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A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids

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We study the boundary value problem $-\mathrm{div}(a(x,\nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1})$ in Ω, $u=0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\mathrm{div}(a(x,\nabla u))$ is a $p(x)$-Laplace type operator, with $1<\beta<\gamma<\inf_{x\in\Omega} p(x)$. We prove that if λ is large enough then there exist at least two non-negative weak solutions. Our approach relies on the variable exponent theory of generalized Lebesgue–Sobolev spaces, combined with adequate variational methods and a variant of the Mountain Pass lemma.

Keywords: $p(x)$-Laplace operator; generalized Lebesgue–Sobolev space; critical point; weak solution; electrorheological fluids

1. Introduction and preliminary results

Most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces, L^p and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as 'smart fluids'), this is not adequate, but rather the exponent p should be able to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function.

This paper is motivated by phenomena which are described by nonlinear boundary value problems of the type

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x,u), & \text{for } x\in\Omega, \\ u=0, & \text{for } x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, $1<p(x)$ and $p(x)\in C(\bar{\Omega})$. The interest in studying such problems consists of the presence of the $p(x)$-Laplace type operator $\mathrm{div}(a(x,\nabla u))$. We remember that the $p(x)$-Laplace operator is defined by $\Delta_{p(x)}u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$. The study of differential equations and variational problems involving $p(x)$-growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids was due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends

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on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For a general account of the underlying physics consult Halsey (1992) and for some technical applications Pfeiffer *et al.* (1999). Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids we refer to Halsey (1992), Acerbi & Mingione (2001), Diening (2002), Ruzicka (2002), Fan *et al.* (2005) and Chabrowski & Fu (2005).

We point out a recent mathematical model developed by Rajagopal & Ruzicka (2001). The model takes into account the delicate interaction between the electromagnetic fields and the moving fluids. Particularly, in the context of continuum mechanics, these fluids are seen as non-Newtonian fluids. The system modelling the phenomenon arising from this study is

$$\begin{cases} \operatorname{div} \boldsymbol{E} = 0 \quad \operatorname{curl} \boldsymbol{E} = 0, \\ \dfrac{\partial v}{\partial t} - \operatorname{div} \mathsf{S}(x, \boldsymbol{E}, \mathcal{E}(v)) + [\nabla v]v + \nabla \pi = g(x, \boldsymbol{E}), \\ \operatorname{div} v = 0, \end{cases} \tag{1.2}$$

where $\boldsymbol{E}(x)$ is the electromagnetic field, $v : \Omega(\subset\mathbb{R}^3) \to \mathbb{R}^3$ is the velocity of the field, $\mathcal{E}(v)$ is the symmetric part of the gradient, S is the extra stress tensor and π is the pressure (according to notations in Rajagopal & Ruzicka (2001)).

The constitutive relation for the extra stress tensor S is

$$\mathsf{S}(x, \boldsymbol{E}, z) = \nu(\boldsymbol{E})(1 + |z|^2)^{(p-2)/2} z + \text{terms of the same growth},$$

for all symmetric 3×3 matrices z and where $p = p(|\boldsymbol{E}|^2)$. The structure of the system allows the determination of $\boldsymbol{E}$ so that it depends on x and thus, $p = p(x)$.

The extra stress tensor S is chosen such that it is a monotone vector field satisfying the ellipticity condition

$$D_z\mathsf{S}(x, \boldsymbol{E}, z)\lambda\otimes\lambda \geq \nu(\boldsymbol{E})(1 + |z|^2)^{(p-2)/2}|\lambda|^2,$$

where $\nu(\boldsymbol{E}) \geq \bar{\nu} > \mathbf{0}$, for any 3×3 symmetric matrices z, λ with null trace.

For the system described above, Rajagopal & Ruzicka established an existence theory which is particularly satisfying in the steady case

$$-\operatorname{div} \mathsf{S}(x, \mathcal{E}(v)) + [\nabla v]v + \nabla \pi = g(x).$$

Our paper can be regarded as a starting point for investigations of models like those described above, since we treat the existence and multiplicity of solutions for problems with $p(x)$ growth as in equation (1.1). We point out that even if our results will be formulated in a variational context, our methods and techniques can be applied to systems as well (see e.g. the work of El Hamidi (2004) for a nice generalization of such results to the study of elliptic systems of gradient type with $p(x)$ growth).

A complete description regarding the development of variable exponent spaces, based on a rich bibliography, can be found in the paper of Diening *et al.* (2004). We resume in what follows some basic facts from the above quoted study. According to that paper, variable exponent Lebesgue spaces had already appeared in the literature for the first time in a article by Orlicz (1931). In the 1950s, this study was carried on by Nakano who made the first systematic study of spaces with variable exponent (called *modular spaces*). Nakano explicitly mentioned variable exponent Lebesgue spaces as an example of more general spaces he considered, see Nakano (1950; p. 284). Later, the Polish mathematicians investigated the modular function spaces (e.g. Musielak 1983). Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context, we refer to the work of Tsenov (1961), Sharapudinov (1978) and Zhikov (1987).

We recall in what follows some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Throughout this paper, we assume that $p(x)>1$, $p(x)\in C^{0,\alpha}(\bar{\Omega})$ with $\alpha\in(0,1)$.

Set

$$C_+(\bar{\Omega}) = \{h; h\in C(\bar{\Omega}), h(x)>1 \text{ for all } x\in\bar{\Omega}\}.$$

For any $h\in C_+(\bar{\Omega})$, we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x)\in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u;\, u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\mathrm{d}x<\infty \right\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{ \mu>0; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces (Kováčik & Rákosník 1991; theorem 2.5), the Hölder inequality holds (Kováčik & Rákosník 1991; theorem 2.1), they are reflexive if and only if $1<p^-\le p^+<\infty$ (Kováčik & Rákosník 1991; corollary 2.7) and continuous functions are dense, if $p^+<\infty$ (Kováčik & Rákosník 1991; theorem 2.11). The inclusion between Lebesgue spaces also generalizes naturally (Kováčik & Rákosník 1991; theorem 2.8): if $0<|\Omega|<\infty$ and p_1, p_2 are variable exponents so that $p_1(x)\le p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega|+1$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/q(x)=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, the Hölder type inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}, \tag{1.3}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *Modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}: L^{p(x)}(\Omega)\to\mathbb{R}$ defined by

$$\rho_{p(x)}(u)=\int_\Omega |u|^{p(x)}\mathrm{d}x.$$

If (u_n), $u\in L^{p(x)}(\Omega)$ and $p^+<\infty$ then the following relations hold true

$$|u|_{p(x)}>1\Rightarrow |u|_{p(x)}^{p^-}\le\rho_{p(x)}(u)\le|u|_{p(x)}^{p^+}, \tag{1.4}$$

$$|u|_{p(x)}<1\Rightarrow |u|_{p(x)}^{p^+}\le\rho_{p(x)}(u)\le|u|_{p(x)}^{p^-}, \tag{1.5}$$

$$|u_n-u|_{p(x)}\to 0 \quad\Leftrightarrow\quad \rho_{p(x)}(u_n-u)\to 0. \tag{1.6}$$

Spaces with $p^+=\infty$ have been studied by Edmunds *et al.* (1999).

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|=|\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega),\|\cdot\|)$ is a separable and reflexive Banach space. We note that if $q\in C_+(\bar\Omega)$ and $q(x)<p^*(x)$ for all $x\in\bar\Omega$, then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x)=Np(x)/(N-p(x))$ if $p(x)<N$ or $p^*(x)=+\infty$ if $p(x)\ge N$. We refer to Kováčik & Rákosník (1991), Edmunds & Rákosník (1992, 2000) and Fan & Zhao (2001) for further properties of variable exponent Lebesgue–Sobolev spaces.

The paper contains two sections. In §2, we describe the problem and we state the main result. Some remarks and connections regarding similar results are also included at the end of this section. In §3, we prove the main result of the paper. We also include some generalizations of standard results involving the generalized Lebesgue–Sobolev spaces in order to offer clarity and strictness to our paper. These auxiliary results aim to be a guide which facilitates the reading of the paper.

2. The main result

Assume that $a(x,\xi):\bar\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is the continuous derivative with respect to ξ of the mapping $A:\bar\Omega\times\mathbb{R}^N\to\mathbb{R}$, $A=A(x,\xi)$, i.e. $a(x,\xi)=\nabla_\xi A(x,\xi)$. Suppose that a and A satisfy the following hypotheses:

(A1) The following equality holds

$$A(x,0)=0,$$

for all $x\in\bar\Omega$.

(A2) There exists a positive constant c_1 such that

$$|a(x,\xi)|\le c_1(1+|\xi|^{p(x)-1}),$$

for all $x\in\bar\Omega$ and $\xi\in\mathbb{R}^N$.

(A3) The following inequality holds

$$0 \le (a(x,\xi) - a(x,\psi)) \cdot (\xi - \psi),$$

for all $x \in \bar{\Omega}$ and $\xi, \psi \in \mathbb{R}^N$, with equality if and only if $\xi = \psi$.

(A4) There exists $k > 0$ such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \le \frac{1}{2} A(x,\xi) + \frac{1}{2} A(x,\psi) - k|\xi - \psi|^{p(x)},$$

for all $x \in \bar{\Omega}$ and $\xi, \psi \in \mathbb{R}^N$.

(A5) The following inequalities hold true

$$|\xi|^{p(x)} \le a(x,\xi) \cdot \xi \le p(x) A(x,\xi),$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N$.

Examples:

(i) Set $A(x,\xi) = (1/p(x))|\xi|^{p(x)}$, $a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \ge 2$. Then we get the $p(x)$-Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

(ii) Set $A(x,\xi) = (1/p(x))[(1+|\xi|^2)^{p(x)/2} - 1]$, $a(x,\xi) = (1+|\xi|^2)^{(p(x)-2)/2}\xi$, where $p(x) \ge 2$. Then we obtain the generalized mean curvature operator

$$\operatorname{div}((1+|\nabla u|^2)^{(p(x)-2)/2} \nabla u).$$

In this paper, we study problem (1.1) in the particular case

$$f(x,t) = \lambda(t^{\gamma-1} - t^{\beta-1}),$$

with $1 < \beta < \gamma < \inf_{x \in \bar{\Omega}} p(x)$ and $t \ge 0$. More precisely, we consider the degenerate boundary value problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1}), & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \\ u \ge 0, & \text{for } x \in \Omega. \end{cases} \tag{2.1}$$

We say that $u \in W_0^{1,p(x)}(\Omega)$ is a *weak solution* of problem (2.1), if $u \ge 0$ a.e. in Ω and

$$\int_\Omega a(x,\nabla u) \cdot \nabla\varphi \,\mathrm{d}x - \lambda \int_\Omega u^{\gamma-1}\varphi \,\mathrm{d}x + \lambda \int_\Omega u^{\beta-1}\varphi \,\mathrm{d}x = 0,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Our main result asserts that problem (2.1) has at least two non-trivial weak solutions provided that $\lambda > 0$ is large enough and operators A and a satisfy conditions (A1)–(A5). More precisely, we prove the following.

Theorem 2.1. *Assume hypotheses (A1)–(A5) are fulfilled. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda > \lambda^*$ *problem (2.1) has at least two distinct non-negative, non-trivial weak solutions, provided that* $p^+ < \min\{N, Np^-/(N-p^-)\}$.

Remark. By theorem 4.3 in Fan & Zhang (2003), problem (2.1) has at least a weak solution in the particular case $a(x,\xi) = |\xi|^{p(x)-1}\xi$. However, the proof in Fan & Zhang (2003) does not state the fact that the solution is non-negative and not even non-trivial in the case when $f(x,0) = 0$.

We point out that our result is inspired by theorem 1.2 in Perera (2003), where a related property is proved in the case of the p-Laplace operators. We point out that the extension from p-Laplace operator to $p(x)$-Laplace operator is not trivial, since the $p(x)$-Laplacian has a more complicated structure than the p-Laplace operator, for example, it is inhomogeneous.

Finally, we mention that a similar study regarding the existence and multiplicity of solutions for a system of equations involving the $p(x)$-Laplace operator can be found in El Hamidi (2004). The arguments used by the author rely on the Mountain Pass theorem and Bartsch's Fountain theorem.

3. Proof of theorem 2.1

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$.

Define the energy functional $I : E \to \mathbb{R}$ by

$$I(u) = \int_\Omega A(x, \nabla u)\mathrm{d}x - \frac{\lambda}{\gamma}\int_\Omega u_+^\gamma \mathrm{d}x + \frac{\lambda}{\beta}\int_\Omega u_+^\beta \mathrm{d}x,$$

where $u_+(x) = \max\{u(x), 0\}$.

We first establish some basic properties of I.

Proposition 3.1. *The functional* I *is well-defined on* E *and* $I \in C^1(E, \mathbb{R})$ *with the derivative given by*

$$\langle I'(u), \varphi\rangle = \int_\Omega a(x, \nabla u)\cdot\nabla\varphi \,\mathrm{d}x - \lambda\int_\Omega u_+^{\gamma-1}\varphi \,\mathrm{d}x + \lambda\int_\Omega u_+^{\beta-1}\varphi\,\mathrm{d}x,$$

for all u, $\varphi \in E$.

To prove proposition 3.1, we define the functional $\Lambda : E \to \mathbb{R}$ by

$$\Lambda(u) = \int_\Omega A(x, \nabla u)\mathrm{d}x, \quad \forall u \in E.$$

Lemma 3.2.

(i) *The functional* Λ *is well-defined on* E.
(ii) *The functional* Λ *is of class* $C^1(E, \mathbb{R})$ *and*

$$\langle \Lambda'(u), \varphi\rangle = \int_\Omega a(x, \nabla u)\cdot\nabla\varphi\,\mathrm{d}x,$$

for all $u, \varphi \in E$.

Proof. (i) For any $x \in \Omega$ and $\xi \in \mathbb{R}^N$, we have

$$A(x,\xi) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} A(x,t\xi)\mathrm{d}t = \int_0^1 a(x,t\xi)\cdot\xi\,\mathrm{d}t.$$

Using hypothesis (A2), we get

$$\begin{aligned} A(x,\xi) &\le c_1 \int_0^1 (1+|\xi|^{p(x)-1}t^{p(x)-1})|\xi|\mathrm{d}t \le c_1|\xi| + \frac{c_1}{p(x)}|\xi|^{p(x)} \\ &\le c_1|\xi| + \frac{c_1}{p^-}|\xi|^{p(x)}, \quad \forall x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^N. \end{aligned} \tag{3.1}$$

The above inequality and (A5) imply

$$0 \le \int_\Omega A(x,\nabla u)\mathrm{d}x \le c_1 \int_\Omega |\nabla u|\mathrm{d}x + \frac{c_1}{p^-}\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x, \quad \forall u \in E.$$

Using inequality (1.3) and relations (1.4) and (1.5), we deduce that Λ is well defined on E.

(ii) *Existence of the Gâteaux derivative.* Let $u, \varphi \in E$. Fix $x \in \Omega$ and $0<|r|<1$. Then, by the mean value theorem, there exists $\nu \in [0,1]$ such that

$$|A(x,\nabla u(x)+r\nabla\varphi(x)) - A(x,\nabla u)|/|r| = |a(x,\nabla u(x)+\nu r\nabla\varphi(x))||\nabla\varphi(x)|.$$

Using condition (A2), we obtain

$$\begin{aligned} |A(x,\nabla u(x)+r\nabla\varphi(x)) - A(x,\nabla u)|/|r| &\le [c_1 + c_1(|\nabla u(u)|+|\nabla\varphi(x)|)^{p(x)-1}]|\nabla\varphi(x)| \\ &\le [c_1 + c_1 2^{p^+}(|\nabla u(x)|^{p(x)-1}+|\nabla\varphi(x)|^{p(x)-1})]|\nabla\varphi(x)|. \end{aligned}$$

Next, by inequality (1.3), we have

$$\int_\Omega c_1|\nabla\varphi|\mathrm{d}x \le |c_1|_{p(x)/(p(x)-1)}\cdot|\nabla\varphi|_{p(x)}$$

and

$$\int_\Omega |\nabla u|^{p(x)-1}|\nabla\varphi|\mathrm{d}x \le ||\nabla u|^{p(x)-1}|_{p(x)/(p(x)-1)}\cdot|\nabla\varphi|_{p(x)}.$$

The above inequalities imply

$$c_1[1+2^{p^+}(|\nabla u(x)|^{p(x)-1}+|\nabla\varphi(x)|^{p(x)-1})]|\nabla\varphi(x)| \in L^1(\Omega).$$

It follows from the Lebesgue theorem that

$$\langle \Lambda'(u),\varphi\rangle = \int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x.$$

Assume $u_n \to u$ in E. Let us define $\theta(x,u) = a(x,\nabla u)$. Using hypothesis (A2) and proposition 2.2 in Fan & Zhang (2003), we deduce that $\theta(x,u_n)\to\theta(x,u)$ in $(L^{q(x)}(\Omega))^N$, where $q(x) = p(x)/\,(p(x)-1)$. By inequality (1.3), we obtain

$$|\langle \Lambda'(u_n) - \Lambda'(u),\varphi\rangle| \le |\theta(x,u_n)-\theta(x,u)|_{q(x)}|\nabla\varphi|_{p(x)},$$

and so

$$\|\Lambda'(u_n)-\Lambda'(u)\| \le |\theta(x,u_n)-\theta(x,u)|_{q(x)} \to 0, \quad \text{as } n\to\infty.$$

The proof of lemma 3.2 is complete. ■

Lemma 3.3. *If $u\in E$ then u_+, $u_-\in E$ and*

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u\le 0], \\ \nabla u, & \text{if } [u>0], \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if } [u\ge 0], \\ \nabla u, & \text{if } [u<0], \end{cases}$$

where $u_\pm(x)=\max\{\pm u(x),0\}$ for all $x\in\Omega$.

Proof. Let $u\in E$ be fixed. Then there exists a sequence $(\varphi_n)\in C_0^\infty(\Omega)$ such that

$$|\nabla(\varphi_n-u)|_{p(x)}\to 0.$$

Since $1<p^-\le p(x)$ for all $x\in\Omega$, it follows that $L^{p(x)}$ is continuously embedded in $L^{p^-}(\Omega)$ and thus,

$$|\nabla(\varphi_n-u)|_{p^-}\to 0.$$

Hence $u\in W_0^{1,p^-}(\Omega)$. We obtain

$$u_+, u_-\in W_0^{1,p^-}(\Omega)\subset W_0^{1,1}(\Omega). \tag{3.2}$$

On the other hand, theorem 7.6 in Gilbarg & Trudinger (1998) implies

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u\le 0], \\ \nabla u, & \text{if } [u>0], \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if } [u\ge 0], \\ \nabla u, & \text{if } [u<0]. \end{cases}$$

By the above equalities, we deduce that

$$|u_+(x)|^{p(x)}\le |u(x)|^{p(x)}, \quad |\nabla u_+(x)|^{p(x)}\le |\nabla u|^{p(x)}, \quad \text{a.e. } x\in\Omega, \tag{3.3}$$

and

$$|u_-(x)|^{p(x)}\le |u(x)|^{p(x)}, \quad |\nabla u_-(x)|^{p(x)}\le |\nabla u|^{p(x)}, \quad \text{a.e. } x\in\Omega. \tag{3.4}$$

Since $u\in E$, we have

$$|u(x)|^{p(x)}, \quad |\nabla u(x)|^{p(x)}\in L^1(\Omega). \tag{3.5}$$

By equations (3.3)–(3.5) and Lebesgue theorem, we obtain that u_+, $u_-\in L^{p(x)}(\Omega)$ and $\rho_{p(x)}(|\nabla u_+|)<\infty$, $\rho_{p(x)}(|\nabla u_-|)<\infty$. It follows that

$$u_+, u_-\in W^{1,p(x)}(\Omega), \tag{3.6}$$

where $W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega); |\nabla u|\in L^{p(x)}(\Omega)\}$ (see Fan & Zhao (2001) for more details).

By equations (3.2) and (3.6), we conclude that

$$u_+, u_-\in W^{1,p(x)}(\Omega)\cap W_0^{1,1}(\Omega).$$

Since $p\in C^{0,\alpha}(\bar{\Omega})$, theorem 2.6 and remark 2.9 in Fan & Zhao (2001) show that $E=W^{1,p(x)}(\Omega)\cap W_0^{1,1}(\Omega)$. Thus, u_+, $u_-\in E$ and the proof of lemma 3.3 is complete. ■

By lemmas 3.2 and 3.3, it is clear that proposition 3.1 holds true.

Remark. If u is a critical point of I then using lemma 3.3 and condition (A5), we have

$$0=\langle I'(u),u_-\rangle=\int_\Omega a(x,\nabla u)\cdot\nabla u_-\mathrm{d}x-\lambda\int_\Omega (u_+)^{\gamma-1}u_-\mathrm{d}x+\lambda\int_\Omega (u_+)^{\beta-1}u_-\mathrm{d}x$$
$$=\int_\Omega a(x,\nabla u)\cdot\nabla u_-\mathrm{d}x=\int_\Omega a(x,\nabla u_-)\cdot\nabla u_-\mathrm{d}x\geq\int_\Omega |\nabla u_-|^{p(x)}\mathrm{d}x.$$

Thus, we deduce that $u\geq 0$. It follows that the non-trivial critical points of I are non-negative solutions of (2.1).

The above remark shows that we can prove theorem 2.1 using the critical points theory. More exactly, we first show that for $\lambda>0$ large enough, the functional I has a global minimizer $u_1\geq 0$ such that $I(u_1)<0$. Next, by means of the Mountain Pass theorem, a second critical point u_2 with $I(u_2)>0$ is obtained.

Lemma 3.4. *The functional Λ is weakly lower semi-continuous.*

Proof. By corollary III.8 in Brezis (1992), it is enough to show that Λ is lower semi-continuous. For this purpose, we fix $u\in E$ and $\epsilon>0$. Since Λ is convex (by condition (A4)), we deduce that for any $v\in E$, the following inequality holds

$$\int_\Omega A(x,\nabla v)\mathrm{d}x\geq\int_\Omega A(x,\nabla u)\mathrm{d}x+\int_\Omega a(x,\nabla u)\cdot(\nabla v-\nabla u)\mathrm{d}x.$$

Using condition (A2) and inequality (1.3), we have

$$\begin{aligned}\int_\Omega A(x,\nabla v)\mathrm{d}x&\geq\int_\Omega A(x,\nabla u)\mathrm{d}x-\int_\Omega |a(x,\nabla u)||\nabla v-\nabla u|\mathrm{d}x\\
&\geq\int_\Omega A(x,\nabla u)\mathrm{d}x-c_1\int_\Omega|\nabla(v-u)|\mathrm{d}x-c_1\int_\Omega|\nabla u|^{p(x)-1}|\nabla(v-u)|\mathrm{d}x\\
&\geq\int_\Omega A(x,\nabla u)\mathrm{d}x-c_2|1|_{q(x)}|\nabla(v-u)|_{p(x)}-c_3||\nabla u|^{p(x)-1}|_{q(x)}|\nabla(v-u)|_{p(x)}\\
&\geq\int_\Omega A(x,\nabla u)\mathrm{d}x-c_4\|v-u\|\geq\int_\Omega A(x,\nabla u)\mathrm{d}x-\epsilon,\end{aligned}$$

for all $v\in E$ with $\|v-u\|<\delta=\epsilon/c_4$, where c_2, c_3, c_4 are positive constants and $q(x)=p(x)/(p(x)-1)$. We conclude that Λ is weakly lower semi-continuous. The proof of lemma 3.4 is complete. ■

Lemma 3.5. *There exists* $\lambda_1 > 0$ *such that*

$$\lambda_1 = \inf_{u \in E, \|u\|>1} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x}{\int_\Omega |u|^{p^-} \mathrm{d}x}.$$

Proof. We know that E is continuously embedded in $L^{p^-}(\Omega)$. It follows that there exists $C > 0$ such that

$$\|u\| \geq C|u|_{p^-}, \quad \forall u \in E.$$

On the other hand, by equation (1.4), we have

$$\int_\Omega |\nabla u|^{p(x)} \mathrm{d}x \geq \|u\|^{p^-}, \quad \forall u \in E \text{ with } \|u\| > 1.$$

Combining the above inequalities, we obtain

$$\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x \geq \frac{C^{p^-}}{p^+} \int_\Omega |u|^{p^-} \mathrm{d}x, \quad \forall u \in E \text{ with } \|u\| > 1.$$

The proof of lemma 3.5 is complete. ∎

Proposition 3.6.

(i) *The functional* I *is bounded from below and coercive.*
(ii) *The functional* I *is weakly lower semi-continuous.*

Proof. (i) Since $1 < \beta < \gamma < p^-$, we have

$$\lim_{t \to \infty} \frac{\frac{1}{\gamma} t^\gamma - \frac{1}{\beta} t^\beta}{t^{p^-}} = 0.$$

Then for any $\lambda > 0$, there exists $C_\lambda > 0$ such that

$$\lambda \left(\frac{1}{\gamma} t^\gamma - \frac{1}{\beta} t^\beta \right) \leq \frac{\lambda_1}{2} t^{p^-} + C_\lambda, \quad \forall t \geq 0,$$

where λ_1 is defined in lemma 3.5.

Condition (A5) and the above inequality show that for any $u \in E$ with $\|u\| > 1$, we have

$$\begin{aligned} I(u) &\geq \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \frac{\lambda_1}{2} \int_\Omega |u|^{p^-} \mathrm{d}x - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2} \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2p^+} \|u\|^{p^-} - C_\lambda \mu(\Omega). \end{aligned}$$

This shows that I is bounded from below and coercive.

(ii) Using lemma 3.4, we deduce that Λ is weakly lower semi-continuous. We show that I is weakly lower semi-continuous. Let $(u_n) \subset E$ be a sequence which converges weakly to u in E. Since Λ is weakly lower semi-continuous, we have

$$\Lambda(u) \le \liminf_{n\to\infty} \Lambda(u_n). \tag{3.7}$$

On the other hand, since E is compactly embedded in $L^{\gamma}(\Omega)$ and $L^{\beta}(\Omega)$, it follows that (u_{n+}) converges strongly to u_+ both in $L^{\gamma}(\Omega)$ and in $L^{\beta}(\Omega)$. This fact together with relation (3.7) imply

$$I(u) \le \liminf_{n\to\infty} I(u_n).$$

Therefore, I is weakly lower semi-continuous. The proof of proposition 3.6 is complete. ■

By proposition 2 and theorem 1.2 in Struwe (1996), we deduce that there exists $u_1 \in E$ a global minimizer of I. The following result implies that $u_1 \neq 0$, provided that λ is sufficiently large.

Proposition 3.7. *There exists* $\lambda^* > 0$ *such that* $\inf_E I < 0$.

Proof. Let $\Omega_1 \subset \Omega$ be a compact subset, large enough and $u_0 \in E$ be such that $u_0(x) = t_0$ in Ω_1 and $0 \le u_0(x) \le t_0$ in $\Omega \backslash \Omega_1$, where $t_0 > 1$ is chosen such that

$$\frac{1}{\gamma} t_0^{\gamma} - \frac{1}{\beta} t_0^{\beta} > 0.$$

We have

$$\begin{aligned}\frac{1}{\gamma}\int_{\Omega} u_0^{\gamma}\mathrm{d}x - \frac{1}{\beta}\int_{\Omega} u_0^{\beta}\mathrm{d}x &\ge \frac{1}{\gamma}\int_{\Omega_1} u_0^{\gamma}\mathrm{d}x - \frac{1}{\beta}\int_{\Omega_1} u_0^{\beta}\mathrm{d}x - \frac{1}{\beta}\int_{\Omega\backslash\Omega_1} u_0^{\beta}\mathrm{d}x \\ &\ge \frac{1}{\gamma}\int_{\Omega_1} u_0^{\gamma}\mathrm{d}x - \frac{1}{\beta}\int_{\Omega_1} u_0^{\beta}\mathrm{d}x - \frac{1}{\beta} t_0^{\beta}\mu(\Omega\backslash\Omega_1) > 0,\end{aligned}$$

and thus $I(u_0) < 0$ for $\lambda > 0$ large enough. The proof of proposition 3.7 is complete. ■

Since proposition 3.7 holds true, it follows that $u_1 \in E$ is a non-trivial weak solution of problem (2.1).

Fix $\lambda \ge \lambda^*$. Set

$$g(x,t) = \begin{cases} 0, & \text{for } t < 0, \\ t^{\gamma-1} - t^{\beta-1}, & \text{for } 0 \le t \le u_1(x), \\ u_1(x)^{\gamma-1} - u_1(x)^{\beta-1}, & \text{for } t > u_1(x), \end{cases}$$

and

$$G(x,t) = \int_0^t g(x,s)\mathrm{d}s.$$

Define the functional $J : E \to \mathbb{R}$ by

$$J(u) = \int_{\Omega} A(x, \nabla u)\mathrm{d}x - \lambda \int_{\Omega} G(x,u)\mathrm{d}x.$$

The same arguments as those used for functional I imply that $J \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), \varphi \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla\varphi \, \mathrm{d}x - \lambda \int_\Omega g(x, u)\varphi \, \mathrm{d}x,$$

for all u, $\varphi \in E$.

On the other hand, we point out that if $u \in E$ is a critical point of J then $u \geq 0$. The proof can be carried out as in the case of functional I.

Next, we prove the following.

Lemma 3.8. *If u is a critical point of J then $u \leq u_1$.*

Proof. We have

$$\begin{aligned} 0 &= \langle J'(u) - I'(u_1), (u-u_1)_+ \rangle \\ &= \int_\Omega (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla(u-u_1)_+ \mathrm{d}x - \lambda \int_\Omega [g(x,u) - (u_1^{\gamma-1} - u_1^{\beta-1})](u-u_1)_+ \mathrm{d}x \\ &= \int_{[u>u_1]} (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla(u-u_1) \mathrm{d}x. \end{aligned}$$

By condition (A3), we deduce that the above equality holds if and only if $\nabla u = \nabla u_1$. It follows that $\nabla u(x) = \nabla u_1(x)$ for all $x \in \omega := \{y \in \Omega; u(y) > u_1(y)\}$. Hence

$$\int_\omega |\nabla(u-u_1)|^{p(x)} \mathrm{d}x = 0,$$

and thus,

$$\int_\Omega |\nabla(u-u_1)_+|^{p(x)} \mathrm{d}x = 0.$$

By relation (1.5), we obtain

$$\|(u-u_1)_+\| = 0.$$

Since $u - u_1 \in E$ by lemma 3.3, we have that $(u-u_1)_+ \in E$. Thus, we obtain that $(u-u_1)_+ = 0$ in Ω, i.e. $u \leq u_1$ in Ω. The proof of lemma 3.8 is complete. ■

In the following, we determine a critical point $u_2 \in E$ of J such that $J(u_2) > 0$ via the Mountain Pass theorem. By the above lemma, we will deduce that $0 \leq u_2 \leq u_1$ in Ω. Therefore,

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1}, \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma} u_2^\gamma - \frac{1}{\beta} u_2^\beta,$$

and thus,

$$J(u_2) = I(u_2), \quad \text{and} \quad J'(u_2) = I'(u_2).$$

More exactly we find

$$I(u_2) > 0 = I(0) > I(u_1), \quad \text{and} \quad I'(u_2) = 0.$$

This shows that u_2 is a weak solution of problem (2.1) such that $0 \le u_2 \le u_1$, $u_2 \ne 0$ and $u_2 \ne u_1$.

In order to find u_2 described above, we prove the following.

Lemma 3.9. *There exist* $\rho \in (0, \|u_1\|)$ *and* $a > 0$ *such that* $J(u) \ge a$, *for all* $u \in E$ *with* $\|u\| = \rho$.

Proof. Let $u \in E$ be fixed, such that $\|u\| < 1$. It is clear that there exists $\delta > 1$ such that

$$\frac{1}{\gamma} t^{\gamma} - \frac{1}{\beta} t^{\beta} \le 0, \quad \forall t \in [0, \delta].$$

For δ given above, we define

$$\Omega_u = \{x \in \Omega; u(x) > \delta\}.$$

If $x \in \Omega \backslash \Omega_u$ with $u(x) < u_1(x)$, we have

$$G(x, u) = \frac{1}{\gamma} u_+^{\gamma} - \frac{1}{\beta} u_+^{\beta} \le 0.$$

If $x \in \Omega \backslash \Omega_u$ with $u(x) > u_1(x)$, then $u_1(x) \le \delta$ and we have

$$G(x, u) = \frac{1}{\gamma} u_1^{\gamma} - \frac{1}{\beta} u_1^{\beta} \le 0.$$

Thus, we deduce that

$$G(x, u) \le 0, \quad \text{on} \quad \Omega \backslash \Omega_u.$$

Provided that $\|u\| < 1$ by condition (A5) and relation (1.5), we get

$$J(u) \ge \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \lambda \int_{\Omega_u} G(x, u) \mathrm{d}x \ge \frac{1}{p^+} \|u\|^{p^+} - \lambda \int_{\Omega_u} G(x, u) \mathrm{d}x. \tag{3.8}$$

Since $p^+ < \min\{N, Np^-/(N - p^-)\}$, it follows that $p^+ < p^*(x)$ for all $x \in \bar{\Omega}$. Then there exists $q \in (p^+, Np^-/(N - p^-))$ such that E is continuously embedded in $L^q(\Omega)$. Thus, there exists a positive constant $C > 0$ such that

$$|u|_q \le C \|u\|, \quad \forall u \in E.$$

Using the definition of G, Hölder's inequality and the above estimate, we obtain

$$\begin{aligned} \lambda \int_{\Omega_u} G(x, u) \mathrm{d}x &= \lambda \int_{\Omega_u \cap [u < u_1]} \left(\frac{1}{\gamma} u_+^{\gamma} - \frac{1}{\beta} u_+^{\beta} \right) \mathrm{d}x \\ &\quad + \lambda \int_{\Omega_u \cap [u > u_1]} \left(\frac{1}{\gamma} u_1^{\gamma} - \frac{1}{\beta} u_1^{\beta} \right) \mathrm{d}x \le \frac{2\lambda}{\gamma} \int_{\Omega_u} u_+^{\gamma} \mathrm{d}x \\ &\le \frac{2\lambda}{\gamma} \int_{\Omega_u} u_+^{p^+} \mathrm{d}x \le \frac{2\lambda}{\gamma} \left(\int_{\Omega_u} u_+^{q} \mathrm{d}x \right)^{p^+/q} [\mu(\Omega_u)]^{1 - p^+/q} \\ &\le C \frac{2\lambda}{\gamma} [\mu(\Omega_u)]^{1 - p^+/q} \|u\|^{p^+}. \end{aligned} \tag{3.9}$$

By equations (3.8) and (3.9), we infer that it is enough to show that $\mu(\Omega_u)\to 0$ as $\|u\|\to 0$ in order to prove lemma 3.9.

Let $\epsilon>0$. We choose $\Omega_\epsilon\subset\Omega$ a compact subset, such that $\mu(\Omega\setminus\Omega_\epsilon)<\epsilon$. We denote by $\Omega_{u,\epsilon}:=\Omega_u\cap\Omega_\epsilon$. Then it is clear that

$$C[\mu(\Omega)]^{1-p^+/q}\|u\|^{p^+}\geq\left(\int_\Omega|u|^q\mathrm{d}x\right)^{p^+/q}\geq\left(\int_{\Omega_{u,\epsilon}}|u|^q\mathrm{d}x\right)^{p^+/q}\geq\delta^{p^+}[\mu(\Omega_{u,\epsilon})]^{p^+/q}.$$

The above inequality implies that $\mu(\Omega_{u,\epsilon})\to 0$ as $\|u\|\to 0$.

Since $\Omega_u\subset\Omega_{u,\epsilon}\cup(\Omega\setminus\Omega_\epsilon)$, we have

$$\mu(\Omega_u)\leq\mu(\Omega_{u,\epsilon})+\epsilon,$$

and $\epsilon>0$ is arbitrary. We find that $\mu(\Omega_u)\to 0$ as $\|u\|\to 0$. This concludes the proof of lemma 3.9. ∎

Lemma 3.10. *The functional J is coercive.*

Proof. For each $u\in E$ with $\|u\|>1$ by condition (A5), relation (1.4) and inequality (1.3), we have

$$\begin{aligned}J(u)&\geq\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x-\lambda\int_{[u>u_1]}G(x,u)\mathrm{d}x-\lambda\int_{[u<u_1]}G(x,u)\mathrm{d}x\\&\geq\frac{1}{p^+}\|u\|^{p^-}-\frac{\lambda}{\gamma}\int_{[u>u_1]}u_1^\gamma\mathrm{d}x+\frac{\lambda}{\beta}\int_{[u>u_1]}u_1^\beta\mathrm{d}x-\frac{\lambda}{\gamma}\int_{[u<u_1]}u_+^\gamma\mathrm{d}x+\frac{\lambda}{\beta}\int_{[u<u_1]}u_+^\beta\mathrm{d}x\\&\geq\frac{1}{p^+}\|u\|^{p^-}-\frac{\lambda}{\gamma}\int_\Omega u_1^\gamma\mathrm{d}x-\frac{\lambda}{\gamma}\int_\Omega u_+^\gamma\mathrm{d}x\geq\frac{1}{p^+}\|u\|^{p^-}-\frac{\lambda}{\gamma}[\mu(\Omega)]^{1-\gamma/p^-}C_1\|u\|^\gamma-C_2\\&\geq\frac{1}{p^+}\|u\|^{p^-}-C_23\|u\|^\gamma-C_2,\end{aligned}$$

where C_1, C_2 and C_3 are positive constants. Since $\gamma<p^-$ the above inequality implies that $J(u)\to\infty$ as $\|u\|\to\infty$, i.e. J is coercive. The proof of lemma 3.10 is complete. ∎

The following result yields a sufficient condition which ensures that a weakly convergent sequence in E converges strongly, too.

Lemma 3.11. *Assume that the sequence (u_n) converges weakly to u in E and*

$$\limsup_{n\to\infty}\int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n-\nabla u)\mathrm{d}x\leq 0.$$

Then (u_n) converges strongly to u in E.

Proof. Using relation (3.1), we have that there exists a positive constant c_5 such that

$$A(x,\xi)\leq c_5(|\xi|+|\xi|^{p(x)}),\quad\forall x\in\bar{\Omega},\ \xi\in\mathbb{R}^N.$$

The above inequality implies

$$A(x,\nabla u_n) \leq c_5(|\nabla u_n| + |\nabla u_n|^{p(x)}), \quad \forall x \in \bar{\Omega},\ n. \tag{3.10}$$

The fact that u_n converges weakly to u in E implies that there exists $R>0$ such that $\|u_n\| \leq R$ for all n. By relation (3.10) and inequalities (1.3)–(1.5), we deduce that $\{\int_\Omega A(x,\nabla u_n)\mathrm{d}x\}$ is bounded. Then, up to a subsequence, we deduce that $\int_\Omega A(x,\nabla u_n)\mathrm{d}x \to c$. By lemma 3.4, we obtain

$$\int_\Omega A(x,\nabla u)\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega A(x,\nabla u_n)\mathrm{d}x = c.$$

On the other hand, since A is convex, we have

$$\int_\Omega A(x,\nabla u)\mathrm{d}x \geq \int_\Omega A(x,\nabla u_n)\mathrm{d}x + \int_\Omega a(x,\nabla u_n)\cdot(\nabla u - \nabla u_n)\mathrm{d}x.$$

Next, by the hypothesis $\limsup_{n\to\infty} \int_\Omega a(x,\nabla u_n)\cdot(\nabla u_n - \nabla u)\mathrm{d}x \leq 0$, we conclude that $\int_\Omega A(x,\nabla u)\mathrm{d}x = c$.

Taking into account that $(u_n + u)/2$ converges weakly to u in E and using lemma 3.4, we have

$$c = \int_\Omega A(x,\nabla u)\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega A\left(x, \nabla\frac{u_n + u}{2}\right)\mathrm{d}x. \tag{3.11}$$

We assume by contradiction that u_n does not converge to u in E. Then by (1.6), it follows that there exist $\epsilon>0$ and a subsequence (u_{n_m}) of (u_n) such that

$$\int_\Omega |\nabla(u_{n_m} - u)|^{p(x)}\mathrm{d}x \geq \epsilon, \quad \forall m. \tag{3.12}$$

By condition (A4), we have

$$\frac{1}{2}A(x,\nabla u) + \frac{1}{2}A(x,\nabla u_{n_m}) - A\left(x,\nabla\frac{u+u_{n_m}}{2}\right) \geq k|\nabla(u_{n_m} - u)|^{p(x)}. \tag{3.13}$$

Relations (3.12) and (3.13) yield

$$\frac{1}{2}\int_\Omega A(x,\nabla u)\mathrm{d}x + \frac{1}{2}\int_\Omega A(x,\nabla u_{n_m})\mathrm{d}x - \int_\Omega A\left(x,\nabla\frac{u+u_{n_m}}{2}\right)$$
$$\geq k\int_\Omega |\nabla(u_{n_m} - u)|^{p(x)}\mathrm{d}x \geq k\epsilon.$$

Letting $m \to \infty$ in the above inequality, we obtain

$$c - k\epsilon \geq \limsup_{m\to\infty} \int_\Omega A\left(x,\nabla\frac{u+u_{n_m}}{2}\right)\mathrm{d}x,$$

and that is a contradiction with (3.11). It follows that u_n converges strongly to u in E and lemma 3.11 is proved. ■

Proof of theorem 2.1. Using lemma 3.9 and the Mountain Pass theorem (see Ambrosetti & Rabinowitz (1973) with the variant given by theorem 1.15 in

Willem (1996)) we deduce that there exists a sequence $(u_n) \subset E$ such that

$$J(u_n) \to c > 0, \quad \text{and} \quad J'(u_n) \to 0, \tag{3.14}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) = u_1\}.$$

By relation (3.14) and lemma 3.10, we obtain that (u_n) is bounded and thus, passing eventually to a subsequence, still denoted by (u_n), we may assume that there exists $u_2 \in E$ such that u_n converges weakly to u_2. Since E is compactly embedded in $L^i(\Omega)$ for any $i \in [1, p^-]$, it follows that u_n converges strongly to u_2 in $L^i(\Omega)$ for all $i \in [1, p^-]$. Hence,

$$\langle \Lambda'(u_n) - \Lambda'(u_2), u_n - u_2 \rangle = \langle J'(u_n) - J'(u_2), u_n - u_2 \rangle$$
$$+ \lambda \int_\Omega [g(x, u_n) - g(x, u_2)](u_n - u_2)\mathrm{d}x = o(1),$$

as $n \to \infty$. By lemma 3.11, we deduce that u_n converges strongly to u_2 in E and using relation (3.14), we find

$$J(u_2) = c > 0, \quad \text{and} \quad J'(u_2) = 0.$$

Therefore, $J(u_2) = c > 0$ and $J'(u_2) = 0$. By lemma 3.8, we deduce that $0 \leq u_2 \leq u_1$ in Ω. Therefore,

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1}, \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma} u_2^\gamma - \frac{1}{\beta} u_2^\beta,$$

and thus,

$$J(u_2) = I(u_2), \quad \text{and} \quad J'(u_2) = I'(u_2).$$

We conclude that u_2 is a critical point of I and thus a solution of (2.1). Furthermore, $I(u_2) = c > 0$ and $I(u_2) > 0 > I(u_1)$. Thus, u_2 is not trivial and $u_2 \neq u_1$. The proof of theorem 2.1 is now complete. ■

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