k(\beta) \to +\infty$ as $k \to \infty$ which are all the eigenvalues of (2). Let $E(\hat{\lambda}_k(\beta))$ be the corresponding eigenspace. If $\xi \in L^s(\Omega)$ with $s > N$, then using the regularity result of Wang [29], we have that $E(\hat{\lambda}_k(\beta)) \subseteq C^1(\overline{\Omega})$. For the eigenvalues $\hat{\lambda}_k(\beta)$ $k \geqslant 2$, we have the following variational characterization

$$\begin{aligned}\hat{\lambda}_k(\beta) &= \inf\left[\frac{\sigma(u)}{||u||_2^2} : u \in \overline{\underset{i\geqslant k}{\oplus} E(\hat{\lambda}_i(\beta))},\ u \neq 0\right] \\ &= \sup\left[\frac{\sigma(u)}{||u||_2^2} : u \in \overset{k}{\underset{i=1}{\oplus}} E(\hat{\lambda}_i(\beta)),\ u \neq 0\right], \quad k \geqslant 2.\end{aligned} \tag{5}$$

In both (3) and (5) the infimum (and the case of (5) also the supremum), is realized on the corresponding eigenspace $E(\hat{\lambda}_k(\beta))$. The first eigenvalue $\hat{\lambda}_1(\beta)$ is simple (that is, $\dim E(\hat{\lambda}_1(\beta) = 1)$, Krein–Rutman theorem) and from (3) it is clear that the nontrivial elements of $E(\hat{\lambda}_1(\beta))$ do not change sign. All the other eigenvalues have nodal (sign changing) eigenfunctions. By $\hat{u}_1(\beta) \in H^1(\Omega)$ we denote the L^2-normalized (that is, $||\hat{u}_1(\beta)||_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(\beta)$. If $\xi \in L^s(\Omega)$ with $s > N$, then $\hat{u}_1(\beta) \in C_+\backslash\{0\}$ (see Wang [29]). Moreover, using the Harnack inequality of Pucci and Serrin [26, p. 163], we have $u_1(z) > 0$ for all $z \in \Omega$. Also, if $\xi^+ \in L^\infty(\Omega)$, then by the Hopf theorem (see, for example, Pucci and Serrin [26, p. 120]), we have $\hat{u}_1(\beta) \in \operatorname{int} C_+$. Finally when $\xi \in L^s(\Omega)$ with $s > \frac{N}{2}$, then all the eigenspaces $E(\hat{\lambda}_k(\beta))$ have the "unique continuation property" (UCP for short), that is, if $u \in E(\hat{\lambda}_k(\beta))$ and vanishes on a set of positive measure, then $u \equiv 0$ (see de Figueiredo and Gossez [7]).

For the second eigenvalue $\hat{\lambda}_2(\beta)$, in addition to the variational characterization (5) we have a minimax expression (see [18,23]) which we will need in the sequel. So, let

$$\partial B_1^{L^2} = \{u \in L^2(\Omega) : ||u||_2 = 1\} \quad \text{and} \quad M = H^1(\Omega) \cap \partial B_1^{L^2}.$$

Proposition 2 *We have* $\hat{\lambda}_2(\beta) = \inf_{\hat{\gamma}\in\hat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \sigma(\hat{\gamma}(t))$, *where*

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1, 1], M) : \hat{\gamma}(-1) = -\hat{u}_1(\beta),\ \hat{\gamma}(1) = \hat{u}_1(\beta)\}.$$

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z, x)| \leqslant a_0(z)(1 + |x|^{r-1}) \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and

$$1 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geqslant 3 \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and consider the C^1-functional $\varphi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\sigma(u) - \int_\Omega F_0(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

Next we recall some basic definitions and facts from Morse theory (critical groups), which we will use in the sequel.

Let X be a Banach space and $\varphi \in C^1(X),\ c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\}, \quad K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{and} \quad K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$ by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Recall that for $k < 0,\ H_k(Y_1, Y_2) = 0$. Consider an isolated critical point $u_0 \in K_\varphi^c$. Then the critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U\backslash\{u_0\}) \quad \text{for every integer } k \geqslant 0.$$

Here U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, Gasinski and Papageogiou [10, p. 628]), implies that the above definition of critical groups at infinity, is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We introduce the following quantities:

$$M(t,u) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi,u)t^k \quad \text{for all } t \in \mathbb{R},\ \text{all } u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi,\infty)t^k \quad \text{for all } t \in \mathbb{R}.$$

The "Morse relation" says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t), \tag{6}$$

where $Q(t) = \sum_{k\geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Suppose that $X = Y \oplus V$ with $\dim Y < \infty$ and $\varphi \in C^1(X)$. We say that $\varphi \in C^1(X)$ has a "local linking" at the origin, if we can find $\rho > 0$ such that

$$\varphi(y) \leqslant 0 \quad \text{for all } y \in Y, \quad ||y||_X \leqslant \rho$$
$$\varphi(v) > 0 \quad \text{for all } v \in V, \quad 0 < ||v||_X \leqslant \rho.$$

In that case, we know that

$$C_{d_Y}(\varphi,0) \neq 0, \quad \text{where } d_Y = \dim Y.$$

By $C^{2-0}(X)$ we denote the $C^1(X)$-functionals whose derivative is locally Lipschitz. The so-called "shifting theorem" which is known to hold for C^2-functionals, was extended to C^{2-0} functionals by Li et al. [14]. We present a particular case of their result suitable for our purposes. Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for almost all $z \in \Omega$, $f_0(z,\cdot) \in C^{2-0}(\mathbb{R})$. Set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the functional $\varphi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\sigma(u) - \int_\Omega F_0(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Then $\varphi_0 \in C^{2-0}(H^1(\Omega))$. Let $X = C^1(\overline{\Omega})$ and suppose that $u \in X$ is a critical point of φ_0. Then $\varphi_0' \in C^1(D, H^1(\Omega))$ and $\varphi_0''(u) \in \mathcal{L}(X, H^1(\Omega))$, with D an X-neighborhood of u. The Morse index $\mu(u)$ of u is the dimension of the maximal subspace of X on which $\varphi_0''(u)$ is negative definite. The nullity of u, denoted by $\nu(u)$, is the dimension of the kernel of $\varphi_0''(u)$. The extended shifting theorem of Li, Li and Liu [14] says:

Proposition 3 *If $\xi \in L^{\frac{N}{2}}(\Omega)$, $\beta \in L^{\infty}(\partial\Omega)$ and $u \in K_{\varphi}$ has finite Morse index $\mu = \mu(u)$ and nullity $\nu = \nu(u)$, then either*
(a) $C_k(\varphi_0, u) = 0$ *for $k \leqslant \mu$ and $k \geqslant \mu + \nu$, or*
(b) $C_k(\varphi_0, u) = \delta_{k,\mu}\mathbb{Z}$ *for all $k \geqslant 0$, or*
(c) $C_k(\varphi, u) = \delta_{k,\mu+\nu}\mathbb{Z}$ *for all $k \geqslant 0$.*

Finally by $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the operator

$$\langle A(u), y\rangle = \int_{\Omega}(Du, Dy)_{\mathbb{R}^N}dz \quad \text{for all } u, \quad y \in H^1(\Omega).$$

Recall that a Banach space X has the "Kadec–Klee property", if the following is true:

$$\text{“}u_n \xrightarrow{w} u \quad \text{in } X \quad \text{and} \quad ||u_n||_X \to ||u||_X \Rightarrow u_n \to u \text{ in } X\text{''}.$$

We know that locally uniformly convex Banach spaces, in particular Hilbert spaces, have the Kadec–Klee property.

3 Three nontrivial solutions

In this section, we prove two multiplicity theorems producing three nontrivial solutions, two of constant sign and the third nodal. The two multiplicity results differ on the conditions on $f(z, \cdot)$ near zero. In the first, it is assumed that $f(z, \cdot)$ is superlinear near zero, while in the second $f(z, \cdot)$ is linear near zero.

First let us state our conditions on the data of problem (1).

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^{\infty}(\Omega)_+$. $H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta \geqslant 0$.

For the first multiplicity theorem, our hypotheses on the reaction $f(z, x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exist functions $w_{\pm} \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega},\\
&f(z, w_+(z)) - \xi(z)w_+(z) \leqslant 0 \leqslant f(z, w_-(z)) - \xi(z)w_-(z) \quad \text{for almost all } z \in \Omega,\\
&A(w_-) \leqslant 0 \leqslant A(w_+) \quad \text{in } H^1(\Omega)^*;
\end{aligned}$$

(ii) if $\rho = \max\{||w_+||_{\infty}, ||w_-||_{\infty}\}$, then there exists $a_{\rho} \in L^{\infty}(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_{\rho}(z) \quad \text{for almost all } z \in \Omega, \quad \text{all } |x| \leqslant \rho;$$

(iii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exist $\delta \in (0, \min\{c_{\pm}, 1\})$ and $q \in (1, 2)$ such that

$$c_1|x|^q \leqslant f(z, x)x \leqslant qF(z, x) \quad \text{for almost all } z \in \Omega, \quad \text{all } |x| \leqslant \delta.$$

Remark 1 Note that no global condition is imposed on $f(z,\cdot)$. In fact the behavior of $f(z,\cdot)$ beyond $w_\pm(z)$ is irrelevant. Hypothesis $H_1(i)$ is satisfied if for example we have

$$\lim_{x\to\pm\infty}\frac{f(z,x)}{x}=-\infty \quad \text{uniformly for almost all } z\in\Omega.$$

Hypothesis $H_1(iii)$ implies the presence of a concave term near zero.
By hypotheses $H_1(i)$, (ii) and since $q<2$, we have

$$f(z,x)x\geqslant c_2x^2-c_3|x|^r \quad \text{for almost all } z\in\Omega, \quad \text{all } x\in[-\rho,\rho] \tag{7}$$

with $c_2>\hat{\lambda}_1(\beta),\ c_3>0,\ r\in(2,2^*)$ and $\rho=\max[||w_+||_\infty,||w_-||_\infty]$. We choose the unilateral growth condition (7) (instead of the more natural one involving a term with $|x|^q$ due to hypothesis $H_1(iii)$), because it can be used also in the second multiplicity theorem and also facilitates our arguments in the existence results of Proposition 6.

Motivated from (7), we can consider the following auxiliary Robin problems

$$\left\{\begin{array}{ll} -\Delta u(z)+\xi(z)u=c_2u(z)-c_3|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}+\beta(z)u=0 & \text{on } \partial\Omega. \end{array}\right\} \tag{8}$$

Proposition 4 *If hypotheses $H(\xi)$ and $H(\beta)$ hold, then problem (8) admits a unique positive solution $\bar{u}\in int\,C_+$ and since (8) is odd, $\bar{v}=-\bar{u}\in -int\,C_+$ is the unique negative solution.*

Proof Let $\psi: H^1(\Omega)\to\mathbb{R}$ be the C^1-functional defined by

$$\begin{aligned}\psi(u)&=\frac{1}{2}\sigma(u)+\frac{\mu}{2}||u||_2^2-\frac{c_2+\mu}{2}||u^+||_2^2+\frac{c_3}{r}||u^+||_r^r \quad \text{for all } u\in H^1(\Omega)\\ &\geqslant\frac{c_0}{2}||u^-||^2+\frac{1}{2}\sigma(u^+)+\frac{c_3}{r}||u^+||_r^r-\frac{c_2}{\mu}||u^+||_2^2 \quad \text{(see (4))}.\end{aligned} \tag{9}$$

Recall that $2<r$. Then using Young's inequality with $\epsilon>0$, we have

$$\frac{c_2}{2}||u^+||_2^2\leqslant\hat{c}||u^+||_r^2\leqslant c_\epsilon+\epsilon||u^+||_r^r \quad \text{for some } \hat{c},\ c_\epsilon>0.$$

Using this estimate in (9) with $\epsilon\in(0,\frac{c_3}{r})$ and because of hypothesis $H(\xi)$ and since $r>2$, we see that ψ is coercive. Moreover, using the Sobolev embedding theorem and the trace theorem, we see that ψ is sequentially weakly lower semicontinuous. So, we can find $\bar{u}\in H^1(\Omega)$ such that

$$\psi(\bar{u})=\inf[\psi(u):u\in H^1(\Omega)]. \tag{10}$$

Let $t > 0$ and $\hat{u}_1(\beta) \in \operatorname{int} C_+$ is the L^2-normalized principal eigenfunction of (2). We have

$$\psi(t\hat{u}_1(\beta)) = \frac{t^2}{2}(\hat{\lambda}_1(\beta) - c_2) + \frac{t^r c_3}{r}||\hat{u}_1(\beta)||_r^r \quad \text{(see (3))}.$$

Since $c_2 > \hat{\lambda}_1(\beta)$ and $r > 2$, for $t \in (0, 1)$ small we have

$$\begin{aligned} &\psi(t\hat{u}_1(\beta)) < 0, \\ &\Rightarrow \psi(\bar{u}) < 0 = \psi(0) \quad \text{(see (10)), hence } \bar{u} \neq 0. \end{aligned}$$

From (10), we have

$$\begin{aligned} \psi'(\bar{u}) &= 0, \\ &\Rightarrow \langle A(\bar{u}, h)\rangle + \int_\Omega (\xi(z) + \mu)\bar{u}hdz + \int_{\partial\Omega} \beta(z)\bar{u}hd\sigma_0 = (c_2 + \mu)\int_\Omega \bar{u}^+ hdz \\ &\quad -c_3 \int_\Omega (\bar{u}^+)^{r-1} hdz \quad \text{for all } h \in H^1(\Omega). \end{aligned} \tag{11}$$

In (11), we choose $h = -\bar{u}^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\sigma(\bar{u}^-) + \mu||\bar{u}^-||_2^2 = 0, \\ &\Rightarrow c_0||\bar{u}^-||^2 \leqslant 0 \text{ see (4)}, \\ &\Rightarrow \bar{u} \geqslant 0, \ \bar{u} \neq 0. \end{aligned}$$

Then relation (11) becomes, for all $h \in H^1(\Omega)$,

$$\begin{aligned} &\langle A(\bar{u}), h\rangle + \int_\Omega \xi(z)\bar{u}hdz + \int_{\partial\Omega} \beta(z)\bar{u}hd\sigma_0 = \int_\Omega (c_2\bar{u} - c_3\bar{u}^{r-1})hdz, \\ &\Rightarrow \bar{u} \text{ is a positive solution of (8).} \end{aligned}$$

From the regularity result of Wang [29, Section 5] and the strong maximum principle, we have $\bar{u} \in \operatorname{int} C_+$.

Next we show the uniqueness of this positive solutions. So, let $\bar{y}$ be another positive solution of (8). As above, we show that $\bar{y} \in \operatorname{int} C_+$. From Lemma 3.3 of Filippakis et al. [8], we know that there exists $t > 0$ such that

$$t\bar{y} \leqslant \bar{u}.$$

Consider the biggest such real number and assume that $t \in (0, 1)$ (if $t = 1$, then $\bar{u} \geqslant \bar{y}$). Note that there exists $\bar{\eta} > 0$ such that $x \longmapsto (c_2 + \bar{\eta})x - c_3 x^{r-1}$ is nondecreasing on $[0, \rho]$. We have

$$\begin{aligned} &-\Delta(t\bar{y}) + (\xi(z) + \bar{\eta})(t\bar{y}) \\ &\quad = t[-\Delta\bar{y} + (\xi(z) + \bar{\eta})\bar{y}] \end{aligned}$$

$$\begin{aligned}
&= t[(c_2+\bar{\eta})\bar{y} - c_3\bar{y}^{r-1}] \text{ (since } \bar{y}\in \operatorname{int} C_+ \text{ is a solution of (8))}\\
&< (c_2+\bar{\eta})(t\bar{y}) - c_3(t\bar{y})^{r-1} \text{ (since } t\in(0,1) \text{ and } 2<r)\\
&\leqslant (c_2+\bar{\eta})\bar{u} - c_3\bar{u}^{r-1} \text{ (since } t\bar{y}\leqslant \bar{u})\\
&= -\Delta\bar{u} + (\xi(z)+\bar{\eta})\bar{u} \text{ (since } \bar{u}\in \operatorname{int} C_+ \text{ is a solution of (8))},\\
&\Rightarrow -\Delta(\bar{u}-t\bar{y}) + (\xi(z)+\bar{\eta})(\bar{u}-t\bar{y}) \geqslant 0,\\
&\Rightarrow \Delta(\bar{u}-t\bar{y}) \leqslant (||\xi^+||_\infty+\bar{\eta})(\bar{u}-t\bar{y}) \text{ (see hypothesis } H(\xi)),\\
&\Rightarrow \bar{u}-t\bar{y}\in \operatorname{int} C_+ \text{ (by the strong maximum principle).}
\end{aligned}$$

This contradicts the maximality of $t > 0$. Hence $t \geqslant 1$ and so

$$\bar{y} \leqslant \bar{u}.$$

Interchanging the roles of $\bar{u}$ and $\bar{y}$ in the above argument, we can also have

$$\begin{aligned}
&\bar{u} \leqslant \bar{y},\\
&\Rightarrow \bar{u} = \bar{y}.
\end{aligned}$$

This proves the uniqueness of the positive solutions $\bar{u} \in \operatorname{int} C_+$ of (8). Since (8) is odd, it follows that $\bar{v} = -\bar{u} \in -\operatorname{int} C_+$ is the unique negative solution of problem (8). □

Remark 2 To prove the uniqueness of the positive solution $\bar{u} \in \operatorname{int} C_+$ of (8), one can alternatively use Picone's identity (see, for example, Gasinski and Papageorgiou [10, p. 783]).

Now let

$$\begin{aligned}
S_+ &= \{u\in H^1(\Omega) : u\in[0,w_+], \quad u\neq 0, u \text{ is a solution of (1)}\}\\
S_- &= \{v\in H^1(\Omega) : v\in[w_-,0], \quad v\neq 0, v \text{ is a solution of (1)}\}.
\end{aligned}$$

Eventually we will establish the nonemptiness of the sets S_+ and S_-. For the moment, we establish some a priori bounds for the elements of S_+ and S_-.

Proposition 5 *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then $\bar{u} \leqslant u$ for all $u \in S_+$ and $v \leqslant \bar{v}$ for all $v \in S_-$.*

Proof Let $u \in S_+$ and consider the following Carathéodory function

$$g_+(z,x) = \begin{cases} 0 & \text{if } x<0\\ (c_2+\mu)x - c_3x^{r-1} & \text{if } 0\leqslant x\leqslant u(z)\\ (c_2+\mu)u(z) - c_3u(z)^{r-1} & \text{if } u(z)<x. \end{cases} \tag{12}$$

Let $G_+(z,x) = \int_0^x g_+(z,s)ds$ and consider the C^1-functional $\psi_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega G_+(z,u)dz \quad \text{for all } u\in H^1(\Omega).$$

From (4) and (11), we see that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_* \in H^1(\Omega)$ such that

$$\psi_+(\bar{u}_*) = \inf[\psi_+(u) : u \in H^1(\Omega)]. \tag{13}$$

As in the proof of Proposition 4 and because $2 < r$ and $c_2 > \hat{\lambda}_1(\beta)$, we have

$$\psi_+(\bar{u}_*) < 0 = \psi_+(0), \quad \text{hence } \bar{u}_* \neq 0.$$

From (13) we have

$$\begin{aligned}
&\psi_+'(\bar{u}_*) = 0,\\
&\quad \Rightarrow \langle A(\bar{u}_*), h\rangle + \int_\Omega (\xi(z)+\mu)\bar{u}_* h dz + \int_{\partial\Omega} \beta(z)\bar{u}_* h d\sigma_0 = \int_\Omega g_+(z, \bar{u}_*) h dz\\
&\quad \text{for all } h \in H^1(\Omega).
\end{aligned} \tag{14}$$

In (14) first we choose $h = -\bar{u}_*^- \in H^1(\Omega)$. Then

$$\begin{aligned}
&\sigma(\bar{u}_*^-) + \mu||\bar{u}_*^-||_2^2 = 0,\\
&\quad \Rightarrow c_0||\bar{u}_*^-||^2 \leqslant 0 \text{ (see (4))},\\
&\quad \Rightarrow \bar{u}_* \geqslant 0, \bar{u}_* \neq 0.
\end{aligned}$$

Next, in (14) we choose $h = (\bar{u}_* - u)^+ \in H^1(\Omega)$. We have

$$\begin{aligned}
&\langle A(\bar{u}_*), (\bar{u}_* - u)^+\rangle + \int_\Omega (\xi(z) + \mu)\bar{u}_*(\bar{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)\bar{u}_*(\bar{u}_* - u)^+ d\sigma_0\\
&\quad = \int_\Omega [(c_2 + \mu)u - c_3 u^{r-1}](\bar{u}_* - u)^+ dz \text{ (see (11))}\\
&\quad \leqslant \int_\Omega [f(z, u) + \mu u](\bar{u}_* - u)^+ dz \text{ (see (7))}\\
&\quad = \langle A(u), (\bar{u}_* - u)^+\rangle + \int_\Omega (\xi(z) + \mu)u(\bar{u}_* - u)^+ dz + \int_{\partial\Omega} \beta(z)u(\bar{u}_* - u)^+ d\sigma\\
&\qquad \text{(since } u \in S_+)\\
&\quad \Rightarrow \sigma((\bar{u}_* - u)^+) + \mu||(\bar{u}_* - u)^+||_2^2 \leqslant 0,\\
&\quad \Rightarrow c_0||(\bar{u}_* - u)^+||^2 \leqslant 0 \text{ (see (4))},\\
&\quad \Rightarrow \bar{u}_* \leqslant u.
\end{aligned}$$

So, we have proved that

$$\bar{u}_* \in [0, u] = \{y \in H^1(\Omega) : 0 \leqslant y(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}, \quad \bar{u}_* \neq 0. \tag{15}$$

Using (12) and (15), Eq. (14) becomes

$$\langle A(\bar{u}_*), h\rangle + \int_\Omega \xi(z)\bar{u}_* h dz + \int_{\partial\Omega} \beta(z)\bar{u}_* h d\sigma_0 = \int_\Omega (c_2\bar{u}_* - c_3\bar{u}_*^{r-1})h dz$$

for all $h \in H^1(\Omega)$,

$\Rightarrow \bar{u}_*$ is a positive solution of problem (8),

$\Rightarrow \bar{u}_* = \bar{u} \in \mathrm{int}\, C_+$ (see Proposition 4),

$\Rightarrow \bar{u} \leqslant u$ for all $u \in S_+$.

For the a priori bound on the negative solutions, given $v \in S$, we consider the Carathéodory function

$$g_-(z,x) = \begin{cases} c_2 v(z) - c_3|v(z)|^{r-2}v(z) & \text{if } x < v(z) \\ c_2 x - c_3|x|^{r-2}x & \text{if } v(z) \leqslant x \leqslant 0 \\ 0 & \text{if } 0 < x. \end{cases}$$

We set $G_-(z,x) = \int_0^x g_-(z,s)ds$ and consider the C^1-functional $\psi_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega G_-(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Working as above, this time with the functional ψ_-, we show that

$$v \leqslant \bar{v} \quad \text{for all } v \in S_-.$$

The proof is now complete. □

Remark 3 These a priori bounds will be useful in producing extremal constant sign solutions which will lead to nodal solutions.

Proposition 6 *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then problem (1) has at least two solutions of constant sign*

$$u_0 \in int\, C_+ \quad and \quad v_0 \in -int\, C_+.$$

Proof First we produce the positive solution.

So, let $\hat{f}_+(z,x)$ be the Carathéodory function defined by

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) + \mu x & \text{if } 0 \leqslant x \leqslant w_+(z) \\ f(z,w_+(z)) + \mu w_+(z) & \text{if } w_+(z) < x. \end{cases} \tag{16}$$

We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_+(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega \hat{F}_+(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

From (4) and (16), it is clear that $\hat{\varphi}_+$ is coercive. Moreover, using the Sobolev embedding theorem and the trace theorem, we see that $\hat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) : u \in H^1(\Omega)]. \tag{17}$$

As before (see the proof of Proposition 4) for $t \in (0, 1)$ small such that $t\bar{u}_1(\beta)(z) \leqslant \delta$, since $q < 2 < r$, we have

$$\begin{aligned}
&\hat{\varphi}_+(t\hat{u}_1(\beta)) < 0,\\
&\quad \Rightarrow \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \text{ (see (17))}, \quad \text{hence } u_0 \neq 0.
\end{aligned}$$

From (17) we have

$$\begin{aligned}
&\hat{\varphi}'_+(u_0) = 0,\\
&\quad \Rightarrow \langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \mu)u_0 h dz + \int_{\partial\Omega} \beta(z)u_0 h d\sigma_0 = \int_\Omega \hat{f}_+(z, u_0)hdz\\
&\qquad \text{for all } h \in H^1(\Omega).
\end{aligned} \tag{18}$$

In (18), first we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\begin{aligned}
&\sigma(u_0^-) + \mu||u_0^-||_2^2 = 0,\\
&\quad \Rightarrow c_0||u_0^-||^2 \leqslant 0 \text{ (see (4))}, \quad \text{hence } u_0 \geqslant 0,\ u_0 \neq 0.
\end{aligned}$$

Next in (17) we choose $h = (u_0 - w_+)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\langle A(u_0), (u_0 - w_+)^+\rangle + \int_\Omega (\xi(z) + \mu)u_0(u_0 - w_+)^+ dz\\
&\quad + \int_{\partial\Omega} \beta(z)u_0(u_0 - w_+)^+ d\sigma_0\\
&= \int_\Omega [f(z, w_+) + \mu w_+](u_0 - w_+)^+ dz \quad \text{(see (16))},\\
&\leqslant \left\langle A(w_+), (u_0 - w_+)^+\right\rangle + \int_\Omega (\xi(z) + \mu)w_+(u_0 - w_+)^+ dz\\
&\quad + \int_{\partial\Omega} \beta(z)u_0(u_0 - w_+)^+ d\sigma_0 \quad \text{(see hypotheses } H_1(i))\\
&\Rightarrow \sigma((u_0 - w_+)^+) + \mu||(u_0 - w_+)^+||_2^2 \leqslant 0,\\
&\Rightarrow c_0||(u_0 - w_+)^+||^2 \leqslant 0 \text{ (see (4))},\\
&\Rightarrow u_0 \leqslant w_+.
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, w_+] = \{u \in H^1(\Omega) : 0 \leqslant u(z) \leqslant w_+(z) \quad \text{for almost all } z \in \Omega\}. \tag{19}$$

Because of (16) and (19), equation (18) becomes

$$\langle A(u_0), h\rangle + \int_\Omega \xi(z) u_0 h dz + \int_{\partial\Omega} \beta(z) u_0 h d\sigma_0 = \int_\Omega f(z, u_0) h dz$$
$$\text{for all } h \in H^1(\Omega),$$
$$\Rightarrow u_0 \text{ is a positive solution of problem (1) (see Papageorgiou and Rădulescu [23]).}$$

We set

$$k_0(z) = \begin{cases} \dfrac{f(z, u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0 \\ 0 & \text{if } u_0(z) = 0. \end{cases}$$

From (19), Proposition 5 and hypothesis $H_1(i)$, we see that $k_0 \in L^\infty(\Omega)$. We have

$$\left\{ \begin{array}{ll} -\Delta u_0(z) = (k_0(z) - \xi(z)) u_0(z) & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{20}$$

Note that $k_0 - \xi \in L^s(\Omega)$ (see hypothesis $H(\xi)$). Then Lemma 5.1 in Wang [29] implies that $u_0 \in L^\infty(\Omega)$ and so $\Delta u_0 \in L^s(\Omega)$. Then Lemma 5.2. of Wang [29] implies that $u_0 \in W^{2,s}(\Omega)$. Since $s > N$ (see hypothesis $H(\xi)$), from the Sobolev embedding theorem, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1+\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0,$$
$$\Rightarrow u_0 \in C_+ \backslash \{0\}.$$

Hypotheses $H_1(ii), (iii)$ imply that there exists $\eta_+ > 0$ such that

$$f(z, x) + \eta_+ x \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } x \in [0, \rho]. \tag{21}$$

Then we have

$$-\Delta u_0(z) + (\xi(z) + \eta_+) u_0(z) \geqslant 0 \quad \text{for almost all } z \in \Omega \text{ (see (21))},$$
$$\Rightarrow \Delta u_0(z) \leqslant (||\xi^+||_\infty + \eta_+) u_0(z) \quad \text{for almost all } z \in \Omega \text{ (see hypothesis } H(\xi)),$$
$$\Rightarrow u_0 \in \operatorname{int} C_+ \text{ (by the strong maximum principle).}$$

To produce the negative solution, we consider the Carathéodory function

$$\hat{f}_-(z, x) = \begin{cases} f(z, w_-(z)) + \mu w_-(z) & \text{if } x < w_-(z) \\ f(z, x) + \mu x & \text{if } w_-(z) \leqslant x \leqslant 0 \\ 0 & \text{if } 0 < x. \end{cases} \tag{22}$$

We set $\hat{F}_-(z,x) = \int_0^x \hat{f}_-(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_- : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_-(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega \hat{F}_-(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Reasoning as in the first part of the proof, using this time (22) and the functional $\hat{\varphi}_-$, we produce a negative solution $v_0 \in -\text{int}\, C_+$. □

Remark 4 So, from this proposition and its proof, we have

$$\emptyset \neq S_+ \subseteq [0, w_+] \cap \text{int}\, C_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq [w_-, 0] \cap (-\text{int}\, C_+).$$

As in Filippakis and Papageorgiou [9] (see also Motreanu et al. [17, p. 421]), we can show that the set S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leqslant u_1, u \leqslant u_2$) and the set S_- is upward directed (that is, if $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v_1 \leqslant v, v_2 \leqslant v$). Moreover, since both sets are bounded, the infimum of S_+ and the supremum of S_- can be taken over countable sets (see Dunford and Schwartz [6, p. 336] and Hu and Papageorgiou [12, p. 178]).

Next we will produce extremal constant sign solutions, that is, the smallest positive solution $u_* \in \text{int}\, C_+$ and the biggest negative solution $v_* \in -\text{int}\, C_+$. Subsequently these extremal constant sign solutions will lead to the existence of a nodal solution.

Proposition 7 *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then problem (1) has a smallest positive solution $u_* \in int\, C_+$ and a biggest negative solution $v_* \in -int\, C_+$.*

Proof From Dunford and Schwartz [6, p. 336] and Hu and Papageorgiou [12, p. 178], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

We have for all $n \geqslant 1$ and for all $h \in H^1(\Omega)$

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma = \int_\Omega f(z,u_n)hdz. \tag{23}$$

In (23) we choose $h = u_n \in H^1(\Omega)$. Then

$$\begin{aligned}
&\sigma(u_n) \leqslant c_4 \text{ for all } n \geqslant 1 \text{ and some } c_4 > 0 \\
&\qquad \text{(recall } 0 \leqslant u_n \leqslant w_+ \text{ for all } n \geqslant 1 \text{ and see hypothesis } H_1(ii)) \\
&\Rightarrow \sigma(u_n) + \mu||u_n||_2^2 \leqslant c_5 \text{ for some } c_5 > 0, \text{ all } n \geqslant 1, \\
&\Rightarrow c_0||u_n||^2 \leqslant c_5 \text{ for all } n \geqslant 1 \text{ (see (4))}, \\
&\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded.}
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } H^1(\Omega) \quad \text{and} \quad u_n \to u_* \quad \text{in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{24}$$

Passing to the limit as $n \to \infty$ in (23) and using (24), we obtain

$$\langle A(u_*), h\rangle + \int_\Omega \xi(z)u_* h dz + \int_{\partial\Omega} \beta(z)u_* h d\sigma_0 = \int_\Omega f(z, u_*)h dz \text{ for all } h \in H^1(\Omega). \tag{25}$$

From Proposition 5, we know that

$$\begin{aligned} &\bar{u} \leqslant u_n \quad \text{for all } n \geqslant 1, \\ &\Rightarrow \bar{u} \leqslant u_*. \end{aligned} \tag{26}$$

From (25) and (26), it follows that

$$\begin{aligned} &u_* \in S_+ \text{ and } u_* = \inf S_+ \text{ (see Papageorgiou and Rădulescu [23])}, \\ &\Rightarrow u_* \in \operatorname{int} C_+ \text{ is the smallest positive solution of problem (1).} \end{aligned}$$

Similarly we produce

$$v_* \in S_- \text{ with } v_* = \sup S_-,$$

the biggest negative solution of problem (1). □

Using these extremal constant solutions, we can produce nodal (sign changing) solutions. To this end, we consider the following truncation-perturbation of the reaction $f(z, \cdot)$:

$$k(z, x) = \begin{cases} f(z, w_-(z)) + \mu w_-(z) & \text{if } x < w_-(z) \\ f(z, x) + \mu x & \text{if } w_-(z) \leqslant x \leqslant w_+(z) \\ f(z, w_+(z)) + \mu w_+(z) & \text{if } w_+(z) < x. \end{cases} \tag{27}$$

This is a Carathéodory function. Let $K(z, x) = \int_0^x k(z, s)ds$ and consider the C^1-functional $\vartheta : H^1(\Omega) \to \mathbb{R}$ defined by

$$\vartheta(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega K(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

Next we compute the critical groups of ϑ at the origin. Our result extends that of Moroz [16], who did a similar computation but under stronger hypotheses on the function $f(z, x)$ and for the space $H_0^1(\Omega)$. In the space $H_0^1(\Omega)$ the Poincaré inequality simplifies the argument.

Proposition 8 *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold and the critical set K_ϑ is finite, then $C_k(\vartheta, 0) = 0$ for all $k \geqslant 0$.*

Proof From hypothesis $H_1(iii)$ and (27), we have

$$K(z,x) \geqslant \frac{c_1}{q}|x|^q - c_6|x|^r \text{ for almost all } z \in \Omega, \quad \text{all } x \in \mathbb{R}, \text{ with } c_6 > 0. \tag{28}$$

Let $u \in H^1(\Omega)$ and $t \in (0,1)$. Then

$$\begin{aligned} \vartheta(tu) &= \frac{t^2}{2}\sigma(u) + \frac{\mu t^2}{2}||u||_2^2 - \int_\Omega K(z,tu)dz \\ &\leqslant \frac{t^2}{2}\sigma(u) + \frac{\mu t^2}{2}||u||_2^2 - \frac{c_1 t^q}{q}||u||_q^q + c_6 t^r ||u||_r^r \text{ (see (28)).} \end{aligned} \tag{29}$$

Since $1 < q < 2 < r$, from (29) it follows that we can find $t^* = t^*(u) \in (0,1)$ small such that

$$\vartheta(tu) < 0 \quad \text{for all } t \in (0, t^*). \tag{30}$$

Let $u \in H^1(\Omega)$ with $0 < ||u|| \leqslant 1$ and $\vartheta(u) = 0$. Then

$$\begin{aligned} \frac{d}{dt}\vartheta(tu)|_{t=1} &= \langle \vartheta'(u), u\rangle \text{ (by the chain rule)} \\ &= \langle A(u), u\rangle + \int_\Omega (\xi(z)+\mu)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma_0 - \int_\Omega k(z,u)u dz \\ &= \left(1-\frac{q}{2}\right)\sigma(u) + \int_\Omega [qK(z,u) - k(z,u)u]dz + \left(1-\frac{q}{2}\right)\mu||u||_2^2 \\ &\quad \text{(since } \vartheta(u) = 0) \end{aligned} \tag{31}$$

Hypotheses $H_1(ii)$, (iii) imply that

$$qK(z,x) - k(z,x)x \geqslant -c_7|x|^r \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{32}$$

Using (32) in (31), we obtain

$$\begin{aligned} \frac{d}{dt}\vartheta(u)|_{t=1} &\geqslant \left(1-\frac{q}{2}\right)(\sigma(u) + \mu||u||_2^2) - c_8||u||^r \text{ for some } c_8 > 0 \\ &\geqslant c_9||u||^2 - c_8||u||^r \text{ with } c_9 = \left(1-\frac{q}{2}\right)c_0 > 0 \text{ (recall } q < 2). \end{aligned} \tag{33}$$

Since $2 < r$, from (33) we see that we can find $\rho \in (0,1)$ small such that

$$\frac{d}{dt}\vartheta(tu)|_{t=1} > 0 \quad \text{for all } u \in H^1(\Omega) \text{ with } 0 < ||u|| \leqslant \rho, \quad \vartheta(u) = 0. \tag{34}$$

We fix $u \in H^1(\Omega)$ with $0 < ||u|| \leqslant \rho$ and $\varphi(u) = 0$. We claim that

$$\vartheta(tu) \leqslant 0 \quad \text{for all } t \in [0,1]. \tag{35}$$

We argue by contradiction. So, suppose that we can find $t_0 \in (0, 1)$ such that $\vartheta(t_0 u) > 0$. Since $\vartheta(u) = 0$ and $\vartheta(\cdot)$ is continuous, from Bolzano's theorem we have

$$t_* = \min\{t \in [t_0, 1] : \vartheta(tu) = 0\} > t_0 > 0.$$

Then

$$\vartheta(tu) > 0 \quad \text{for all } t \in [t_0, t_*). \tag{36}$$

We set $v = t_* u$. Then $0 < ||v|| \leqslant ||u|| \leqslant \rho$ and $\vartheta(v) = 0$. Then from (34) we have

$$\frac{d}{dt}\vartheta(tv)|_{t=1} > 0. \tag{37}$$

From (36) we have

$$\begin{aligned} \vartheta(v) &= \vartheta(t_* u) = 0 < \vartheta(tu) \quad \text{for all } t \in [t_0, t_*), \\ &\Rightarrow \frac{d}{dt}\vartheta(tv)|_{t=1} = t_* \frac{d}{dt}\vartheta(tu)|_{t=t_*} = t_* \lim_{t \to t_*^-} \frac{\vartheta(tu)}{t - t_*} \leqslant 0. \end{aligned} \tag{38}$$

Comparing (37) and (38), we reach a contradiction. This proves (35) for all $u \in H^1(\Omega)$ with $0 < ||u|| \leqslant \rho$ and $\vartheta(u) = 0$.

Also, we have

$$\vartheta(tu) < 0 \quad \text{for all } t \in (0, 1) \text{ and all } u \in H^1(\Omega), \quad 0 < ||u|| \leqslant \rho, \quad \vartheta(u) < 0. \tag{39}$$

Indeed, note that due to the continuity of ϑ, we can find $s \in (0, 1)$ such that

$$\vartheta(tu) < 0 \quad \text{for all } t \in (1 - s, 1].$$

Suppose that there exists $t_0 \in (0, 1 - s]$ such that $\vartheta(t_0 u) = 0$ and $\vartheta(tu) < 0$ for all $t \in (t_0, 1]$. Let $u_0 = t_0 u$. Then $0 < ||u_0|| \leqslant \rho$ and $\vartheta(u_0) = 0$. So, from (34) we have

$$\frac{d}{dt}\vartheta(tu_0)|_{t=1} > 0 \tag{40}$$

On the other hand, we have

$$\begin{aligned} \vartheta(tu) &= \vartheta(tu) - \vartheta(t_0 u) < 0 \quad \text{for all } t \in (t_0, 1], \\ &\Rightarrow \frac{d}{dt}\vartheta(tu)|_{t=t_0} = \frac{d}{dt}\vartheta(tu_0)|_{t=1} \leqslant 0, \end{aligned}$$

which contradicts (40). Therefore (39) holds.

We can always choose $\rho \in (0, 1)$ small such that $K_\vartheta \cap \overline{B}_\rho = \{0\}$. Let $h : [0, 1] \times (\vartheta^0 \cap \overline{B}_\rho) \to \vartheta^0 \cap \overline{B}_\rho$ be the deformation defined by

$$h(t, u) = (1 - t)u \quad \text{for all } t \in [0, 1], \quad \text{all } u \in \vartheta^0 \cap \overline{B}_\rho.$$

From (35) and (39), we see that this deformation is well-defined. Moreover, with this deformation we show that

$$\vartheta^0 \cap \overline{B}_\rho \quad \text{is contractible in itself.} \tag{41}$$

Let $u \in \overline{B}_\rho$ with $\vartheta(u) > 0$. We claim that there exists unique $t(u) \in (0, 1)$ such that

$$\vartheta(t(u)u) = 0. \tag{42}$$

From (30) and using Bolzano's theorem, we see that such a $t(u) \in (0, 1)$ exists. So, we have to show the uniqueness of $t(u)$. Arguing by contradiction suppose we can find $0 < t_1 = t(u)_1 < t_2 = t(u)_2 < 1$ such that $\vartheta(t_1 u) = \vartheta(t_2 u) = 0$.

From (35), we have

$$\begin{aligned} \vartheta(tt_2 u) &\leqslant 0 \quad \text{for all } t \in [0, 1], \\ &\Rightarrow \frac{t_1}{t_2} \in (0, 1) \text{ is a maximizer of } t \longmapsto \vartheta(tt_2 u), \\ &\Rightarrow \frac{t_1}{t_2} \frac{d}{dt}\vartheta(tt_2, u)|_{t=\frac{t_1}{t_2}} = \frac{d}{dt}\vartheta(tt_1 u)|_{t=1} = 0, \end{aligned}$$

which contradicts (34). This proves the uniqueness of $t(u) \in (0, 1)$ in (42). We have

$$\vartheta(tu) < 0 \quad \text{for all } t \in (0, t(u)) \qquad \text{and} \qquad \vartheta(tu) > 0 \quad \text{for all } t \in (t(u), 1].$$

We consider the function $\eta_1 : \overline{B}_\rho \backslash \{0\} \to (0, 1]$ defined by

$$\eta_1(u) = \begin{cases} 1 & \text{if } u \in \overline{B}_\rho \backslash \{0\}, \quad \vartheta(u) \leqslant 0 \\ t(u) & \text{if } u \in \overline{B}_\rho \backslash \{0\}, \quad \vartheta(u) > 0. \end{cases}$$

It is straightforward to check that η_1 is continuous. Then consider the map $\eta_2 : \overline{B}_\rho \backslash \{0\} \to (\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}$ defined by

$$\eta_2(u) = \begin{cases} u & \text{if } u \in \overline{B}_\rho \backslash \{0\}, \quad \vartheta(u) \leqslant 0 \\ \eta_1(u)u & \text{if } u \in \overline{B}_\rho \backslash \{0\}, \quad \vartheta(u) > 0. \end{cases}$$

Then η_2 is continuous and

$$\eta_2|_{(\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}} = id|_{(\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}}.$$

So, we have that $(\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}$ is a retract of $\overline{B}_\rho \backslash \{0\}$ and the latter is contractible. Therefore $(\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}$ is contractible. This fact, (41) and Proposition 4.9 and 4.10 of Granas and Dugundji [11, p. 389], imply

$$\begin{aligned} &H_k(\vartheta^0 \cap \overline{B}_\rho, (\vartheta^0 \cap \overline{B}_\rho) \backslash \{0\}) = 0 \quad \text{for all } k \geqslant 0, \\ &\Rightarrow C_k(\vartheta, 0) = 0 \quad \text{for all } k \geqslant 0. \end{aligned}$$

The proof is now complete. □

Now we are ready to produce nodal solutions. In what follows, $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are the extremal constant sign solutions of problem (1) produced in Proposition 7.

Proposition 9 *If hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold, then problem* (1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}).$$

Proof Let $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of (1) produced in Proposition 7. We introduce the following truncation-perturbation of the reaction $f(z, \cdot)$:

$$\tau(z, x) = \begin{cases} f(z, v_*(z)) + \mu v_*(z) & \text{if } x < v_*(z) \\ f(z, x) + \mu x & \text{if } v_*(z) \leqslant x \leqslant u_*(z) \\ f(z, u_*(z)) + \mu u_*(z) & \text{if } u_*(z) < x. \end{cases} \tag{43}$$

This is a Carathéodory function. Let $T(z, x) = \int_0^x \tau(z, s)ds$ and consider the C^1-functional $\psi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\tau(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega T(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

In addition, we consider the positive and negative truncation of $\tau(z, \cdot)$, that is, the Carathéodory functions

$$\tau_\pm(z, x) = \tau(z, \pm x^\pm).$$

We set $T_\pm(z, x) = \int_0^x \tau_\pm(z, s)ds$ and consider the C^1-functionals $\psi_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{2}\sigma(u) + \frac{\mu}{2}||u||_2^2 - \int_\Omega T_\pm(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

Claim 1 *We have*

$$K_\psi \subseteq [v_*, u_*], \quad K_{\psi_+} = \{0, u_*\}, \quad K_{\psi_-} = \{0, v_*\}.$$

Let $u \in K_\psi$. Then we have

$$\begin{aligned} \psi'(u) = 0 \Rightarrow \langle A(u), h\rangle + \int_\Omega (\xi(z) + \mu)uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma_0 \\ = \int_\Omega \tau(z, u)hdz, \ \forall h \in H^1(\Omega). \end{aligned} \tag{44}$$

In (44) we choose $h = (u - u_*)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\langle A(u), (u-u_*)^+\rangle + \int_\Omega (\xi(z)+\mu)u(u-u_*)^+dz + \int_{\partial\Omega} \beta(z)u(u-u_*)^+d\sigma_0 \\
&= \int_\Omega [f(z,u_*)+\mu u_*](u-u_*)^+dz \text{ (see (43))} \\
&= \langle A(u_*), (u-u_*)^+\rangle + \int_\Omega (\xi(z)+\mu)u_*(u-u_*)^+dz + \int_{\partial\Omega} \beta(z)u_*(u-u_*)^+d\sigma_0 \\
&\quad \text{(recall that } u_* \in S_+) \\
&\Rightarrow \sigma((u-u_*)^+) + \mu||(u-u_*)^+||_2^2 = 0, \\
&\Rightarrow c_0||(u-u_*)^+||^2 \leqslant 0 \text{ (see (4))}, \\
&\Rightarrow u \leqslant u_*.
\end{aligned}$$

Similarly, choosing $h = (v_* - u)^+ \in H^1(\Omega)$ in (44), we show that

$$\begin{aligned}
&v_* \leqslant x, \\
&\Rightarrow K_\psi \subseteq [v_*, u_*] \text{ (since } u \in K_\psi \text{ is arbitrary).}
\end{aligned}$$

In a similar fashion, we show that

$$K_{\psi_+} \subseteq [0, u_*] \quad \text{and} \quad K_{\psi_-} \subseteq [v_*, 0].$$

The extremality of the constant sign solutions $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ implies that

$$K_{\psi_+} = \{0, u_*\} \quad \text{and} \quad K_{\psi_-} = \{0, v_*\}.$$

This proves Claim 1.

Claim 2 *$u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of ψ.*

From (4) and (39), it is clear that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we ca find $\hat{u}_* \in H^1(\Omega)$ such that

$$\psi_+(\hat{u}_*) = \inf[\psi_+(u) : u \in H^1(\Omega)]. \tag{45}$$

Using hypothesis $H_1(iii)$ and since $q < 2 < r$, we see that for $t \in (0, 1)$ small we have

$$\begin{aligned}
&\psi_+(t\hat{u}_1(\beta)) < 0, \\
&\Rightarrow \psi_+(\hat{u}_*) < 0 = \psi_+(0) \text{ (see (45))}, \quad \text{hence } \hat{u}_* \neq 0.
\end{aligned}$$

From (45) we have

$$\begin{aligned}
&\hat{u}_* \in K_{\psi_+} \setminus \{0\}, \\
&\Rightarrow \hat{u}_* = u_* \in \operatorname{int} C_+ \text{ (see Claim 1).}
\end{aligned}$$

Note that $\psi|_{C_+} = \psi_+|_{C_+}$. So, $u_* \in \operatorname{int} C_+$ is a local $C^1(\overline{\Omega})$-minimizer of ψ. Then using Proposition 3 in Papageorgiou and Rădulescu [23] (generalized version of the classical result established by Brezis and Nirenberg [4]), u_0 is also a local $H^1(\Omega)$-minimizer of ψ.

Similarly for $v_* \in -\operatorname{int} C_+$ using this time the functional ψ_-.

This proves Claim 2.

Due to (43) and the extremality of $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$, every nontrivial critical point of ψ distinct from u_* and v_* is necessarily a nodal solution of (1) (see Claim 1). So, we may assume that K_φ is finite. Also, without any loss of generality, we may assume that $\psi(v_*) \leqslant \psi(u_*)$ (the reasoning is similar if the opposite inequality holds). By Claim 2, $u_* \in \operatorname{int} C_+$ is a local minimizers of ψ. So, we can find $\rho \in (0, 1)$ small such that

$$\psi(v_*) \leqslant \psi(u_*) < \inf[\psi(u) : ||u - u_*|| = \rho] = m_\rho, \quad ||v_* - u_*|| > \rho \tag{46}$$

(see Aizicovici et al. [1], proof of Proposition 29). Recall that ψ is coercive. So, it satisfies the PS-condition. This fact and (46) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_\psi \subseteq [v_*, u_*] \quad \text{(see Claim 1) and } m_\rho \leqslant \psi(y_0). \tag{47}$$

From (46) and (47), we see that $y_0 \notin \{v_*, u_*\}$. Therefore, if we can show that $y_0 \neq 0$, then y_0 is a nodal solution of (1). Since y_0 is a critical point of ψ of mountain pass type, we have

$$C_1(\psi, y_0) \neq 0 \quad \text{(see Chang [5, p. 89])}. \tag{48}$$

On the other hand, from Proposition 8, we know that

$$C_k(\psi, 0) = 0 \quad \text{for all } k \geqslant 0. \tag{49}$$

Comparing (48) and (49), we conclude that $y_0 \neq 0$. So, y_0 is a nodal solution of (1) and as before $y_0 \in C^1(\overline{\Omega})$. □

In fact we can improve the conclusion of Proposition 9, proved we strengthen a little the conditions on $f(z, \cdot)$. So, now we assume the following:

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function such that $f(z, 0) = 0$, hypotheses $H_2(i), (ii), (iii)$ are the same as the corresponding hypotheses $H_1(i), (ii), (iii)$ and

(iv) there exists $\hat{\vartheta} > 0$ such that for almost all $z \in \Omega$, $x \longmapsto f(z, x) + \hat{\vartheta} x$ is nondecreasing on $[-\rho, \rho]$.

Remark 5 Evidently this extra condition on $f(z, \cdot)$ is satisfied if for example, for almost all $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and $f'_x(z, \cdot)$ is $L^\infty(\Omega)$-bounded on $[-\rho, \rho]$.

Proposition 10 *If hypotheses $H(\xi)$, $H(\beta)$ and H_2 hold, then problem* (1) *admits a nodal solution*

$$y_0 \in int_{C^1(\overline{\Omega})}[v_*, u_*].$$

Proof From Proposition 9, we already have a nodal solution $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$. Let $\hat{\vartheta} > 0$ be as postulated by hypothesis $H_2(iv)$. Then

$$\begin{aligned}
&-\Delta y_0(z) + (\xi(z) + \hat{\vartheta}) y_0(z)\\
&\quad = f(z, u_*(z)) + \hat{\vartheta} y_0(z)\\
&\quad \leqslant f(z, u_*(z)) + \hat{\vartheta} u_*(z) \quad (\text{since } y_0 \leqslant x_*, \ \text{see hypothesis } H_2(iv))\\
&\quad = -\Delta u_*(z) + (\xi(z) + \hat{\vartheta}) u_*(z) \quad (\text{since } u_* \in S_+),\\
&\quad \Rightarrow \Delta(u_* - y_0)(z) \leqslant (||\xi^+||_\infty + \hat{\varphi})(u_* - y_0)(z) \quad \text{for almost all } z \in \Omega\\
&\qquad (\text{see hypothesis } H(\xi)),\\
&\quad \Rightarrow u_* - y_0 \in \operatorname{int} C_+ \ (\text{by the strong maximum principle}).
\end{aligned}$$

In a similar fashion, we show that

$$y_0 - v_* \in \operatorname{int} C_+.$$

Therefore finally we have $y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]$. □

Now we can formulate our first multiplicity result.

Theorem 11 *Assume that hypotheses $H(\xi)$, $H(\beta)$ and H_1 hold. Then problem (1) admits at least three nontrivial solutions*

$$u_0 \in \mathit{int}\, C_+, \quad v_0 \in -\mathit{int}\, C_+ \ \mathit{and} \ y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \ \mathit{nodal}.$$

Moreover, if hypotheses H_2 hold, then $y_0 \in int_{C^1(\overline{\Omega})}[v_0, u_0]$.

Next we modify the behavior of $f(z, \cdot)$ near zero and assume that $f(z, \cdot)$ is linear near zero. In this way we change the geometry of the problem. Nevertheless, for the new setting we prove again a three solutions theorem proving sign information for all the solutions.

The new hypotheses on the reaction $f(z, x)$ are the following:

H_3 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exist functions $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega},\\
&f(z, w_+(z)) - \xi(z) w_+(z) \leqslant 0 \leqslant f(z, w_-(z)) - \xi(z) w_-(z) \ \text{for almost all } z \in \Omega,\\
&A(w_-) \leqslant 0 \leqslant A(w_+) \ \text{in } H^1(\Omega)^*;
\end{aligned}$$

(ii) if $\rho = \max\{||w_+||_\infty, ||w_-||_\infty\}$, then there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega, \quad \text{all } |x| \leqslant \rho;$$

(iii) there exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1(\beta) \leqslant \eta(z) \text{ for almost all } z \in \Omega, \text{ strictly on a set of positive measure,}$$
$$\eta(z) \leqslant \liminf_{x \to 0} \frac{f(z,x)}{x} \leqslant \limsup_{x \to 0} \frac{f(z,x)}{x} \leqslant \hat{\eta}(z) \text{ uniformly for almost all } z \in \Omega.$$

Under this new geometry near zero for the reaction $f(z, \cdot)$, the previous results on the existence of constant sign solutions remain valid with very minor changes in their proofs. So, we have:

Proposition 12 *If hypotheses $H(\xi)$, $H(\beta)$ and H_3 hold, then problem (1) has at least two solutions of constant sign*

$$u_0 \in int\, C_+ \quad and \quad v_0 \in -int\, C_+.$$

Proof The proof is similar to that of Proposition 6. Again we consider the C^1-functional $\hat{\varphi}_+$ (see the proof of Proposition 6) and use the direct method. The only thing that differs in the present proof, is how we show that

$$\hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) : u \in H^1(\Omega)] < 0 = \hat{\varphi}_+(0). \tag{50}$$

By virtue of hypothesis $H_3(iii)$, given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) \in (0, c_+]$ such that

$$f(z,x) \geqslant (\eta(z) - \epsilon)x \quad \text{for almost all } z \in \Omega, \quad \text{all } x \in [0,\delta],$$
$$\Rightarrow F(z,x) \geqslant \frac{1}{2}(\eta(z) - \epsilon)x^2 \quad \text{for almost all } z \in \Omega, \quad \text{all } x \in [0,\delta]. \tag{51}$$

For $t \in (0,1)$ small such that $t\hat{u}_1(\beta)(z) \in (0,\delta]$ for all $z \in \overline{\Omega}$. Then

$$\hat{\varphi}_+(t\hat{u}_1(\beta)) \leqslant \frac{t^2}{2}\sigma(\hat{u}_1(\beta)) - \frac{t^2}{2}\int_\Omega \eta(z)\hat{u}_1(\beta)^2 dz + \frac{t^2\epsilon}{2}||\hat{u}_1(\beta)||_2^2 \quad \text{(see (51))}$$
$$= \frac{t^2}{2}\left(\int_\Omega [\hat{\lambda}_1(\beta) - \eta(z)]\hat{u}_1(\beta)^2 dz + \epsilon\right)$$
$$\text{(recall that } ||\hat{u}_1(\beta)||_2 = 1).$$

Since $\hat{u}_1(\beta) \in \text{int}\, C_+$, we have

$$\mathcal{I} = \int_\Omega (\eta(z) - \hat{\lambda}_1(\beta))\hat{u}_1(\beta)^2 dz > 0.$$

Then

$$\hat{\varphi}_1(t\hat{u}_1(\beta)) \leqslant \frac{t^2}{2}[-\mathcal{I} + \epsilon].$$

So, choosing $\epsilon \in (0, \mathcal{I})$, we infer that

$$\hat{\varphi}_+(t\hat{u}_1(\beta)) < 0,$$
$$\Rightarrow \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \quad \text{(see (50))}, \text{ hence } u_0 \neq 0.$$

Then as in proof of Proposition 6, we show that $u_0 \in \text{int}\, C_+$ is a solution of problem (1).

Similarly, working with the functional $\hat{\varphi}_-$ (see the proof of Proposition 6), we produce a negative solution $v_0 \in -\text{int}\, C_+$. □

In the present setting, we see that given $\epsilon > 0$, we can find $c_{10} = c_{10}(\epsilon) > 0$ such that

$$f(z, x)x \geqslant (\eta(z) - \epsilon)x^2 - c_{10}|x|^r \quad \text{for almost all } z \in \Omega, \text{ all } x \in [-\rho, \rho]. \tag{52}$$

This leads to the following auxiliary problem

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = (\eta(z) - \epsilon)u(z) - c_{10}|u(z)|^{r-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{53}$$

With essentially identical proofs to those of Propositions 4 and 5, we have the following result.

Proposition 13 *If hypotheses $H(\xi)$, $H(\beta)$ and H_3 hold, then problem (53) admits a unique positive solution $\bar{u} \in \text{int}\, C_+$, a unique negative solution $\bar{v} = -\bar{u} \in -\text{int}\, C_+$ and*

$$\bar{u} \leqslant u \textit{ for all } u \in S_+ \quad \textit{and} \quad v \leqslant \bar{v} \textit{ for all } v \in S_-.$$

As in Proposition 7, using this time Proposition 13, we generate extremal constant sign solutions.

Proposition 14 *If hypotheses $H(\xi)$, $H(\beta)$ and H_3 hold, then problem (1) admits extremal constant sign solutions, that is, there exists a smallest positive solution $u_* \in \text{int}\, C_+$ and a biggest negative solution $v_* \in -\text{int}\, C_+$.*

Then we can produce a nodal solution. To do this we need to strengthen further the condition on $f(z, \cdot)$ near zero, without altering the geometry of the problem.

So, the new hypotheses on $f(z, x)$, are the following:

H_4 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$, hypotheses $H_3(i)$, (ii) are the same as the corresponding hypotheses $H_3(i)$, (ii) and

(iii) there exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_2(\beta) < \eta(z) \text{ for almost all } z \in \Omega,$$
$$\eta(z) \leqslant \liminf_{x \to 0} \frac{f(z, x)}{x} \leqslant \limsup_{x \to 0} \frac{f(z, x)}{x} \leqslant \hat{\eta}(z) \text{ uniformly for almost all } z \in \Omega.$$

Proposition 15 *If hypotheses $H(\xi)$, $H(\beta)$ and H_4 hold, then problem* (1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}).$$

Proof We use the extremal constant sign solutions $u_* \in \text{int}\, C_+$ and $v_* \in -\text{int}\, C_+$ produced in Proposition 15 and argue as in the proof of Proposition 9. Then via the mountain pass theorem (see Theorem 1), we produce a solution

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}), \quad y_0 \neq \{v_*, u_*\}.$$

We need to show that $y_0 \neq 0$ to conclude that y_0 is nodal.

From Theorem 1, we have

$$m_\rho \leqslant \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \psi(\gamma(t)) = \psi(y_0), \tag{54}$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = v_*, \gamma(1) = u_*\}$ (see also (46). According to (54), in order to establish the nontriviality of y_0 and therefore conclude that y_0 is nodal, it suffices to produce a path $\gamma_* \in \Gamma$ such that $\psi|_{\gamma_*} < 0 = \psi(0)$. To this end, we consider the following Banach manifolds

$$M = H^1(\Omega) \cap \partial B_1^{L^2} \quad \text{and} \quad M_c = M \cap C^1(\overline{\Omega}).$$

Here $\partial B_1^{L^2} = \{u \in L^2(\Omega) : ||u||_2 = 1\}$. Evidently M_c is dense in M. We introduce the following sets of paths

$$\begin{aligned} \hat{\Gamma} &= \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1(\beta), \hat{\gamma}(1) = \hat{u}_1(\beta)\} \\ \hat{\Gamma}_c &= \{\hat{\gamma} \in C([-1,1], M_c) : \hat{\gamma}(-1) = -\hat{u}_1(\beta), \hat{\gamma}(1) = \hat{u}_1(\beta)\} \\ &\quad \text{(recall that } \hat{u}_1(\beta) \in \text{int}\, C_+). \end{aligned}$$

Claim 3 $\hat{\Gamma}_c$ *is dense in* $\hat{\Gamma}$.

Let $\hat{\gamma} \in \hat{\Gamma}$ and $\epsilon > 0$. We consider the multifunction $R_\epsilon : [-1,1] \to 2^{C^1(\overline{\Omega})}$ defined by

$$R_\epsilon(t) = \begin{cases} \{u \in C^1(\overline{\Omega}) : ||u - \hat{\gamma}(t)|| < \epsilon\} & \text{if } -1 < t < 1 \\ \{\pm \hat{u}_1(\beta)\} & \text{if } t = \pm 1. \end{cases}$$

Evidently $R_\epsilon(\cdot)$ has nonempty and convex values. In addition

$$\begin{aligned} &R_\epsilon(t) \text{ is open for all } t \in (-1,1), \\ &R_\epsilon(\pm 1) \text{ are singletons.} \end{aligned}$$

From Papageorgiou and Kyritsi [20, p. 458], we have that the multifunction $R_\epsilon(\cdot)$ is lower semicontinuous. So, we can apply Theorem 3.1''' of Michael [15] (see also

Hu and Papageorgiou [12, p. 97]) and find a continuous path $\hat{\gamma}_\epsilon : [-1, 1] \to C^1(\overline{\Omega})$ such that

$$\hat{\gamma}_\epsilon(t) \in R_\epsilon(t) \quad \text{for all } t \in [-1, 1].$$

Let $\epsilon_n = \frac{1}{n}$ and let $\hat{\gamma}_n = \hat{\gamma}_{\epsilon_n}$, $n \geqslant 1$ be as above. We have

$$||\hat{\gamma}_n(t) - \hat{\gamma}(t)|| < \frac{1}{n} \quad \text{for all } t \in [-1, 1], \text{ all } n \geqslant 1. \tag{55}$$

Since $\hat{\gamma}(t) \in \partial B_1^{L^2}$ for all $t \in [-1, 1]$, from (55) we see that $n \geqslant 1$ big, we have $||\hat{\gamma}_n(t)||_2 \neq 0$ for all $t \in [-1, 1]$. So, we may assume that $||\hat{\gamma}_n(t)||_2 \neq 0$ for all $t \in [-1, 1]$, all $n \geqslant 1$. We set

$$\hat{\gamma}_n^0(t) = \frac{\hat{\gamma}_n(t)}{||\gamma_n(t)||_2} \quad \text{for all } t \in [-1, 1], \text{ all } n \geqslant 1. \tag{56}$$

We have $\hat{\gamma}_n^0 \in C([-1, 1], M_c)$ and $\hat{\gamma}_n^0(\pm\hat{u}_1(\beta)) = \pm\hat{u}_1(\beta)$. From (55) and (56) we have

$$\begin{aligned} ||\hat{\gamma}_n^0(t) - \hat{\gamma}(t)|| &\leqslant ||\hat{\gamma}_n^0(t) - \hat{\gamma}_n(t)|| + ||\hat{\gamma}_n(t) - \hat{\gamma}(t)|| \\ &\leqslant \frac{|1 - ||\hat{\gamma}_n(t)||_2|}{||\hat{\gamma}_n(t)||_2} ||\hat{\gamma}_n(t)|| + \frac{1}{n} \text{ (see (55)).} \end{aligned} \tag{57}$$

We have

$$\begin{aligned} \max_{-1 \leqslant t \leqslant 1} |1 - ||\hat{\gamma}_n(t)||_2| &= \max_{-1 \leqslant t \leqslant 1} |||\hat{\gamma}(t)||_2 - ||\hat{\gamma}_n(t)||_2| \text{ (recall } \hat{\gamma}(t) \in \partial B_1^{L^1} \\ &\quad \text{for all } t \in [-1, 1]) \\ &\leqslant \max_{-1 \leqslant t \leqslant 1} ||\hat{\gamma}(t) - \hat{\gamma}_n(t)||_2 \text{ (by the triangle inequality)} \\ &\leqslant \max_{-1 \leqslant t \leqslant 1} ||\hat{\gamma}(t) - \hat{\gamma}_n(t)|| \leqslant \frac{1}{n} \text{ for all } n \geqslant 1 \text{ (see (55)).} \end{aligned} \tag{58}$$

From (57) and (58) it follows that

$$\begin{aligned} &\hat{\gamma}_n^0 \to \hat{\gamma} \text{ in } C([-1, 1], M) \text{ as } n \to \infty, \\ &\Rightarrow \hat{\Gamma}_c \text{ is dense in } \hat{\Gamma}. \end{aligned}$$

This proves Claim 3.

Invoking Proposition 2 and the Claim 3, given $\delta_0 > 0$, we can find $\hat{\gamma}_0 \in \hat{\Gamma}_c$ such that

$$\max_{-1 \leqslant t \leqslant 1} \sigma(\hat{\gamma}_0(t)) \leqslant \hat{\lambda}_2(\beta) + \delta_0. \tag{59}$$

Since $\hat{\gamma}_0 \in \hat{\Gamma}_c$ and $u_* \in \operatorname{int} C_+,\ v_* \in -\operatorname{int} C_+$, we can find $\vartheta \in (0,1)$ small such that

$$\begin{aligned}&\vartheta \hat{\gamma}_0(t) \in [v_*, u_*] \text{ and } \vartheta|\hat{\gamma}_0(t)(z)|, \quad \vartheta|D\hat{\gamma}_0(t)(z)| \leqslant \delta \\ &\text{for all } t \in [-1,1],\ \text{all } z \in \overline{\Omega}.\end{aligned} \tag{60}$$

Here for the first inclusion in (60) we have used Lemma 3.3 of Filippakis et al. [8] and $\delta > 0$ is as in (51). Of course we can always take $\delta > 0$ such that $\delta \leqslant \min\{\min_{\overline{\Omega}} u_*, \min_{\overline{\Omega}} v_*\}$. We have

$$\begin{aligned}\psi(\vartheta\hat{\gamma}_0(t)) &= \frac{\vartheta^2}{2}\sigma(\hat{\gamma}_0(t)) + \frac{\mu\vartheta^2}{2}||\hat{\gamma}_0(t)||_2^2 - \int_\Omega T(z, \vartheta\hat{\gamma}_0(z))dz \\ &= \frac{\vartheta^2}{2}\sigma(\hat{\gamma}_0(t)) - \int_\Omega F(z, \vartheta\hat{\gamma}_0(t))dz \quad \text{(see (43) and (60))} \\ &\leqslant \frac{\vartheta^2}{2}\sigma(\hat{\gamma}_0(t)) - \frac{\vartheta^2}{2}\int_\Omega \eta(z)\hat{\gamma}_0(t)^2 dz + \frac{\vartheta^2\epsilon}{2} \\ &\qquad \text{(see (51) and (60) and recall that } ||\hat{\gamma}_0(t)||_2 = 1 \quad \text{for all } t \in [-1,1]) \\ &\leqslant \frac{\vartheta^2}{2}\left[\int_\Omega (\hat{\lambda}_2(\beta) - \eta(z))\hat{\gamma}_0(t)^2 dz + \epsilon\right] \quad \text{for all } t \in [-1,1].\end{aligned} \tag{61}$$

From hypothesis $H_4(iii)$, we have

$$\mathcal{I} = \int_\Omega [\eta(z) - \hat{\lambda}_2(\beta)]\hat{\gamma}_0(t)^2 dz > 0.$$

So, choosing $\epsilon \in (0, \mathcal{I}]$ from (61), we have

$$\psi(\vartheta\hat{\gamma}_0(t)) < 0 \quad \text{for all } t \in [-1,1].$$

We set $\hat{\gamma} = \vartheta\hat{\gamma}_0$. Then this is a continuous path in $H^1(\Omega)$ joining $-\vartheta\hat{u}_1(\beta)$ and $\vartheta\hat{u}_1(\beta)$ and

$$\psi|_{\hat{\gamma}} < 0. \tag{62}$$

Next we produce a continuous path in $H^1(\Omega)$ joining $\vartheta\hat{u}_1(\beta)$ and u_* and along which the functional ψ is negative. To this end, recall that

$$\tau = \psi_+(u_*) = \inf_{H^1(\Omega)^+} \psi_+ < 0 = \psi_+(0), \tag{63}$$

$$K_{\psi_+} = \{0, u_*\} \text{ (see Claim 1 in the proof of Proposition 9)}, \tag{64}$$

ψ_+ satisfies the PS-condition (being coercive, see (4) and (43)).

So, we can apply the second deformation theorem (see Gasinski and Papageorgiou [10, p. 628]) and produce a deformation $h : [0, 1] \times (\psi_+^0 \backslash K_{\psi_+}^0) \to \psi_+^0$ such that

$$h(0, u) = u \quad \text{for all } u \in \psi_+^0 \backslash K_{\psi_+}^0, \tag{65}$$

$$h(1, \psi_+^0 \backslash K_{\psi_+}^0) \subseteq \psi_+^\tau = \{u_*\} \quad \text{(see (63) and (64))} \tag{66}$$

$$\psi_+(h(t, u)) \leqslant \psi_+(h(s, u)) \quad \text{for all } s, t \in [0, 1], s \leqslant t, \text{ all } u \in \psi_+^0 \backslash K_{\psi_+}^0. \tag{67}$$

We have

$$\begin{aligned} \psi_+(\vartheta \hat{u}_1(\beta)) &= \psi(\vartheta \hat{u}_1(\beta)) = \psi(\hat{\gamma}(1)) < 0 \quad \text{(see (62))}, \\ &\Rightarrow \vartheta \hat{u}_1(\beta) \in \psi_+^0 \backslash K_{\psi_+}^0. \end{aligned} \tag{68}$$

Therefore we can define

$$\hat{\gamma}_+(t) = h(t, \vartheta \hat{u}_1(\beta))^+ \text{ for all } t \in [0, 1]. \tag{69}$$

We have

$$\begin{aligned} &\hat{\gamma}_+(0) = h(0, \vartheta \hat{u}_1(\beta))^+ = \vartheta \hat{u}_1(\beta) \quad \text{(see (65) and recall that } \hat{u}_1(\beta) \in \operatorname{int} C_+), \\ &\hat{\gamma}_+(1) = h(1, \vartheta \hat{u}_1(\beta))^+ = u_* \quad \text{(see (66) and recall that } u_* \in \operatorname{int} C_+), \\ &\psi_+(\hat{\gamma}_+(t)) \leqslant \psi_+(\vartheta \hat{u}_1(\beta)) < 0 \quad \text{for all } t \in [0, 1] \text{ (see (67) and (68)).} \end{aligned}$$

So, $\hat{\gamma}_+$ is a continuous path in $H^1(\Omega)$ joining $\vartheta \hat{u}_1(\beta)$ and u_* and such that

$$\psi_+|_{\hat{\gamma}_+} < 0.$$

From (69) we see that $\hat{\gamma}_+(t)(z) \geqslant 0$ for almost all $z \in \Omega$, all $t \in [0, 1]$. Hence

$$\psi|_{\hat{\gamma}_+} < 0. \tag{70}$$

Similarly, we produce another continuous path $\hat{\gamma}_-$ in $H^1(\Omega)$ which joins $-\vartheta \hat{u}_1(\beta)$ and v_* and such that

$$\psi|_{\hat{\gamma}_-} < 0. \tag{71}$$

We concatenate $\hat{\gamma}_-, \hat{\gamma}, \hat{\gamma}_+$ and generate $\gamma_* \in \Gamma$ such that

$$\begin{aligned} &\psi|_{\gamma_*} < 0 \text{ (see (62), (70), (71))}, \\ &\Rightarrow y_0 \neq 0 \text{ and so } y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}) \text{ is a nodal solution of (1).} \end{aligned}$$

This completes the proof. □

Again, if we strengthen the conditions on $f(z, \cdot)$, we can improve the conclusion of the above proposition.

The new conditions on the reaction $f(z, x)$, are the following:

H_5 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses $H_5(i), (ii), (iii)$ are the same as the corresponding hypotheses $H_4(i), (ii), (iii)$ and

(iv) there exists $\hat{\vartheta} > 0$ such that for almost all $z \in \Omega$, the mapping $x \to f(z,x) + \hat{\vartheta} x$ is nondecreasing on $[-\rho, \rho]$.

This is a mild extra requirement on the reaction $f(z,\cdot)$ and it is satisfied if for example $f(z,\cdot)$ is differentiable and $f'_x(z,\cdot)$ is L^∞-bounded on $[-\rho, \rho]$. This hypothesis leads to strong comparison results for v_*, y_0, u_*. Indeed we have

$$\begin{aligned}
&-\Delta u_*(z) + (\xi(z) + \hat{\vartheta})u_*(z) = f(z, u_*(z)) + \hat{\vartheta} u_*(z) \quad \text{for a.a. } z \in \Omega,\\
&\quad -\Delta y_0(z) + (\xi(z) + \hat{\vartheta}) y_0(z) = f(z, y_0(z)) + \hat{\vartheta} y_0(z) \quad \text{for a.a. } z \in \Omega,\\
&\Longrightarrow \Delta(u_* - y_0)(z) \leqslant (\xi(z) + \hat{\vartheta})(u_* - y_0)(z) \quad \text{for a.a. } z \in \Omega,\\
&\Longrightarrow \Delta(u_* - y_0)(z) \leqslant (\|\xi\|_\infty + \hat{\vartheta})(u_* - y_0)(z) \quad \text{for a.a. } z \in \Omega,\\
&\Longrightarrow u_* - y_0 \in \operatorname{int} C_+.
\end{aligned}$$

Similarly, we show that $y_0 - v_* \in \operatorname{int} C_+$.

Therefore we can improve the conclusion of Proposition 15. We will need this stronger result in Sect. 4.

Proposition 16 *If hypotheses $H(\xi)$, $H(\beta)$ and H_4 hold, then problem* (1) *admits a nodal solution*

$$y_0 \in int_{C^1(\overline{\Omega})}[v_*, u_*].$$

So, now we can formulate our second multiplicity theorem.

Theorem 17 *If hypotheses $H(\xi)$, $H(\beta)$ and H_5 hold, then problem* (1) *admits at least three nontrivial solutions*

$$u_0 \in int\, C_+, \quad v_0 \in -int\, C_+ \text{ and } y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal}.$$

Moreover, if hypotheses H_4 hold, then $y_0 \in int_{C^1(\overline{\Omega})}[v_0, u_0]$.

Note that in this multiplicity result at zero we avoid any interaction with $\hat{\lambda}_2(\beta)$ (see hypothesis $H_4(iii)$). We can allow partial interaction (nonuniform nonresonance) at the expense of strengthening the behavior of $f(z,\cdot)$ near $x = 0$.

So, we impose the following conditions on the reaction $f(z,x)$.

H_6 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) there exist functions $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega},\\
&f(z, w_+(z)) - \xi(z) w_+(z) \leqslant 0 \leqslant f(z, w_-(z)) - \xi(z) w_-(z) \quad \text{for almost all } z \in \Omega\\
&A(w_-) \leqslant 0 \leqslant A(w_+) \text{ in } H^1(\Omega)^*;
\end{aligned}$$

(ii) if $\rho = \max\{||w_+||_\infty, ||w_-||_\infty\}$, then there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(iii) $f(z,\cdot)$ is locally Lipschitz and differentiable at $x = 0$ and

$$\eta(z) \leqslant f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \text{ uniformly for almost all } z \in \Omega$$

with $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant \hat{\lambda}_2(\beta)$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure.

So, we see from hypothesis $H_6(iii)$ that now at zero we allow partial interaction with $\hat{\lambda}_2(\beta)$ (nonuniform nonresonance), while in hypothesis $H_4(iii)$ we required uniform nonresonance (recall that in that hypothesis we had $\eta(z) > \hat{\lambda}_2(\beta)$ for a.a. $z \in \Omega$). The proof now changes and uses tools from Morse theory.

Proposition 18 *If hypotheses $H(\xi)$, $H(\beta)$ and H_6 hold, then problem* (1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega}).$$

Proof In this case there is a $C^1(\overline{\Omega})$-neighborhood D of $u = 0$ such that $\psi' \in C^1(D, H^1(\Omega))$ and $\psi''(0) \in \mathcal{L}(C^1(\Omega), H^1(\Omega))$. Using hypothesis $H_6(iii)$, we see that we can find $\delta \in (0,1)$ small such that

$$\psi(u) \leqslant 0 \quad \text{for all } u \in \bigoplus_{i=1}^{2} E(\hat{\lambda}_i(\beta)), \quad ||u|| \leqslant \delta.$$

On the other hand from (7) we see that we can have

$$\psi(u) > 0 \quad \text{for all } u \in \overline{\bigoplus_{i \geqslant 3} E(\hat{\lambda}_i(\beta))}, \quad ||u|| \leqslant \delta.$$

So, ψ has a local linking at $u = 0$, hence

$$C_{d_2}(\psi, 0) \neq 0 \quad \text{with } d_2 = \dim \bigoplus_{i=1}^{2} E(\hat{\lambda}_i(\beta)) \geqslant 2.$$

Invoking Proposition 3, we infer that

$$C_1(\psi, 0) = 0. \tag{72}$$

On the other hand, from the proof of Proposition 9 (see (48)), we have

$$C_1(\psi, y_0) \neq 0 \tag{73}$$

Comparing (72) and (73), we conclude that $y_0 \neq 0$ and so $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$ is a nodal solution for problem (1). □

4 Four nontrivial solutions

In this section we improve the regularity on $f(z,\cdot)$ and using Morse theory we are able to produce a second nodal solution, for a total of four nontrivial solutions all with precise sign information.

The new hypotheses on the reaction $f(z,x)$ are the following.

H_7 : $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that for almost all $z\in\Omega$ $f(z,0)=0,\ f(z,\cdot)\in C^1(\mathbb{R})$ and

(i) there exist functions $w_\pm\in H^1(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{aligned}&w_-(z)\leqslant c_-<0<c_+\leqslant w_+(z)\quad\text{for all } z\in\overline{\Omega},\\&f(z,w_+(z))-\xi(z)w_+(z)\leqslant 0\leqslant f(z,w_-(z))-\xi(z)w_-(z)\quad\text{for almost all } z\in\Omega,\\&A(w_-)\leqslant 0\leqslant A(w_+)\quad\text{in } H^1(\Omega)^*;\end{aligned}$$

(ii) if $\rho=\max\{||w_+||_\infty,||w_-||_\infty\}$, then there exists $a_\rho\in L^\infty(\Omega)_+$ such that

$$|f'_x(z,x)|\leqslant a_\rho(z)\quad\text{for almost all } z\in\Omega,\ |x|\leqslant\rho;$$

(iii) there exist an integer $m\geqslant 2$ and $\delta_0>0$ such that $\hat{\lambda}_m(\beta)x^2\leqslant f(z,x)x$ for almost all $z\in\Omega$, all $|x|\leqslant\delta_0$ if $m=2$, then the inequality is strict on a set of positive measure and $f'_x(z,0)=\lim_{x\to 0}\frac{f(z,x)}{x}\leqslant\hat{\lambda}_{m+1}(\beta)$ uniformly for almost all $z\in\Omega$ and the inequality is strict on a set of positive measure.

Remark 6 From hypothesis $H_7(ii)$ and the mean value theorem, we see that we can find $\hat{\vartheta}>0$ such that for almost all $z\in\Omega$, the mapping $x\longmapsto f(z,x)+\hat{\vartheta}x$ is nondecreasing on $[-\rho,\rho]$. We know that Morse theory is more effective in the framework of C^2-functionals. For this reason we strengthened the regularity of $f(z,\cdot)$.

Theorem 19 *If hypotheses $H(\xi)$, $H(\beta)$ and H_7 hold, then problem (1) admits at least four nontrivial solutions*

$$u_0\in int\,C_+,\, v_0\in -int\,C_+\quad and\quad y_0,\hat{y}\in int_{C^1(\overline{\Omega})}[v_0,u_0]\ nodal.$$

Proof From Theorem 17 and Proposition 18, we know that there are at least three nontrivial solutions

$$u_0\in \text{int}\,C_+,\, v_0\in -\text{int}\,C_+\quad\text{and}\quad y_0\in \text{int}\,C_{C^1(\overline{\Omega})}[v_0,u_0]\ \text{nodal}.$$

Let ψ be the functional introduced in the proof of Proposition 15. We have that $\psi\in C^{2-0}(H^1(\Omega))$. Hypothesis $H_7(iii)$ implies that given $\epsilon>0$, we can find $\delta=\delta(\epsilon)\,(-,\delta_0]$ such that

$$F(z,x)\leqslant\frac{1}{2}(f'_x(z,0)+\epsilon)x^2\quad\text{for almost all } z\in\Omega,\ \text{all } |x|\leqslant\delta. \tag{74}$$

Without any loss of generality, we may assume that $\delta_0 \leqslant \min\{\min_{\overline{\Omega}} u^*, \min_{\overline{\Omega}}(-v_*)\}$ (recall that $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$). Let $\widehat{H}_m = \oplus_{i \geqslant m+1} E(\hat{\lambda}_i(\beta))$ and let $u \in C^1(\overline{\Omega}) \cap \widehat{H}_m$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta$. Then

$$\begin{aligned}\psi(u) &= \frac{1}{2}\sigma(u) - \int_\Omega F(z,u)dz \quad \text{(see (43))}\\ &\geqslant \frac{1}{2}\sigma(u) - \frac{1}{2}\int_\Omega f_x'(z,0)u^2dz - \frac{\epsilon}{2}||u||_2^2 \quad \text{(see (74))}\\ &\geqslant \frac{1}{2}\left(c_{11} - \frac{\epsilon}{\hat{\lambda}_1(\beta)}\right)||u||^2 \text{ for some } c_{11} > 0 \\ &\qquad \text{(see hypothesis } H_7(iii) \text{ and (3)).}\end{aligned} \tag{75}$$

Also, if $\bar{H}_m = \oplus_{i=1}^m E(\hat{\lambda}_i(\beta))$ and $u \in C^1(\overline{\Omega}) \cap \bar{H}_m$ with $||u||_{C^1(\overline{\Omega})} \leqslant 0$, then

$$\psi(u) \leqslant \frac{1}{2}\sigma(u) - \frac{\hat{\lambda}_m(\beta)}{2}||u||_2^2 \leqslant 0 \quad \text{(see hypothesis } H_7(iii) \text{ and (5)).} \tag{76}$$

From (75) and (76) it follows that ψ has a local linking at the origin and so

$$C_{d_m}(\psi|_{c^1(\overline{\Omega})}, 0) \neq 0 \text{ with } d_m = \dim \bar{H}_m. \tag{77}$$

From Palais [19] (see also Chang [5, p. 14]), we have

$$\begin{aligned}C_k(\psi|_{C^1(\overline{\Omega})}, 0) &\neq 0 = C_k(\psi, 0) \quad \text{for all } k \geqslant 0,\\ &\Rightarrow C_{d_m}(\psi, 0) \neq 0 \quad \text{(see (77)).}\end{aligned}$$

Since $\mu(0) = \dim \overline{H}_{m-1}$ and $\nu(0) = \dim E(\hat{\lambda}_m(\beta))$, from Proposition 3 we have

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{78}$$

Suppose that $K_\psi = \{0, u_0, v_0, y_0\}$. We can always assume that u_0 and v_0 are the extremal constant solutions (that is, $u_0 = u_* \in \operatorname{int} C_+$ and $v_0 = v_* \in -\operatorname{int} C_+$, see Proposition 7). From the proof of Proposition 15 (see Claim 2), we know that u_0 and v_0 are local minimizers of ψ. Hence

$$C_k(\psi, u_0) = C_k(\psi, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{79}$$

Recall that y_0 is a critical point of ψ of mountain pass type (see the proof of Proposition 15). So, from Theorem 2.7 of Li et al. [14], we have

$$C_k(\psi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{80}$$

Finally recall that ψ is coercive (see (4)) and (43). Hence

$$C_k(\psi, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{81}$$

From (78), (79), (80), (81) and the Morse relation with $t = -1$ (see (6)), we have

$$\begin{aligned} &(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0, \\ &\Rightarrow (-1)^{d_m} = 0, \text{ a contradiction.} \end{aligned}$$

So, there exists $\hat{y} \in K_\psi,\ \hat{y} \notin \{0, u_0, v_0, y_0\}$. We have

$$\hat{y} \in [v_0, u_0] \cap C^1(\overline{\Omega}) \quad \text{(see Claim 1 in the proof of Proposition 15).}$$

Therefore $\hat{y}$ is a nodal solution of (1). Moreover, as before (see the proof of Proposition 16), using the strong maximum principle, we show that $\hat{y} \in \text{int}_{C^1(\overline{\Omega})}[v_0, u_0]$. □

5 A special case

Consider the following Robin problem:

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = \lambda u(z) - g(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u(z) = 0 & \text{on } \partial\Omega, \lambda \in \mathbb{R} \end{array} \right\} \tag{82}$$

The hypotheses on the perturbation $g(z, x)$ are the following:

$H_8 : g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(ii) $\lim_{x\to+\infty} \frac{g(z,x)}{x} = +\infty$ uniformly for almost all $z \in \Omega$;
(iii) $\lim_{x\to 0} \frac{g(z,x)}{x} = 0$ uniformly for almost all $z \in \Omega$;
(iv) for every $\rho > 0$, there exists $\hat{\vartheta}_\rho > 0$ such that for almost all $z \in \Omega$, $x \longmapsto \hat{\vartheta}_\rho x - g(z, x)$ is nondecreasing on $[-\rho, \rho]$.

Remark 7 If $g(z, x) = g(z) = |x|^{r-2}x$ with $r > 2$, then we have the equidiffusive logistic equation with an indefinite and unbounded potential.

Using Theorem 17, we have:

Theorem 20 *If hypotheses $H(\xi)$, $H(\beta)$ and $H_8(i), (ii), (iii)$ hold and $\lambda > \hat{\lambda}_2(\beta)$, then problem (82) has at least three nontrivial solutions*

$$u_0 \in int\, C_+, v_0 \in -int\, C_+ \quad and \quad y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal}.$$

Moreover, if in addition hypothesis $H_8(iv)$ holds, then

$$y_0 \in int_{C^1(\overline{\Omega})}[v_0, u_0].$$

Remark 8 Such a multiplicity result, was first proved by Ambrosetti and Mancini [3] with subsequent improvements by Ambrosetti and Lupo [2] and Struwe [27,28], for the Dirichlet problem, with $\xi \equiv 0$ and with stronger conditions on the perturbation g. None of the aforementioned works produced a nodal solution. The extension to Neumann problems (that is, $\beta \equiv 0$) with a potential term, was proved by Papageorgiou and Smyrlis [25]. The extension to p-Laplacian equations with $\xi \equiv 0$ and Robin boundary condition, can be found in the recent work of Papageorgiou and Rădulescu [23].

By strengthening the regularity on $f(z, \cdot)$, we can improve Theorem 20 by producing a second nodal solution.

The new hypotheses on the reaction $f(z, x)$ are the following:

H_9 : $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$ $g(z, 0) = 0,\ g(z, \cdot) \in C^1(\mathbb{R})$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|g'_x(z, x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega,\ \text{all } |x| \leqslant \rho;$$

(ii) $\lim_{x\to\pm\infty} \frac{g(z,x)}{x} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) $g'_x(z, 0) = \lim_{x\to 0} \frac{g(z,x)}{x} = 0$ uniformly for almost all $z \in \Omega$.

Using Theorem 19, we have:

Theorem 21 *If hypotheses $H(\xi)$, $H(\beta)$ and H_9 hold and $\lambda > \hat{\lambda}_2(\beta)$, then problem* (82) *admits at least four nontrivial solutions*

$$u_0 \in int\, C_+,\ v_0 \in -int\, C_+ \quad and \quad y_0, \hat{y} \in int_{C_1(\overline{\Omega})}[v_0, u_0]\ nodal.$$

Remark 9 This theorem extends Theorem 14 of Papageorgiou and Rădulescu [23], where $\xi \equiv 0$ and the conditions on the perturbation $g(z, x)$ are a little stronger.

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