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# NORMALIZED SOLUTIONS FOR LOWER CRITICAL CHOQUARD EQUATIONS WITH CRITICAL SOBOLEV PERTURBATION*

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**Abstract.** We study normalized solutions for the following Choquard equations with lower critical exponent and a local perturbation $-\Delta u+\lambda u=\gamma(I_\alpha*|u|^{\frac{\alpha}{N}+1})|u|^{\frac{\alpha}{N}-1}u+\mu|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N, \int_{\mathbb{R}^N} |u|^2dx=c^2$, where $\gamma$, $\mu$, $c$ are given positive numbers and $2<q\le\frac{2N}{N-2}$. The frequency $\lambda$ appears as a real Lagrange parameter and is part of the unknowns. By introducing new arguments and under different assumptions on $q$, $c$, $\gamma$, and $\mu$, we prove several nonexistence and existence results. In particular, we consider the case $q=\frac{2N}{N-2}$, which corresponds to equations involving double critical exponents. We also describe some qualitative properties of the solutions with prescribed mass and of the associated Lagrange multipliers $\lambda$.

**Key words.** normalized solutions, Choquard-type equations, critical exponents, variational method, constraint manifold

**MSC codes.** Primary, 35J20; Secondary, 35J61, 35Q40

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## 1. Introduction.

In recent years, the normalized solutions for various classes of local or nonlocal problems have been widely investigated, and there are many results, both for their particular interest from a physical point of view and for their relevance in models arising in nonlinear optics and Bose–Einstein condensation. We recall that the Choquard equation was first introduced in the pioneering work of Fröhlich [9] and Pekar [29] for the modeling of quantum polaron:

$$-\Delta u+u=\left(\frac{1}{|x|}*|u|^2\right)u \text{ in } \mathbb{R}^3. \tag{1.1}$$

As pointed out by Fröhlich and Pekar, this model corresponds to the study of free electrons in an ionic lattice interacting with phonons associated to deformations of the lattice or with the polarization that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree–Fock theory of one component plasma, Choquard used (1.1) to describe an electron trapped in its own hole.

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In general, the associated Schrödinger-type evolution equation

$$i\partial_t\psi - \Delta\psi = (W * |\psi|^2)\psi,\ (t,x) \in \mathbb{R}\times\mathbb{R}^3, \tag{1.2}$$

is a model large system of nonrelativistic bosonic atoms and molecules under an attractive interaction that is weaker and has a longer range than that of the nonlinear Schrödinger equation, where the interaction potential $W$ is formally Dirac's delta at the origin [10]. Equation (1.2) arises as a mean-field limit of a bosonic system with attractive two-body interactions which can be taken rigorously in many cases [10, 15].

The Choquard equation is also known as the Schrödinger–Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein–Klein–Gordon and Einstein–Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [8] and Giulini and Großardt [12]. Penrose [30, 31] proposed (1.1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon. As pointed out by Lieb [21], Choquard used (1.1) to study steady states of the one component plasma approximation in the Hartree–Fock theory.

Consider the Choquard-type equations with a local perturbation,

$$i\partial_t\psi - \Delta\psi = \gamma(I_\alpha * |\psi|^p)|\psi|^{p-2}\psi + \mu|\psi|^{q-2}\psi,\ (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{1.3}$$

where $2_\alpha := \frac{N+\alpha}{N} \le p \le 2_\alpha^* := \frac{N+\alpha}{N-2}$ and $2 < q \le 2^* := \frac{2N}{N-2}$, the parameters $\gamma, \mu \in \mathbb{R}$, $I_\alpha$ is the Riesz potential of order $\alpha \in (0, N)$ defined by

$$I_\alpha = \frac{A(N,\alpha)}{|x|^{N-\alpha}} \quad \text{with} \quad A(N,\alpha) = \frac{\Gamma(\frac{N-\alpha}{2})}{\pi^{N/2}2^\alpha\Gamma(\frac{\alpha}{2})} \text{ for each } x \in \mathbb{R}^N\backslash\{0\},$$

and $*$ is the convolution product on $\mathbb{R}^N$.

An important topic on (1.3) is to study their standing wave solutions. A standing wave solution of (1.3) is a solution of the form $\psi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda\in\mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{C}$ satisfies the stationary equation

$$-\Delta u + \lambda u = \gamma(I_\alpha * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u \text{ in } \mathbb{R}^N. \tag{1.4}$$

There are two different ways to deal with (1.4) according to the role of $\lambda$:

(i) the frequency $\lambda$ is a fixed and assigned parameter;

(ii) the frequency $\lambda$ is an unknown of the problem.

For case (i), one can see that solutions of (1.4) can be obtained as critical points of the functional defined in $H^1(\mathbb{R}^N)$ by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda|u|^2)dx - \frac{\gamma}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N}|u|^q dx.$$

This case has attracted much attention in the last years, depending on $p, q, \gamma$, and $\mu$; see, for example, [2, 11, 20, 26, 27, 28, 35] and the references therein.

Alternatively, one can search for solutions to (1.4) with the frequency $\lambda$ unknown. In this case, the real parameter $\lambda$ appears as a Lagrange multiplier, and $L^2$-norms of solutions are prescribed, which are usually called normalized solutions. This way seems particularly meaningful from the physical point of view, since, in addition to being a conserved quantity for (1.3) with the time dependent, the mass has often a clear physical meaning. For example, it represents the power supply in nonlinear optics or the total number of atoms in Bose–Einstein condensation. Moreover, this way turns out to be useful also from the purely mathematical perspective, since it gives a better insight into the properties of the stationary solutions for (1.3), such as stability or instability [4, 6]. For these reasons, in this paper we focus on this issue. More precisely, we are interested in finding solutions to the following constrained nonlocal problem:

$$\begin{cases} -\Delta u + \lambda u = \gamma(I_\alpha * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u \qquad \text{in} \quad \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c^2, \end{cases} \tag{1.5}$$

where $c > 0$ is given.

From the mathematical point of view, problem (1.5) is nonlocal since the appearance of the term $(I_\alpha * |u|^p)|u|^{p-2}u$ indicates that (1.5) is not a pointwise identity. This kind of problem has been paid much attention after the pioneering work of Lions [23], in which an abstract functional analysis framework was introduced. Nowadays, since physicists are interested in normalized solutions, mathematical researchers began to focus on solutions having a prescribed $L^2$-norm, that is, solutions which satisfy $\|u\|_{L^2} = c > 0$ for a priori given $c$. To the best of our knowledge, the study of solutions with prescribed norm was initiated by Lieb [21] and P.-L. Lions [24].

Solutions of problem (1.5) can be found by looking for critical points of the energy functional $E_{p,q} : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_{p,q}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\gamma}{2p}\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx$$

on the constraint

$$S(c) = \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2 dx = c^2 \right\}.$$

It is straightforward that $E_{p,q}$ is a well-defined and $C^1$ functional on $S(c)$ for $2_\alpha \le p \le 2^*_\alpha$ and $2 \le q \le 2^*$.

We note that the number and properties of the normalized solutions to problem (1.5) are strongly affected by further assumptions on the exponents $p, q$ and the parameters $\gamma, \mu$. For example, when the problem only involves Hartree nonlinearity, that is, for $\gamma = 1$ and $\mu = 0$, then (1.4) becomes the following classical Choquard equation:

$$-\Delta u + \lambda u = (I_\alpha * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{1.6}$$

If $N = 3$ and $p = \alpha = 2$, Lieb [21] proved the existence and uniqueness of normalized solutions for (1.6) by using symmetrization techniques, and Lions [25] studied the existence and stability issues of normalized solutions for (1.6). If $N \ge 3$ and $\overline{p} := \frac{N+\alpha+2}{N} < p < 2^*_\alpha$, Li and Ye [16] concluded that (1.6) has a mountain-pass–type normalized solution for each $c > 0$.

We need to point out that (1.6) has no solutions in $H^1(\mathbb{R}^N)$ when either $p \le 2_\alpha$ or $p \ge 2^*_\alpha$ for fixed $\lambda > 0$ (see [26]), where $p = 2_\alpha$ and $2^*_\alpha$ are critical exponents that come from the Hardy–Littlewood–Sobolev inequality (see Lemma 2.2). So it is interesting to study normalized solutions of (1.6) with $p = 2_\alpha$ or $2^*_\alpha$ under a local perturbation, such as problem (1.5). In two recent papers, Li [18, 19] considered problem (1.5) with $\gamma = 1, p = 2^*_\alpha$, and $2 < q < 2^*$. He concluded that problem (1.5) has one radial solution when $\overline{q} := 2 + \frac{4}{N} < q < 2^*$ and two radial solutions when

$2 < q < \overline{q}$. Moreover, qualitative properties and stability of solutions are described. Note that the upper critical exponent $2_\alpha^*$ plays a role similar to the Sobolev critical exponent in the local semilinear equations. The results obtained in [18, 19] can be regarded as a generalization of those in [13, 14, 33, 37], which studied normalized solutions of the Schrödinger equation with mixed nonlinearities,

$$-\Delta u + \lambda u = |u|^{2^*-2}u + \mu|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geq 3$, $2 < q < 2^*$, and $\mu > 0$. However, so far there seems to be no result concerned with normalized solutions of the lower critical Choquard equations with a local perturbation. To the best of our knowledge, it is completely different from the case of the upper critical exponent, since the lower critical exponent $2_\alpha$ seems to be a new feature for the Choquard equation, which is related to a new phenomenon of "bubbling at infinity" [27]. Inspired by this fact, in this paper we will study the nonexistence, existence, and qualitative properties of solutions for problem (1.5) with $p = 2_\alpha$ and $2 < q \leq 2^*$. The whole study can be considered as a counterpart of the Brezis–Nirenberg problem for nonlocal elliptic equation in the context of normalized solutions. Compared with the study developed if $p = 2_\alpha^*$, the new abstract setting seems to be more challenging, and thus new methods and ideas need to be explored. More details will be discussed in the next subsection.

**1.1. Main results.** Before stating our main results, we agree that when $q = 2^*$ is involved, we always assume that $N \geq 3$. For the other cases, we require $N \geq 2$. Next, we give the definition of a ground state in the following sense.

DEFINITION 1.1. *We say that u is a ground state of problem* (1.5) *if it is a solution of problem* (1.5) *having minimal energy among all the solutions:*

$$E'_{p,q}|_{S(c)}(u) = 0 \text{ and } E_{p,q}(u) = \inf\{E_{p,q}(v) \mid E'_{p,q}|_{S(c)}(v) = 0 \text{ and } v \in S(c)\}.$$

First of all, we give the following result when the functional $E_{p,q}$ is bounded from below on $S(c)$ and the minimization problem

$$\sigma(c) = \inf_{u \in S(c)} E_{p,q}(u) \tag{1.7}$$

is achieved.

THEOREM 1.2. *Let $\gamma > 0$, $p = 2_\alpha$, and $2 < q < \overline{q}$. Then there exists $\mu_0 > 0$ such that for every $\mu > \mu_0$, the infimum*

$$\sigma(c) < -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha}$$

*is achieved by $u_0 \in S(c)$ with the following properties:*

(i) *$u_0$ is a real-valued positive function in $\mathbb{R}^N$, which is radially symmetric and non-increasing;*

(ii) *$u_0$ is a ground state of problem* (1.5) *with some $\lambda_c > \frac{\gamma}{2_\alpha} S_\alpha^{-2_\alpha} c^{\frac{2\alpha}{N}}$.*

*Remark* 1.3. By Theorem 1.2, the set of ground states

$$Z_{2_\alpha,q}(c) := \{u \in S(c) \mid E_{2_\alpha,q}(u) = \sigma(c)\}$$

is not empty and compact, up to translation, where $E_{2_\alpha,q} = E_{p,q}$ with $p = 2_\alpha$. Hence, if we are able to deduce the global well-posedness of solutions for the Cauchy problem of (1.3) in $H^1(\mathbb{R}^N)$, then by using the strategy in [6], the set $Z_{2_\alpha,q}(c)$ is orbitally stable. Namely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\psi_0 \in H^1(\mathbb{R}^N)$ with $\inf_{v\in Z_{2_\alpha,q}(c)} \|\psi_0 - v\|_{H^1(\mathbb{R}^N)} < \delta$, we have

$$\inf_{v\in Z_{2_\alpha,q}(c)} \|\psi(t,\cdot) - v\|_{H^1} < \varepsilon \quad \forall t > 0,$$

where $\psi(t,\cdot)$ denotes the solution of (1.3) with initial datum $\psi_0$.

To prove that the minimum in (1.7) is achieved, we shall use the concentration-compactness principle by Lions [25]. Specifically, it suffices to exclude the vanishing and dichotomy of the minimizing sequence, respectively. Usually, the dichotomy can be easily excluded by using the strict subadditivity inequality

$$\sigma(c) < \sigma(\eta) + \sigma(c-\eta) \text{ for any } 0 < \eta < c.$$

However, it is difficult to exclude the vanishing by using common arguments because of the presence of the lower critical nonlocal term $\int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha}dx$. In order to overcome this difficulty, we need to make an estimate of $\sigma(c)$ by controlling the parameter $\mu$.

Next, we state the following nonexistence result when $p = 2_\alpha$ and $q = \overline{q}$.

THEOREM 1.4. *Let $\gamma > 0$, $p = 2_\alpha$, and $q = \overline{q}$. Then for $0 < \mu < \frac{N+2}{N\overline{S}c^{4/N}}$, problem (1.5) has no solution for any $\lambda \in \mathbb{R}$.*

We now turn to the case of $p = 2_\alpha$ and $\overline{q} < q \leq 2^*$ such that the functional $E_{p,q}$ is unbounded from below on $S(c)$. Then it will not be possible to find a global minimizer. In order to seek for critical points of $E_{p,q}$ restricted to $S(c)$, we shall use the Pohozaev manifold $\mathcal{M}_q(c)$ as a natural constraint of $E_{p,q}$ that contains all the critical points of $E_{p,q}$ restricted to $S(c)$. This manifold is defined by

$$\mathcal{M}_q(c) := \{u \in S(c) \mid Q_q(u) = 0\},$$

where $Q_q(u) = 0$ is the Pohozaev-type identity corresponding to (3). For more details, we refer the reader to section 2. As we shall see, the functional $E_{p,q}$ restricted to $\mathcal{M}_q(c)$ is bounded from below.

Set

$$m_q(c) := \inf_{u\in\mathcal{M}_q(c)} E_{p,q}(u) \tag{1.8}$$

and

$$c_* := \left(\frac{(N+\alpha)(N(q-2)-4)S_\alpha^{\frac{N+\alpha}{N}}}{\gamma N^2(q-2)}\right)^{\frac{N(N(q-2)-4)}{2N(q-2)\alpha+4Nq-8(N+\alpha)}} \cdot \left(\frac{2q}{\mu N\overline{S}(q-2)}\right)^{\frac{2N}{N(q-2)\alpha+2Nq-4(N+\alpha)}}. \tag{1.9}$$

We have the following result.

THEOREM 1.5. *Let $\gamma > 0$, $p = 2_\alpha$, $\overline{q} < q < 2^*$, and $0 < c < c_*$. Then there exists $\overline{\mu} > 0$ such that for every $\mu > \overline{\mu}$, problem (1.5) has a ground state $\overline{u} \in H^1(\mathbb{R}^N)$ for* *some* $\bar{\lambda} > 0$*, which is real-valued, positive, radially symmetric and nonincreasing in* $\mathbb{R}^N$*. Moreover, the following estimates hold:*

$$E_{2_\alpha,q}(\bar{u}) = m_q(c) \geq \frac{N(q-2)-4}{2N(q-2)}\left(\frac{2q}{\mu N\overline{S}(q-2)}c^{-\frac{2N-q(N-2)}{2}}\right)^{\frac{4}{N(q-2)-4}} - \frac{\gamma}{22_\alpha}S_\alpha^{-\frac{N+\alpha}{N}}c^{22_\alpha} > 0$$

*and*

$$\bar{\lambda} > \frac{2N-q(N-2)}{N(q-2)}\left(\frac{2q}{\mu N\overline{S}(q-2)}\right)^{\frac{4}{N(q-2)-4}} c^{-\frac{4(q-2)}{N(q-2)-4}}.$$

To prove the above result, we shall apply the minimax method to $E_{2_\alpha,q}$ restricted to $\mathcal{M}_q(c)$, introduced by Bartsch and Soave [3], to obtain a $(PS)$-sequence $\{u_n\} \subset \mathcal{M}_q(c)$ for $E_{2_\alpha,q}$ at level $m_q(c)$. However, the compactness of $\{u_n\}$ is a delicate problem due to the absence of the nonlocal term in the Pohozaev-type identity $Q_q(u) = 0$, arising from the feature of the lower critical exponent $2_\alpha$. In view of this fact, we introduce a method of adding mass term, which can effectively solve the weak limit being not 0 and the strong convergence of $\{u_n\}$.

Next, let us consider the double critical case, that is, $p = 2_\alpha$ and $q = 2^*$. Setting

$$c^* := \left(\frac{N+\alpha}{N\gamma}\right)^{\frac{N}{2\alpha}} S_\alpha^{\frac{N+\alpha}{2\alpha}},$$

we have following result.

THEOREM 1.6. *Let* $p = 2_\alpha$, $q = 2^*$, $0 < c < \min\{c_*, c^*\}$, *and*

$$\gamma \geq \left[\frac{\alpha N S_\alpha^{\frac{N+\alpha}{\alpha}}}{2S^{N/2}(N+\alpha)}\right]^{\frac{\alpha}{N}} \mu^{\frac{\alpha(N-2)}{2N}}. \tag{1.10}$$

*Then there exists* $\tilde{\mu} \geq \overline{\mu}$ *such that for every* $\mu > \tilde{\mu}$*, problem* (1.5) *has a ground state* $\tilde{u} \in H^1(\mathbb{R}^N)$ *for some* $\tilde{\lambda} > 0$*, which is real-valued, positive, radially symmetric, and nonincreasing in* $\mathbb{R}^N$*. Moreover, we have*

$$E_{2_\alpha,2^*}(\tilde{u}) = m_{2^*}(c) \geq \frac{1}{N}\left(\mu^{-1}S^{\frac{N}{N-2}}\right)^{\frac{N-2}{2}} - \frac{\gamma N}{2(N+\alpha)}S_\alpha^{-\frac{N+\alpha}{N}}c^{22_\alpha} > 0$$

*and*

$$0 < \tilde{\lambda} \leq \gamma S_\alpha^{-\frac{N+\alpha}{N}},$$

*where* $E_{2_\alpha,2^*} = E_{p,q}$ *with* $p = 2_\alpha$ *and* $q = 2^*$.

In the proof of Theorem 1.6, due to the interaction of the double critical terms, the method of adding mass term used in Theorem 1.5 is invalid for the case of $q = 2^*$, although the existence of the $(PS)$-sequence for the functional $E_{2_\alpha,2^*}$ can be proved. Fortunately, with the help of the solutions obtained in Theorem 1.5, by the Sobolev subcritical approximation method combined with the new estimate trick, we show that $m_{2^*}(c)$ is achieved in $S(c)$.

*Remark* 1.7. (i) In Theorems 1.2, 1.5, and 1.6, the parameter $\mu$ is required to be large enough. It is not clear to us if there is a solution when $\mu > 0$ is small. We believe it would be interesting to investigate in that direction.

(ii) In Theorems 1.2 and 1.5, we do not require any assumption on the parameter $\gamma > 0$ when $2 < q < 2^*$. However, for $q = 2^*$ we need the technical condition (1.10) in Theorem 1.6 in order to decrease the energy level because of the presence of the double critical terms.

(iii) It should be pointed out that the instability of the standing wave $\psi(t,x) = e^{-i\lambda t}u(x)$ is open, where $u$ is a ground state obtained in Theorems 1.5 and 1.6.

In order to explore the behavior of solutions as $p \to 2_\alpha^+$, we give the following result.

THEOREM 1.8. *Let $\gamma, \mu > 0$. Assume that one of the three following conditions holds:*

(i) $N \geq 2$, $2_\alpha < p < \overline{p}$, $\overline{q} < q < 2^*$;

(ii) $3 \leq N \leq 5$, $2_\alpha < p < \overline{p}$, *and* $q = 2^*$;

(iii) $N \geq 6$, $2_\alpha < p \leq \frac{N+\alpha-2}{N-2}$, $q = 2^*$, *and* $\gamma > \gamma_0$ *for some* $\gamma_0 > 0$.

*Then there exists $\hat{c} > 0$ such that for any $0 < c < \hat{c}$, problem* (1.5) *has two solutions $(u^\pm, \lambda^\pm) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ satisfying $E_{p,q}(u^+) < 0 < E_{p,q}(u^-)$. In particular, $u^+$ is a ground state of problem* (1.5). *Moreover, we have*

$$\|\nabla u^+\|_2 \to 0 \text{ and } \|u^+\|_q \to 0 \text{ as } p \to 2_\alpha^+.$$

The above theorem can be proved by using some ideas developed in [19, 33, 37, 40].

*Remark* 1.9. By Theorem 1.8, we find that the negative energy solution will disappear as $p \to 2_\alpha^+$, which is consistent with the results obtained in Theorems 1.5 and 1.6.

Finally, we summarize some properties of the mappings $\sigma(c)$ and $m_q(c)$ as follows.

THEOREM 1.10. (i) *Under the assumptions of Theorem* 1.2*, the mapping $c \mapsto \sigma(c)$ is a continuous and strictly decreasing mapping.*

(ii) *Under the assumptions of Theorems* 1.5 *and* 1.6*, the mapping $c \mapsto m_q(c)$ is continuous and strictly decreasing.*

The paper is organized as follows. In section 2 we recall some classical inequalities, and we present some preliminary results. In section 3 we treat the case $2 < q < \overline{q}$ and prove Theorem 1.2. In section 4, we give the proof of Theorem 1.4. In section 5, we study the case $\overline{q} < q \leq 2^*$ and prove Theorems 1.5 and 1.6. Finally, in section 6, we give the proof of Theorem 1.10.

## 2. Preliminary results.

For convenience, we set

$$A(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad B(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$$

and

$$C(u) = \int_{\mathbb{R}^N} |u|^q dx, \quad D(u) = \int_{\mathbb{R}^N} |u|^2 dx.$$

Then,

$$E_{2_\alpha,q}(u) = \frac{1}{2}A(u) - \frac{\gamma}{22_\alpha}B(u) - \frac{\mu}{q}C(u). \tag{2.1}$$

LEMMA 2.1 (Gagliardo–Nirenberg inequality [38]). *Let $r \in (2, 2^*)$ if $N \geq 3$. Then there exists a sharp constant $\overline{S}(N, r) > 0$ such that*

$$\|u\|_r \leq \overline{S}^{1/r} \|\nabla u\|_2^{\beta} \|u\|_2^{1-\beta}, \tag{2.2}$$

*where $\beta = N(\frac{1}{2} - \frac{1}{r})$.*

LEMMA 2.2 (Hardy–Littlewood–Sobolev inequality [22]). *Let $N \geq 1$, $\alpha \in (0, N)$, and $s \in (1, N/\alpha)$. Then for any $\varphi \in L^s(\mathbb{R}^N)$, it holds that $I_\alpha * \varphi \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$, and*

$$\int_{\mathbb{R}^N} |I_\alpha * \varphi|^{\frac{Ns}{N-\alpha s}} dx \leq \bar{C} \left( \int_{\mathbb{R}^N} |\varphi|^s dx \right)^{\frac{N}{N-\alpha s}}, \tag{2.3}$$

*where the constant $\bar{C} > 0$ depends only on $N, \alpha$, and $s$.*

*Remark* 2.3. By the semigroup identity for the Riesz potential $I_\alpha = I_{\alpha/2} * I_{\alpha/2}$, for $p \in [2_\alpha, 2_\alpha^*]$, the Hardy–Littlewood–Sobolev inequality (2.3) can be rewritten as

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx = \int_{\mathbb{R}^N} \left| I_{\alpha/2} * |u|^p \right|^2 dx \leq \bar{C} \left( \int_{\mathbb{R}^N} |u|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N}}. \tag{2.4}$$

For $p = 2_\alpha$, this inequality can be restated as the following minimization problem:

$$S_\alpha = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 dx \mid u \in H^1(\mathbb{R}^N) \backslash \{0\} \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha} dx = 1 \right\} > 0, \tag{2.5}$$

which is achieved by the function

$$V_\varepsilon(x) = \bar{C} \left( \frac{\varepsilon}{\varepsilon^2 + |x - y|^2} \right)^{\frac{N}{2}} \tag{2.6}$$

for some given constants $\bar{C} \in \mathbb{R}, y \in \mathbb{R}^N$, and $\varepsilon \in (0, +\infty)$ (see [22, Theorem 4.3]).

LEMMA 2.4 (Sobolev inequality [39]). *For $N \geq 3$, there exists an optimal constant $S > 0$ depending only on the dimension $N$ such that*

$$S\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2 \quad \forall u \in D^{1,2}(\mathbb{R}^N),$$

*where $D^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm*

$$\|u\|_{D^{1,2}} := \|\nabla u\|_2.$$

Next, we show a splitting property for the functional $B$, which is similar to the Brezis–Lieb-type lemma for nonlocal nonlinearities [1, 2].

LEMMA 2.5. *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. If $u_n \to u$ a.e. in $\mathbb{R}^N$, then*

$$B(u_n - u) = B(u_n) - B(u) + o(1).$$

We establish the following Pohozaev identity.

LEMMA 2.6. *Let $u$ be a weak solution to the equation*

$$-\Delta u + \lambda u = \gamma(I_\alpha * u^{2_\alpha})|u|^{2_\alpha - 2}u + \mu|u|^{q-2}u.$$

*Then $u$ satisfies the Pohozaev identity*

$$\frac{N-2}{2}A(u) + \frac{\lambda N}{2}D(u) = \frac{\gamma N}{2}B(u) + \frac{\mu N}{q}C(u). \tag{2.7}$$

*As a consequence it satisfies*

$$A(u) - \frac{\mu N(q-2)}{2q}C(u) = 0. \tag{2.8}$$

*Proof.* The proof is similar to that of [26, Proposition 3.1]; we omit it here. ∎

Following the idea of Soave [32] and Cingolani and Jeanjean [7], we will introduce a natural constraint manifold $\mathcal{M}_q(c)$ that contains all the critical points of the functional $E_{2_\alpha,q}$ restricted to $S(c)$. For each $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ and $t > 0$, we set

$$u_t(x) := t^{N/2}u(tx).$$

Then a direct calculation shows that $\|u_t\|_2^2 = \|u\|_2^2, A(u_t) = t^2A(u), B(u_t) = B(u)$, and $C(u_t) = t^{\frac{N(q-2)}{2}}C(u)$. Define the fibering map $t \in (0,\infty) \mapsto g_u(t) := E_{2_\alpha,q}(u_t)$ given by

$$g_u(t) = \frac{t^2}{2}A(u) - \frac{\gamma}{2p}B(u) - \frac{\mu t^{\frac{N(q-2)}{2}}}{q}C(u).$$

By calculating the first and second derivatives of $g_u(t)$, we have

$$g_u'(t) = tA(u) - \frac{\mu N(q-2)}{2q}t^{\frac{N(q-2)-2}{2}}C(u) \tag{2.9}$$

and

$$g_u''(t) = A(u) - \frac{\mu N(q-2)(N(q-2)-2)}{4q}t^{\frac{N(q-2)-4}{2}}C(u).$$

Moreover, we notice that

$$\frac{d}{dt}E_{2_\alpha,q}(u_t) = g_u'(t) = \frac{Q_q(u_t)}{t},$$

where

$$Q_q(u) := \frac{d}{dt}|_{t=1}E_{2_\alpha,q}(u_t) = A(u) - \frac{\mu N(q-2)}{2q}C(u).$$

Actually the condition $Q_q(u) = 0$ corresponds to the Pohozaev identity (2.8). Then we define the Pohozaev manifold by

$$\mathcal{M}_q(c) := \{u \in S(c) \mid Q_q(u) = 0\} = \{u \in S(c) \mid g_u'(1) = 0\},$$

which appears as a natural constraint. We also recognize that for any $u \in S(c)$, the dilated function $u_t(x) = t^{N/2}u(tx)$ belongs to the constraint manifold $\mathcal{M}_q(c)$ if and only if $t \in \mathbb{R}$ is a critical value of the fibering map $t \in (0,\infty) \mapsto g_u(t)$, namely, $g_u'(t) = 0$. Thus, it is natural to split $\mathcal{M}_q(c)$ into three parts corresponding to local minima, local maxima, and points of inflection. Following [36], we define

$$\mathcal{M}_q^+(c) = \{u \in S(c) \mid g_u'(1) = 0, g_u''(1) > 0\};$$
$$\mathcal{M}_q^-(c) = \{u \in S(c) \mid g_u'(1) = 0, g_u''(1) < 0\};$$
$$\mathcal{M}_q^0(c) = \{u \in S(c) \mid g_u'(1) = 0, g_u''(1) = 0\}.$$

Then for $u \in \mathcal{M}_q(c)$, we deduce that

$$g_u''(1) = A(u) - \frac{\mu N(q-2)(N(q-2)-2)}{4q}C(u)$$
$$= 2A(u) - \frac{\mu N^2(q-2)^2}{4q}C(u) \tag{2.10}$$
$$= \frac{4-N(q-2)}{2}A(u). \tag{2.11}$$

Furthermore, following the argument of Soave [32], we have the following lemma.

LEMMA 2.7. *Assume that $\mathcal{M}_q^0(c) = \emptyset$. Then $\mathcal{M}_q(c)$ is a smooth submanifold of codimension 2 of $H^1(\mathbb{R}^N)$ and a submanifold of codimension 1 in $S(c)$.*

Next, we shall give a general minimax theorem to establish the existence of a Palais–Smale sequence.

DEFINITION 2.8 ([34, Definition 3.1]). *Let $\Theta$ be a closed subset of a metric space $X$. We say that a class $\mathcal{F}$ of compact subsets of $X$ is a homotopy-stable family with closed boundary $\Theta$ provided that*
(i) *every set in $\mathcal{F}$ contains $\Theta$;*
(ii) *for any set $H \in \mathcal{F}$ and any $\eta \in C([0,1] \times X, X)$ satisfying $\eta(s,x) = x$ for all $(s,x) \in (\{0\} \times X) \cup ([0,1] \times \Theta)$, we have that $\eta(\{1\} \times H) \in \mathcal{F}$.*

LEMMA 2.9 ([34, Theorem 3.2]). *Let $\varphi$ be a $C^1$-functional on a complete connected $C^1$-Finsler manifold $X$ (without boundary), and consider a homotopy-stable family $\mathcal{F}$ of compact subsets of $X$ with a closed boundary $\Theta$. Set*

$$d = d(\varphi, \mathcal{F}) = \inf_{H \in \mathcal{F}} \max_{u \in H} \varphi(u)$$

*and suppose that $\sup \varphi(\Theta) < d$. Then for any sequence of sets $\{H_n\}$ in $\mathcal{F}$ such that $\lim_{n\to\infty} \sup_{H_n} \varphi = d$, there exists a sequence $\{u_n\}$ in $X$ such that* (i) $\lim_{n\to\infty} \varphi(u_n) = d$; (ii) $\lim_{n\to\infty} \|\varphi'(u_n)\| = 0$; (iii) $\lim_{n\to\infty} dist(u_n, H_n) = 0$.
*Moreover, if $\varphi'$ is uniformly continuous, then $u_n$ can be chosen to be in $H_n$ for each $n$.*

## 3. The case $2 < q < \overline{q}$.

LEMMA 3.1. *Let $\gamma, \mu > 0$ and $2 < q < \overline{q}$. Then the following statements are true.*
(i) *The functional $E_{2_\alpha,q}$ is bounded below and coercive on $S(c)$.*
(ii) *$\sigma(c) < \sigma(\eta) + \sigma(c-\eta)$ for $0 < \eta < c$, where $\sigma(c)$ is defined as (1.7).*

*Proof.* (i) By (2.1), (2.2), and (2.5) one has

$$E_{2_\alpha,q}(u) = \frac{1}{2}A(u) - \frac{\gamma}{22_\alpha}B(u) - \frac{\mu}{q}C(u)$$
$$\geq \frac{1}{2}A(u) - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} - \frac{\mu\overline{S}}{q}c^{\frac{2N-q(N-2)}{2}}A(u)^{\frac{N(q-2)}{4}},$$

which implies that the functional $E_{2_\alpha,q}$ is bounded below and coercive on $S(c)$.

(ii) Let $\{u_n\} \subset S(c)$ be a bounded minimizing sequence for $\sigma(c)$. Then for any $\theta > 0$ and $u \in S(c)$, it holds that $\theta u \in S(\theta^2 c)$ and

$$E_{2_\alpha,q}(\theta u_n) - \theta^2 E_{2_\alpha,q}(u_n) = \frac{\theta^2 - \theta^{22_\alpha}}{2p}\gamma B(u_n) + \frac{\theta^2 - \theta^q}{q}\mu C(u_n).$$

If we choose $\theta > 1$, then it is clear that $E_{2_\alpha,q}(\theta u_n) < \theta^2 E_{2_\alpha,q}(u_n)$ by using the above inequality. This implies that $\sigma(\theta^2 c) \leq \theta^2\sigma(c)$, where the equality holds if and only if $B(u_n) + C(u_n) \to 0$ as $n \to \infty$. But this is not possible, since otherwise we find that

$$0 > \sigma(c) = \lim_{n\to\infty} E_{2_\alpha,q}(u_n) \geq \lim\inf_{n\to\infty} \frac{1}{2}A(u_n) \geq 0.$$

Without loss of generality, we may assume that $0 < \eta < c - \eta$; then

$$\sigma(c) < \frac{c}{c-\eta}\sigma(c-\eta) = \sigma(c-\eta) + \frac{\eta}{c-\eta}\sigma(c-\eta) < \sigma(c-\eta) + \sigma(\eta).$$

We complete the proof. ∎

LEMMA 3.2. *Assume that $\gamma > 0$ and $2 < q < \overline{q}$. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that $E_{2_\alpha,q}(u_n) \to \sigma(c)$ and $\|u_n\|_2 \to c$. Then there exists a constant $\mu_0 > 0$ such that for every $\mu > \mu_0$, the sequence $\{u_n\}$ is relatively compact in $H^1(\mathbb{R}^N)$ up to translations, that is, there exist a subsequence, still denoted by $\{u_n\}$, a sequence of points $\{y_n\} \subset \mathbb{R}^N$, and a function $u_0 \in S(c)$ such that $u_n(\cdot + y_n) \to u_0$ strongly in $H^1(\mathbb{R}^N)$.*

*Proof.* First of all, we claim that there exists a constant $\mu_0 > 0$ such that for every $\mu > \mu_0$,

$$\sigma(c) < -\frac{\gamma}{22_\alpha} S_\alpha^{-2_\alpha} c^{22_\alpha}. \tag{3.1}$$

Indeed, let us fix a $u \in S(c)$. Then there exists a constant $\mu_0 > 0$ sufficiently large such that

$$\frac{1}{2}A(u) + \frac{\gamma}{22_\alpha}(S_\alpha^{-2_\alpha} c^{22_\alpha} - B(u)) - \frac{\mu_0}{q}C(u) < 0.$$

Thus for every $\mu > \mu_0$, we have

$$\begin{aligned}\sigma(c) \leq E_{2_\alpha,q}(u) &= \frac{1}{2}A(u) - \frac{\gamma}{22_\alpha}B(u) - \frac{\mu}{q}C(u) \\ &= \frac{1}{2}A(u) + \frac{\gamma}{22_\alpha}(S_\alpha^{-2_\alpha} c^{22_\alpha} - B(u)) - \frac{\mu}{q}C(u) - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha} c^{22_\alpha} \\ &< -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha} c^{22_\alpha}.\end{aligned}$$

From Lemma 3.1 it follows that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Then according to the concentration-compactness principle [21], there exists a subsequence, still denoted by $\{u_n\}$, satisfying one of the following three possibilities:

(i) Vanishing:

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0 \text{ for } R > 0.$$

(ii) Dichotomy: there exists $0 < \eta < c$ and $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ bounded in $H^1(\mathbb{R}^N)$ such that as $n \to \infty$,

$$\|u_n - (u_n^{(1)} + u_n^{(2)})\|_r \to 0 \text{ for } 2 \le r < 2^*, \text{ and } dist(\text{supp}u_n^{(1)}, \text{supp}u_n^{(2)}) \to +\infty;$$

$$\left| \int_{\mathbb{R}^N} |u_n^{(1)}|^2 dx - \eta \right| \to 0 \text{ and } \left| \int_{\mathbb{R}^N} |u_n^{(2)}|^2 dx - (c - \eta) \right| \to 0;$$

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_n^{(1)}|^2 - |\nabla u_n^{(2)}|^2) dx \ge 0.$$

(iii) Compactness: there exists $y_n \in \mathbb{R}^N$ such that for all $\varepsilon > 0$, there exists $R > 0$,

$$\int_{B_R(y_n)} |u_n|^2 dx \ge c - \varepsilon.$$

We firstly verify that the vanishing cannot occur. Suppose the contrary. Then $u_n \to 0$ strongly in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Using this together with (2.1) and (2.5) gives

$$\begin{aligned}
\sigma(c) + o_n(1) = E_{2_\alpha,q}(u_n) &= \frac{1}{2}A(u_n) - \frac{\gamma}{22_\alpha}B(u_n) - \frac{\mu}{q}C(u_n) \\
&\ge \frac{1}{2}A(u_n) - \frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} + o_n(1) \\
&\ge -\frac{1}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha} + o_n(1),
\end{aligned}$$

which contradicts (3.1).

Also dichotomy cannot occur, since otherwise

$$\sigma(c) = \lim_{n\to\infty} E_{2_\alpha,q}(u_n) \ge \lim_{n\to\infty} E_{2_\alpha,q}(u_n^{(1)}) + \lim_{n\to\infty} E_{2_\alpha,q}(u_n^{(2)}) \ge \sigma(\eta) + \sigma(c - \eta),$$

where we have used Lemma 2.5. This contradicts Lemma 3.1. So the compactness holds, namely, there exist subsequences $\{u_n\}, \{y_n\} \subset \mathbb{R}^N$, and $u_0 \in S(c)$ such that $\overline{u}_n := u_n(\cdot + y_n) \to u_0$ in $L^s(\mathbb{R}^N)$ for $2 \le s < 2^*$, which implies that $B(\overline{u}_n) \to B(u_0)$ by (2.4). Hence, we have

$$\sigma(c) \le E_{2_\alpha,q}(u_0) \le \liminf_{n\to\infty} E_{2_\alpha,q}(\overline{u}_n) = \liminf_{n\to\infty} E_{2_\alpha,q}(u_n) = \sigma(c).$$

This shows that $u_0$ is a minimizer for $\sigma(c)$. We complete the proof. ∎

**Now, we give the proof of Theorem 1.2.** It follows from Lemma 3.2 that for every $\mu > \mu_0$, there exists a global minimizer $u_0$ for $E_{2_\alpha,q}$ on $S(c)$. In other words, the infimum $\sigma(c) < -\frac{\gamma}{22_\alpha}S_\alpha^{-2_\alpha}c^{22_\alpha}$ is achieved by $u_0 \in S(c)$ which is a ground state of problem (1.5). Let $|u_0|^*$ denote the Schwartz rearrangement of $|u_0|$. Then

$$D(|u_0|^*) = D(u_0) = c^2,\ A(|u_0|^*) \le A(u_0),\ C(|u_0|^*) = C(|u_0|^*). \tag{3.2}$$

By Riesz's rearrangement inequality (see [22, section 3.7]), we get

$$B(|u_0|^*) \ge B(u_0). \tag{3.3}$$

This implies that $|u_0|^* \in S(c)$ and $E_{2_\alpha,q}(|u_0|^*) \leq E_{2_\alpha,q}(u_0) = \sigma(c)$. By the definition of $\sigma(c)$, we know that $\sigma(c)$ is achieved by the real-valued positive and radially symmetric nonincreasing function. Moreover, since $u_0$ is a critical point of $E_{2_\alpha,q}$ restricted to $S(c)$, there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $E'_{2_\alpha,q}(u_0) + \lambda_c u_0 = 0$. Then we have

$$\begin{aligned}\lambda_c c^2 &= -A(u_0) + \gamma B(u_0) + \mu C(u_0)\\ &= -2\sigma(c) + \frac{\gamma(2_\alpha - 1)}{2_\alpha} B(u_0) + \frac{\mu(q-2)}{q} C(u_0)\\ &> -2\sigma(c),\end{aligned}$$

which implies that

$$\lambda_c > \frac{\gamma}{2_\alpha} S_\alpha^{-2_\alpha} c^{2(2_\alpha - 1)},$$

where we have used (3.1). We complete the proof.

## 4. The case $q = \overline{q}$.

*Proof of Theorem* 1.4. Let $u \in S(c)$ and $t > 0$. By (2.2) and (2.9) we have

$$\begin{aligned}g'_u(t) &= tA(u) - \frac{\mu N}{(N+2)} tC(u)\\ &\geq tA(u) - \frac{\mu N\overline{S}}{(N+2)} c^{\frac{4}{N}} tA(u)\\ &= \left[1 - \frac{\mu N\overline{S}}{(N+2)} c^{\frac{4}{N}}\right] tA(u)\\ &> 0 \text{ if } \mu < \frac{(N+2)}{N\overline{S}} c^{-\frac{4}{N}}.\end{aligned}$$

This shows that the fiber map $g_u(t)$ is strictly increasing and so the functional $E_{2_\alpha,\overline{q}}$ has no critical point on $S(c)$ for $\mu < \frac{(N+2)}{N\overline{S}} c^{-\frac{4}{N}}$. In other words, problem (1.5) has no solution for any $\lambda \in \mathbb{R}$. We complete the proof of Theorem 1.4. ∎

## 5. The case $\overline{q} < q \leq 2^*$.

For the case of $\overline{q} < q \leq 2^*$, the functional $E_{2_\alpha,q}$ will be no longer bounded from below on $S(c)$, and so it will not be possible to obtain a global minimizer. In this section, in order to find critical points of $E_{2_\alpha,q}$, we shall restrict it to a natural constraint manifold $\mathcal{M}_q(c)$ defined in section 2, on which $E_{2_\alpha,q}$ is bounded below.

LEMMA 5.1. *Assume that* $\gamma, \mu > 0$ *and* $\overline{q} < q \leq 2^*$. *Then there exists a unique* $t^-(u) > 0$ *such that* $u_{t^-(u)} \in \mathcal{M}_q(c) = \mathcal{M}_q^-(c)$ *and*

$$E_{2_\alpha,q}(u_{t^-(u)}) = \sup_{t>0} E_{2_\alpha,q}(u_t) > 0.$$

*Proof.* Clearly, $\mathcal{M}_q(c) = \mathcal{M}_q^-(c)$ by (2.11). Fix $u \in S(c)$. Let

$$h(t) := \frac{2q}{\mu N(q-2)} A(u) t^{\frac{4-N(q-2)}{2}} \text{ for } t > 0.$$

Then $u_t \in \mathcal{M}_q(c)$ if and only if $h(t) = C(u)$. A direct calculation shows that $\lim_{t\to 0^+} h(t) = \infty$, $\lim_{t\to+\infty} h(t) = 0$, and $h(t)$ is decreasing on $(0, +\infty)$. This implies that there exists a unique $t^-(u) > 0$ such that $u_{t^-(u)} \in \mathcal{M}_q(c)$. Moreover, we also obtain that $g_u'(t) > 0$ on $(0, t^-(u))$ and $g_u'(t) < 0$ on $(t^-(u), +\infty)$, which indicates that

$$E_{2_\alpha,q}(u_{t^-(u)}) = \sup_{t>0} E_{2_\alpha,q}(u_t) > 0.$$

We complete the proof. ∎

LEMMA 5.2. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q \leq 2^*$. Then the functional $E_{2_\alpha,q}$ is bounded below by a positive constant and coercive on $\mathcal{M}_q(c)$ for $0 < c < c_*$, where $c_*$ is as in* (1.9).

*Proof.* Clearly, for $u \in \mathcal{M}_q(c)$ one has

$$A(u) - \frac{\mu N(q-2)}{2q} C(u) = 0, \tag{5.1}$$

which implies that

$$A(u) \geq \left( \frac{2q}{\mu N \overline{S}(q-2)} c^{-\frac{2N - q(N-2)}{2}} \right)^{\frac{4}{N(q-2)-4}}, \tag{5.2}$$

where we have used Lemma 2.1. Then it follows from (2.1), (2.5), (5.1), and (5.2) that

$$\begin{aligned}
E_{2_\alpha,q}(u) &= \frac{1}{2} A(u) - \frac{\gamma}{22_\alpha} B(u) - \frac{\mu}{q} C(u) \\
&= \frac{N(q-2)-4}{2N(q-2)} A(u) - \frac{\gamma}{22_\alpha} B(u) \\
&\geq \frac{N(q-2)-4}{2N(q-2)} A(u) - \frac{\gamma}{22_\alpha} S_\alpha^{-\frac{N+\alpha}{N}} c^{22_\alpha} \\
&\geq \frac{N(q-2)-4}{2N(q-2)} \left( \frac{2q}{\mu N \overline{S}(q-2)} c^{-\frac{2N-q(N-2)}{2}} \right)^{\frac{4}{N(q-2)-4}} - \frac{\gamma}{22_\alpha} S_\alpha^{-\frac{N+\alpha}{N}} c^{22_\alpha} \\
&> 0,
\end{aligned}$$

provided that

$$\begin{aligned}
c < c_* = &\left( \frac{(N+\alpha)(N(q-2)-4) S_\alpha^{\frac{N+\alpha}{N}}}{\gamma N^2 (q-2)} \right)^{\frac{N(N(q-2)-4)}{2N\alpha(q-2)+4Nq-8(N+\alpha)}} \\
&\cdot \left( \frac{2q}{\mu N \overline{S}(q-2)} \right)^{\frac{2N}{N\alpha(q-2)+2Nq-4(N+\alpha)}}.
\end{aligned}$$

We complete the proof. ∎

We now work in the subspace of functions in $H^1(\mathbb{R}^N)$ which are radially symmetric with respect to 0, denoted by $H_r^1(\mathbb{R}^N)$, and we define

$$S_r(c) := S(c) \cap H_r^1(\mathbb{R}^N) \text{ and } \mathcal{M}_q^r(c) := \mathcal{M}_q(c) \cap H_r^1(\mathbb{R}^N). \tag{5.3}$$

LEMMA 5.3. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q \leq 2^*$. Then for $0 < c < c_*$, we have*

$$m_q(c) := \inf_{u \in \mathcal{M}_q(c)} E_{2_\alpha,q}(u) = \inf_{u \in \mathcal{M}_q^r(c)} E_{2_\alpha,q}(u) > 0.$$

*Proof.* The proof follows the idea of [14]. Since $\mathcal{M}_q^r(c) \subset \mathcal{M}_q(c)$, we have

$$\inf_{u\in\mathcal{M}_q(c)} E_{2_\alpha,q}(u) \le \inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u).$$

Next, let us prove that $\inf_{u\in\mathcal{M}_q(c)} E_{2_\alpha,q}(u) \ge \inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u)$. By Lemma 5.1, we have

$$\inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u) = \inf_{u\in S(c)} \sup_{0<t\le t^-(u)} E_{2_\alpha,q}(u_t). \tag{5.4}$$

Let $u \in S(c)$ and $|u|^* \in S_r(c)$ be the Schwartz rearrangement of $|u|$. Then by (3.2) and (3.3), for all $t > 0$, we have

$$\begin{aligned} E_{2_\alpha,q}((|u|^*)_t) &= \frac{t^2}{2}A(|u|^*) - \frac{\gamma}{22_\alpha}B(|u|^*) - \frac{\mu t^{N(q-2)/2}}{q}C(|u|^*) \\ &\le \frac{t^2}{2}A(u) - \frac{\gamma}{22_\alpha}B(u) - \frac{\mu t^{N(q-2)/2}}{q}C(u) \\ &= E_{2_\alpha,q}(u_t). \end{aligned} \tag{5.5}$$

Note that $g'_{|u|^*}(0) = g'_u(0) = 0$ and $g''_{|u|^*}(t) \le g''_u(t)$ for $t > 0$. This implies that $0 < t^-(|u|^*) \le t^-(u)$. Hence, it follows from (5.5) that

$$\sup_{0<t\le t^-(|u|^*)} E_{2_\alpha,q}((|u|^*)_t) \le \sup_{0<t\le t^-(u)} E_{2_\alpha,q}(u_t).$$

Using this, together with (5.4), leads to

$$\inf_{u\in\mathcal{M}_q(c)} E_{2_\alpha,q}(u) \ge \inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u),$$

implying that $\inf_{u\in\mathcal{M}_q(c)} E_{2_\alpha,q}(u) = \inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u) > 0$, where we have also used Lemma 5.2. We complete the proof. ◻

Next, we establish the existence of a Palais–Smale sequence $\{u_n\} \subset \mathcal{M}_q^r(c)$ for $E_{2_\alpha,q}$ restricted to $S_r(c)$ at level $m_q(c)$. Our arguments are inspired by [7]. Observe that $\Theta = \emptyset$ is admissible. Now we define the functional $J : S_r(c) \mapsto \mathbb{R}$ by

$$J(u) = E_{2_\alpha,q}(u_{t^-(u)}).$$

LEMMA 5.4. *The map $u \in S_r(c) \mapsto t^-(u) \in \mathbb{R}$ is of class $C^1$.*

*Proof.* By a direct application of the implicit function theorem on the $C^1$ function $F : \mathbb{R}\times S_r(c) \to \mathbb{R}$ defined by $F(t,u) = g'_u(t)$, we easily reach the conclusion. ◻

By the above lemma, we obtain that the functional $J$ is of class $C^1$.

LEMMA 5.5. *The map $\Psi : T_uS_r(c) \to T_{u_{t^-(u)}}S_r(c)$ defined by $\psi \to \psi_{t^-(u)}$ is an isomorphism, where $T_uS_r(c)$ denotes the tangent space to $S_r(c)$ in $u$.*

*Proof.* For $\psi \in T_uS_r(c)$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} u_{t^-(u)}(x)\psi_{t^-(u)}(x)dx &= \int_{\mathbb{R}^N} (t^-(u))^{N/2}u(t^-(u)x)(t^-(u))^{N/2}\psi(t^-(u)x)dx \\ &= \int_{\mathbb{R}^N} u(y)\psi(y)dy = 0, \end{aligned}$$

which implies that $\psi_{t^-(u)} \in T_{u_{t^-(u)}} S_r(c)$, and thus the map $\Psi$ is well defined. Moreover, for all $\psi_1, \psi_2 \in T_u S_r(c)$ and all $k \in \mathbb{R}$, it holds that

$$\begin{aligned}\Psi(\psi_1 + \psi_2) &= (\psi_1 + \psi_2)_{t^-(u)} \\ &= (t^-(u))^{N/2}(\psi_1(t^-(u)x) + \psi_2(t^-(u)x)) \\ &= (\psi_1)_{t^-(u)} + (\psi_2)_{t^-(u)} \\ &= \Psi(\psi_1) + \Psi(\psi_2)\end{aligned}$$

and $\Psi(k\psi_1) = (k\psi_1)_{t^-(u)} = k(\psi_1)_{t^-(u)} = k\Psi(\psi_1)$. This shows that the map $\Psi$ is linear. Finally, let us claim that the map $\Psi$ is a bijection. For all $\psi_1, \psi_2 \in T_u S_r(c)$ with $\psi_1 \neq \psi_2$, by the fact of $t_u^- > 0$, we have

$$\Psi(\psi_1) = (t^-(u))^{N/2}\psi_1(t^-(u)x) \neq (t^-(u))^{N/2}\psi_2(t^-(u)x) = \Psi(\psi_2).$$

Moreover, let $\chi \in T_{u_{t^-(u)}} S_r(c)$. Clearly, $((t^-(u))^{-N/2}\chi(\frac{x}{t^-(u)}))_{t^-(u)} = \chi(x)$ and

$$\begin{aligned}\int_{\mathbb{R}^N} (t^-(u))^{-N/2}\chi\left(\frac{x}{t^-(u)}\right) u(x)dx &= \int_{\mathbb{R}^N} \chi(y)(t^-(u))^{N/2}u(t^-(u)y)dy \\ &= \int_{\mathbb{R}^N} \chi(y)u_{t^-(u)}(y)dy = 0,\end{aligned}$$

leading to $(t^-(u))^{-N/2}\chi(\frac{x}{t^-(u)}) \in T_u S_r(c)$. So, $\Psi$ is a bijection. We complete the proof. ◻

LEMMA 5.6. *It holds that*

$$(J)'(u)[\psi] = E'_{2_\alpha,q}(u_{t^-(u)})[\psi_{t^-(u)}]$$

*for any* $u \in S_r(c)$ *and* $\psi \in T_u S_r(c)$.

*Proof.* The proof is similar to that of [7, Lemma 3.15]; we omit it here. ◻

LEMMA 5.7. *Let* $\mathcal{F}$ *be a homotopy-stable family of compact subsets of* $S_r(c)$ *with closed boundary* $\Theta$, *and let*

$$e_{\mathcal{F}} := \inf_{H\in\mathcal{F}} \max_{u\in H} J(u).$$

*Assume that* $\Theta$ *is contained in a connected component of* $\mathcal{M}_q^r(c)$ *and*

$$\max\{\sup J(\Theta), 0\} < e_{\mathcal{F}} < \infty.$$

*Then there exists a Palais–Smale sequence* $\{u_n\} \subset \mathcal{M}_q^r(c)$ *for* $E_{2_\alpha,q}$ *restricted to* $S_r(c)$ *at level* $e_{\mathcal{F}}$.

*Proof.* Using Lemmas 5.5 and 5.6, similar to the arguments of [7, Lemma 3.16], we easily obtain the conclusion. ◻

In view of Lemma 5.3 we have

$$e_{\overline{\mathcal{F}}} = \inf_{H\in\overline{\mathcal{F}}} \max_{u\in H} J(u) = \inf_{u\in S_r(c)} J(u) = \inf_{u\in\mathcal{M}_q^r(c)} E_{2_\alpha,q}(u) = \inf_{u\in\mathcal{M}_q(c)} E_{2_\alpha,q}(u) = m_q(c).$$

Then the following lemma follows directly from Lemma 5.7.

LEMMA 5.8. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q \leq 2^*$. Then for $0 < c < c_*$, there exists a Palais–Smale sequence $\{u_n\} \subset \mathcal{M}_q^r(c)$ for $E_{2_\alpha,q}$ restricted to $S_r(c)$ at the level $m_q(c) > 0$.*

LEMMA 5.9. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q < 2^*$. Let $\{u_n\} \subset \mathcal{M}_q^r(c)$ be a bounded Palais–Smale sequence for $E_{2_\alpha,q}$ restricted to $S_r(c)$ at level $m_q(c) > 0$. Then there exists a constant $\overline{\mu} > 0$ such that for every $\mu > \overline{\mu}$, up to a subsequence, $u_n \to \bar{u}$ strongly in $H_r^1(\mathbb{R}^N)$ for $0 < c < c_*$.*

*Proof.* Since $\{u_n\} \subset \mathcal{M}_q^r(c)$ is a bounded Palais–Smale sequence of $E_{2_\alpha,q}$ restricted to $S_r(c)$, there exists $\bar{u} \in H_r^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup \bar{u}$ weakly in $H_r^1(\mathbb{R}^N)$, $u_n \to \bar{u}$ strongly in $L^r(\mathbb{R}^N)$ for $2 < r < 2^*$, and a.e. in $\mathbb{R}^N$. Moreover, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that for every $\varphi \in H_r^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + \lambda_n u_n \varphi) dx - \gamma \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_n^{2_\alpha} u_n^{2_\alpha - 2}}{|x-y|^{N-\alpha}} u_n \varphi dx$$
$$-\mu \int_{\mathbb{R}^N} |u_n|^{q-2} u_n \varphi dx = o(1)\|\varphi\|,$$

which implies that

$$\lambda_n c^2 = \lambda_n D(u_n) = \gamma B(u_n) + \mu C(u_n) - A(u_n) + o(1). \tag{5.6}$$

This indicates that $\{\lambda_n\}$ is bounded. Then we can assume that $\lambda_n \to \bar{\lambda} \in \mathbb{R}$. It follows from (5.6) and the fact of $Q_q(u_n) = o(1)$ that

$$\bar{\lambda} c^2 = \lim_{n\to\infty} \lambda_n c^2 = \lim_{n\to\infty} \left[\gamma B(u_n) + \frac{2N - q(N-2)}{2q} \mu C(u_n)\right] \geq 0, \tag{5.7}$$

which shows that $\bar{\lambda} \geq 0$. We now claim that $\bar{\lambda} \neq 0$. Otherwise, it follows from (5.7) that $B(u_n) = C(u_n) = o(1)$, and together with $Q_q(u_n) = o(1)$, leading to $A(u_n) = o(1)$. Thus, we have $E_{2_\alpha,q}(u_n) = o(1)$, which contradicts $m_q(c) > 0$. Hence, $\bar{\lambda} > 0$.

Next, we claim that $\bar{u} \neq 0$. Otherwise, we have $A(u_n) = C(u_n) = o(1)$, which implies that

$$l := \lim_{n\to\infty} \gamma B(u_n) = \lim_{n\to\infty} \lambda_n D(u_n) = \bar{\lambda} c > 0,$$

where we have used (5.6). By (2.5) we deduce that $l \geq \gamma^{-\frac{N}{\alpha}} \left(\bar{\lambda} S_\alpha\right)^{\frac{N+\alpha}{\alpha}}$. Then we get

$$\begin{aligned} m_q(c) + \frac{\bar{\lambda}}{2} c^2 &= \lim_{n\to\infty} \left(E_{2_\alpha,q}(u_n) + \frac{\lambda_n}{2} D(u_n)\right) \\ &= \lim_{n\to\infty} \left(\frac{\lambda_n}{2} D(u_n) - \frac{\gamma}{22_\alpha} B(u_n)\right) \\ &= \lim_{n\to\infty} \frac{\alpha\gamma}{2(N+\alpha)} B(u_n) \\ &\geq \frac{\alpha}{2(N+\alpha)} \gamma^{-\frac{N}{\alpha}} \left(\bar{\lambda} S_\alpha\right)^{\frac{N+\alpha}{\alpha}}. \end{aligned} \tag{5.8}$$

On the other hand, we use $v_\varepsilon := c \frac{V_\varepsilon}{\|V_\varepsilon\|_2}$ to estimate $m_q(c) + \frac{\bar{\lambda}}{2} c^2$, where $V_\varepsilon$ is defined as (2.6). We note that $\|V_\varepsilon\|_2$ is a positive constant independent $\varepsilon$,

$$\int_{\mathbb{R}^N} |\nabla V_\varepsilon|^2 dx = \bar{C} \varepsilon^2, \quad \text{and} \quad \int_{\mathbb{R}^N} |V_\varepsilon|^q dx = \bar{C} \varepsilon^{\frac{(q-2)N}{2}}.$$

Clearly, $\|v_\varepsilon\|_2 = c$. By Lemma 5.1, there exists a unique constant $t_\varepsilon^- > 0$ independent of $\varepsilon$ such that $(v_\varepsilon)_{t_\varepsilon^-} \in \mathcal{M}_q(c)$ and

$$E_{2_\alpha,q}((v_\varepsilon)_{t_\varepsilon^-}) = \sup_{t\geq 0} E_{2_\alpha,q}((v_\varepsilon)_t),$$

leading to

$$m_q(c) \leq \sup_{t\geq 0} E_{2_\alpha,q}((v_\varepsilon)_t). \tag{5.9}$$

Moreover, a direct calculation shows that

$$\begin{aligned}
&\frac{\bar\lambda c^2}{2\|V_\varepsilon\|_2^2}\int_{\mathbb{R}^N}|V_\varepsilon|^2dx - \frac{\gamma c^{22_\alpha}}{22_\alpha\|V_\varepsilon\|_2^{22_\alpha}}\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|V_\varepsilon|^{2_\alpha}|V_\varepsilon|^{2_\alpha}}{|x-y|^{N-\alpha}}dxdy\\
&= \frac{\bar\lambda c^2}{2\|V_\varepsilon\|_2^2}\int_{\mathbb{R}^N}|V_\varepsilon|^2dx - \frac{\gamma c^{22_\alpha}}{22_\alpha\|V_\varepsilon\|_2^{22_\alpha}}S_\alpha^{-2_\alpha}\left(\int_{\mathbb{R}^N}|V_\varepsilon|^2dx\right)^{2_\alpha}\\
&\leq \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}\left(\bar\lambda S_\alpha\right)^{\frac{N+\alpha}{\alpha}}.
\end{aligned} \tag{5.10}$$

Then it follows from (5.9) and (5.10) that there exists a constant $\bar\mu > 0$ such that for every $\mu > \bar\mu$,

$$\begin{aligned}
m_q(c) + \frac{\bar\lambda}{2}c^2 &\leq E_{2_\alpha,q}((v_\varepsilon)_{t_\varepsilon^-}) + \frac{\bar\lambda}{2}c^2\\
&= E_{2_\alpha,q}((v_\varepsilon)_{t_\varepsilon^-}) + \frac{\bar\lambda}{2}\int_{\mathbb{R}^N}|v_\varepsilon|^2dx\\
&= \frac{(t_\varepsilon^-)^2}{2}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^2dx + \frac{\bar\lambda}{2}\int_{\mathbb{R}^N}|v_\varepsilon|^2dx - \frac{\gamma}{22_\alpha}\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v_\varepsilon|^{2_\alpha}|v_\varepsilon|^{2_\alpha}}{|x-y|^{N-\alpha}}dxdy\\
&\quad -\frac{\mu(t_\varepsilon^-)^{\frac{(q-2)N}{2}}}{q}\int_{\mathbb{R}^N}|v_\varepsilon|^qdx\\
&= \frac{(t_\varepsilon^-)^2c^2}{2\|V_\varepsilon\|_2^2}\int_{\mathbb{R}^N}|\nabla V_\varepsilon|^2dx + \frac{\bar\lambda c^2}{2\|V_\varepsilon\|_2^2}\int_{\mathbb{R}^N}|V_\varepsilon|^2dx\\
&\quad -\frac{\gamma c^{22_\alpha}}{22_\alpha\|V_\varepsilon\|_2^{22_\alpha}}\int\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|V_\varepsilon|^{2_\alpha}|V_\varepsilon|^{2_\alpha}}{|x-y|^{N-\alpha}}dxdy - \frac{\mu c^q(t_\varepsilon^-)^{\frac{(q-2)N}{2}}}{q\|V_\varepsilon\|_2^q}\int_{\mathbb{R}^N}|V_\varepsilon|^qdx.\\
&< \frac{c^2C(t_\varepsilon^-)^2}{2\|V_\varepsilon\|_2^2}\varepsilon^2 - \frac{\mu c^qC(t_\varepsilon^-)^{\frac{(q-2)N}{2}}}{q\|V_\varepsilon\|_2^q}\varepsilon^{\frac{(q-2)N}{2}} + \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}\left(\bar\lambda S_\alpha\right)^{\frac{N+\alpha}{\alpha}}\\
&< \frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}\left(\bar\lambda S_\alpha\right)^{\frac{N+\alpha}{\alpha}},
\end{aligned} \tag{5.11}$$

which contradicts (5.8). So $\bar u \neq 0$.

By weak convergence, we have

$$-\Delta\bar u + \bar\lambda\bar u = \gamma(I_\alpha * |\bar u|^{2_\alpha})|\bar u|^{2_\alpha-2}\bar u + \mu|\bar u|^{q-2}\bar u \quad \text{in } \mathbb{R}^N,$$

and so $Q_{2_\alpha,q}(\bar u) = 0$. Let $v_n := u_n - \bar u$. Then it holds that

$$A(u_n) = A(\bar u) + A(v_n) + o(1) \tag{5.12}$$

and

$$B(u_n) = B(\bar{u}) + B(v_n) + o(1)$$

by Lemma 2.5. Since $Q_q(u_n) = o(1)$ and $C(u_n) = C(\bar{u}) + o(1)$, by (5.12) we deduce that

$$A(\bar{u}) + A(v_n) = \frac{\mu N(q-2)}{2p} C(\bar{u}) + o(1).$$

Note that $Q_q(\bar{u}) = A(\bar{u}) - \frac{\mu N(q-2)}{22_\alpha} C(\bar{u}) = 0$. Then $A(v_n) = o(1)$. Using this, together with $C(v_n) = o(1)$ and

$$A(v_n) + \lambda_n D(v_n) = \gamma B(v_n) + \mu C(v_n) + o(1),$$

gives

$$\bar{l} := \lim_{n\to\infty} \lambda_n D(v_n) = \lim_{n\to\infty} \gamma B(v_n),$$

implying that either $\bar{l} = 0$ or $\bar{l} \geq \gamma^{-\frac{N}{\alpha}} \left(\bar{\lambda} S_\alpha\right)^{\frac{N+\alpha}{\alpha}}$, where we have used (2.5). If $\bar{l} = 0$, clearly $u_n \to \bar{u}$ in $H^1(\mathbb{R}^N)$. If $\bar{l} \geq \gamma^{-\frac{N}{\alpha}} \left(\bar{\lambda} S_\alpha\right)^{\frac{N+\alpha}{\alpha}}$, then we have

$$\begin{aligned}
m_q(c) + \frac{\bar{\lambda}}{2} c^2 &= m_q(c) + \frac{1}{2} \lim_{n\to\infty} \lambda_n D(u_n) \\
&\geq m_q(c) + \frac{1}{2} \lim_{n\to\infty} \lambda_n D(v_n) \\
&= E_{2_\alpha,q}(\bar{u}) + \lim_{n\to\infty} \left( E_{2_\alpha,q}(v_n) + \frac{\lambda_n}{2} D(v_n) \right) \\
&= E_{2_\alpha,q}(\bar{u}) + \lim_{n\to\infty} \left( \frac{\lambda_n}{2} D(v_n) - \frac{\gamma}{22_\alpha} B(v_n) \right) \\
&= E_{2_\alpha,q}(\bar{u}) + \lim_{n\to\infty} \frac{\gamma\alpha}{2(N+\alpha)} B(v_n) \\
&\geq E_{2_\alpha,q}(\bar{u}) + \frac{\alpha}{2(N+\alpha)} \gamma^{-\frac{N}{\alpha}} \left(\lambda S_\alpha\right)^{\frac{N+\alpha}{\alpha}} \\
&> \frac{\alpha}{2(N+\alpha)} \gamma^{-\frac{N}{\alpha}} \left(\lambda S_\alpha\right)^{\frac{N+\alpha}{\alpha}}, \qquad (5.13)
\end{aligned}$$

where we have used the fact of $E_{2_\alpha,q}(\bar{u}) > 0$. In fact, similar to the argument of Lemma 5.2, by the fact that $Q_q(\bar{u}) = 0$ and $\|\bar{u}\|_2 \leq c$, we easily obtain $E_{2_\alpha,q}(\bar{u}) > 0$ for $0 < c < c_*$. Finally, repeating the same argument as proving $\bar{u} \neq 0$, we use $v_\varepsilon$ to deduce that for every $\mu > \bar{\mu}$,

$$m_q(c) + \frac{\bar{\lambda}}{2} c^2 < \frac{\alpha}{2(N+\alpha)} \gamma^{-\frac{N}{\alpha}} \left(\bar{\lambda} S_\alpha\right)^{\frac{N+\alpha}{\alpha}},$$

contradicting (5.13). Therefore, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^N)$. The proof is complete. ∎

**Proof of Theorem 1.5.** By Lemmas 5.2 and 5.8, for $0 < c < c_*$, there exists a bounded Palais–Smale sequence $\{u_n\} \subset \mathcal{M}_q^r(c)$ for $E_{2_\alpha,q}$ restricted to $S_r(c)$ at the level $m_q(c)$. Thus it follows from Lemma 5.9 that there exists a constant $\bar{\mu} > 0$ such that for every $\mu > \bar{\mu}$, up to a subsequence, $u_n \to \bar{u}$ strongly in $H^1(\mathbb{R}^N)$ for

$0 < c < c_*$, and together with Lemma 5.3, showing that $\bar{u}$ is a ground state solution of problem (1.5) for some $\bar{\lambda} > 0$, which is radially symmetric. Let $|\overline{u}|^*$ be the Schwartz symmetrization rearrangement of $|\overline{u}|$. By (3.2), we have $Q_q(|\overline{u}|^*) \le 0$, and there exists a unique $t \in (0,1]$ such that $Q_q((|\overline{u}|^*)_t) = 0$. Thus $(|\overline{u}|^*)_t \in \mathcal{M}_q(c)$, and we have

$$\begin{aligned} E_{2_\alpha,q}(|\overline{u}|^*) &= \frac{\mu(N(q-2)-4)t^{\frac{N(q-2)}{2}}}{4q}C(|\overline{u}|^*) - \frac{\gamma}{22_\alpha}B(|\overline{u}|^*) \\ &\le \frac{\mu(N(q-2)-4)}{4q}C(\overline{u}) - \frac{\gamma}{22_\alpha}B(\overline{u}) = E_{2_\alpha,q}(\overline{u}) = m_q(c). \end{aligned}$$

By the definition of $m_q(c)$, we know that $m_q(c)$ is achieved by the real-valued positive and radially symmetric nonincreasing function.

Moreover, it follows from (2.7), (2.8), and (5.2) that

$$\begin{aligned} \bar{\lambda}c^2 &= -A(\bar{u}) + \gamma B(\bar{u}) + \mu C(\bar{u}) \\ &= \frac{2N - q(N-2)}{N(q-2)}A(\bar{u}) + \gamma B(\bar{u}) \\ &> \frac{2N - q(N-2)}{N(q-2)}\left(\frac{2q}{\mu N\overline{S}(q-2)}c^{-\frac{2N-q(N-2)}{2}}\right)^{\frac{4}{N(q-2)-4}}, \end{aligned}$$

which implies that

$$\bar{\lambda} > \frac{2N - q(N-2)}{N(q-2)}\left(\frac{2q}{\mu N\overline{S}(q-2)}\right)^{\frac{4}{N(q-2)-4}} c^{-\frac{4(q-2)}{N(q-2)-4}}.$$

We complete the proof. □

At the end of this section, we study the case of $q = 2^*$. First of all, we give a property on $m_q(c)$ as follows.

LEMMA 5.10. *Assume that $\gamma, \mu > 0$ and $\overline{q} < q < 2^*$. Then we have*

$$m_{2^*}(c) \ge \limsup_{q\to 2^*} m_q(c) > 0.$$

*Proof.* The proof is similar to that of [17, Lemma 3.1]; we omit it here. □

LEMMA 5.11 ([5, Radial Lemma A.IV]). *Let $N \ge 3$ and $1 \le s < +\infty$. If $u \in L^s(\mathbb{R}^N)$ is a radial nonincreasing function (that is, $0 \le u(x) \le u(y)$ if $|x| \ge |y|$), then one has*

$$|u(x)| \le |x|^{-N/s}\left(\frac{N}{|S^{N-1}|}\right)^{1/s}\|u\|_s \quad \text{if } x \ne 0,$$

*where $|S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$.*

LEMMA 5.12. *Assume that $q = 2^*, 0 < c < \min\{c_*, c^*\}$, and*

$$\gamma \ge \left[\frac{\alpha N S_\alpha^{\frac{N+\alpha}{\alpha}}}{2S^{N/2}(N+\alpha)}\right]^{\frac{\alpha}{N}} \mu^{\frac{\alpha(N-2)}{2N}}.$$

*Then there exists $\tilde{\mu} \ge \bar{\mu}$ such that for every $\mu > \tilde{\mu}$, the infimum $m_{2^*}(c) > 0$ is achieved by $\tilde{u}$, where $\bar{\mu}$ is as in Lemma 5.9. In addition, there exists $\tilde{\lambda} > 0$ such that $(\tilde{u}, \tilde{\lambda})$ is a solution to problem (1.5).*

*Proof.* Let $q_n \to 2^{*-}$ as $n \to \infty$; by Theorem 1.5 and Lemma 5.10, for every $\mu > \bar{\mu}$, there exists a sequence of positive and radially nonincreasing functions $\{u_n := u_{q_n}\} \subset \mathcal{M}^r_{q_n}(c)$ such that

$$E_{2_\alpha, q_n}(u_n) = m_{q_n}(c) \leq m_{2^*}(c) + 1,$$

which implies that $\{u_n\}$ is bounded. Then there exists $\hat{u} \in H^1_r(\mathbb{R}^N)$ such that $u_n \rightharpoonup \hat{u}$ in $H^1_r(\mathbb{R}^N)$, $u_n \to \hat{u}$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, and $u_n \to \hat{u}$ a.e. on $\mathbb{R}^N$. Moreover, by the Lagrange multipliers rule there exists $\lambda_n \in \mathbb{R}$ such that for every $\varphi \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx + \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx - \gamma \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^{2_\alpha} u_n^{2_\alpha - 2}}{|x-y|^{N-\alpha}} u_n \varphi dx \\ &-\mu \int_{\mathbb{R}^N} |u_n|^{q_n - 2} u_n \varphi dx = o(1)\|\varphi\|. \end{aligned} \tag{5.14}$$

In particular, it holds that

$$\lambda_n c^2 = -\|\nabla u_n\|_2^2 + \gamma \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^{2_\alpha} u_n^{2_\alpha}}{|x-y|^{N-\alpha}} dx + \mu \int_{\mathbb{R}^N} |u_n|^{q_n} dx + o(1),$$

which implies that $\{\lambda_n\}$ is bounded. So there exists $\hat{\lambda} \in \mathbb{R}$ such that $\lambda_n \to \hat{\lambda}$ as $n \to \infty$. Since $Q_{q_n}(u_n) = o(1)$, one has

$$\hat{\lambda} c^2 = \lim_{n\to\infty} \lambda_n c^2 = \lim_{n\to\infty} \left[ \gamma B(u_n) + \frac{2N - q_n(N-2)}{2q_n} \mu \int_{\mathbb{R}^N} |u_n|^{q_n} dx \right] \geq 0,$$

leading to $\hat{\lambda} \geq 0$. If $\hat{\lambda} = 0$, then we get

$$B(u_n) = \int_{\mathbb{R}^N} |u_n|^{q_n} dx = o(1),$$

and further $A(u_n) = o(1)$. This shows that $E_{q_n}(u_n) = o(1)$, which contradicts $\liminf_{n\to\infty} m_{q_n}(c) > 0$. Thus, $\hat{\lambda} > 0$.

Since $q_n \to 2^{*-}$ as $n \to \infty$ and $\varphi \in L^s(\mathbb{R}^N)$ for $s \in (1, \infty)$, by the Young and Hölder inequalities and Lemma 5.11, there exists a constant $C_1 > 0$ independent of $n$ such that

$$\begin{aligned} \left| |u_n|^{q_n - 2} u_n \varphi \right| &\leq C_1 (|u_n|^{2-1} + |u_n|^{2^* - 1}) |\varphi| \\ &\leq C_1 (|x|^{\frac{2-N}{2}(2-1)} + |x|^{\frac{2-N}{2}(2^*-1)}) |\varphi| \in L^1(\mathbb{R}^N), \end{aligned}$$

and together with the Lebesgue dominated convergence theorem, leading to

$$\int_{\mathbb{R}^N} |u_n|^{q_n - 2} u_n \varphi dx \to \int_{\mathbb{R}^N} |\hat{u}|^{2^* - 2} \hat{u} \varphi dx \text{ as } n \to \infty.$$

Combining this with (5.14), we have

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx + \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx - \gamma \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_n^{2_\alpha} u_n^{2_\alpha - 2}}{|x-y|^{N-\alpha}} u_n \varphi dx \\ &-\mu \int_{\mathbb{R}^N} |u_n|^{q_n - 2} u_n \varphi dx \\ \to &\int_{\mathbb{R}^N} \nabla \hat{u} \nabla \varphi dx + \hat{\lambda} \int_{\mathbb{R}^N} \hat{u} \varphi dx - \gamma \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{u}^{2_\alpha} \hat{u}^{2_\alpha - 2}}{|x-y|^{N-\alpha}} \hat{u} \varphi dx \\ &-\mu \int_{\mathbb{R}^N} |\hat{u}|^{2^* - 2} \hat{u} \varphi dx \end{aligned}$$

as $n \to \infty$. That is, $\hat{u}$ is a weak solution of

$$-\Delta u + \hat{\lambda} u = \gamma(I_\alpha * |u|^{2_\alpha})|u|^{2_\alpha - 2}u + \mu|u|^{2^*-2}u \ \forall x \in \mathbb{R}^N.$$

Thus, $Q_{2^*}(\hat{u}) = 0$.

Next, we claim that $\hat{u} \neq 0$. Otherwise, by using $Q_{q_n}(u_n) = 0$ and the Young inequality,

$$|u_n|^{q_n} \le \frac{2^* - q_n}{2^* - s}|u_n|^s + \frac{q_n - s}{2^* - s}|u_n|^{2^*} \text{ for } q_n < s < 2^*,$$

we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx &= \frac{\mu N(q_n - 2)}{2q_n} \int_{\mathbb{R}^N} |u_n|^{q_n} dx \\ &\le \frac{\mu N(q_n - 2)(q_n - s)}{2q_n(2^* - s)} \int_{\mathbb{R}^N} |u_n|^{2^*} dx + o(1) \\ &\le \frac{\mu N(q_n - 2)(q_n - s)}{2q_n(2^* - s)} \left( \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{S} \right)^{\frac{N}{N-2}} + o(1) \\ &\le \mu \left( \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{S} \right)^{\frac{N}{N-2}} + o(1), \end{aligned} \tag{5.15}$$

where we have also used the fact of $\int_{\mathbb{R}^N} |u_n|^s dx = o(1)$ for $q_n < s < 2^*$. Since $\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx > 0$, it follows from (5.15) that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \ge \mu^{-\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{5.16}$$

By Lemma 5.10, (5.16), and the fact of $Q_{q_n}(u_n) = o(1)$, one has

$$\begin{aligned} m_{2^*}(c) &\ge \limsup_{n\to\infty} m_{q_n}(c) \\ &= \limsup_{n\to\infty} \left[ \frac{1}{2}A(u_n) - \frac{\gamma}{22_\alpha}B(u_n) - \frac{2}{N(q_n - 2)}A(u_n) \right] \\ &= \limsup_{n\to\infty} \left[ \left( \frac{1}{2} - \frac{2}{N(q_n - 2)} \right) A(u_n) - \frac{\gamma}{22_\alpha}B(u_n) \right] \\ &= \limsup_{n\to\infty} \left[ \left( \frac{1}{2} - \frac{2}{N(2^* - 2)} \right) A(u_n) - \frac{\gamma}{22_\alpha}B(u_n) \right] \\ &\ge \frac{1}{N}A(u_n) - \frac{\gamma}{22_\alpha}B(u_n) \\ &\ge \frac{1}{N}\mu^{-\frac{N-2}{2}} S^{\frac{N}{2}} - \frac{\gamma}{22_\alpha} S_\alpha^{-\frac{N+\alpha}{N}} c^{22_\alpha}. \end{aligned}$$

Moreover, using the similar estimate on (5.11), we get

$$m_{2^*}(c) < \frac{\alpha}{2(N + \alpha)} \gamma^{-\frac{N}{\alpha}} S_\alpha^{\frac{N+\alpha}{\alpha}} - \frac{1}{2}c^2.$$

But since

$$\gamma \ge \left( \frac{\mu^{\frac{N-2}{2}} \alpha N S_\alpha^{\frac{N+\alpha}{\alpha}}}{2S^{N/2}(N + \alpha)} \right)^{\frac{\alpha}{N}} \text{ and } 0 < c \le c^* := \left( \frac{N + \alpha}{N\gamma} \right)^{\frac{N}{2\alpha}} S_\alpha^{\frac{N+\alpha}{2\alpha}},$$

we have

$$\frac{1}{N}\mu^{-\frac{N-2}{2}}S^{\frac{N}{2}}-\frac{\gamma}{22_\alpha}S_\alpha^{-\frac{N+\alpha}{N}}c^{22_\alpha}\geq\frac{\alpha}{2(N+\alpha)}\gamma^{-\frac{N}{\alpha}}S_\alpha^{\frac{N+\alpha}{\alpha}}-\frac{1}{2}c^2,$$

which is a contradiction. So $\hat{u}\neq 0$.

Now, let us prove that $m_{2^*}(c)$ is achieved. Set $\int_{\mathbb{R}^N}|\hat{u}|^2dx=a^2\leq c^2$, and define

$$\tilde{u}(x)=\left(\frac{a}{c}\right)^{\frac{N-2}{2}}\hat{u}\left(\frac{a}{c}x\right).$$

A direct calculation shows that

$$\int_{\mathbb{R}^N}|\tilde{u}|^2dx=c^2,\ C(\tilde{u})=C(\hat{u}),\ A(\tilde{u})=A(\hat{u}), \tag{5.17}$$

and

$$B(\tilde{u})=\int_{\mathbb{R}^N}(I_\alpha*|\tilde{u}|^{2_\alpha})|\tilde{u}|^{2_\alpha}dx=\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}B(\hat{u})\geq B(\hat{u}).$$

Clearly, $\tilde{u}\in S_r(c)$. Moreover, using (5.17) and the fact of $Q_{2^*}(\hat{u})=0$ gives

$$A(\tilde{u})=A(\hat{u})=\frac{\mu N(2^*-2)}{22^*}C(\hat{u})=\frac{\mu N(2^*-2)}{22^*}C(\tilde{u}),$$

which implies that $\tilde{u}\in\mathcal{M}_{2^*}^r(c)$.

Since

$$\frac{N-2}{2}A(\hat{u})+\frac{\hat{\lambda}N}{2}\int_{\mathbb{R}^N}|\hat{u}|^2dx-\frac{\gamma N}{2}B(\hat{u})-\frac{\mu N}{2^*}C(\hat{u})=0,$$

we have

$$\frac{N-2}{2}A(\tilde{u})+\frac{\hat{\lambda}N}{2}\left(\frac{a}{c}\right)^2\int_{\mathbb{R}^N}|\tilde{u}|^2dx=\frac{\gamma N}{2}\left(\frac{a}{c}\right)^{\frac{2(N+\alpha)}{N}}B(\tilde{u})+\frac{\mu N}{2^*}C(\tilde{u}). \tag{5.18}$$

Then by Lemma 5.10 and (5.18) one has

$$\begin{aligned}
m_{2^*}(c)\leq&E_{2_\alpha,2^*}(\tilde{u})\\
=&\frac{1}{2}A(\tilde{u})-\frac{\gamma N}{2(N+\alpha)}B(\tilde{u})-\frac{\mu}{2^*}C(\tilde{u})\\
=&-\frac{\hat{\lambda}N}{2(N-2)}\left(\frac{a}{c}\right)^2\int_{\mathbb{R}^N}|\tilde{u}|^2dx+\frac{\gamma N}{2(N-2)}\left(\frac{a}{c}\right)^{\frac{2(N+\alpha)}{N}}B(\tilde{u})\\
&+\frac{\mu N}{2^*(N-2)}C(\tilde{u})-\frac{\gamma N}{2(N+\alpha)}B(\tilde{u})-\frac{\mu}{2^*}C(\tilde{u})\\
=&\frac{\gamma N}{2}\left(\frac{1}{N-2}-\frac{1}{N+\alpha}\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right)B(\hat{u})+\frac{2\mu}{2^*(N-2)}C(\hat{u})\\
&-\frac{\hat{\lambda}N}{2(N-2)}\int_{\mathbb{R}^N}|\hat{u}|^2dx
\end{aligned}$$

$$
\begin{aligned}
&=\frac{\gamma N}{2}\left(\frac{1}{N-2}-\frac{1}{N+\alpha}\right)B(\hat{u})+\frac{2\mu}{2^*(N-2)}C(\hat{u})\\
&\quad+\frac{\gamma N}{2(N+\alpha)}\left(1-\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right)B(\hat{u})-\frac{\hat{\lambda}N}{2(N-2)}\int_{\mathbb{R}^N}|\hat{u}|^2dx\\
&\le\liminf_{n\to\infty}\left[\frac{\gamma N}{2}\left(\frac{1}{N-2}-\frac{1}{N+\alpha}\right)B(u_n)+\frac{2\mu}{q_n(N-2)}\int_{\mathbb{R}^N}|u_n|^{q_n}dx\right]\\
&\quad+\frac{\gamma N}{2(N+\alpha)}\left(1-\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}\right)B(\hat{u})-\frac{\hat{\lambda}N}{2(N-2)}a^2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&E_{2_\alpha,q_n}(u_n)+\frac{\lambda_n N}{2(N-2)}\int_{\mathbb{R}^N}|u_n|^2dx\\
&=\frac{1}{2}A(u_n)-\frac{\gamma N}{2(N+\alpha)}B(u_n)-\frac{\mu}{q_n}\int_{\mathbb{R}^N}|u_n|^{q_n}dx+\frac{\lambda_n N}{2(N-2)}\int_{\mathbb{R}^N}|u_n|^2dx\\
&=\frac{\gamma N}{2(N-2)}B(u_n)+\frac{\mu N}{q_n(N-2)}\int_{\mathbb{R}^N}|u_n|^{q_n}dx-\frac{\gamma N}{2(N+\alpha)}B(u_n)-\frac{\mu}{q_n}\int_{\mathbb{R}^N}|u_n|^{q_n}dx\\
&=\frac{\gamma N}{2}\left(\frac{1}{N-2}-\frac{1}{N+\alpha}\right)B(u_n)+\frac{\mu}{q_n}\left(\frac{N}{N-2}-1\right)\int_{\mathbb{R}^N}|u_n|^{q_n}dx\\
&=\frac{\gamma N}{2}\left(\frac{1}{N-2}-\frac{1}{N+\alpha}\right)B(u_n)+\frac{2\mu}{q_n(N-2)}\int_{\mathbb{R}^N}|u_n|^{q_n}dx.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
m_{2^*}(c)&\le E_{2_\alpha,2^*}(\tilde{u})\\
&\le\liminf_{n\to\infty}E_{2_\alpha,q_n}(u_n)\\
&\quad+\frac{\hat{\lambda}N}{2(N-2)}\left(c^2-a^2\right)-\frac{\gamma N}{2(N+\alpha)}\left(\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}-1\right)B(\hat{u})\\
&=\liminf_{n\to\infty}m_{q_n}(c)+\frac{\hat{\lambda}N}{2(N-2)}\left(c^2-a^2\right)-\frac{\gamma N}{2(N+\alpha)}\left(\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}-1\right)B(\hat{u})\\
&\le\limsup_{n\to\infty}m_{q_n}(c)\le m_{2^*}(c),
\end{aligned}
$$

provided that

$$
\frac{\hat{\lambda}N}{2(N-2)}\left(c^2-a^2\right)-\frac{\gamma N}{2(N+\alpha)}\left(\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}-1\right)B(\hat{u})\le 0.
$$

If $a=c$, then it is clear that $m_{2^*}(c)$ is achieved by $\tilde{u}$ (in fact $\tilde{u}=\hat{u}$ here) for all $\gamma>0$. If $a<c$, then $m_{2^*}(c)$ is achieved by $\tilde{u}$ when

$$
\gamma\ge\frac{\hat{\lambda}\left(c^2-a^2\right)(N+\alpha)}{(N-2)\left(\left(\frac{a}{c}\right)^{-\frac{2(N+\alpha)}{N}}-1\right)B(\hat{u})}.
$$

Note that

$$
\gamma\ge\left(\frac{\alpha N S_\alpha^{\frac{N+\alpha}{\alpha}}}{2S^{N/2}(N+\alpha)}\right)^{\frac{\alpha}{N}}\mu^{\frac{\alpha(N-2)}{2N}}.
$$

Then there exists $\tilde{\mu} \geq \bar{\mu}$ such that for every $\mu > \tilde{\mu}$, the infimum $m_{2^*}(c)$ is achieved by $\tilde{u}$.

Since the infimum $m_{2^*}(c)$ is achieved by $\tilde{u}$, there exist $\tilde{\lambda}$ and $\eta$ such that

$$-\Delta\tilde{u} + \tilde{\lambda}\tilde{u} - \gamma(I_\alpha * |\tilde{u}|^{2_\alpha})|\tilde{u}|^{2_\alpha-2}\tilde{u} - \mu|\tilde{u}|^{2^*-2}\tilde{u} = \eta\left[-2\Delta\tilde{u} - 2^*\mu|\tilde{u}|^{2^*-2}\tilde{u}\right], \tag{5.19}$$

namely,

$$-(1-2\eta)\Delta\tilde{u} + \tilde{\lambda}\tilde{u} = \gamma(I_\alpha * |\tilde{u}|^{2_\alpha})|\tilde{u}|^{2_\alpha-2}\tilde{u} + \mu(1-2^*\eta)|\tilde{u}|^{2^*-2}\tilde{u}.$$

Similar to Lemma 2.6, $\tilde{u}$ satisfies the Pohozaev identity as follows:

$$(1-2\eta)A(\tilde{u}) - \mu(1-2^*\eta)C(\tilde{u}) = 0.$$

Using this, together with $Q_{2^*}(\tilde{u}) = A(\tilde{u}) - \mu C(\tilde{u}) = 0$, we deduce that

$$\eta\mu(2^*-2)C(\tilde{u}) = 0,$$

leading to $\eta = 0$. Moreover, it follows from (5.19) and $Q_{2^*}(\tilde{u}) = 0$ that

$$\tilde{\lambda} = \frac{1}{c^2}\left[-A(\tilde{u}) + \gamma B(\tilde{u}) + \mu C(\tilde{u})\right] = \frac{\gamma}{c^2}B(\tilde{u}) > 0. \tag{5.20}$$

The proof is complete. □

**Proof of Theorem 1.6.** By Lemma 5.12, we obtain that $\tilde{u}$ is a ground state solution to problem (1.5) for some $\tilde{\lambda} > 0$. Moreover, by (5.20) one has

$$\tilde{\lambda} = \frac{1}{c^2}\left[-A(\tilde{u}) + \gamma B(\tilde{u}) + \mu C(\tilde{u})\right] = \frac{\gamma}{c^2}B(\tilde{u}) \leq \gamma S_\alpha^{-\frac{N+\alpha}{N}}.$$

We complete the proof.

## 6. Qualitative properties of the mappings $\sigma(c)$ and $m_q(c)$.

This section is devoted to the proof of Theorem 1.10.

(i) Assume that $c_n \to c$ as $n \to \infty$. It follows from the definition of $\sigma(c)$ that for any $\varepsilon > 0$, there exists $u_n \in S(c_n)$ such that

$$E_{2_\alpha,q}(u_n) \leq \sigma(c) + \varepsilon. \tag{6.1}$$

Let $v_n := \frac{c}{c_n}u_n$. Taking into account that $v_n \in S(c)$ and $\frac{c}{c_n} \to 1$, we have

$$\sigma(c) \leq E_{2_\alpha,q}(v_n) = E_{2_\alpha,q}(u_n) + o(1). \tag{6.2}$$

Combining (6.1) and (6.2), one has

$$\sigma(c) \leq \sigma(c_n) + \varepsilon + o(1).$$

By reversing the argument we get

$$\sigma(c_n) \leq \sigma(c) + \varepsilon + o(1).$$

Therefore, since $\varepsilon > 0$ is arbitrary, we deduce that $\sigma(c_n) \to \sigma(c)$ as $n \to \infty$.

By Lemma 3.1(ii), we deduce that the map $c \mapsto \sigma(c)$ is strictly decreasing.

(ii) Similar to the argument of (i), we easily obtain that $c \mapsto m_q(c)$ is a continuous mapping. Next, let us prove that the function $c \mapsto m_q(c)$ is strictly decreasing. Let

$0 < c_1 < c_2$. According to Lemma 5.1 and the definition of $m_q(c)$, there exists $\{u_n\} \subset \mathcal{M}_q(c_1)$ such that

$$E_{2_\alpha,q}(u_n) = \max_{t>0} E_{2_\alpha,q}((u_n)_t) \le m_q(c_1) + \frac{1}{n}.$$

Let

$$w_n := \left(\frac{c_2}{c_1}\right)^{\frac{2-N}{2}} u_n\left(\left(\frac{c_2}{c_1}\right)^{-1} x\right).$$

Then we have

$$A(w_n) = A(u_n), \quad B(w_n) = \left(\frac{c_2}{c_1}\right)^{\frac{2(N+\alpha)}{N}} B(u_n)$$

and

$$C(w_n) = \left(\frac{c_2}{c_1}\right)^{\frac{2N-q(N-2)}{2}} C(u_n), \quad D(w_n) = c_2^2.$$

Moreover, by Lemma 5.1, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}_q(c_2)$ and

$$E_{2_\alpha,q}((w_n)_{t_n}) = \max_{t>0} E_{2_\alpha,q}((w_n)_t).$$

Hence, we have

$$\begin{aligned}
m_q(c_2) &\le E_{2_\alpha,q}((w_n)_{t_n}) \\
&= E_{2_\alpha,q}((u_n)_{t_n}) - \frac{\gamma}{2p}\left(\left(\frac{c_2}{c_1}\right)^{\frac{2(N+\alpha)}{N}} - 1\right) B(u_n) \\
&\quad - \frac{\mu (t_n)^{\frac{N(q-2)}{2}}}{q}\left(\left(\frac{c_2}{c_1}\right)^{\frac{2N-q(N-2)}{2}} - 1\right) C(u_n) \\
&\le m_q(c_1) + \frac{1}{n} - \frac{\gamma}{2p}\left(\left(\frac{c_2}{c_1}\right)^{\frac{2(N+\alpha)}{N}} - 1\right) B(u_n) \\
&\quad - \frac{\mu (t_n)^{\frac{N(q-2)}{2}}}{q}\left(\left(\frac{c_2}{c_1}\right)^{\frac{2N-q(N-2)}{2}} - 1\right) C(u_n),
\end{aligned}$$

which implies that $m_q(c_2) < m_q(c_1)$. We complete the proof.

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