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Ground state solutions for quasilinear Schrödinger equations with variable potential and superlinear reaction

Sitong Chen, Vicenţiu D. Rădulescu, Xianhua Tang and Binlin Zhang

Abstract. This paper is concerned with the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - \frac{1}{2}\Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$ and $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$ is superlinear at infinity. By using variational and some new analytic techniques, we prove the above problem admits ground state solutions under mild assumptions on V and g. Moreover, we establish a minimax characterization of the ground state energy. Especially, we impose some new conditions on V and more general assumptions on g. For this, some new tricks are introduced to overcome the competing effect between the quasilinear term and the superlinear reaction. Hence our results improve and extend recent theorems in several directions.

1. Introduction and main results

In this paper, we consider the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)\,u - \frac{1}{2}\Delta(u^2)\,u = g(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $V\colon \mathbb{R}^N \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$ and $V_\infty := \lim_{|y|\to\infty} V(y) \geq V(x)$ for all $x \in \mathbb{R}^N$;

(G1) $\lim_{|t|\to 0} g(t)/t = 0$ and $\lim_{|t|\to\infty} |g(t)|/|t|^{2\cdot 2^*-1} = 0$;

(G2) $\lim_{|t|\to\infty} G(t)/|t|^2 = +\infty$, where $G(t) = \int_0^t g(s)\mathrm{d}s$;

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(G3) there exists a constant $p > 2$ such that $\frac{g(t)t+NG(t)}{|t|^{p-1}t}$ is nondecreasing on both $(-\infty, 0)$ and $(0, \infty)$.

This quasilinear version of the nonlinear Schrödinger equation arises in several models of different physical phenomena, such as in the study of superfluid films in plasma physics, in condensed matter theory, and a model of self-trapped electrons in quadratic or hexagonal lattices, see e.g. [9], [10], [11], [14], [15]. After the work of Poppenberg [16], equations like (1.1) have received much attention in mathematical analysis and applications in recent years, we refer to [1], [2], [5], [7], [12], [17], [18], [22], [25], [28] and so on.

In general, problem (1.1) has an energy functional of the form

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1+u^2)\,|\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\,u^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\,\mathrm{d}x.$$

Since Φ is not well defined in general in $H^1(\mathbb{R}^N)$, we employ an argument developed by Colin and Jeanjean [5], and make the change of variables by $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+|f(t)|^2}} \quad \text{on } [0, +\infty), \qquad f(-t) = -f(t) \quad \text{on } (-\infty, 0].$$

After the change of variables from Φ, we obtain the following functional:

$$I(v) = \Phi(u) = \Phi(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x) f^2(v)\right] \mathrm{d}x - \int_{\mathbb{R}^N} G(f(v))\,\mathrm{d}x. \tag{1.2}$$

Note that

$$0 < f'(t) \le 1, \quad |f(t)| \le |t|, \quad \forall t \in \mathbb{R} \tag{1.3}$$

and

$$|f(t)| \le 2^{1/4}|t|^{1/2}, \quad f(t)/2 \le f'(t)t \le f(t), \quad \forall t \in \mathbb{R}. \tag{1.4}$$

Under (V1) and (G1), we have $I \in \mathcal{C}^1(H^1(\mathbb{R}^N), \mathbb{R})$, and critical points of I are solutions of the semi-linear equation

$$-\Delta v + V(x)\,f(v)\,f'(v) = g(f(v))\,f'(v), \quad x \in \mathbb{R}^N. \tag{1.5}$$

Moreover, v is a solution of (1.5) if and only if $u = f(v)$ solves (1.1), see [5] and [12]. A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

A typical tool to deal with (1.5) is to use the mountain pass theorem. To this end, one usually requires that g is superlinear near zero and super-cubic near infinity, and satisfies the Ambrosetti–Rabinowitz type condition

(AR) there exists $\mu > 4$ such that $g(t)t \ge \mu G(t) \ge 0$ for all $t \in \mathbb{R}$.

In fact, under these conditions, it is easy to obtain a bounded (PS) sequence for the functional I. If g further satisfies the following monotonicity condition:

(MN) $g(t)/|t|^3$ is nondecreasing for $t \in \mathbb{R} \setminus \{0\}$, the authors in [26], [27], through a convenient change of variables, proved the existence for ground state solutions of (1.1) by the Nehari manifold technique. In [13], Liu, Wang and Wang used a minimization on a Nehari-type constraint for I to get ground state solutions of (1.1) with $g(u) = |u|^{q-2}u$ for $4 \le q < 2 \cdot 2^*$, their argument does not depend on any change of variables. However, these methods mentioned above do not work for (1.1) in the case when $g(u) = |u|^{q-2}u$ with $2 < q < 4$ due to the competing effect between $\Delta(u^2)u$ and $g(u)$, because it is more difficult to get a bounded (PS) sequence (or a bounded minimizing sequence on a Nehari-type constraint) for I. Based on a constrained minimization procedure, Ruiz and Siciliano [18] proved that (1.1) with $g(u) = |u|^{q-2}u$ for $2 < q < 4$ has a ground state solution, if V satisfies (V1) and

(V2′) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V > 0$ and $t \mapsto t^{(N+2)/(N+q)} V(t^{1/(N+q)}x)$ is concave on $(0, \infty)$ for any $x \in \mathbb{R}^N$.

In their arguments, the constraint is a new manifold that is defined by a condition which is a combination of the Nehari equation and the Pohožaev equality rather than Nehari manifold. Later, Wu and Wu [24] obtained a similar result by using the change of variables, Jeanjean's monotonicity trick [8] and the Pohožaev identity, where V satisfies (V1) and

(V2″) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V > 0$, $V(x) = V(|x|)$ and $t^{3-q}\nabla V(tx) \cdot x$ is nonincreasing on $t \in (0, \infty)$ for any $x \in \mathbb{R}^N$.

We would like to point out that the strategies used in [18], [24] rely heavily on the condition $\inf_{\mathbb{R}^N} V > 0$ and the algebraic form $g(u) = |u|^{q-2}u$, see Proposition 3.3 and Proof of Theorem 2.1 in [18], and Lemma 2.6 in [24], it is difficult to generalize the above two results to (1.1) with a general interaction function $g(u)$ even for the case that $g(u) = |u|^{q_1-2}u + |u|^{q_2-2}u$, with $2 < q_1 < q_2 \le 4$.

In the present paper, we shall establish the existence of ground state solutions for (1.1) under (G1)–(G3). To state our results, we need the following new decay condition on V:

(V2) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, the set $\{x \in \mathbb{R}^3 : |\nabla V(x) \cdot x| \ge \epsilon\}$ has finite Lebesgue measure for every $\epsilon > 0$, and one of the following cases holds:

- (I) $\nabla V(x) \cdot x \le (p-2)V(x) + \frac{(N-2)^2}{2|x|^2}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;
- (II) $\|\max\{\nabla V(x) \cdot x - (p-2)V(x), 0\}\|_{N/2} \le 2S$, where $S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \|\nabla u\|_2^2 / \|u\|_{2^*}^2$.

We would like to mention that (V2) is much weaker than (V2′) and (V2″). In fact, when $g(u) = |u|^{p-2}u$ with $2 < p < 4$, it is easy to check that for any $x \in \mathbb{R}^N$,

$$t^{(N+2)/(N+p)}\, V(t^{1/(N+p)}x) \quad \text{is concave on } t \in (0, \infty)$$

is equivalent to

$$[(N+2)V(tx) + \nabla V(tx) \cdot (tx)]/t^{p-2} \quad \text{is nonincreasing on } t \in (0, \infty). \tag{1.6}$$

Moreover,

$$t^{3-p}\, \nabla V(tx) \cdot x \quad \text{is nonincreasing on } t \in (0, \infty)$$

implies that

$$t^{2-p}V(tx) \quad \text{is nonincreasing on } t \in (0,\infty). \tag{1.7}$$

Hence we can deduce from (1.6) or (1.7) that $\nabla V(x)\cdot x \le (p-2)V(x)$, see (2.14) and Remark 1.6 for more details.

Now, we are in a position to state the first result of the present paper.

Theorem 1.1. *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then problem* (1.1) *has a ground state solution with positive energy.*

Next, we further provide a minimax characterization of the ground state energy. For this purpose, inspired by [18], we introduce the following monotonicity condition on V:

(V3) $V \in \mathcal{C}^1(\mathbb{R}^N,\mathbb{R})$, the set $\{x \in \mathbb{R}^3 : |\nabla V(x)\cdot x| \ge \epsilon\}$ has finite Lebesgue measure for every $\epsilon > 0$, and $t \mapsto [(N+2)V(tx) + \nabla V(tx)\cdot(tx)]/t^{p-2}$ is nonincreasing on $[0,\infty)$ for any $x \in \mathbb{R}^N$, where $p > 2$ is given by (G3).

Similar to [24], Lemma 2.1, we define the Pohožaev type functional of (1.5):

$$\mathcal{P}(v) := \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x - N\int_{\mathbb{R}^N}G(f(v))\,\mathrm{d}x$$

for all $v \in H^1(\mathbb{R}^N)$. It is well known that any solution v of (1.5) satisfies $\mathcal{P}(v) = 0$ and $\langle I'(v), f(v)/f'(v)\rangle = 0$, where

$$\begin{aligned}\langle I'(v), f(v)/f'(v)\rangle = &\int_{\mathbb{R}^N}\left(1+\frac{f^2(v)}{1+f^2(v)}\right)|\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}V(x)f^2(v)\,\mathrm{d}x\\ &- \int_{\mathbb{R}^N}g(f(v))f(v)\,\mathrm{d}x.\end{aligned} \tag{1.8}$$

Motivated by this fact, we define a new functional on $H^1(\mathbb{R}^N)$:

$$\begin{aligned}J(v) &:= \langle I'(v), f(v)/f'(v)\rangle + \mathcal{P}(v)\\ &= \frac{N}{2}\|\nabla v\|_2^2 + \int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V(x)+\nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x\\ &\quad - \int_{\mathbb{R}^N}[g(f(v))f(v) + NG(f(v))]\,\mathrm{d}x\end{aligned} \tag{1.9}$$

and define the Nehari–Pohožaev manifold of I by

$$\mathcal{M} := \left\{v \in H^1(\mathbb{R}^N)\setminus\{0\} : J(v) = 0\right\}.$$

Then every non-trivial solution of (1.5) is contained in $\mathcal{M}$. Our second main result is as follows.

Theorem 1.2. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then problem* (1.1) *has a ground state solution* $\bar{u} = f(\bar{v}) \in H^1(\mathbb{R}^N)$ *such that*

$$I(\bar{v}) = \inf_{\mathcal{M}} I = \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I(v_t) > 0,$$

where $v_t(x) = f^{-1}(tf(v(t^{-1}x))$.

Applying Theorem 1.2 to the "limit equation" of (1.1),

$$-\Delta u + V_\infty u - \frac{1}{2}\Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N, \tag{1.10}$$

we have the following corollary.

Corollary 1.3. *Assume that* (G1)–(G3) *hold. Then problem* (1.10) *has a ground state solution* $\bar{u}^\infty = f(\bar{v}^\infty) \in H^1(\mathbb{R}^N)$ *such that*

$$I^\infty(\bar{v}^\infty) = \inf_{\mathcal{M}^\infty} I^\infty = \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I^\infty(v_t) > 0,$$

where

$$I^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V_\infty f^2(v)\right] \mathrm{d}x - \int_{\mathbb{R}^N} G(f(v))\,\mathrm{d}x, \quad \forall v \in H^1(\mathbb{R}^N), \tag{1.11}$$

$$\begin{aligned} J^\infty(v) := {}& \frac{N}{2}\|\nabla v\|_2^2 + \int_{\mathbb{R}^N} \frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x + \frac{N+2}{2}V_\infty \|f(v)\|_2^2 \\ & - \int_{\mathbb{R}^N} [g(f(v))f(v) + NG(f(v))]\,\mathrm{d}x, \quad \forall v \in H^1(\mathbb{R}^N) \end{aligned} \tag{1.12}$$

and

$$\mathcal{M}^\infty := \left\{v \in H^1(\mathbb{R}^N) \setminus \{0\} : J^\infty(v) = 0\right\}. \tag{1.13}$$

Remark 1.4. The results in [18], [24] are special cases of Theorems 1.1 and 1.2 as the function $g(u) = |u|^{q-2}u$ with $2 < q < 4$ satisfies (G1)–(G3) with $p = q$, and (V2′) or (V2″) implies (V2) and (V3). Therefore, our results extend and improve the previous results for (1.1) and other related results in the literature.

Remark 1.5. There are many functions satisfying (G1)–(G3); for example,

(i) $g(t) = a|t|^{q_1-2}t + b|t|^{q_2-2}t$ satisfies (G1)–(G3) with $p = q_1$, where $a \in \mathbb{R}$, $b > 0$ and $2 < q_1 < q_2 \le 4$;

(ii) $g(t) = \frac{|t|t}{\ln(e+|t|)}$ satisfies (G1)–(G3) with $p = 5/2$.

There are also many functions which satisfy (V1)–(V3). For example,

$$V(x) = a - \frac{b}{|x|^\beta + 1} \quad \text{with } a > b > 0 \text{ and } \beta > 0$$

satisfies (V1)–(V3) for any $p > 2$. In particular, (V1) and (V2) are satisfied by many non-monotonic functions, for example,

$$V(x) = a - \frac{b\sin^2|x|^2}{|x|^2 + 1}$$

with $(p-2)a > 2b > 0$ and $(p-2)a-(p+4)b+(N-2)^2/2 \geq 0$; or

$$V(x) = a - \frac{b\sin^2|x|^\beta}{|x|^\beta+1}$$

with $\beta > 0$, $(p-2)a > b\beta > 0$, and $(p-2)(a-b)-3b\beta \geq 0$.

Remark 1.6. For readers' convenience, we give the proof of (1.7). If $t^{3-p}\nabla V(tx)\cdot x$ is nonincreasing on $t \in (0,\infty)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{2-p}V(tx)\right] &= (2-p)\,t^{1-p}V(tx) + t^{2-p}\nabla V(tx)\cdot x \\ &= (2-p)\,t^{1-p}\int_0^t s^{p-3}s^{3-p}\nabla V(sx)\cdot x\,\mathrm{d}s + t^{2-p}\nabla V(tx)\cdot x \\ &\leq (2-p)\,t^{1-p}t^{3-p}\nabla V(tx)\cdot x\int_0^t s^{p-3}\,\mathrm{d}s + t^{2-p}\nabla V(tx)\cdot x = 0.\end{aligned}$$

This shows that (1.7) holds.

To prove Theorem 1.1, we will use Jeanjean's monotonicity trick [8], that is an approximation procedure to obtain a bounded (PS)-sequence for I, instead of starting directly from an arbitrary (PS)-sequence. Below we give a sketch of the proof of this result.

Firstly, for $\lambda \in [1/2,1]$ we consider a family of functionals $I_\lambda\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by $I_\lambda(v) = A(v) - \lambda B(v)$ with

$$A(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla v|^2 + (V(x)+2a)f^2(v)\right]\mathrm{d}x, \quad B(v) = \int_{\mathbb{R}^N}\left[G(f(v))+af^2(v)\right]\mathrm{d}x,$$

where $a > 0$ is a constant satisfying $G(t)+at^2 \geq 0$ for all $t \in \mathbb{R}$ (the constant a can be found easily under (G1) and (G2)). For simplicity and without loss of generality, we assume that $a = 0$ here. Then

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(x)f^2(v)\right)\mathrm{d}x - \lambda\int_{\mathbb{R}^N}G(f(v))\,\mathrm{d}x, \quad \forall v \in H^1(\mathbb{R}^N). \tag{1.14}$$

These functionals have a mountain pass geometry, and denoting the corresponding mountain pass levels by c_λ. Moreover, in view of Jeanjean's monotonicity theorem, I_λ has a bounded (PS)-sequence $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ at level c_λ for almost every $\lambda \in [1/2,1]$.

Secondly, we use the global compactness lemma to show that the bounded sequence $\{v_n(\lambda)\}$ converges weakly to a nontrivial critical point of I_λ. To do that, we have to establish the following strict inequality:

$$c_\lambda < \inf_{\mathcal{K}_\lambda^\infty} I_\lambda^\infty, \tag{1.15}$$

where I_λ^∞ is the associated limited functional defined by

$$I_\lambda^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V_\infty f^2(v)\right)\mathrm{d}x - \lambda\int_{\mathbb{R}^N}G(f(v))\,\mathrm{d}x, \quad \forall v \in H^1(\mathbb{R}^N) \tag{1.16}$$

and $\mathcal{K}_\lambda^\infty := \{w \in H^1(\mathbb{R}^N)\setminus\{0\} : (I_\lambda^\infty)'(w) = 0\}$.

A classical way to show (1.15) is to find a positive function $w_\lambda^\infty \in \mathcal{K}_\lambda^\infty$ such that $I_\lambda^\infty(w_\lambda^\infty) = \inf_{\mathcal{K}_\lambda^\infty} I_\lambda^\infty$ when nonconstant potential $V(x) \le V_\infty$. However, it seems to be impossible to obtain this w_λ^∞ mentioned above only under (G1)–(G3). So the usual arguments cannot be applied here to obtain (1.15). To overcome this difficulty, we follow a strategy introduced in [20] where semilinear Schrödinger equations were studied. However, we have to face some new difficulties caused by the change of variables by $v = f^{-1}(u)$. These difficulties enforce the implementation of new ideas and techniques. More precisely, we first make the scaling $v_t(x) = f^{-1}(tf(v(t^{-1}x)))$ and show that there exists $\bar{v}^\infty$ such that

$$\bar{v}^\infty \in \mathcal{M}^\infty, \qquad I^\infty(\bar{v}^\infty) = \inf_{\mathcal{M}^\infty} I^\infty, \tag{1.17}$$

then by using the translation invariance for $\bar{v}^\infty$ and the crucial inequality

$$I_\lambda^\infty(v) \ge I_\lambda^\infty(v_t) + \frac{1-t^{N+p}}{N+p} J_\lambda^\infty(v) + \frac{1}{2} H(t,v) \tag{1.18}$$

(see Lemma 3.2) and some new analytic techniques (see Lemma 3.4), we can find $\bar{\lambda} \in [1/2, 1)$ such that

$$c_\lambda < m_\lambda^\infty := \inf_{\mathcal{M}_\lambda^\infty} I_\lambda^\infty, \quad \forall \lambda \in (\bar{\lambda}, 1], \tag{1.19}$$

where $H(t,v) > H(1,v) = 0$ for $t \ne 0$ and $v \ne 0$ (see (2.4)),

$$\mathcal{M}_\lambda^\infty = \left\{ v \in H^1(\mathbb{R}^N) \setminus \{0\} : J_\lambda^\infty(v) = \langle (I_\lambda^\infty)'(v), f(v)/f'(v) \rangle + \mathcal{P}_\lambda^\infty(v) = 0 \right\},$$

and $\mathcal{P}_\lambda^\infty(v) = 0$ is the corresponding Pohožave type identity. In particular, any information on sign of $\bar{v}^\infty$ is not required in our arguments.

Finally, we choose two sequences $\{\lambda_n\} \subset (\bar{\lambda}, 1]$ and $\{v_{\lambda_n}\} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\lambda_n \to 1$ and $I'_{\lambda_n}(v_{\lambda_n}) = 0$, by using (1.19) and the global compactness lemma, we get a nontrivial critical point $\bar{v}$ of the functional I.

We would like to mention that a crucial step in the proof of Theorem 1.1 is to prove (1.17), which is a corollary of Theorem 1.2. Inspired of [4], [19], [21], we shall prove Theorem 1.2 following this scheme:

Step i). We verify $\mathcal{M} \ne \emptyset$ and establish the minimax characterization of $m := \inf_{\mathcal{M}} I > 0$.

Step ii). We prove that m is achieved.

Step iii). We show that the minimizer of I on $\mathcal{M}$ is a critical point.

More precisely, in step i), we first establish the key inequality

$$I(v) \ge I(v_t) + \frac{1-t^{N+p}}{N+p} J(v) + \frac{1}{2} H(t,v), \quad \forall v \in H^1(\mathbb{R}^N) \setminus \{0\},\ t > 0 \tag{1.20}$$

in Lemma 2.2; then we construct a saddle point structure with respect to the fibre $\{v_t : t > 0\} \subset H^1(\mathbb{R}^N)$ for $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, see Lemma 2.4, finally based on these constructions we obtain the minimax characterization of m, see Lemma 2.5. In step ii), we first choose a minimizing sequence $\{v_n\}$ of I on $\mathcal{M}$, and show that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, then with the help of the key inequality (1.20) and a concentration-compactness argument, we prove that there exist $\hat{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$

and $\hat{t} > 0$ such that $v_n \rightharpoonup \hat{v}$ in $H^1(\mathbb{R}^N)$ up to translations and extraction of a subsequence, and $\hat{v}_{\hat{t}} \in \mathcal{M}$ is a minimizer of $\inf_{\mathcal{M}} I$, see Lemmas 2.9 and 2.10. This step is most difficult since there are no global compactness and any information on $I'(v_n)$. To finish step iii), inspired by Lemma 2.13 in [20], we use the key inequality (1.20), the deformation lemma and intermediary theorem for continuous functions, which overcome the difficulty that $\mathcal{M}$ may not be a $\mathcal{C}^1$-manifold of $H^1(\mathbb{R}^N)$ due to the lack of the smoothness of g, see Lemma 2.11.

Throughout the paper we make use of the following notations:

- $H^1(\mathbb{R}^N)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)\, \mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

- $L^s(\mathbb{R}^N)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s\, \mathrm{d}x\right)^{1/s}$.
- For any $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, $v_t(x) = f^{-1}(tf(v(t^{-1}x)))$ for $t > 0$.
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$.
- $C_1, C_2, \ldots$ denote positive constants possibly different in different places.

Under (V1), there exists $\gamma_0 > 0$ such that

$$\gamma_0 \|u\|^2 \le \int_{\mathbb{R}^N} \left[|\nabla u|^2 + V(x)u^2\right] \mathrm{d}x. \tag{1.21}$$

Under (G1), (1.3) and (1.4), for any $\varepsilon > 0$ and some $q \in (2, 2^*)$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} [|g(f(v))f(v)| + |G(f(v))|]\, \mathrm{d}x \\ &\qquad \le \varepsilon \left[\|f(v)\|_2^2 + \|v\|_{2^*}^{2^*}\right] + C_\varepsilon \|v\|_q^q, \quad \forall v \in H^1(\mathbb{R}^N). \end{aligned} \tag{1.22}$$

The rest of the paper is organized as follows. In Section 2, we study the existence of ground state solutions for (1.1) by using the Nehari–Pohožaev manifold $\mathcal{M}$, and give the proof of Theorem 1.2. In Section 3, based on Jeanjean's monotonicity trick, we consider the existence of ground state solutions for (1.1), and complete the proof of Theorem 1.1.

2. Proof of Theorem 1.2

Lemma 2.1. *Assume that* (V1), (V3), (G1) *and* (G3) *hold. Then, for all* $t \ge 0$, $x \in \mathbb{R}^N$,

$$h_1(t, x) := V(x) - t^{N+2} V(tx) - \frac{1 - t^{N+p}}{N + p} [(N + 2)V(x) + \nabla V(x) \cdot x] \ge 0, \tag{2.1}$$

and for all $t \geq 0$, $\tau \in \mathbb{R}$,

$$h_2(t,\tau) := t^N G(t\tau) - G(\tau) + \frac{1-t^{N+p}}{N+p}\,[g(\tau)\tau + NG(\tau)] \geq 0. \tag{2.2}$$

Proof. For any $x \in \mathbb{R}^N$, by (V3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}h_1(t,x) = t^{N+p-1}\Big[(N+2)V(x) + \nabla V(x)\cdot x - \frac{(N+2)V(tx)+\nabla V(tx)\cdot(tx)}{t^{p-2}}\Big] \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases}$$

which, together with the continuity of $h_1(t,x)$ on t, implies that $h_1(t,x) \geq h_1(1,x) = 0$ for all $t \geq 0$ and $x \in \mathbb{R}^N$, i.e., (2.1) holds. It is easy to see that $h_2(t,0) \geq 0$ for all $t \geq 0$. For $\tau \neq 0$, by (G3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}h_2(t,\tau) = t^{N+p-1}|\tau|^p\Big[\frac{g(t\tau)t\tau + NG(t\tau)}{|t\tau|^p} - \frac{g(\tau)\tau+NG(\tau)}{|\tau|^p}\Big] \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases}$$

which, together with the continuity of $h_2(t,\tau)$ on t, implies that $h_2(t,\tau) \geq h_2(1,\tau) = 0$ for all $t \geq 0$ and $\tau \in \mathbb{R}\setminus\{0\}$. This shows that (2.2) holds. □

For $t > 0$ and $v \in H^1(\mathbb{R}^N)$, we let

$$H(t,v) := \int_{\mathbb{R}^N}\Big\{\Big[1 - \frac{t^N(1+t^2f^2(v))}{1+f^2(v)}\Big] - \frac{1-t^{N+p}}{N+p}\Big[N + \frac{2f^2(v)}{1+f^2(v)}\Big]\Big\}|\nabla v|^2\,\mathrm{d}x. \tag{2.3}$$

It is easy to check that

$$H(t,v) > H(1,v) = 0, \quad \forall t \in [0,1)\cup(1,\infty),\ v \in H^1(\mathbb{R}^N)\setminus\{0\}. \tag{2.4}$$

Indeed, by a simple calculation, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(t,v) = t^{N-1}\big\{&t^2(t^{p-2}-1)\left[N\|\nabla v\|_2^2 + 2(\|\nabla v\|_2^2 - \|\nabla f(v)\|_2^2)\right] \\ &+ N(t^2-1)\,\|\nabla f(v)\|_2^2\big\},\end{aligned}$$

which, together with

$$\|\nabla v\|_2 \geq \|\nabla f(v)\|_2, \quad \forall v \in H^1(\mathbb{R}^N), \tag{2.5}$$

implies that (2.4) holds.

Inspired by [3], [4], we establish the following inequality.

Lemma 2.2. *Assume that* (V1), (V3), (G1) *and* (G3) *hold. Then*

$$I(v) \geq I(v_t) + \frac{1-t^{N+p}}{N+p}\,J(v) + \frac{1}{2}H(t,v), \quad \forall v \in H^1(\mathbb{R}^N)\setminus\{0\},\ t > 0. \tag{2.6}$$

Proof. Since $v_t(x) = f^{-1}(tf(v(t^{-1}x)))$, then $f(v_t(x)) = tf(v(t^{-1}x))$. Note that

$$I(v_t) = \frac{t^N}{2}\int_{\mathbb{R}^N}\frac{1+t^2f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N}V(tx)\,f^2(v)\,\mathrm{d}x - t^N\int_{\mathbb{R}^N}G(tf(v))\,\mathrm{d}x. \tag{2.7}$$

Thus, it follows from (1.2), (1.9), (2.7) and the definitions of h_1 and h_2 that

$$
\begin{aligned}
&I(v) - I(v_t)\\
&= \frac{1}{2}\|\nabla v\|_2^2 - \frac{t^N}{2}\int_{\mathbb{R}^N} \frac{1+t^2f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\left[V(x) - t^{N+2}V(tx)\right]f^2(v)\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\left[t^N G(tf(v)) - G(f(v))\right]\mathrm{d}x\\
&= \frac{1-t^{N+p}}{N+p}\left\{\frac{N}{2}\|\nabla v\|_2^2 + \int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x\right.\\
&\qquad\qquad + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V(x) + \nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x\\
&\qquad\qquad \left. - N\int_{\mathbb{R}^N}[g(f(v))f(v) + NG(f(v))]\,\mathrm{d}x\right\}\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left\{\left[1 - \frac{t^N(1+t^2f^2(v))}{1+f^2(v)}\right] - \frac{1-t^{N+p}}{N+p}\left[N + \frac{2f^2(v)}{1+f^2(v)}\right]\right\}|\nabla v|^2\,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\left\{V(x) - t^{N+2}V(tx) - \frac{1-t^{N+p}}{N+p}\left[(N+2)V(x) + \nabla V(x)\cdot x\right]\right\}f^2(v)\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\left\{t^N G(tf(v)) - G(f(v)) + \frac{1-t^{N+p}}{N+p}[g(f(v))f(v) + NG(f(v))]\right\}\mathrm{d}x\\
&= \frac{1-t^{N+p}}{N+p}J(v) + \frac{1}{2}H(t,v) + \frac{1}{2}\int_{\mathbb{R}^N} h_1(t,x)f^2(v)\,\mathrm{d}x + \int_{\mathbb{R}^N} h_2(t,f(v))\,\mathrm{d}x\\
&\geq \frac{1-t^{N+p}}{N+p}J(v) + \frac{1}{2}H(t,v), \quad \forall v\in H^1(\mathbb{R}^N),\ t>0.
\end{aligned}
$$

As desired. □

From Lemma 2.2, we have the following corollary.

Corollary 2.3. *Assume that* (V1), (V3), (G1) *and* (G3) *hold. Then, for* $v\in\mathcal{M}$,

$$
I(v) = \max_{t>0} I(v_t). \tag{2.8}
$$

Lemma 2.4. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then for any* $v\in H^1(\mathbb{R}^N)\setminus\{0\}$, *there exists a unique* $t_v>0$ *such that* $v_{t_v}\in\mathcal{M}$.

Proof. Let $v\in H^1(\mathbb{R}^N)\setminus\{0\}$ be fixed and define a function $\zeta(t) := I(v_t)$ on $(0,\infty)$. Clearly, by (1.9) and (2.7), we have

$$
\begin{aligned}
\zeta'(t) = 0 \ \Leftrightarrow\ & \frac{Nt^{N-1}}{2}\int_{\mathbb{R}^N}\frac{1+t^2f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x + t^{N+1}\int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x\\
&+ \frac{t^{N+1}}{2}\int_{\mathbb{R}^N}\left[(N+2)V(tx) + \nabla V(tx)\cdot(tx)\right]f^2(v)\,\mathrm{d}x\\
&- t^{N-1}\int_{\mathbb{R}^N}\left[g(tf(v))tf(v) + NG(tf(v))\right]\mathrm{d}x = 0\\
\Leftrightarrow\ & J(v_t) = 0 \ \Leftrightarrow\ v_t\in\mathcal{M}.
\end{aligned} \tag{2.9}
$$

It is easy to verify, using (V1), (V3), (1.22) and (2.7), that $\lim_{t\to 0}\zeta(t)=0$, $\zeta(t)>0$ for $t>0$ small and $\zeta(t)<0$ for t large. Therefore $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at some $t_v>0$ so that $\zeta'(t_v)=0$ and $v_{t_v}\in\mathcal{M}$.

Next we claim that t_v is unique for any $v\in H^1(\mathbb{R}^N)\setminus\{0\}$. In fact, for some $v\in H^1(\mathbb{R}^N)\setminus\{0\}$, if there exist two positive constants $t_1\neq t_2$ such that $v_{t_1},v_{t_2}\in\mathcal{M}$, i.e., $J(v_{t_1})=J(v_{t_2})=0$, then (2.4) and (2.6) imply

$$\begin{aligned} I(v_{t_1}) &> I(v_{t_2})+\frac{t_1^{N+p}-t_2^{N+p}}{(N+p)t_1^{N+p}}J(v_{t_1})=I(v_{t_2})\\ &> I(v_{t_1})+\frac{t_2^{N+p}-t_1^{N+p}}{(N+p)t_2^{N+p}}J(v_{t_2})=I(v_{t_1}). \end{aligned}$$

This contradiction shows that $t_v>0$ is unique for any $v\in H^1(\mathbb{R}^N)\setminus\{0\}$. □

From Corollary 2.3 and Lemma 2.4, we have $\mathcal{M}\neq\emptyset$ and the following lemma.

Lemma 2.5. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then*

$$\inf_{v\in\mathcal{M}} I(v)=m=\inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}}\max_{t>0} I(v_t).$$

Lemma 2.6. *Assume that V satisfies* (V1) *and* (V3). *Then there exists $\gamma_1>0$ such that*

$$\frac{N}{2}\|\nabla u\|_2^2+\int_{\mathbb{R}^N}\big[(N+2)V(x)+\nabla V(x)\cdot x\big]u^2\,\mathrm{d}x\geq\gamma_1\|u\|^2,\quad\forall u\in H^1(\mathbb{R}^N). \tag{2.10}$$

Proof. Arguing as in the proof of Lemma 3.8 in [21], we can obtain the above conclusion. □

Lemma 2.7. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then*

(i) *there exists $\rho_0>0$ such that $\|\nabla v\|_2\geq\rho_0$ for all $v\in\mathcal{M}$;*

(ii) $m=\inf_{\mathcal{M}} I>0$.

Proof. (i). Since $J(v)=0$ for $v\in\mathcal{M}$, by (G1), (1.3), (1.4), (1.9), (2.5), (2.10) and the Sobolev embedding inequality, one has

$$\begin{aligned} &\frac{N}{4}\|\nabla v\|_2^2+\frac{\gamma_1}{2}\|f(v)\|^2\\ &\quad\leq\frac{N}{4}\|\nabla v\|_2^2+\frac{N}{4}\|\nabla f(v)\|_2^2+\frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V(x)+\nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x\\ &\quad\leq\frac{N}{2}\|\nabla v\|_2^2+\int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x\\ &\qquad+\frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V(x)+\nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x\\ &\quad=\int_{\mathbb{R}^N}[g(f(v))f(v)+NG(f(v))]\,\mathrm{d}x\leq\frac{\gamma_1}{4}\|f(v)\|_2^2+C_1\|v\|_{2^*}^{2^*}\\ &\quad\leq\frac{\gamma_1}{4}\|f(v)\|_2^2+C_1S^{-2^*/2}\|\nabla v\|_2^{2^*},\quad\forall v\in\mathcal{M}, \end{aligned} \tag{2.11}$$

which implies

$$\|\nabla v\|_2 \geq \left(\frac{N}{4C_1}\right)^{(N-2)/4} S^{N/4} := \rho_0, \quad \forall v \in \mathcal{M}. \tag{2.12}$$

(ii). Note that

$$\begin{aligned} I(v) - \frac{1}{N+p}J(v) &= \frac{p-2}{2(N+p)}\|\nabla v\|_2^2 + \frac{1}{N+p}\|\nabla f(v)\|_2^2 \\ &\quad + \frac{1}{2(N+p)}\int_{\mathbb{R}^N}[(p-2)V(x) - \nabla V(x)\cdot x]\, f^2(v)\,\mathrm{d}x \\ &\quad + \frac{1}{N+p}\int_{\mathbb{R}^N}[g(f(v))f(v) - pG(f(v))]\,\mathrm{d}x, \quad \forall v \in H^1(\mathbb{R}^N). \end{aligned} \tag{2.13}$$

By (2.1) and (2.2), one has

$$(p-2)V(x) - \nabla V(x)\cdot x \geq 0, \quad \forall x \in \mathbb{R}^N \tag{2.14}$$

and

$$g(t)t - pG(t) \geq 0, \quad \forall t \in \mathbb{R}. \tag{2.15}$$

Since $J(v) = 0$ for $v \in \mathcal{M}$, then it follows from (2.12), (2.13), (2.14) and (2.15) that

$$I(v) = I(v) - \frac{1}{N+p}J(v) \geq \frac{p-2}{2(N+p)}\|\nabla v\|_2^2 \geq \frac{p-2}{2(N+p)}\rho_0^2, \quad \forall v \in \mathcal{M}.$$

This shows that $m = \inf_{\mathcal{M}} I > 0$. □

The following lemma is a known result which can be proved by a standard argument.

Lemma 2.8. *Assume that* (V1), (V3), (G1) *and* (G2) *hold. If* $v_n \rightharpoonup \bar{v}$ *in* $H^1(\mathbb{R}^N)$, *then*

$$I(v_n) = I(\bar{v}) + I(v_n - \bar{v}) + o(1) \quad \text{and} \quad J(v_n) = J(\bar{v}) + J(v_n - \bar{v}) + o(1).$$

Lemma 2.9. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then* $m^\infty \geq m$.

Proof. Since $V(x) \equiv V_\infty$ satisfies (V1) and (V3), the conclusions for I in this section are true for I^∞. By Lemma 2.4, we have $\mathcal{M}^\infty \neq \emptyset$. Arguing indirectly, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$. Then there exists v_ε^∞ such that

$$v_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > I^\infty(v_\varepsilon^\infty). \tag{2.16}$$

In view of Lemma 2.4, there exists $t_\varepsilon > 0$ such that $(v_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Since $V_\infty \geq V(x)$ for all $x \in \mathbb{R}^N$, it follows from (1.2), (1.11), (2.16) and Corollary 2.3 that

$$m^\infty + \frac{\varepsilon}{2} > I^\infty(v_\varepsilon^\infty) \geq I^\infty((v_\varepsilon^\infty)_{t_\varepsilon}) \geq I((v_\varepsilon^\infty)_{t_\varepsilon}) \geq m = m^\infty + \varepsilon.$$

This contradiction shows that $m^\infty \geq m$. □

Lemma 2.10. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. Then* m *is achieved.*

Proof. In view of Lemmas 2.4 and 2.7, we have $\mathcal{M} \neq \emptyset$ and $m > 0$. Let $\{v_n\} \subset \mathcal{M}$ be such that $I(v_n) \to m$. Since $J(v_n) = 0$, then it follows from (2.13), (2.14) and (2.15) that

$$m + o(1) = I(v_n) = I(v_n) - \frac{1}{N+p} J(v_n) \geq \frac{p-2}{2(N+p)} \|\nabla v_n\|_2^2.$$

This shows that $\{\|\nabla v_n\|_2\}$ is bounded. In view of (2.11), we have

$$\frac{\gamma_1}{4} \|f(v_n)\|^2 \leq C_1 S^{-2^*/2} \|\nabla v_n\|_2^{2^*},$$

which implies $\{\|f(v_n)\|\}$ is bounded. Then it follows from (1.3), (1.4) and the Sobolev embedding inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} v_n^2 \,\mathrm{d}x &= \int_{|v_n|\leq 1} v_n^2 \,\mathrm{d}x + \int_{|v_n|>1} v_n^2 \,\mathrm{d}x \\ &\leq C_2 \int_{|v_n|\leq 1} |f(v_n)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} |v_n|^{2^*} \,\mathrm{d}x \leq C_2 \|f(v_n)\|_2^2 + S^{-2^*/2} \|\nabla v_n\|_2^{2^*}. \end{aligned} \tag{2.17}$$

Hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in $H^1(\mathbb{R}^N)$, $v_n \to \bar{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $v_n \to \bar{v}$ a.e. in $\mathbb{R}^N$. There are two possible cases: i) $\bar{v} = 0$, and ii) $\bar{v} \neq 0$.

Case i). $\bar{v} = 0$, i.e., $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Then $v_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq s < 2^*$ and $v_n \to 0$ a.e. in $\mathbb{R}^N$. By (V1) and (V3), it is easy to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} [V_\infty - V(x)] f^2(v_n) \,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v_n) \,\mathrm{d}x = 0. \tag{2.18}$$

By (1.2), (1.9), (1.11), (1.12) and (2.18), one can get

$$I^\infty(v_n) \to m, \quad J^\infty(v_n) \to 0. \tag{2.19}$$

From (1.4), (1.22), (2.5), (2.10) and (2.12), one has

$$\begin{aligned} &\frac{N}{4}\rho_0^2 + \frac{\gamma_1}{2}\|f(v_n)\|^2 \\ &\quad\leq \frac{N}{4}\|\nabla v_n\|_2^2 + \frac{N}{4}\|\nabla f(v_n)\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x)\cdot x] f^2(v_n) \,\mathrm{d}x \\ &\quad\leq \frac{N}{2}\|\nabla v_n\|_2^2 + \int_{\mathbb{R}^N} \frac{f^2(v_n)}{1+f^2(v_n)} |\nabla v_n|^2 \,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^N} [(N+2)V(x) + \nabla V(x)\cdot x] f^2(v_n) \,\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N} [g(f(v_n)) f(v_n) + NG(f(v_n))] \,\mathrm{d}x \\ &\quad\leq (1+N)\varepsilon \left(\|f(v_n)\|_2^2 + \|v_n\|_{2^*}^{2^*}\right) + (1+N) C_\varepsilon \|v_n\|_q^q. \end{aligned} \tag{2.20}$$

Using (2.20) and Lions' concentration compactness principle Lemma 1.21 in [23], we can prove that there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |v_n|^2 \, dx > \delta$. Let $\hat{v}_n(x) = v_n(x + y_n)$. Then $\|\hat{v}_n\| = \|v_n\|$, and by (2.19), one has

$$J^\infty(\hat{v}_n) = o(1), \quad I^\infty(\hat{v}_n) \to m, \quad \int_{B_1(0)} |\hat{v}_n|^2 \, dx > \delta. \tag{2.21}$$

Therefore, there exists $\hat{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{v}_n \rightharpoonup \hat{v}, & \text{in } H^1(\mathbb{R}^N), \\ \hat{v}_n \to \hat{v}, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N), \ \forall s \in [1, 2^*), \\ \hat{v}_n \to \hat{v}, & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{2.22}$$

Let $w_n = \hat{v}_n - \hat{v}$. Then (2.22) and Lemma 2.8 yield

$$I^\infty(\hat{v}_n) = I^\infty(\hat{v}) + I^\infty(w_n) + o(1), \quad J^\infty(\hat{v}_n) = J^\infty(\hat{v}) + J^\infty(w_n) + o(1). \tag{2.23}$$

Let

$$\Psi^\infty(v) := I^\infty(v) - \frac{1}{N+p} J^\infty(v), \quad \forall v \in H^1(\mathbb{R}^N). \tag{2.24}$$

From (1.2), (1.9), (2.21), (2.23) and (2.24), one has

$$\Psi^\infty(w_n) = m - \Psi^\infty(\hat{v}) + o(1), \quad J^\infty(w_n) = -J^\infty(\hat{v}) + o(1). \tag{2.25}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then we have

$$I^\infty(\hat{v}) = m, \quad J^\infty(\hat{v}) = 0. \tag{2.26}$$

Next, we assume that $w_n \neq 0$. We claim that $J^\infty(\hat{v}) \leq 0$. Otherwise, if $J^\infty(\hat{v}) > 0$, then (2.25) implies $J^\infty(w_n) < 0$ for large n. Applying Lemma 2.4 to I^∞, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}^\infty$ for large n. Applying Lemma 2.2 to I^∞, from (1.11), (1.12), (2.4), (2.24), (2.25) and Lemma 2.9, we derive

$$\begin{aligned} m - \Psi^\infty(\hat{v}) + o(1) = \Psi^\infty(w_n) &= I^\infty(w_n) - \frac{1}{N+p} J^\infty(w_n) \\ &\geq I^\infty\left((w_n)_{t_n}\right) - \frac{t_n^{N+p}}{N+p} J^\infty(w_n) \geq m^\infty \geq m, \end{aligned}$$

which is a contradiction due to $\Psi^\infty(\hat{v}) > 0$. This shows that $J^\infty(\hat{v}) \leq 0$. Applying Lemmas 2.2 and 2.4 to I^∞, there exists $t_\infty > 0$ such that $\hat{v}_{t_\infty} \in \mathcal{M}^\infty$, moreover, it follows from (1.11), (1.12), (2.4), (2.15), (2.21), (2.24), Fatou's lemma and Lemma 2.9 that

$$\begin{aligned} m &= \lim_{n\to\infty} \left[I^\infty(\hat{v}_n) - \frac{1}{N+p} J^\infty(\hat{v}_n) \right] \geq I^\infty(\hat{v}) - \frac{1}{N+p} J^\infty(\hat{v}) \\ &\geq I^\infty\left(\hat{v}_{t_\infty}\right) - \frac{t_\infty^{N+p}}{N+p} J^\infty(\hat{v}) \geq m^\infty \geq m, \end{aligned}$$

which implies (2.26) holds also.

In view of Lemma 2.4, there exists $\hat{t} > 0$ such that $\hat{v}_{\hat{t}} \in \mathcal{M}$. Applying Corollary 2.3 to I^∞, we deduce from (V1), (1.2), (1.11) and (2.26) that

$$m \le I(\hat{v}_{\hat{t}}) \le I^\infty(\hat{v}_{\hat{t}}) \le I^\infty(\hat{v}) = m.$$

This shows that m is achieved at $\hat{v}_{\hat{t}} \in \mathcal{M}$.

Case ii). $\bar{v} \neq 0$. In this case, analogous to the proof of (2.26), by using I and J instead of I^∞ and J^∞, we can deduce that $I(\bar{v}) = m$ and $J(\bar{v}) = 0$. □

Lemma 2.11. *Assume that* (V1), (V3) *and* (G1)–(G3) *hold. If $\bar{v} \in \mathcal{M}$ and $I(\bar{v}) = m$, then $\bar{v}$ is a critical point of I.*

Proof. From (G1), (G2), (1.9) and (2.9), we can deduce that there exist $T_1 \in (0,1)$ and $T_2 \in (1,\infty)$ such that $J(\bar{v}_{T_1}) > 0$ and $J(\bar{v}_{T_2}) < 0$. Similar to the proof of Lemma 2.15 in [20], we can prove this lemma by using

$$I(\bar{v}_t) \le I(\bar{v}) - \frac{1}{2}H(t,\bar{v}) = m - \frac{1}{2}H(t,\bar{v}), \quad \forall t > 0,$$

$$\text{and} \quad \varepsilon := \min\Big\{\frac{1}{4}H(T_1,\bar{v}), \frac{1}{4}H(T_2,\bar{v}), 1, \frac{\varrho\delta}{8}\Big\}.$$ □

Proof of Theorem 1.2. In view of Lemmas 2.5, 2.10 and 2.11, there exists $\bar{v} \in \mathcal{M}$ such that

$$I(\bar{v}) = m = \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} I(v_t), \quad I'(\bar{v}) = 0.$$

This shows that $\bar{v}$ is a ground state solution of (1.1) such that $I(\bar{v}) = \inf_{\mathcal{M}} I > 0$. □

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

Proposition 3.1 ([8])**.** *Let X be a Banach space and let $K \subset \mathbb{R}^+$ be an interval. We consider a family $\{\mathcal{I}_\lambda\}_{\lambda\in K}$ of $\mathcal{C}^1$-functional on X of the form*

$$\mathcal{I}_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in K,$$

where $B(u) \ge 0$, $\forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) > \max\{\mathcal{I}_\lambda(v_1), \mathcal{I}_\lambda(v_2)\}, \tag{3.1}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1],X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in K$, there is a bounded $(\mathrm{PS})_{c_\lambda}$ sequence for $\mathcal{I}_\lambda$, that is, there exists a sequence such that

i) *$\{u_n(\lambda)\}$ is bounded in X;*

ii) *$\mathcal{I}_\lambda(u_n(\lambda)) \to c_\lambda$;*

iii) *$\mathcal{I}'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X.*

Moreover, c_λ is nonincreasing and left continuous on $\lambda \in [1/2, 1]$.

If $I'_\lambda(v) = 0$, then v satisfies the Pohožaev identity

$$\mathcal{P}_\lambda(v) := \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N}[NV(x)+\nabla V(x)\cdot x]\, f^2(v)\,\mathrm{d}x - N\lambda\int_{\mathbb{R}^N} G(f(v))\,\mathrm{d}x = 0,$$

where $I_\lambda(v)$ is defined by (1.14). For $\lambda \in [1/2, 1]$ and $v \in H^1(\mathbb{R}^N)$, we set $J_\lambda(v) = \langle I'_\lambda(v), f(v)/f'(v)\rangle + \mathcal{P}_\lambda(v)$, then

$$\begin{aligned} J_\lambda(v) = &\frac{N}{2}\|\nabla v\|_2^2 + \int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}|\nabla v|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}[(N+2)V(x)+\nabla V(x)\cdot x]f^2(v)\,\mathrm{d}x \\ &- \lambda\int_{\mathbb{R}^N}[g(f(v))f(v) + NG(f(v))]\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Correspondingly, for $\lambda \in [1/2, 1]$ and $v \in H^1(\mathbb{R}^N)$ we define

$$\begin{aligned} J_\lambda^\infty(v) = &\frac{N}{2}\|\nabla v\|_2^2 + \int_{\mathbb{R}^N}\frac{f^2(v)}{1+f^2(v)}\,|\nabla v|^2\,\mathrm{d}x + \frac{N+2}{2}V_\infty\,\|f(v)\|_2^2 \\ &- \lambda\int_{\mathbb{R}^N}[g(f(v))f(v) + NG(f(v))]\,\mathrm{d}x. \end{aligned}$$

Set

$$\mathcal{M}_\lambda^\infty := \{v \in H^1(\mathbb{R}^N)\setminus\{0\} : J_\lambda^\infty(v) = 0\}, \quad m_\lambda^\infty := \inf_{v\in\mathcal{M}_\lambda^\infty} I_\lambda^\infty(v). \tag{3.3}$$

By (2.3) and Lemma 2.2, we have the following lemma.

Lemma 3.2. *Assume that* (G1) *and* (G3) *hold. Then*

$$I_\lambda^\infty(v) \geq I_\lambda^\infty(v_t) + \frac{1-t^{N+p}}{N+p}J_\lambda^\infty(v) + \frac{1}{2}H(t,v), \qquad \forall v \in H^1(\mathbb{R}^N),\ t > 0. \tag{3.4}$$

In view of Theorem 1.1, $I_1^\infty = I^\infty$ has a minimizer $v^\infty \neq 0$ on $\mathcal{M}_1^\infty = \mathcal{M}^\infty$, i.e.,

$$v^\infty \in \mathcal{M}_1^\infty, \quad (I_1^\infty)'(v^\infty) = 0 \quad \text{and} \quad m_1^\infty = I_1^\infty(v^\infty), \tag{3.5}$$

where m_λ^∞ is defined by (3.3). Since (1.10) is autonomous, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $V(x) \leq V_\infty$ but $V(x) \not\equiv V_\infty$, then there exist $\bar{x} \in \mathbb{R}^N$ and $\bar{r} > 0$ such that

$$V_\infty - V(x) > 0, \quad |v^\infty(x)| > 0 \quad a.e.\ |x - \bar{x}| \leq \bar{r}. \tag{3.6}$$

Lemma 3.3. *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then*

(i) *there exists $T > 0$, independent of λ, such that $I_\lambda((v^\infty)_T) < 0$ for all $\lambda \in [1/2, 1]$;*

(ii) *there exists a positive constant κ_0, independent of λ, such that for all $\lambda \in [1/2, 1]$,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq \kappa_0 > \max\{I_\lambda(0), I_\lambda((v^\infty)_T)\},$$

where

$$\Gamma = \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = (v^\infty)_T\right\};$$

(iii) *c_λ and m_λ^∞ is non-increasing on $\lambda \in [1/2, 1]$;*

(iv) *$\limsup_{\lambda\to\lambda_0} c_\lambda = c_{\lambda_0}$ for $\lambda_0 \in (1/2, 1]$.*

The proof of Lemma 3.3 is standard, so we omit it.

Lemma 3.4. *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then there exists* $\bar{\lambda} \in [1/2, 1)$ *such that* $c_\lambda < m_\lambda^\infty$ *for* $\lambda \in (\bar{\lambda}, 1]$.

Proof. It is easy to check that $I_\lambda((v^\infty)_t)$ is continuous on $t \in (0, \infty)$. Hence for any $\lambda \in [1/2, 1]$, we can choose $t_\lambda \in (0, T)$ such that $I_\lambda((v^\infty)_{t_\lambda}) = \max_{t\in(0,T]} I_\lambda((v^\infty)_t)$. Set

$$\gamma_0(t) = \begin{cases} (v^\infty)_{(tT)}, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Then $\gamma_0 \in \Gamma$, where Γ is defined by Lemma 3.3 (ii). Moreover,

$$I_\lambda((v^\infty)_{t_\lambda}) = \max_{t\in[0,1]} I_\lambda(\gamma_0(t)) \geq c_\lambda. \tag{3.7}$$

Using (2.15), it is easy to check that the function $G(t)/t|t|^{p-1}$ is nondecreasing on both $t \in (-\infty, 0)$ and $(0, +\infty)$. Since $t_\lambda \in (0, T)$, then we have

$$\frac{G(t_\lambda f(v^\infty))}{t_\lambda^p} \leq \frac{G(Tf(v^\infty))}{T^p}. \tag{3.8}$$

Let

$$\zeta_0 := \min\{3\bar{r}/8(1 + |\bar{x}|), 1/4\}. \tag{3.9}$$

Then it follows from (3.6) and (3.9) that

$$|x - \bar{x}| \leq \frac{\bar{r}}{2} \quad \text{and} \quad s \in [1 - \zeta_0, 1 + \zeta_0] \Rightarrow |sx - \bar{x}| \leq \bar{r}. \tag{3.10}$$

Let

$$\begin{aligned} \bar{\lambda} := \max\Big\{\frac{1}{2}, 1 - \frac{(1-\zeta_0)^{N+2} \min_{s\in[1-\zeta_0,1+\zeta_0]} \int_{\mathbb{R}^N} [V_\infty - V(sx)] f^2(v^\infty)\, dx}{2T^N \int_{\mathbb{R}^N} G(Tf(v^\infty))\, dx}, \\ 1 - \frac{\min\{H(1-\zeta_0, v^\infty), H(1+\zeta_0, v^\infty)\}}{2T^N \int_{\mathbb{R}^N} G(Tf(v^\infty))\, dx}\Big\}. \end{aligned} \tag{3.11}$$

Then it follows from (2.4), (3.6) and (3.10) that $1/2 \leq \bar{\lambda} < 1$. We have two cases to distinguish:

Case i). $t_\lambda \in [1 - \zeta_0, 1 + \zeta_0]$. From (1.14), (1.16), (3.4)–(3.8), (3.10), (3.11) and Lemma 3.3 (iv), we have

$$\begin{aligned} m_\lambda^\infty &\geq m_1^\infty = I_1^\infty(v^\infty) \geq I_1^\infty((v^\infty)_{t_\lambda}) \\ &= I_\lambda((v^\infty)_{t_\lambda}) - (1-\lambda)t_\lambda^N \int_{\mathbb{R}^N} G(t_\lambda f(v^\infty))\, dx + \frac{t_\lambda^{N+2}}{2} \int_{\mathbb{R}^N} [V_\infty - V(t_\lambda x)] f^2(v^\infty)\, dx \\ &\geq c_\lambda - (1-\lambda)T^N \int_{\mathbb{R}^N} G(Tf(v^\infty))\, dx \\ &\quad + \frac{(1-\zeta_0)^{N+2}}{2} \min_{s\in[1-\zeta_0,1+\zeta_0]} \int_{\mathbb{R}^N} [V_\infty - V(sx)] f^2(v^\infty)\, dx \\ &> c_\lambda, \quad \forall \lambda \in (\bar{\lambda}, 1]. \end{aligned}$$

Case ii). $t_\lambda \in (0, 1-\zeta_0) \cup (1+\zeta_0, T)$. From (V1), (1.3), (1.14), (1.16), (3.4), (3.5), (3.7), (3.11) and Lemma 3.3 (iii), we have

$$\begin{aligned}
m_\lambda^\infty \ge m_1^\infty &= I_1^\infty(v^\infty) \ge I_1^\infty\left((v^\infty)_{t_\lambda}\right) + \frac{1}{2}H(t_\lambda, v^\infty) \\
&= I_\lambda\left((v^\infty)_{t_\lambda}\right) - (1-\lambda)t_\lambda^N \int_{\mathbb{R}^N} G(t_\lambda f(v^\infty))\,\mathrm{d}x \\
&\quad + \frac{t_\lambda^{N+2}}{2}\int_{\mathbb{R}^N} [V_\infty - V(t_\lambda x)] f^2(v^\infty)\,\mathrm{d}x + \frac{1}{2}H(t_\lambda, v^\infty) \\
&\ge c_\lambda - (1-\lambda)T^N \int_{\mathbb{R}^N} G(Tf(v^\infty))\,\mathrm{d}x + \frac{1}{2}\min\left\{H(1-\zeta_0, v^\infty), H(1+\zeta_0, v^\infty)\right\} \\
&> c_\lambda, \quad \forall\, \lambda \in (\bar{\lambda}, 1].
\end{aligned}$$

In both cases, we obtain that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$. □

Lemma 3.5 (Lemma 3.3 in [6])**.** *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Let* $\{v_n\}$ *be a bounded (PS) sequence for* I_λ*, for* $\lambda \in [1/2, 1]$*. Then there exist a subsequence of* $\{v_n\}$*, still denoted by* $\{v_n\}$*, an integer* $l \in \mathbb{N} \cup \{0\}$*, a sequence* $\{y_n^k\}$ *and* $w^k \in H^1(\mathbb{R}^N)$ *for* $1 \le k \le l$*, such that*

(i) $v_n \rightharpoonup v_0$ *with* $I_\lambda'(v_0) = 0$;

(ii) $w^k \neq 0$ *and* $(I_\lambda^\infty)'(w^k) = 0$ *for* $1 \le k \le l$;

(iii) $\left\|v_n - v_0 - \sum_{k=1}^l w^k(\cdot + y_n^k)\right\| \to 0$;

(iv) $I_\lambda(v_n) \to I_\lambda(v_0) + \sum_{i=1}^l I_\lambda^\infty(w^i)$;

where we agree that in the case $l = 0$ *the above holds without* w^k*.*

Lemma 3.6. *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then for almost every* $\lambda \in (\bar{\lambda}, 1]$*, there exists* $v_\lambda \in H^1(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$I_\lambda'(v_\lambda) = 0, \quad I_\lambda(v_\lambda) = c_\lambda.$$

Proof. Under (V1), (V2) and (G1)–(G3), Lemma 3.3 implies that $I_\lambda(v)$ satisfies the assumptions of Proposition 3.1 with $X = H^1(\mathbb{R}^N)$, $K = [\bar{\lambda}, 1]$ and $\mathcal{I}_\lambda = I_\lambda$. So for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists a bounded sequence $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ (for simplicity, we denote the sequence by $\{v_n\}$ instead of $\{v_n(\lambda)\}$) such that

$$I_\lambda(v_n) \to c_\lambda > 0, \quad I_\lambda'(v_n) \to 0.$$

By Lemma 3.5, there exist a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and $v_\lambda \in H^1(\mathbb{R}^N)$, an integer $l \in \mathbb{N} \cup \{0\}$, and $w^1, \ldots, w^l \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$v_n \rightharpoonup v_\lambda \ \text{ in } H^1(\mathbb{R}^N), \quad I_\lambda'(v_\lambda) = 0, \tag{3.12}$$

$$(I_\lambda^\infty)'(w^k) = 0, \quad I_\lambda^\infty(w^k) \ge m_\lambda^\infty, \quad 1 \le k \le l, \tag{3.13}$$

$$\text{and} \quad c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^l I_\lambda^\infty(w^k). \tag{3.14}$$

Since $\|v_n\| \nrightarrow 0$, we deduce from (3.13) and (3.14) that if $v_\lambda = 0$ then $l \geq 1$ and

$$c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k) \geq m_\lambda^\infty,$$

which contradicts with Lemma 3.4. Thus $v_\lambda \neq 0$. Since $I'_\lambda(v_\lambda) = 0$, then we have $J_\lambda(v_\lambda) = \langle I'_\lambda(v_\lambda), f(v_\lambda)/f'(v_\lambda)\rangle + \mathcal{P}_\lambda(v_\lambda) = 0$. It follows from (1.14), (2.15) and (3.2) that

$$\begin{aligned}
I_\lambda(v_\lambda) &= I_\lambda(v_\lambda) - \frac{1}{N+p} J_\lambda(v_\lambda) \\
&\geq \frac{p-2}{2(N+p)} \|\nabla v_\lambda\|_2^2 + \frac{1}{N+p} \|\nabla f(v_\lambda)\|_2^2 \\
&\quad - \frac{1}{2(N+p)} \int_{\mathbb{R}^N} [\nabla V(x) \cdot x - (p-2)V(x)]\, f^2(v_\lambda)\, \mathrm{d}x.
\end{aligned} \tag{3.15}$$

If Case (I) of (V2) holds, then it follows from Hardy's inequality that

$$\int_{\mathbb{R}^N} [\nabla V(x) \cdot x - (p-2)V(x)] f^2(v_\lambda)\, \mathrm{d}x \leq \frac{(N-2)^2}{2} \int_{\mathbb{R}^N} \frac{f^2(v_\lambda)}{|x|^2}\, \mathrm{d}x \leq 2\|\nabla f(v_\lambda)\|_2^2. \tag{3.16}$$

If Case (II) of (V2) holds, then it follows from Sobolev's embedding inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N} [\nabla V(x) \cdot x - (p-2)V(x)] f^2(v_\lambda)\, \mathrm{d}x \\
&\quad \leq \left(\int_{\mathbb{R}^N} |\max\{[\nabla V(x) \cdot x - (p-2)V(x)], 0\}|^{N/2}\, \mathrm{d}x \right)^{2/N} \\
&\qquad \cdot \left(\int_{\mathbb{R}^N} |f(v_\lambda)|^{2N/(N-2)}\, \mathrm{d}x \right)^{(N-2)/N} \\
&\quad \leq \frac{\|\max\{[\nabla V(x) \cdot x - (p-2)V(x)], 0\}\|_{N/2}}{S} \|\nabla f(v_\lambda)\|_2^2 \leq 2\|\nabla f(v_\lambda)\|_2^2.
\end{aligned} \tag{3.17}$$

Thus, from (3.15) and (3.16) or (3.17), we deduce

$$I_\lambda(v_\lambda) \geq \frac{p-2}{2(N+p)} \|\nabla v_\lambda\|_2^2 > 0. \tag{3.18}$$

By (3.13), (3.14) and (3.18), we have

$$c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k) > l\, m_\lambda^\infty,$$

which, together with Lemma 3.4, implies that $l = 0$ and $I_\lambda(v_\lambda) = c_\lambda$. □

Lemma 3.7. *Assume that* (V1), (V2) *and* (G1)–(G3) *hold. Then there exists* $\bar{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$I'(\bar{v}) = 0, \quad 0 < I(\bar{v}) = c_1. \tag{3.19}$$

Proof. In view of Lemma 3.3 (iv) and Lemma 3.6, there exist two sequences $\{\lambda_n\} \subset [\bar{\lambda}, 1]$ and $\{v_{\lambda_n}\} \subset H^1(\mathbb{R}^N) \setminus \{0\}$, denoted by $\{v_n\}$, such that

$$\lambda_n \to 1, \quad c_{\lambda_n} \to c_1, \quad I'_{\lambda_n}(v_n) = 0, \quad 0 < I_{\lambda_n}(v_n) = c_{\lambda_n}.$$

Then we have $J_{\lambda_n}(v_n) = 0$. Similar to the proof of (3.18), one can get

$$c_1 + o(1) = c_{\lambda_n} = I_{\lambda_n}(v_n) - \frac{1}{N+p} J_{\lambda_n}(v_n) \geq \frac{p-2}{2(N+p)} \|\nabla v_n\|_2^2.$$

This shows that $\{\|\nabla v_n\|_2\}$ is bounded. Since $\langle I'_{\lambda_n}(v_n), f(v_n)/f'(v_n)\rangle = 0$, it follows from (G1), (1.3), (1.4), (1.8), (1.21) and the Sobolev embedding inequality that

$$\begin{aligned}\gamma_0 \|f(v_n)\|^2 &\leq \int_{\mathbb{R}^N} \left(1 + \frac{f^2(v_n)}{1+f^2(v_n)}\right) |\nabla v_n|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) f^2(v_n) \, \mathrm{d}x \\ &= \lambda_n \int_{\mathbb{R}^N} g(f(v_n)) f(v_n) \, \mathrm{d}x \leq \frac{\gamma_0}{2} \|f(v_n)\|_2^2 + C_3 S^{2^*/2} \|\nabla v_n\|_2^2,\end{aligned}$$

which, together with (2.17), implies that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. The rest of the proof is similar to that of Lemma 3.6, so we omit it. □

Proof of Theorem 1.1. Let

$$\mathcal{K} := \left\{v \in H^1(\mathbb{R}^N) \setminus \{0\} : I'(v) = 0\right\}, \quad \hat{m} := \inf_{v \in \mathcal{K}} I(v).$$

Then Lemma 3.7 shows that $\mathcal{K} \neq \emptyset$ and $\hat{m} \leq c_1$. Similar to the proof of (3.18), we have $I(v) = I_1(v) \geq 0$ for all $v \in \mathcal{K}$, and so $\hat{m} \geq 0$. Let $\{v_n\} \subset \mathcal{K}$ be such that $I'(v_n) = 0$ and $I(v_n) \to \hat{m}$. Similar to the proof of Lemma 3.7, we can deduce that there exists $\hat{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $I'(\hat{v}) = 0$ and $I(\hat{v}) = \hat{m}$. □

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Sitong Chen: School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P. R. China.
E-mail: mathsitongchen@163.com

Vicenţiu D. Rădulescu: Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland; *and* Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania.
E-mail: radulescu@inf.ucv.ro

Xianhua Tang: School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, P. R. China.
E-mail: tangxh@mail.csu.edu.cn

Binlin Zhang (corresponding author): College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, P. R. China; *and* Department of Mathematics, Heilongjiang Institute of Technology, Harbin 150050, P. R. China.
E-mail: zhangbinlin2012@163.com

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