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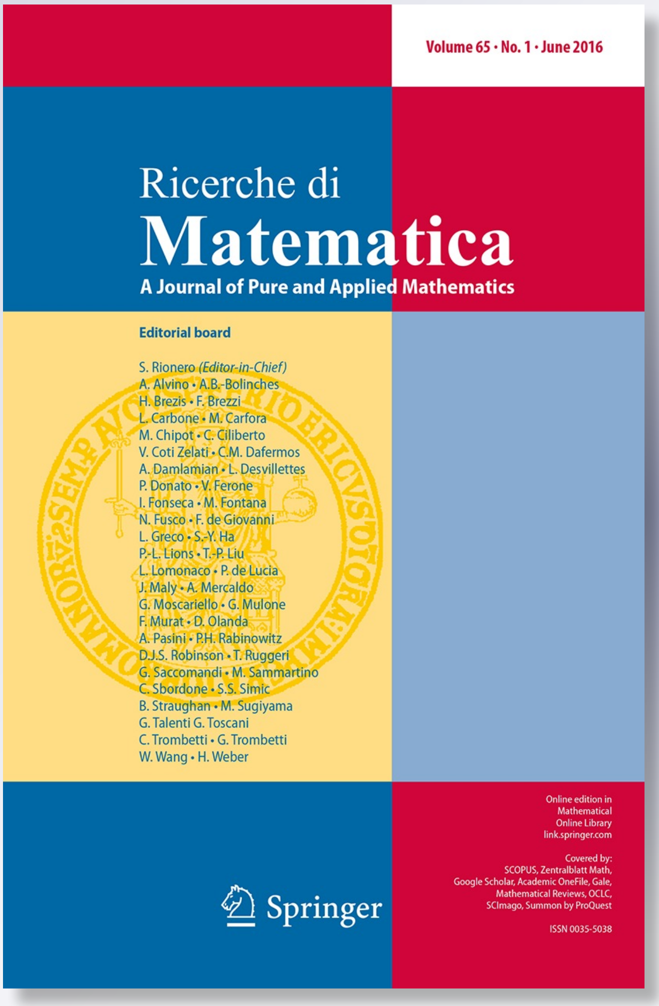

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Infinitely many solutions for a nonlinear difference equation with oscillatory nonlinearity

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Abstract In this paper, we study a discrete nonlinear boundary value problem that involves a nonlinear term oscillating near the origin and a power-type nonlinearity u^p. By using variational methods, we establish the existence of a sequence of non-negative weak solutions that converges to 0 if $p \geq 1$. In the sublinear case, we prove that for all n positive integer, the problem has at least n weak solutions if the parameter lies in a certain range.

Keywords Difference equations · Discrete Laplacian · Oscillatory nonlinearities · Variational methods

Mathematics Subject Classification 39A14 · 47J30

Dedicated with esteem to Professor Hugo Beirão da Veiga on his 70th anniversary.

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1 Introduction and preliminary results

Let $n \geq 2$ be an integer number and denote $\mathbb{Z}[1, n] := \{1, \ldots, n\}$. The discrete Laplace operator is defined by

$$\Delta u(k) = \nabla(\nabla u(k+1)),$$

where ∇ is the backward difference operator, namely

$$\nabla u(k) = u(k) - u(k-1) \quad \text{for all } k \in \mathbb{Z}[1, n].$$

In this paper, we are interested in the existence of solutions solutions $u = (u(1), \ldots, u(n)) \in \mathbb{R}_+^T$ of the following problem

$$\begin{cases} -\Delta u(k) = \lambda a(k) u(k)^p + f(u(k)) & \text{for all } k \in \mathbb{Z}[1, n], \\ u(0) = u(n+1) = 0, \end{cases} \qquad (P_\lambda)$$

where $a = (a(1), \ldots, a(n)) \in \mathbb{R}^n$, $f : [0, +\infty) \to \mathbb{R}$ is continuous, $p > 0$ and $\lambda \in \mathbb{R}$.

This problem is in relationship with the study of the properties of solitons in photorefractive media, see Krolikowski et al. [6]. We also refer to Eisenberg et al. [3] for the first experimental study of discrete spatial solitons in nonlinear waveguide arrays with Kerr nonlinearity. Soon thereafter, waveguides with a negative diffraction were obtained, which enabled defocusing of light and paved the way to the discovery of the discrete diffraction-managed spatial solitons. We refer to Pankov et al. [14] for related results and for the qualitative analysis of solutions of discrete nonlinear Schrödinger equations with saturable nonlinearity.

A thorough qualitative analysis of nonlinear discrete problems by using variational methods is developed in the recent works by Rădulescu [15] and Rădulescu and Repovš [16]. See also Molica Bisci and Repovš [7,8].

Problem (P_λ) is the discrete version of the semilinear elliptic equation studied in [5]. Moreover, this problem was recently extended by Molica Bisci, Rădulescu and Servadei [9,10] to general classes of quasilinear elliptic equations.

Motivated by the studies in [5,9], we focus in the present paper on the case of nonlinear difference equations. We are concerned in the study of the number of solutions of problem (P_λ) and of their behavior in the case when f oscillates near the origin. Usually, equations involving oscillatory nonlinearities give infinitely many distinct solutions (see [11,12]), but the presence of an additional term may alter the situation.

Define the vector space

$$H = \{v = (v(0), v(1), \ldots, v(n), v(n+1)) \in \mathbb{R}^{n+2} \text{ such that } v(0) = v(n+1) = 0\}.$$

Then H is a n-dimensional Hilbert space (see [1]) with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{n+1} \nabla u(k) \nabla v(k), \quad \forall u, v \in H.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{n+1} |\nabla u(k)|^2 \right)^{1/2}.$$

For all $u \in H$ we set

$$\|v\|_\infty = \max_{k \in \mathbb{Z}[1,n]} |v(k)|. \tag{1.1}$$

Since H is finite-dimensional, the norms $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent on H.

Definition 1.1 We say that $u \in H$ is a weak solution for the problem (P_λ) if

$$\sum_{k=1}^{n+1} \nabla u(k) \nabla v(k) - \lambda \sum_{k=1}^{n} a(k) u(k)^p v(k) - \sum_{k=1}^{n} f(u(k)) v(k) = 0, \tag{1.2}$$

for all $v \in H$.

Remark 1.2 Note that (1.2) can be obtained by multiplying (P_λ) with $v(k)$ for all $k \in \mathbb{Z}[1, n]$ and summing up from $k = 0$ to $k = n + 1$. By taking into account that $v(0) = v(n + 1) = 0$ and using some simple computations we deduce the variational characterization of weak solutions from (1.2).

2 Main results

Throughout this paper, we assume that $f : [0, +\infty) \to \mathbb{R}$ is a continuous function and we denote for all $s \in (0, +\infty)$, $F(s) := \int_0^s f(t)dt$.

We assume that f oscillates near the origin, namely the following conditions are fulfilled:

(f_1^0) $-\infty < \liminf_{s \to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s \to 0^+} \frac{F(s)}{s^2} > \frac{1}{n}$;

(f_2^0) $l_0 := \liminf_{s \to 0^+} \frac{f(s)}{s} < 0$.

Example 2.1 Let $\alpha > 1$, $\beta \in \mathbb{R}$ and $\gamma > 0$. Define $f_0 : [0, +\infty) \to \mathbb{R}$ by

$$f_0(s) = \begin{cases} 0 & \text{if } s = 0, \\ s(1 + \alpha \sin(\beta s^{-\gamma})) & \text{if } s > 0, \end{cases}$$

Then f_0 satisfies assumptions (f_1^0) and (f_2^0).

Remark 2.2 Hypotheses (f_1^0) and (f_2^0) imply that

$$f(0) = 0. \tag{2.1}$$

We point out that condition (f_1^0) allows us to deduce some information about the number of solutions for problem (P_λ), while (f_2^0) yields the existence of the solutions.

The main results in this paper distinguish between the superlinear case $p \geq 1$ and the sublinear setting that corresponds to $p \in (0, 1)$.

Theorem 2.3 *Let $a = (a(1), \ldots, a(n)) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $p \geq 1$. Assume that $f \in C([0, +\infty); \mathbb{R})$ satisfies conditions (f_1^0) and (f_2^0). If either*

(i) $p = 1$, $l_0 \in (-\infty, 0)$ and $\lambda a(k) < \lambda_0$ for all $k \in \mathbb{Z}[1, n]$ and some $\lambda_0 \in (0, -l_0)$ or
(ii) $p = 1$, $l_0 = -\infty$ and $\lambda \in \mathbb{R}$ is arbitrary or
(iii) $p > 1$ and $\lambda \in \mathbb{R}$ is arbitrary,

then there exists a sequence $\{u_i\}_i$ in H of non-negative, distinct weak solutions of problem (P_λ) such that

$$\lim_{i \to +\infty} \|u_i\| = \lim_{i \to +\infty} \|u_i\|_\infty = 0. \tag{2.2}$$

Theorem 2.4 *Let $a = (a(1), \ldots, a(n)) \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $0 < p < 1$. Assume that $f \in C([0, +\infty); \mathbb{R})$ satisfies conditions (f_1^0) and (f_2^0). Then, for every $n \in \mathbb{N}$, there exists $\Lambda_n > 0$ such that problem (P_λ) has at least n distinct weak solutions $u_{1,\lambda}, \ldots, u_{n,\lambda} \in H$ such that*

$$\|u_{i,\lambda}\| < \frac{1}{i} \quad \text{and} \quad \|u_{i,\lambda}\|_\infty < \frac{1}{i}, \text{ for any } i = 1, \ldots, n, \tag{2.3}$$

provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

3 An auxiliary problem

Consider the problem

$$\begin{cases} -\Delta u(k) + c(k)u(k) = g(k, u(k)), \ k \in \mathbb{Z}[1, n], \\ u(0) = u(n+1) = 0\,. \end{cases} \qquad (P_g^c)$$

Here, we assume that $c = (c(1), \ldots, c(n)) \in \mathbb{R}^n$ is such that

$$\min_{k \in \mathbb{Z}[1,n]} c(k) > 0, \tag{3.1}$$

while $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions

$$g(k, 0) = 0 \quad \text{for every } k \in \mathbb{Z}[1, n]; \tag{3.2}$$

there exists $M_g > 0$ such that

$$|g(k, s)| \leq M_g \quad \text{for every } k \in \mathbb{Z}[1, n] \text{ and all } s \geq 0; \tag{3.3}$$

there exist δ and η, with $0 < \delta < \eta$ such that

$$g(k, s) \leq 0 \quad \text{for every } k \in \mathbb{Z}[1, n] \text{ and all } s \in [\delta, \eta]. \tag{3.4}$$

We extend the function g by taking $g(k, s) = 0$ for every $k \in \mathbb{Z}[1, n]$ and $s \leq 0$.

Definition 3.1 By a weak solution for problem (P_g^c) we understand a vector $u \in H$ such that for all $v \in H$

$$\sum_{k=1}^{n+1} \nabla u(k) \nabla v(k) + \sum_{k=1}^{n} c(k)u(k)v(k) - \sum_{k=1}^{n} g(k, u(k))v(k) = 0.$$

Let $E_{c,g} : H \to \mathbb{R}$ be the energy functional associated to problem (P_g^c), namely

$$E_{c,g}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{k=1}^{n} c(k)u(k)^2 - \sum_{k=1}^{n} G(k, u(k)), \; u \in H, \tag{3.5}$$

where $G(k, s) := \int_0^s g(k, t)dt$ for any $s \in \mathbb{R}$ and $k \in \mathbb{Z}[1, n]$.

Then $E_{c,g}$ is well-defined, of class $C^1(H; \mathbb{R})$ and

$$\langle E'_{c,g}(u), v\rangle = \langle u, v\rangle + \sum_{k=1}^{n} c(k)u(k)v(k) - \sum_{k=1}^{n} g(k, u(k))v(k), \; \forall u, v \in H.$$

Thus, the weak solutions of (P_g^c) coincide with the critical points of $E_{c,g}$.

Finally, we introduce the set W^η defined as follows

$$W^\eta := \{u \in H : \; \|u\|_\infty \leq \eta\},$$

where η is a positive parameter given in (3.4).

Since $g(k, 0) = 0$ for every $k \in \mathbb{Z}[1, n]$ by (3.2), then $u \equiv 0$ is clearly a weak solution of problem (P_g^c).

Theorem 3.2 *Assume that* $c = (c(1), \ldots, c(n)) \in \mathbb{R}^n$ *satisfies* (3.1) *and that* $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ *is a Carathéodory function satisfying* (3.2)*,* (3.3) *and* (3.4)*. Then*

(a) the functional $E_{c,g}$ *is bounded from below on* W^η *attaining its infimum at some* $\tilde{u} \in W^\eta$*;*

(b) $\tilde{u}(k) \in [0, \delta]$ *for every* $k \in \mathbb{Z}[1, n]$*, where* δ *is the positive parameter given in* (3.4)*;*
(c) $\tilde{u}$ *is a non-negative weak solution of problem* (P_g^c)*.*

Proof (a) Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in the finite-dimensional space H, the set W^η is compact in H. Combining this fact with the continuity of $E_{c,g}$, we infer that $E_{c,g}\Big|_{W^\eta}$ attains its infimum at $\tilde{u} \in W^\eta$.

(b) Let δ be as in assumption (3.4) and let $M := \{k \in \mathbb{Z}[1, n] : \ \tilde{u}(k) \notin [0, \delta]\}$. Hence, arguing by contradiction, we suppose that $M \neq \emptyset$. Define the truncation function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) := \min\{s_+, \delta\}$, where $s_+ = \max\{s, 0\}$ and set $w := \gamma \circ \tilde{u}$. Since $\gamma(0) = 0$, we have $w(0) = w(n+1) = 0$, so $w \in H$. Besides, $0 \le w(k) \le \delta$ for every $k \in \mathbb{Z}[1, n]$. By assumption (3.4) we know that $\delta < \eta$, and so $w \in W^\eta$. We introduce the sets $M_- := \{k \in M : \ \tilde{u}(k) < 0\}$ and $M_+ := \{k \in M : \ \tilde{u}(k) > \delta\}$. Thus, $M = M_- \cup M_+$ and we have that

$$w(k) = \begin{cases} \tilde{u}(k) & \text{for all } k \in \mathbb{Z}[1, n] \setminus M, \\ 0 & \text{for all } k \in M_-, \\ \delta & \text{for all } k \in M_+. \end{cases}$$

Moreover, we have

$$\begin{aligned} E_{c,g}(w) - E_{c,g}(\tilde{u}) &= \frac{1}{2}\left(\|w\|^2 - \|\tilde{u}\|^2\right) + \frac{1}{2}\sum_{k=1}^{n} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &\quad - \sum_{k=1}^{n}[G(k, w(k)) - G(k, \tilde{u}(k))] \\ &=: \frac{1}{2}J_1 + \frac{1}{2}J_2 - J_3. \end{aligned} \tag{3.6}$$

Since γ is a Lipschitz function with Lipschitz constant 1, and $w = \gamma \circ \tilde{u}$, we have

$$\begin{aligned} J_1 = \|w\|^2 - \|\tilde{u}\|^2 &= \sum_{k=1}^{n+1}[|\nabla w(k)|^2 - |\nabla \tilde{u}(k)|^2] \\ &= \sum_{k=1}^{n+1}\left[|w(k) - w(k-1)|^2 - |\tilde{u}(k) - \tilde{u}(k-1)|^2\right] \le 0. \end{aligned} \tag{3.7}$$

Since $\min_{k \in \mathbb{Z}[1,n]} c(k) > 0$ by (3.1), we have

$$\begin{aligned} J_2 &= \sum_{k=1}^{n} c(k)[(w(k))^2 - (\tilde{u}(k))^2] = \sum_{k \in M} c(k)[(w(k))^2 - (\tilde{u}(k))^2] \\ &= -\sum_{k \in M_-} c(k)(\tilde{u}(k))^2 + \sum_{k \in M_+} c(k)[\delta^2 - (\tilde{u}(k))^2] \le 0. \end{aligned} \tag{3.8}$$

Next, we estimate J_3. Due to the fact that $g(k,s)=0$ for all $s\le 0$ and for every $k\in\mathbb{Z}[1,n]$, we have

$$\sum_{k\in M_-}\left[G(k,w(k))-G(k,\tilde{u}(k))\right]=0. \tag{3.9}$$

Moreover, by the mean value theorem, for every $k\in M_+$, there exists $\theta(k)\in[\delta,\tilde{u}(k)]\subset[\delta,\eta]$ such that

$$G(k,w(k))-G(k,\tilde{u}(k))=G(k,\delta)-G(k,\tilde{u}(k))=g(k,\theta(k))(\delta-\tilde{u}(k)).$$

Thus, taking into account hypothesis (3.4) and definition of M_+, we have

$$\sum_{k\in M_+}\left[G(k,w(k))-G(k,\tilde{u}(k))\right]\ge 0. \tag{3.10}$$

Hence, by (3.9) and (3.10), we obtain

$$J_3=\sum_{k\in M_+}\left[G(k,w(k))-G(k,\tilde{u}(k))\right]\ge 0. \tag{3.11}$$

Combining relations (3.7), (3.8), (3.11) with (3.6), we get

$$E_{c,g}(w)-E_{c,g}(\tilde{u})\le 0. \tag{3.12}$$

On the other hand, since $w\in W^\eta$, it is easy to see that $E_{c,g}(w)\ge E_{c,g}(\tilde{u})=\inf_{u\in W^\eta}E_{c,g}(u)$. By this and (3.12) we get that every term in $E_{c,g}(w)-E_{c,g}(\tilde{u})$ should be zero. In particular, from J_2 and due to (3.1), we have

$$\sum_{k\in M_-}c(k)(\tilde{u}(k))^2=\sum_{k\in M_+}c(k)[\delta^2-(\tilde{u}(k))^2]=0,$$

which implies that

$$\tilde{u}(k)=\begin{cases}0 & \text{for every } k\in M_-\\ \delta & \text{for every } k\in M_+.\end{cases}$$

In view of the definition of the sets M_- and M_+, we deduce that $M_-=M_+=\emptyset$, which contradicts $M_-\cup M_+=M\neq\emptyset$.

(c) Fix $v\in H$ arbitrarily and let $\varepsilon_0:=\frac{\eta-\delta}{\|v\|_\infty+1}>0$, where δ and η are given as in (3.4). Moreover, let $I:[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}$ be the function defined as $I(\varepsilon):=E_{c,g}(\tilde{u}+\varepsilon v)$. First of all, thanks to (b), for any $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$ we have

$$|\tilde{u}(k)+\varepsilon v(k)|\le\tilde{u}(k)+\frac{\eta-\delta}{\|v\|_\infty+1}\|v\|_\infty\le\eta,$$

for every $k \in \mathbb{Z}[1, n]$. Thus, $\tilde{u} + \varepsilon v \in W^\eta$. Consequently, due to (a), we have $I(\varepsilon) \geq I(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, that is, 0 is an interior minimum point for I. Then $I'(0) = 0$ and $\langle E'_{c,g}(\tilde{u}), v\rangle = 0$. Taking into account that $v \in H$ is arbitrary and using the definition of $E_{c,g}$, we obtain that $\tilde{u}$ is a weak solution of problem (P_g^c). Moreover, due to (b), $\tilde{u}$ is non-negative in $\mathbb{Z}[1, n]$. □

Theorem 3.2 does not guarantee that the solution $\tilde{u}$ of problem (P_g^c) is not the trivial one. In spite of this, by Theorem 3.2 we will derive the existence of nontrivial solutions for the original problem (P_λ), provided that the nonlinear term f is chosen appropriately. Finally, we define the continuous truncation function $\tau_\eta : [0, +\infty) \to \mathbb{R}$ as follows

$$\tau_\eta(s) := \min\{\eta, s\} \quad \text{for every } s \geq 0, \tag{3.13}$$

where η is the positive constant given in assumption (3.4).

4 Oscillation near the origin

In order to prove Theorems 2.3 and 2.4, we consider problem (P_g^c), where $c = (c(1), \ldots, c(n)) \in \mathbb{R}^n$ fulfills (3.1) and $g : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function which satisfies the following assumptions

$$\begin{aligned} &g(k, 0) = 0 \text{ for all } k \in \mathbb{Z}[1, n], \text{ and there exist} \\ &\overline{s} > 0 \text{ and } M > 0 \text{ such that } \max_{s\in[0,\overline{s}]} |g(k, s)| \leq M, \text{ for all } k \in \mathbb{Z}[1, n]; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\text{there exist two sequences } \{\delta_i\}_i \text{ and } \{\eta_i\}_i \text{ with } 0 < \eta_{i+1} < \delta_i < \eta_i \text{ such that} \\ &\lim_{i\to+\infty} \eta_i = 0 \quad \text{and} \quad g(k, s) \leq 0 \text{ for every } k \in \mathbb{Z}[1, n] \text{ and all } s \in [\delta_i, \eta_i],\ i \in \mathbb{N}; \end{aligned} \tag{4.2}$$

$$-\infty < \liminf_{s\to 0^+} \frac{G(k, s)}{s^2} \text{ and } \limsup_{s\to 0^+} \frac{G(k, s)}{s^2} > \frac{1}{n} \text{ uniformly for all } k \in \mathbb{Z}[1, n]. \tag{4.3}$$

Proof of Theorem 2.3 We first show that under suitable assumptions, problem (P_λ) has infinitely many distinct weak solutions, provided that $p \geq 1$. We will consider separately the cases $p = 1$ and $p > 1$ and in both situations the strategy will consist in using Theorem 3.2.

We start by proving assertion (i). In this setting we suppose that $p = 1$ and $l_0 \in (-\infty, 0)$. Let $\lambda \in \mathbb{R}$ be such that $\lambda a(k) < \lambda_0$ for all $k \in \mathbb{Z}[1, n]$ and some $0 < \lambda_0 < -l_0$. Fix $\overline{\lambda}_0 \in (\lambda_0, -l_0)$ and let

$$c(k) := \overline{\lambda}_0 - \lambda a(k) \quad \text{and} \quad g(k, s) := f(s) + \overline{\lambda}_0 s, \tag{4.4}$$

for all $(k, s) \in \mathbb{Z}[1, n] \times [0, +\infty)$. The first step consist in proving that the vector c and the function g given in (4.4) satisfy the assumptions (3.1), (4.1), (4.2) and (4.3). Note that $c \in \mathbb{R}^n$ and $\min_{k\in\mathbb{Z}[1,n]} c(k) > \overline{\lambda}_0 - \lambda_0 > 0$, which obviously implies (3.1). By (2.1) we know that $f(0) = 0$. Thus, using the regularity of f, we obtain that g is a continuous function in $\mathbb{Z}[1, n] \times [0, +\infty)$ and $g(k, 0) = 0$ for all $k \in \mathbb{Z}[1, n]$. Next, the continuity of $s \mapsto g(\cdot, s)$ and the Weierstrass theorem yield (4.1). Moreover, since for any $k \in \mathbb{Z}[1, n]$ and $s > 0$ we have $G(k, s)/s^2 = \overline{\lambda}_0/2 + F(s)/s^2$, hypothesis (f_1^0) immediately implies (4.3).

Next, we show that g satisfies (4.2). By (f_2^0), there exists a sequence $\{s_i\}_i \subset (0, 1)$ converging to 0 such that $\lim_{i\to+\infty} \frac{f(s_i)}{s_i} = l_0$. Since $\overline{\lambda}_0 < -l_0$ by assumption, there exists $\overline{\varepsilon} > 0$ such that $\overline{\lambda}_0 + \overline{\varepsilon} < -l_0$. By this and the above relation we get that for $i \geq i^* \in \mathbb{N}$,

$$f(s_i) < -\overline{\lambda}_0 s_i. \tag{4.5}$$

Thus we obtain that $g(k, s_i) = f(s_i) + \overline{\lambda}_0 s_i < 0$. Consequently, by the continuity of f, there is a neighborhood of s_i, say (δ_i, η_i) and there are two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0, 1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to+\infty} \eta_i = 0$ and $g(k, s) = \overline{\lambda}_0 s + f(s) \leq 0$ for any $k \in \mathbb{Z}[1, n]$ and all $s \in [\delta_i, \eta_i]$ and $i \geq i^*$. In this way, hypothesis (4.2) is verified for g on every interval $[\delta_i, \eta_i]$, $i \in \mathbb{N}$. In the sequel, since $\eta_i \to 0$ as $i \to +\infty$, by (4.2), without any loss of generality, we may assume that

$$0 < \delta_i < \eta_i < \overline{s}, \tag{4.6}$$

for i sufficiently large, where $\overline{s} > 0$ is given by (4.1). For every $i \in \mathbb{N}$, let $g_i : \mathbb{Z}[1, n] \times [0, +\infty) \to \mathbb{R}$ be the truncation function defined by

$$g_i(k, s) := g(k, \tau_{\eta_i}(s)) \text{ and } G_i(k, s) := \int_0^s g_i(k, t)dt, \tag{4.7}$$

for every $k \in \mathbb{Z}[1, n]$ and $s \geq 0$, where τ_{η_i} is the function defined in (3.13) with $\eta = \eta_i$. Let $E_i : H \to \mathbb{R}$ be the energy functional associated with problem $(P_{g_i}^c)$, that is $E_i := E_{c,g_i}$, where E_{c,g_i} is the functional given in (3.5) with $g = g_i$. We note that the function g_i verifies all the assumptions of Theorem 3.2 for $i \in \mathbb{N}$ large enough with $[\delta_i, \eta_i]$. Indeed, thanks to the regularity of g, the continuity of τ_η and the fact that $g(k, 0) = 0$ for all $k \in \mathbb{Z}[1, n]$, the function g_i is Carathéodory and such that $g_i(k, 0) = 0$ for every $k \in \mathbb{Z}[1, n]$. Moreover, by (4.1), (4.6) and (4.7), g_i satisfies (3.2) and (3.3). Finally, condition (3.4) is satisfied thanks to (4.2). Hence, as a consequence of Theorem 3.2, for every $i \in \mathbb{N}$, there exists $u_i \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_i(u) = E_i(u_i); \tag{4.8}$$

$$u_i(k) \in [0, \delta_i] \text{ for every } k \in \mathbb{Z}[1, n]; \tag{4.9}$$

$$u_i \text{ is a non-negative weak solution of } (P_{g_i}^c). \tag{4.10}$$

Using the definition of τ_η, relation (4.7) and the fact that $0 \leq u_i(k) \leq \delta_i < \eta_i$ for every $k \in \mathbb{Z}[1,n]$, we have $g_i(k, u_i(k)) = g(k, \tau_{\eta_i}(u_i(k))) = g(k, u_i(k))$ for every $k \in \mathbb{Z}[1,n]$. Thus, by the above relation and (4.10), u_i is a non-negative weak solution not only for $(P^c_{g_i})$ but also for problem (P^c_g). In the sequel, we prove that there are infinitely many distinct elements in the sequence $\{u_i\}_i$. In order to see this, the first step consists in proving that

$$E_i(u_i) < 0 \quad \text{for } i \in \mathbb{N} \text{ large enough and} \tag{4.11}$$

$$\lim_{i\to+\infty} E_i(u_i) = 0. \tag{4.12}$$

Due to (f^0_1) and (4.4), we have that $\limsup_{s\to0^+} \frac{G(k,s)}{s^2} > \frac{\overline{\lambda}_0}{2} + \frac{1}{n}$. In particular, there exists a sequence $\{\tilde{s}_i\}_i$, with

$$0 < \tilde{s}_i \leq \delta_i \text{ for all } i \in \mathbb{N} \text{ and} \tag{4.13}$$

$$G(k, \tilde{s}_i) > \left(\frac{1}{n} + \frac{\overline{\lambda}_0}{2}\right)\tilde{s}_i^2. \tag{4.14}$$

Now, let us fix $i \in \mathbb{N}$ sufficiently large and let us define the function $w_i \in H$ by $w_i(k) := \tilde{s}_i$ for every $k \in \mathbb{Z}[1,n]$. Then $\|w_i\|_\infty = \tilde{s}_i \leq \delta_i < \eta_i < 1$ by (4.2) and (4.13). Hence, $w_i \in W^{\eta_i}$. This yields that for every $k \in \mathbb{Z}[1,n]$, we have

$$G_i(k, w_i(k)) = G_i(k, \tilde{s}_i) = \int_0^{\tilde{s}_i} g_i(k,t)dt = G(k, \tilde{s}_i). \tag{4.15}$$

By this and taking into account (3.1), (4.4), (4.14), (4.15), for i sufficiently large we have

$$\begin{aligned} E_i(w_i) &= \frac{1}{2}\sum_{k=1}^{n+1} |\nabla w_i(k-1)|^2 + \frac{1}{2}\sum_{k=1}^{n} c(k)(w_i(k))^2 - \sum_{k=1}^{n} G_i(k, w_i(k)) \\ &< (\tilde{s}_i)^2 + \frac{1}{2}\overline{\lambda}_0 T(\tilde{s}_i)^2 - n\left(\frac{1}{n} + \frac{\overline{\lambda}_0}{2}\right)(\tilde{s}_i)^2 < 0. \end{aligned}$$

Consequently, using also (4.8) for i sufficiently large, the above estimation and $w_i \in W^{\tilde{s}_i} \subset W^{\eta_i}$ show that

$$E_i(u_i) = \min_{u\in W^{\eta_i}} E_i(u) \leq E_i(w_i) < 0, \tag{4.16}$$

which proves in particular (4.11). Next, we prove (4.12). For every $i \in \mathbb{N}$ sufficiently large, by using the definition of G_i, the mean value theorem, (4.1), (4.2), (4.6), (4.7) and (4.9), we have

$$E_i(u_i) \geq -\sum_{k=1}^{n} G_i(k, u_i(k)) = -\sum_{k=1}^{n} G(k, u_i(k))$$
$$\geq -\sum_{k=1}^{n} \max_{s\in[0,\overline{s}]} |g(k,s)|u_i(k) \geq -\delta_i T M.$$

Since $\lim_{i\to+\infty} \delta_i = 0$, the above estimate and (4.16) leads to (4.12).

Finally, it is easy to see that relation (2.2) is an immediate consequence of (4.9) combined with $\lim_{i\to+\infty} \delta_i = 0$, and to the fact that norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent. Thus, we get the existence of infinitely many distinct nontrivial non-negative solutions $\{u_i\}_i$ for problem (P_g^c) satisfying condition (2.2). Due to the choice of c and g in (4.4) and taking into account that $p = 1$, it is easy to see that u_i is a weak solution of problem (P_λ) and this ends the proof of assertion (i) in Theorem 2.3 in the case $p = 1$.

Now, let us consider assertion (ii). At this purpose, let $p = 1$, $l_0 = -\infty$ and $\lambda \in \mathbb{R}$ be arbitrary fixed. In this setting we choose $\overline{\lambda}_0 \in (\lambda_0, -l_0)$ and

$$c(k) := \overline{\lambda}_0 \quad \text{and} \quad g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_0)s \text{ for all } (k,s) \in \mathbb{Z}[1,n] \times [0,+\infty).$$

This case can be dealt with in a similar way as (i), using relation $f(s_i) < -(|\lambda| \cdot \|a\|_\infty + \overline{\lambda}_0)s_i$, instead of $f(s_i) < -\overline{\lambda}_0 s_i$, for i large enough, and taking into account that for every $k \in \mathbb{Z}[1,n]$ and $s \geq 0$ one has $g(k,s) = f(s) + (\lambda a(k) + \overline{\lambda}_0)s \leq f(s) + (|\lambda| \cdot \|a\|_\infty + \overline{\lambda}_0)s$.

Now, let us prove assertion (iii). At this purpose, let $p > 1$ and $\lambda \in \mathbb{R}$ be arbitrary fixed. Let us also fix a number $\overline{\lambda}_0 \in (0, -l_0)$ and choose

$$c(k) := \overline{\lambda}_0 \quad \text{and} \quad g(k,s) := \lambda a(k)s^p + \overline{\lambda}_0 s + f(s) \tag{4.17}$$

for all $(k,s) \in \mathbb{Z}[1,n] \times [0,+\infty)$. Also in this setting our aim is to prove that c and g given in (4.17) satisfy the conditions (3.1), (4.1), (4.2) and (4.3). Clearly, (3.1) is satisfied and also thanks to (f_1^0), (f_2^0) we have $g(k,0) = 0$ for all $k \in \mathbb{Z}[1,n]$. Moreover, since $a \in \mathbb{R}^n$ the continuity of $s \mapsto g(\cdot, s)$ and the Weierstrass theorem yield that (4.1) holds true. Furthermore, since $p > 1$ and $\frac{G(k,s)}{s^2} = \lambda \frac{a(k)}{p+1} s^{p-1} + \frac{\overline{\lambda}_0}{2} + \frac{F(s)}{s^2}$, for all $k \in \mathbb{Z}[1,n]$ and $s \in (0,+\infty)$, hypothesis (f_1^0) implies (4.3). In the sequel, note that for all $k \in \mathbb{Z}[1,n]$ and every $s \in [0,+\infty)$, we have

$$g(k,s) \leq |\lambda| \cdot \|a\|_\infty s^p + \overline{\lambda}_0 s + f(s). \tag{4.18}$$

As a consequence of this and of (f_2^0) we get

$$\liminf_{s\to 0^+} \frac{g(k,s)}{s} \leq \overline{\lambda}_0 + l_0 < 0, \tag{4.19}$$

for all $k \in \mathbb{Z}[1,n]$, thanks to the choice of p. In particular, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 as $i \to +\infty$ such that $g(k,s_i) < 0$ for $i \in \mathbb{N}$ large enough and for all $k \in \mathbb{Z}[1,n]$. Thus, by using the continuity of $s \mapsto g(\cdot, s)$, there exist two sequences $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to+\infty} \eta_i = 0$ and $g(k,s) \le 0$, for every $k \in \mathbb{Z}[1,n]$ and all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$ large enough. Summarizing, we deduce that hypothesis (4.2) hold true.

Finally, an argument analogous to that used in (i) proves that problem (P_g^c) is equivalent to problem (P_λ) through the choice (4.17) and so, we get the existence of infinitely many distinct nontrivial solutions $\{u_i\}_i$ for problem (P_λ) satisfying (2.2). This concludes the proof of Theorem 2.3. □

Proof of Theorem 2.4 Let $\overline{\lambda}_0 \in (0, -l_0)$, where $l_0 < 0$ is given in assumption (f_2^0) and let us choose

$$c(k) := \overline{\lambda}_0 \quad \text{and} \quad g(k,s,\lambda) := \lambda a(k) s^p + \overline{\lambda}_0 s + f(s), \tag{4.20}$$

for all $(k,s) \in \mathbb{Z}[1,n] \times [0,+\infty)$, $\lambda \in \mathbb{R}$. Note that for all $k \in \mathbb{Z}[1,n]$ and every $s \in [0,+\infty)$, we have $g(k,s,\lambda) \le |\lambda| \cdot \|a\|_\infty s^p + \overline{\lambda}_0 s + f(s)$. Next, on account of (f_2^0), there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 as $i \to +\infty$ such that $f(s_i) < -\overline{\lambda}_0 s_i$, for $i \in \mathbb{N}$ large enough. Consequently, we have $g(k,s_i,0) = \overline{\lambda}_0 s_i + f(s_i) < 0$, for $i \in \mathbb{N}$ large enough and for all $k \in \mathbb{Z}[1,n]$. Thus, due to the continuity of $s \mapsto g(\cdot,s,\cdot)$ we get that there exist three sequences $\{\delta_i\}_i$, $\{\eta_i\}_i$, $\{\lambda_i\}_i \subset (0,1)$ such that,

$$0 < \eta_{i+1} < \delta_i < s_i < \eta_i < 1, \quad \lim_{i\to+\infty} \eta_i = 0, \tag{4.21}$$

and for $i \in \mathbb{N}$ large enough,

$$g(k,s,\lambda) \le 0, \quad \text{for all } k \in \mathbb{Z}[1,n],\ \lambda \in [-\lambda_i, \lambda_i] \text{ and } s \in [\delta_i, \eta_i]. \tag{4.22}$$

For any $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $g_i : \mathbb{Z}[1,n] \times [0,+\infty) \times [-\lambda_i, \lambda_i] \to \mathbb{R}$ be the function defined by

$$g_i(k,s,\lambda) := g(k, \tau_{\eta_i}(s), \lambda) \tag{4.23}$$

and $G_i(k,s,\lambda) := \int_0^s g_i(k,t,\lambda)dt$, for all $k \in \mathbb{Z}[1,n]$ and $s \ge 0$. In the sequel, let us prove that c given in (4.20) and g_i satisfy all the assumptions of Theorem 3.2. Due to relation (2.1), it is easy to see that g_i satisfies condition (3.2). Also, the assumption (3.1) is trivially verified. Moreover, the regularity of g and the continuity of τ_η show that g_i is a Carathéodory function. Also, thanks to (4.23), (3.13), the continuity of $s \mapsto g(\cdot,s,\cdot)$ and the Weierstrass Theorem give that g_i satisfies (3.3). Finally, (4.22) and (4.23) yield (3.4) for i large enough. Hence, g_i satisfies all the assumptions of Theorem 3.2 for i large. Next, for any $i \in \mathbb{N}$, let $E_{i,\lambda} : H \to \mathbb{R}$ be the energy associated with the problem $(P^c_{g_i(\cdot,\cdot,\lambda)})$, that is,

$$E_{i,\lambda} := E_{c, g_i(\cdot,\cdot,\lambda)}, \tag{4.24}$$

where $E_{c,g_i(\cdot,\cdot,\lambda)}$ is the functional given in (3.5) with $g = g_i(\cdot,\cdot,\lambda)$. So, Theorem 3.2 allows us to deduce that, for $i \in \mathbb{N}$ sufficiently large and $\lambda \in [-\lambda_i, \lambda_i]$, there exists $u_{i,\lambda} \in W^{\eta_i}$ such that

$$\min_{u\in W^{\eta_i}} E_{i,\lambda}(u) = E_{i,\lambda}(u_{i,\lambda}), \tag{4.25}$$

$$u_{i,\lambda}(k) \in [0,\delta_i] \text{ for all } k \in \mathbb{Z}[1,n] \tag{4.26}$$

and

$$u_{i,\lambda} \text{ is a non-negative weak solution of } (P^c_{g_i(\cdot,\cdot,\lambda)}). \tag{4.27}$$

Since for i sufficiently large

$$0 \le u_{i,\lambda}(k) \le \delta_i < \eta_i, \tag{4.28}$$

for all $k \in \mathbb{Z}[1,n]$ by (4.21) and (4.26), we get $g_i(k, u_{i,\lambda}(k), \lambda) = g(k, u_{i,\lambda}(k), \lambda)$. Thus, using (4.20) we obviously have that $u_{i,\lambda}$ is a non-negative weak solution of (P_λ), provided i is large and $|\lambda| \le \lambda_i$.

In the sequel, we prove that for any $n \in \mathbb{N}$ problem (P_λ) admits at least n distinct solutions, for suitable values of λ. We first observe that due to the choice of c and g_i and (4.28), the functional $E_{i,\lambda}$ is given by

$$E_{i,\lambda}(u) = E_{i,0}(u) - \lambda \sum_{k=1}^{n} a(k)\frac{|u(k)|^{p+1}}{p+1}, \quad \text{for any } u \in H. \tag{4.29}$$

For $\lambda = 0$, the function $g_i(\cdot,\cdot,\lambda) = g_i(\cdot,\cdot,0)$ verifies the hypotheses (3.1), (4.1), (4.2) and (4.3). More precisely, $g_i(\cdot,\cdot,0)$ is exactly the function appearing in (4.7) and $E_i := E_{i,0}$ is the energy functional associated with problem $(P^c_{g_i(\cdot,\cdot,0)})$. Thus by (4.25)–(4.27), the elements $u_i := u_{i,0}$ also verify

$$E_i(u_i) = \min_{u\in W^{\eta_i}} E_i(u) \le E_i(w_i) < 0 \quad \text{for all } i \in \mathbb{N}, \tag{4.30}$$

where $w_i \in W^{\eta_i}$ is given in the proof of Theorem 2.3, see for instance (4.16).

In the sequel, let $\{\theta_i\}_i$ be an increasing sequence with negative terms such that $\lim_{i\to+\infty}\theta_i = 0$. On account of (4.30), up to a subsequence, we may assume that

$$\theta_{i-1} < E_i(u_i) \le E_i(w_i) < \theta_i, \text{ for } i \ge i^*, \text{ with } i^* \in \mathbb{N}. \tag{4.31}$$

Now, for any $i \ge i^*$ let

$$\lambda_i' := \frac{(p+1)(E_i(u_i) - \theta_{i-1})}{(\|a\|_\infty + 1)n} \quad \text{and} \quad \lambda_i'' := \frac{(p+1)(\theta_i - E_i(w_i))}{(\|a\|_\infty + 1)n}. \tag{4.32}$$

Note that λ_i' and λ_i'' are strictly positive, due to (4.31) and they are independent of λ. Now, for any fixed $n \in \mathbb{N}$, let

$$\Lambda_n := \min\{\lambda_{i^*+1}, \dots, \lambda_{i^*+n}, \lambda'_{i^*+1}, \dots, \lambda'_{i^*+n}, \lambda''_{i^*+1}, \dots, \lambda''_{i^*+n}\}.$$

On account of (4.31), $\Lambda_n > 0$ and it is independent of λ. Moreover, if $|\lambda| \le \Lambda_n$, then $|\lambda| \le \lambda_i$ for any $i = i^* + 1, \dots, i^* + n$. Consequently, for any $\lambda \in \mathbb{R}$ with $|\lambda| \le \Lambda_n$, we have that $u_{i,\lambda}$ is a non-negative weak solution of problem (P_λ), for any $i = i^* + 1, \dots, i^* + n$. In the sequel, we show that these solutions are distinct. For this purpose, note that $u_{i,\lambda} \in W^{\eta_i}$ by (4.28) and so for any $\lambda \in \mathbb{R}$ with $|\lambda| \le \Lambda_n$ we have

$$E_i(u_i) = \min_{u \in W^{\eta_i}} E_i(u) \le E_i(u_{i,\lambda}). \tag{4.33}$$

Thus by (4.29) and (4.33), for any λ with $|\lambda| \le \Lambda_n$ we obtain

$$\begin{aligned} E_{i,\lambda}(u_{i,\lambda}) &\ge E_i(u_i) - \frac{|\lambda|}{p+1}\|a\|_\infty \eta_i^{p+1} n \\ &\ge E_i(u_i) - \frac{\lambda_i'}{p+1}\|a\|_\infty n > \theta_{i-1}, \end{aligned} \tag{4.34}$$

for any $i = i^* + 1, \dots, i^* + n$, due to (4.21), (4.28), the choice of Λ_n and the definition of λ_i'. On the other hand, by (4.29), (4.30) and using the fact that $\|w_i\|_\infty = \tilde{s}_i \le \delta_i < \eta_i < 1$ (see the proof of Theorem 2.3), for any λ with $|\lambda| \le \Lambda_n$ we deduce that

$$\begin{aligned} E_{i,\lambda}(u_{i,\lambda}) &\le E_i(w_i) + \frac{|\lambda|}{p+1}\|a\|_\infty n \\ &\le E_i(w_i) + \frac{\lambda_i''}{p+1}\|a\|_\infty n < \theta_i, \end{aligned} \tag{4.35}$$

for all $i = i^* + 1, \dots, i^* + n$, again thanks to the choice of Λ_n and the definition of λ_i''. In conclusion, by (4.34), (4.35) and the properties of $\{\theta_i\}_i$, we deduce that for every $i = i^* + 1, \dots, i^* + n$ and $\lambda \in [-\Lambda_n, \Lambda_n]$, we have

$$\theta_{i-1} < E_{i,\lambda}(u_{i,\lambda}) < \theta_i < 0, \tag{4.36}$$

which yields that $E_{1,\lambda}(u_{1,\lambda}) < \dots < E_{n,\lambda}(u_{n,\lambda}) < 0$. But $u_{i,\lambda} \in W^{\eta_1}$ for every $i = i^* + 1, \dots, i^* + n$, so $E_{i,\lambda}(u_{i,\lambda}) = E_{1,\lambda}(u_{i,\lambda})$, see relation (4.23). Therefore, from above, we obtain that for every $\lambda \in [-\Lambda_n, \Lambda_n]$, $E_{1,\lambda}(u_{1,\lambda}) < \dots < E_{1,\lambda}(u_{n,\lambda}) < 0 = E_{1,\lambda}(0)$. These inequalities show that the elements $u_{1,\lambda}, \dots, u_{n,\lambda}$ are all distinct and non-trivial, provided $\lambda \in [-\Lambda_n, \Lambda_n]$.

Finally, it remains to prove conclusion (2.3). For this, by (4.21), (4.28), (4.29), (4.36) and the continuity of f we have that

$$\begin{aligned}\frac{1}{2}\|u_{i,\lambda}\|^2 &< \theta_i + \frac{|\lambda|}{p+1}\|a\|_\infty \delta_i^{p+1} n + \sum_{k=1}^{n}\int_0^{\delta_i} |f(s)|ds \\ &< \frac{\Lambda_n}{p+1}\|a\|_\infty \delta_i n + n \max_{s\in[0,1]} |f(s)|\delta_i,\end{aligned}$$

for any $i = i^*+1, \dots, i^*+n$ and $|\lambda| \le \Lambda_n$. Hence, we obtain $\|u_{i,\lambda}\| \le \tilde{c}\delta_i^{1/2}$, where

$$\tilde{c} = 2^{-1}\left(\frac{\Lambda_n}{p+1}\|a\|_\infty n + n \max_{s\in[0,1]} |f(s)|\right) > 0.$$

Since $\delta_i \to 0$ as $i \to +\infty$, without loss of generality, we may assume that

$$\delta_i \le \min\{\tilde{c}^{-2}, 1\}\frac{1}{i^2}, \tag{4.37}$$

which gives that $\|u_{i,\lambda}\| \le \frac{1}{i}$, for any $i = i^*+1, \dots, i^*+n$, provided $|\lambda| \le \Lambda_n$. In conclusion, by (4.28) and (4.37) we obtain that $\|u_{i,\lambda}\|_\infty \le \frac{1}{i^2} < \frac{1}{i}$, for any $i = i^*+1, \dots, i^*+n$, with $|\lambda| \le \Lambda_n$.

This concludes the proof of Theorem 2.4. □

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