Dedicated to the memory of Professor Nicu Boboc

LOW AND HIGH PERTURBATIONS OF NONHOMOGENEOUS EIGENVALUE PROBLEMS WITH ABSORPTION

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We are concerned with the mathematical analysis of a class of nonlinear eigenvalue problems driven by a nonhomogeneous differential operator. The features of this paper are the presence of an absorption term and the lack of compactness due to the study in the whole Euclidean space. The main result establishes the following properties: (i) the problem does not have solutions in the case of low perturbations of the reaction; (ii) the problem admits at least two nontrivial entire solutions in the case of high perturbations of the reaction. In both cases, the perturbations is considered in terms of the values of a suitable positive parameter. The proofs rely on simple variational methods and the arguments developed in this paper can be extended to other classes of nonlinear eigenvalue problems with nonstandard growth.

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1. INTRODUCTION AND THE MAIN RESULT

The present paper is motivated by our previous work [13], where we have studied existence and multiplicity properties of solutions to the following quasilinear problem

$$(1.1)\quad \begin{cases} -\mathrm{div}(|\nabla u|^{m-2}\nabla u)+|u|^{m-2}u = \lambda|u|^{q-2}u-h(x)|u|^{p-2}u, & \text{if } x\in\mathbb{R}^N\\ u\geq 0, & \text{if } x\in\mathbb{R}^N, \end{cases}$$

where $h(x)$ is a positive continuous function on $\mathbb{R}^N$ ($N\geq 3$) satisfying the condition

$$(1.2)\qquad \int_{\mathbb{R}^N}\frac{1}{h(x)^{q/(p-q)}}\,dx<\infty,$$

$\lambda>0$ is a positive parameter and $2\leq m<q<p<m^\star=Nm/(N-m)$, $m<N$.

Problem (1.1) is the generalized Lane-Emden-Fowler equation and it arises in the description of several patterns in mathematical physics, for instance in boundary-layer models of viscous fluids (see Wong [16]). This equation goes back to the pioneering paper by Lane [7] in 1869 and is originally motivated by Lane's interest in computing both the temperature and the density of mass on the surface of the sun. Problem (1.1) describes the behavior of the density of a gas sphere in hydrostatic equilibrium and the index p, which is called the polytropic index in astrophysics and is related to the ratio of the specific heats of the gas. The study developed in [13] is in connection with related contributions of Alama and Tarantello [1] (case of bounded domain) and Chabrowski [4] (case of unbounded domains). Related contributions have been established by Papageorgiou, Rădulescu and Repovš [9, 10], and Ramos Quoirin and Umezu [12]. We also refer to the recent monograph [11], which includes several relevant contributions to the study of Lane-Emden-Fowler equations.

The aim of the present paper is to extend the analysis developed in [13] to the case of nonhomogeneous differential operators. We study the following problem:

$$(1.3)\qquad \begin{cases} -\mathrm{div}(a(|\nabla u|^2)\nabla u) + a(u^2)u + h(x)|u|^{p-2}u = \lambda |u|^{q-2}u, & \text{in} \quad \mathbb{R}^N \\ u \geq 0, & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $h(x) : \mathbb{R}^N \to (0, \infty)$ $(N \geq 3)$ is a continuous function satisfying the condition

$$(1.4)\qquad \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}}\, dx < \infty,$$

λ is a positive parameter and

$$(1.5)\qquad 2 < q < p < 2^\star = \frac{2N}{N-2}\,.$$

Throughout this paper we assume that $a : \mathbb{R}^+ \to \mathbb{R}^*$ is a function of class C^1 that satisfies the following ellipticity and growth conditions of Leray-Lions type: there exist positive numbers γ and Γ such that

$$(1.6)\qquad \gamma \leq a(t^2) \leq \Gamma \text{ for all } t \geq 0$$

and

$$(1.7)\qquad \left(\gamma - \frac{1}{2}\right) a(t) \leq t a'(t) \leq \Gamma a(t) \text{ for all } t \geq 0.$$

To the best of our knowledge, differential operators of the type $\mathrm{div}(a(|\nabla u|^2)\nabla u)$ with potentials a satisfying hypotheses (1.6) and (1.7), have been introduced by Omari and Zanolin [8].

Basic examples of operators generated by the potential a with the above properties include the Laplace operator (for $a(t) \equiv 1$) but also combinations between the Laplace operator and the mean curvature operator, which is generated by

$$a(t)=\begin{cases}\dfrac{1}{\sqrt{1+t}} & \text{if } t\in[0,1]\\ \dfrac{1}{8\sqrt{2}}(t-2)^2+\dfrac{7}{8\sqrt{2}} & \text{if } t\in(1,2)\\ \dfrac{7}{8\sqrt{2}} & \text{if } t\in[2,\infty).\end{cases}$$

Under hypotheses (1.6) and (1.7), we can deduce by straightforward computation the following properties:

(i) The mapping $t\mapsto ta(t^2)$ is increasing.

(ii) The nonlinear operator $\mathbb{R}^N\ni x\mapsto a(|x|^2)x$ is strictly monotone, that is, there exists $\alpha>0$ such that

$$\langle a(|x|^2)x-a(|y|^2)y,x-y\rangle\geq\alpha\,|x-y|^2 \text{ for all } x,y\in\mathbb{R}^N.$$

We refer to [8] for more details.

In this paper we use standard notations and terminology. We denote by $H^1(\mathbb{R}^N)$ the Sobolev space equipped with the norm

$$\|u\|_{H^1(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)\,dx\right)^{1/2}.$$

For simplicity we will often denote the above norm by $\|u\|$.

We denote by $L^h_p(\mathbb{R}^N)$ (where $1\leq p<\infty$) the weighted Lebesgue space

$$L^h_p(\mathbb{R}^N)=\left\{u:\mathbb{R}^N\to\mathbb{R};\ \int_{\mathbb{R}^N}h(x)|u|^p\,dx<\infty\right\},$$

where $h(x)$ is a positive continuous function on $\mathbb{R}^N$, equipped with the norm

$$\|u\|^p_{h,p}=\left(\int_{\mathbb{R}^N}h(x)|u|^p\,dx\right)^{1/p}.$$

If $h(x)\equiv1$ on $\mathbb{R}^N$, the norm is denoted by $\|\cdot\|_p$.

In this paper we seek weak solutions for problem (1.3) in a subspace of $H^1(\mathbb{R}^N)$. Let E be the weighted Sobolev space defined by

$$E=\left\{u\in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N}h(x)|u|^p\,dx<\infty\right\},$$

equipped with the norm

$$\|u\|^2_E=\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)\,dx+\left(\int_{\mathbb{R}^N}h(x)|u|^p\,dx\right)^{2/p}.$$

We define a *weak solution* for problem (1.3) as a function $u \in E \setminus \{0\}$ with $u(x) \geq 0$ a.e. $x \in \mathbb{R}^N$ satisfying

$$\int_{\mathbb{R}^N} a(|\nabla u|^2)\nabla u \nabla v dx + \int_{\mathbb{R}^N} a(u^2) uv dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv dx \\ = \lambda \int_{\mathbb{R}^N} |u|^{q-2}uv dx$$

for all u, $v \in E$.

In this case, we say that λ is an eigenvalue of problem (1.3) and the corresponding solution $u \in E$ is an eigenfunction corresponding to this eigenvalue. This definition is in accordance with the definition introduced by Fučik, Nečas, Souček, and Souček [5, p. 117] in the case of nonlinear eigenvalue problems.

The main result of the present paper establishes the following nonexistence and multiplicity property.

THEOREM 1. *Assume that hypotheses* (1.4), (1.5), (1.6) *and* (1.7) *are fulfilled. Then there exist positive numbers* λ_* *and* λ^* *such that the following properties hold:*

(i) *problem* (1.3) *does not have any solution if* $\lambda \in (0, \lambda_*)$;

(ii) *problem* (1.3) *has at least two solutions for all* $\lambda \in (\lambda^*, \infty)$.

In particular, this result establishes the existence of a continuous spectrum in a neighbourhood of infinity.

We point out that similar results can be obtained if the C^1-potential $a : \mathbb{R}^+ \to \mathbb{R}^*$ satisfies the more general hypotheses: there exist positive numbers γ, $\Gamma > 0$, $\kappa \in [0, 1]$ and $p \in (1, \infty)$ such that

$$\gamma\,(\kappa + t)^{m-2} \leq a(t^2) \leq \Gamma\,(\kappa + t)^{m-2} \text{ for all } t \geq 0 \tag{1.8}$$

and

$$\left(\gamma - \frac{1}{2}\right) a(t) \leq ta'(t) \leq \Gamma a(t) \text{ for all } t \geq 0. \tag{1.9}$$

2. NONEXISTENCE FOR LOW PERTURBATIONS

This section is devoted to the proof of the first part of Theorem 1.

Let $\mathcal{J} : E \to \mathbb{R}$ be the variational functional defined by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (A(|\nabla u|^2) + A(|u|^2))\, dx + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\, dx,$$

where $A(t) := \int_0^t a(s)ds$.

Then $\mathcal{J} \in C^1(E, \mathbb{R})$ and its Gâteux derivative is given by

$$\begin{aligned} \langle \mathcal{J}'(u), v\rangle &= \int_{\mathbb{R}^N} (a(|\nabla u|^2)\nabla u \nabla v + a(|u|^2)uv)\, dx + \int_{\mathbb{R}^N} h(x)|u|^{p-2}uv\, dx \\ &-\lambda \int_{\mathbb{R}^N} |u|^{q-2}uv\, dx \end{aligned}$$

for any u, $v \in E$.

Since the problem has a variational structure, then solutions of problem (1.3) are critical points of the energy functional $\mathcal{J}$.

We first prove that if $u \in E$ is a solution of problem (1.3), then λ should be large enough. Indeed, if u is a solution then

$$\int_{\mathbb{R}^N} a(|\nabla u|^2)|\nabla u|^2 dx + \int_{\mathbb{R}^N} a(u^2)u^2 dx + \int_{\mathbb{R}^N} h(x)|u|^p dx = \lambda \int_{\mathbb{R}^N} |u|^q dx. \tag{2.1}$$

Next, we apply the Young inequality

$$st \leq \frac{s^\alpha}{\alpha} + \frac{t^\beta}{\beta}, \quad \text{for all } s, t > 0$$

where α, $\beta > 1$ satisfy $1/\alpha + 1/\beta = 1$.

Taking $a = h(x)^{q/p}|u|^q$, $b = \lambda/[h(x)]^{q/p}$, $\alpha = p/q$ and $\beta = p/(p-q)$ we obtain that

$$h(x)^{q/p}|u|^q \frac{\lambda}{h(x)^{q/p}} \leq \frac{q}{p}(h(x)^{q/p}|u|^q)^{p/q} + \frac{p-q}{p}\left(\frac{\lambda}{h(x)^{q/p}}\right)^{p/(p-q)}.$$

Integrating over $\mathbb{R}^N$ we have

$$\lambda \int_{\mathbb{R}^N} |u|^q\, dx \leq \frac{q}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx + \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}}\, dx.$$

Combining this inequality with (2.1) we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2)\, dx &\leq \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}}\, dx \\ &+ \frac{q-p}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx. \end{aligned}$$

Since $q < p$ it follows that

$$\frac{q-p}{p}\int_{\mathbb{R}^N} h(x)|u|^p\, dx < 0,$$

hence

$$\int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2)\, dx \leq \frac{p-q}{p}\lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{1}{h(x)^{q/(p-q)}}\, dx. \tag{2.2}$$

By the Sobolev embedding theorem, there exists $C_q > 0$ such that for all $u \in H^1(\mathbb{R}^N)$ that

$$C_q \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{2/q} \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx.$$

So, by (1.6),

$$C_q \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{2/q} \leq \frac{1}{\gamma} \int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2) \, dx. \tag{2.3}$$

By relation (2.1),

$$\int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2) \, dx \leq \lambda \int_{\mathbb{R}^N} |u|^q \, dx. \tag{2.4}$$

Combining (2.3) and (2.4) we obtain

$$C_q \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{2/q} \leq \frac{\lambda}{\gamma} \int_{\mathbb{R}^N} |u|^q \, dx. \tag{2.5}$$

Therefore

$$(C_q \gamma \lambda^{-1})^{q/(q-2)} \leq \int_{\mathbb{R}^N} |u|^q \, dx.$$

We deduce that

$$C_q (C_q \gamma \lambda^{-1})^{2/(q-2)} \leq \frac{1}{\gamma} \int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2) \, dx.$$

Combining this inequality with (2.2) we obtain

$$\begin{aligned} C_q (C_q \gamma \lambda^{-1})^{2/(q-2)} &\leq \frac{1}{\gamma} \int_{\mathbb{R}^N} (a(|\nabla u|^2)|\nabla u|^2 + a(u^2)u^2) \, dx \\ &\leq \frac{p-q}{p\gamma} \lambda^{p/(p-q)} \int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}} \, dx. \end{aligned}$$

We conclude that if u is an eigenfunction of problem (1.3), then the corresponding eigenvalue satisfies $\lambda > \lambda_*$, where

$$\lambda_* := \left[C_q^{q/(q-2)} \frac{p\gamma}{p-q} \left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}} \, dx \right)^{-1} \right]^{(p-q)(q-2)/(q(p-2))}.$$

This concludes the proof of part (i) in Theorem 1.

3. CASE OF HIGH PERTURBATIONS

This section is devoted to the proof of (ii) in Theorem 1.

3.1. **Auxiliary results**

We establish some properties in order to use the direct method of the calculus of variations.

LEMMA 1. *The functional $\mathcal{J}$ is coercive.*

Proof. We recall the following elementary inequality, whose proof relies on elementary arguments: for every $k_1 > 0$, $k_2 > 0$ and $0 < s < r$ we have

$$k_1|t|^s - k_2|t|^r \leq C_{rs}k_1\left(\frac{k_1}{k_2}\right)^{s/(r-s)}, \quad \text{for all } t \in \mathbb{R}, \tag{3.1}$$

where $C_{rs} > 0$ is a constant depending on r and s.

In (3.1) we take: $k_1 = \frac{\lambda}{q}$, $k_2 = \frac{h(x)}{2p}$, $s = q$ and $r = p$. It follows that for all $x \in \mathbb{R}^N$

$$\begin{aligned}
\frac{\lambda}{q}|u(x)|^q - \frac{h(x)}{2p}|u(x)|^p &\leq C_{pq}\frac{\lambda}{q}\left(\frac{\lambda/q}{h(x)/2p}\right)^{(q/(p-q))} \\
&= C_{pq}\lambda^{(p/(p-q))}\frac{1}{qh(x)^{q/(p-q)}}\left(\frac{2p}{q}\right)^{q/(p-q)} \\
&= C(p,q)\frac{1}{h(x)^{q/(p-q)}}.
\end{aligned}$$

By integration we deduce that over $\mathbb{R}^N$ it follows that

$$\int_{\mathbb{R}^N}\left(\frac{\lambda}{q}|u|^q - \frac{h(x)}{2p}|u|^p\right)dx \leq C(p,q)\lambda^{(p/(p-q))}\int_{\mathbb{R}^N}\frac{dx}{h(x)^{q/(p-q)}}\,dx.$$

Next, by the integrability hypothesis (1.2), we find $C_1 > 0$ such that

$$\int_{\mathbb{R}^N}\left(\frac{\lambda}{q}|u|^q - \frac{h(x)}{2p}|u|^p\right)dx \leq C_1.$$

It follows that

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(A(|\nabla u|^2) + A(|u|^2))\,dx + \frac{1}{p}\int_{\mathbb{R}^N}h(x)|u|^p\,dx \\
&- \frac{\lambda}{q}\int_{\mathbb{R}^N}|u|^q\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}(A(|\nabla u|^2)+A(|u|^2))\,dx - \left[\int_{\mathbb{R}^N}\left(\frac{\lambda}{q}|u|^q - \frac{h(x)}{2p}|u|^p\right)\right]dx \\
&- \int_{\mathbb{R}^N}\frac{h(x)}{2p}|u|^p + \frac{1}{p}\int_{\mathbb{R}^N}h(x)|u|^p\,dx \\
&\geq \frac{\gamma}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\,dx - C_1 + \frac{1}{p}\int_{\mathbb{R}^N}h(x)|u|^p\,dx,
\end{aligned} \tag{3.2}$$

hence $\mathcal{J}$ is coercive. □

Lemma 2. *Let $\{u_n\}$ be a sequence in E such that $\mathcal{J}(u_n)$ is bounded. Then, going eventually to a subsequence, which converges weakly in E to u_0, we have*

$$\mathcal{J}(u_0) \leq \liminf_{n\to\infty} \mathcal{J}(u_n).$$

Proof. We first use (3.2) in order to show that the sequence $\{u_n\}$ is bounded in E. From now on, we use similar ideas as those developed in Rădulescu [13] (proof of Lemma 2). □

3.2. **Proof of Theorem 1 completed**

We prove that problem (1.3) has at least two nonnegative solutions, provided that λ is sufficiently large. One of the solutions is obtained *via* the direct method of the calculus of variations, while the second solution is deduced by applying a mountain pass argument. Finally, these solutions are different because they have different energy levels.

Using Lemma 1, Lemma 2 and the direct method of the calculus of variations we obtain $u \in E$ such that

$$\mathcal{J}(u) = \inf_{v\in E} \mathcal{J}(v),$$

hence u is a solution of problem (1.3).

We prove that u is a nontrivial solution. For this purpose it is enough to show that $\inf_{v\in E} \mathcal{J}(v) < 0$ if λ is large enough.

Consider the constrained minimization problem

$$\lambda^* = \inf\left\{\frac{q}{2}\int_{\mathbb{R}^N} (A(|\nabla u|^2) + A(|u|^2))\,dx + \frac{q}{p}\int_{\mathbb{R}^N} h(x)|u|^p\,dx;\ u \in E,\right.$$
$$\left.\int_{\mathbb{R}^N} |u|^q\,dx = 1\right\}.$$

We observe that $\lambda^* > 0$. Indeed, if $u \in E$ and $\int_{\mathbb{R}^N} |u|^q\,dx = 1$, then by Hölder's inequality we obtain

$$1 = \int_{\mathbb{R}^N} |u|^q\,dx \leq \left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}}\right)^{(p-q)/p} \cdot \left(\int_{\mathbb{R}^N} h(x)|u|^p\,dx\right)^{q/p}.$$

It follows that

$$\lambda^* \geq \frac{q}{p}\left(\int_{\mathbb{R}^N} \frac{dx}{h(x)^{q/(p-q)}}\right)^{(q-p)/p} > 0.$$

Fix $\lambda > \lambda^*$. Then there is $u_1 \in E$ with $\int_{\mathbb{R}^N} |u_1|^q\,dx = 1$ such that

$$\lambda\int_{\mathbb{R}^N} |u_1|^q\,dx = \lambda > \frac{q}{2}\int_{\mathbb{R}^N} (A(|\nabla u_1|^2) + A(|u_1|^2))\,dx + \frac{q}{p}\int_{\mathbb{R}^N} h(x)|u_1|^p\,dx,$$

hence

$$\mathcal{J}(u_1) = \frac{1}{2}\int_{\mathbb{R}^N}(A(|\nabla u_1|^2) + A(|u_1|^2))\,dx + \frac{1}{p}\int_{\mathbb{R}^N} h(x)|u_1|^p\,dx \\ -\frac{\lambda}{q}\int_{\mathbb{R}^N}|u_1|^q\,dx < 0$$

and consequently $\inf_{u\in E}\mathcal{J}(u) < 0$.

We conclude that for all $\lambda > \lambda^*$, problem (1.3) has a nontrivial weak solution $u_1 \in E$. Moreover, the associated energy is negative, namely $\mathcal{J}(u_1) < 0$. Since $\mathcal{J}(u_1) = \mathcal{J}(|u_1|)$ we may assume that $u_1 \geq 0$ in $\mathbb{R}^N$.

Next, we are concerned with the existence of a second solution for problem (1.3). For this purpose we combine the mountain pass theorem of Ambrosetti and Rabinowitz [2] with a truncation argument.

Fix $\lambda \geq \lambda^*$ and consider the truncation function

$$g(x,t) = \begin{cases} 0, & \text{for} \quad t < 0 \\ \lambda t^{q-1} - h(x)t^{p-1}, & \text{for} \quad 0 \leq t \leq u_1(x) \\ \lambda u_1(x)^{q-1} - h(x)u_1(x)^{p-1}, & \text{for} \quad t > u_1(x) \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s)\,ds.$$

Consider the energy functional $\mathcal{F} : E \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(A(|\nabla u|^2) + A(|u|^2))\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx.$$

Using similar arguments as for $\mathcal{J}$ we obtain that $\mathcal{F} \in C^1(E,\mathbb{R})$ and

$$\langle \mathcal{F}'(u), v\rangle = \int_{\mathbb{R}^N}(a(|\nabla u|^2)\nabla u\nabla v + a(|u|^2)uv)\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx,$$

for all u, $v \in E$.

Moreover, if u is a critical point of $\mathcal{F}$, then $u \geq 0$ in $\mathbb{R}^N$.

We now prove that every critical point of $\mathcal{F}$ is dominated by u_1.

LEMMA 3. *Let u be a critical point of $\mathcal{F}$. Then $u \leq u_1$.*

Proof. For every $v \in E$ we define the positive part $v^+(x) = \max\{v(x), 0\}$. By Gilbarg and Trudinger [6, Theorem 7.6] we deduce that if $v \in E$ then $v^+ \in E$. We have

$$\begin{aligned} 0 &= \langle \mathcal{F}'(u) - \mathcal{J}'(u_1), (u-u_1)^+\rangle \\ &= \int_{\mathbb{R}^N}(a(|\nabla u|^2)\nabla u - a(|\nabla u_1|^2)\nabla u_1)\nabla(u-u_1)^+\,dx + \end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} (a(u^2)u - a(u_1^2)u_1)(u-u_1)^+ \, dx - \\
&\int_{\mathbb{R}^N} [g(x,u) - \lambda u_1^{q-1} + h(x)u_1^{p-1}](u-u_1)^+ \, dx \\
= \;& \int_{[u>u_1]} (a(|\nabla u|^2)\nabla u - a(|\nabla u_1|^2)\nabla u_1)(\nabla u - \nabla u_1) \, dx + \\
&\int_{[u>u_1]} (a(u^2)u - a(u_1^2)u_1)(u-u_1) \, dx.
\end{aligned}$$

Since the mapping $\mathbb{R}^N \ni x \mapsto a(|x|^2)x$ is strictly monotone, there exists $\alpha > 0$ such that

$$\begin{aligned}
0 &= \langle \mathcal{F}'(u) - \mathcal{J}'(u_1), (u-u_1)^+ \rangle \\
&\geq \alpha \int_{[u>u_1]} |\nabla(u-u_1)|^2 \, dx + \alpha \int_{[u>u_1]} (u-u_1)^2 \, dx \geq 0.
\end{aligned}$$

We conclude that $u \leq u_1$.

Next, we prove the existence of a nonnegative critical point $u_2 \in E$ of $\mathcal{F}$ such that $\mathcal{F}(u_2) > 0$. By Lemma 3 we have $0 \leq u_2 \leq u_1$ in Ω. Therefore

$$g(x,u_2) = \lambda u_2^{q-1} - h(x)u_2^{p-1} \quad \text{and} \quad G(x,u_2) = \frac{\lambda}{q}u_2^q - \frac{h(x)}{p}u_2^p,$$

hence

$$\mathcal{F}(u_2) = \mathcal{J}(u_2) \quad \text{and} \quad \mathcal{F}'(u_2) = \mathcal{J}'(u_2).$$

More precisely, we prove in what follows that

$$\mathcal{J}(u_2) > 0 = \mathcal{J}(0) > \mathcal{J}(u_1) \qquad \text{and} \qquad \mathcal{J}'(u_2) = 0\,.$$

This will conclude the proof of the main result.

We first establish that $\mathcal{F}$ satisfies one of the geometric assumptions of the mountain pass theorem.

LEMMA 4. *There exists $\rho \in (0, \|u_1\|)$ and $a > 0$ such that $\mathcal{F}(u) \geq a$, for all $u \in E$ with $\|u\| = \rho$.*

Proof. By hypothesis (1.6) we obtain for all $u \in E$

$$\begin{aligned}
\mathcal{F}(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (A(|\nabla u|^2) + A(|u|^2)) \, dx - \int_{\mathbb{R}^N} G(x,u) \, dx \\
&\geq \frac{\gamma}{2}\|u\|^2 - \int_{[u>u_1]} G(x,u) \, dx - \int_{[u<u_1]} G(x,u) \, dx \\
&= \frac{\gamma}{2}\|u\|^2 - \frac{\lambda}{q}\int_{[u>u_1]} u_1^q \, dx + \frac{1}{p}\int_{[u>u_1]} h(x)u_1^p \, dx - \frac{\lambda}{q}\int_{[u>u_1]} u^q \, dx +
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p}\int_{[u>u_1]} h(x)u^p\, dx \\
\geq\ & \frac{\gamma}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} |u|^q\, dx.
\end{aligned}
$$

By hypothesis (1.5), E is continuously embedded into $L^q(\mathbb{R}^N)$. Thus, there exists $L > 0$ such that for all $u \in E$

$$|u|_q \leq L\,\|u\|\,.$$

Therefore

$$\mathcal{F}(u) \geq \frac{\gamma}{2}\,\|u\|^2 - L_1\|u\|^q = \|u\|^2\left[\frac{\gamma}{2} - L_1\|u\|^{q-2}\right],$$

where L_1 is a positive constant.

Since $q > m$, our conclusion follows. □

Lemma 5. *The functional $\mathcal{F}$ is coercive.*

Proof. Fix $u \in E$. Using (1.6) we obtain

$$
\begin{aligned}
\mathcal{F}(u) \ \geq\ & \frac{\gamma}{2}\,\|u\|^2 - \frac{\lambda}{q}\int_{[u>u_1]} u_1^q\, dx + \frac{1}{p}\int_{[u>u_1]} h(x)u_1^p\, dx - \frac{\lambda}{q}\int_{[u>u_1]} u^q\, dx + \\
&+\frac{1}{p}\int_{[u>u_1]} h(x)u^p\, dx \\
\geq\ & \frac{\gamma}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} u_1^q\, dx \\
=\ & \frac{\gamma}{2}\,\|u\|^2 - L_2\,,
\end{aligned}
$$

where L_2 is a positive constant.

We conclude that $\mathcal{F}(u) \to +\infty$ as $\|u\| \to \infty$, hence $\mathcal{F}$ is coercive. □

We use Lemma 4 in combination with the mountain pass theorem in the version established in Willem [15, Theorem 1.15]. Thus, there exists a sequence $(u_n) \subset E$ such that

$$\mathcal{F}(u_n) \to c > 0 \quad \text{and} \quad \mathcal{F}'(u_n) \to 0 \quad \text{as } n \to \infty, \tag{3.3}$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{F}(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

By relation (3.3) and Lemma 5 we deduce that the sequence $(u_n) \subset E$ is bounded. It follows that, up to a subsequence, we can assume that there exists

$u_2 \in E$ such that u_n converges weakly to u_2. Standard arguments based on Sobolev embeddings show that

$$\lim_{n\to\infty} \langle \mathcal{F}'(u_n), v\rangle = \langle \mathcal{F}'(u_2), v\rangle \,,$$

for any $v \in C_0^\infty(\mathbb{R}^N)$. Taking into account that $E \subset H^1(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ the above information implies that u_2 is a weak solution of problem (1.3).

We conclude that problem (1.3) has two nontrivial weak solutions. The proof of Theorem 1 is now complete. $\square$

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