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MOUNTAIN PASS SOLUTIONS FOR NONLOCAL EQUATIONS

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Abstract. This work is devoted to study the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, we prove the existence of at least one nontrivial weak solution, and under additional assumptions, the existence of infinitely many weak solutions. In order to apply mountain pass results, we require rather general assumptions on on the local operator. Finally, a concrete application is presented.

1. Introduction

The purpose of this paper is to investigate existence and multiplicity properties for the following class of nonlocal Dirichlet problems:

$$(D^p_{M,f}) \qquad \begin{cases} -\left[M\left(\int_\Omega |\nabla u(x)|^p\,dx\right)\right]^{p-1}\Delta_p u = f(x,u) & \text{in } \Omega,\\ u|_{\partial\Omega}=0. \end{cases}$$

Here and in the sequel, Ω is a bounded domain in $(\mathbf{R}^N,|\cdot|)$ with smooth boundary $\partial\Omega$, $p>1$, and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the usual p-Laplace operator. Further, $f\colon \bar\Omega\times\mathbf{R}\to\mathbf{R}$ and $M\colon [0,+\infty)\to[0,+\infty)$ are suitable continuous maps.

Nonlocal boundary value problems like problem $(D^p_{M,f})$ model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader, for instance, to Alves and Corrêa [1], Alves and Figueiredo [3], Andrade and Ma [5], Chipot and Lovat [16], Chipot and Rodrigues [17], and Vasconcellos [28]. Problem $(D^p_{M,f})$ is called *nonlocal* because of the presence of the term $M\left(\int_\Omega |\nabla u(x)|^p\,dx\right)$, which implies that the quasilinear partial differential equation in $(D^p_{M,f})$ is no longer a pointwise identity. This phenomenon creates several mathematical difficulties in the qualitative analysis of such equations.

Moreover, problem $(D^p_{M,f})$ is related to the stationary analogue of the hyperbolic equation

$$(1) \qquad u_{tt} - \left(a + b\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = f(x,u) \quad \text{in } \Omega,$$

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where Δ is the usual Laplace operator. Equation (1) is a general version of the Kirchhoff equation

$$\rho\, u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L u_x^2\, dx\right) u_{xx} = 0, \tag{2}$$

presented by Kirchhoff [21]. This equation is as an extension of the classical d'Alembert's wave equation by considering the effects of changes in the length of the strings during the vibrations. The parameters in equation (2) have the following meanings: E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of cross-section, and P_0 is the initial tension.

The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [13] and Pohozaev [25]. However, equation (1) received much attention only after the paper by Lions [23], where an abstract framework to the problem was proposed. Some related results can be found, for example, in Arosio and Panizzi [7], Autuori, Colasuonno and Pucci [8], Autuori, Pucci and Salvatori [12], Counsin, Frota, Lar'kin and Medeiros [19], and D'Ancona and Spagnolo [20]. We also point out that Arosio and Panizzi [7] studied the Cauchy–Dirichlet problem related to (1) in the Hadamard sense as a special case of an abstract second order Cauchy problem in a Hilbert space. D'Ancona and Spagnolo [20] considered Kirchhoff's equation as a quasi-linear hyperbolic Cauchy problem that describes the transverse oscillations of a stretched string. Perera and Zhang [24] obtained a nontrivial solution of $(D^p_{M,f})$ (for $p = 2$) via Yang index and critical groups. Chipot and Lovat [16] pointed out that this kind of problems models several physical and biological systems, where u describes a process depending on the average of itself (for example, population density). We also refer to Colasuonno and Pucci [18], which covers for the first time in the literature the degenerate case of the Kirchhoff equation for the existence of solutions in the elliptic case. In other words, paper [18] includes the degenerate setting that corresponds to $M(0) = 0$. For completeness we refer the reader to some recent interesting results obtained by Autuori and Pucci in [9, 10, 11] studying Kirchhoff equations by using different approaches.

In our context, problem $(D^p_{M,f})$ is studied exploiting classical variational methods. More precisely we apply the celebrated mountain pass theorem (briefly MPT) to this kind of equations motivated by the current literature where the MPT has been intensively applied to find solutions to quasilinear elliptic equations of p-Laplacian type.

Technically, this approach is realizable checking that the associated energy functional verifies the usual compactness Palais–Smale condition. Motivated by this fact, we suppose some restrictions on the behavior of the continuous function M.

More precisely, we assume that there exists a constant m_0 such that

(C^1_M) $0 < m_0 \leq M(t), \ \ \forall\, t \in [0, +\infty)$.

In addition to the above hypotheses, we require that

(C^2_M) there exists $t_0 \geq 0$ such that

$$\widehat{M}(t) \geq t[M(t)]^{p-1},$$

for every $t \in [t_0, +\infty)$, where $\widehat{M}(t) := \int_0^t [M(s)]^{p-1}\, ds$.

As proved in Lemma 4.2, in our setting, the potential $\widehat{M}$ has a sublinear growth. This fact permit to overcome some technical difficulties treating problem $(D^p_{M,f})$ due to the presence of the nonlocal term

$$M\Big(\int_\Omega |\nabla u(x)|^p\,dx\Big).$$

Specifically, studying p-superlinear problems as in the classical literature for the p-Laplace operator, we are conduced to assume M nonincreasing.

In the first part of this manuscript, under the previous assumptions, requiring conditions on the nonlinear part f (among others the Ambrosetti–Rabinowitz relation) we prove the existence of at least one nontrivial weak solution to problem $(D^p_{M,f})$; see Theorem 4.1. This result is related to [2, Theorem 3] where the authors studied problem $(D^2_{M,f})$.

We just observe that in our context, on the contrary of the cited result, we do not require

$$\lim_{t\to 0}\frac{f(x,t)}{t}=0,$$

uniformly with respect to $x\in\bar{\Omega}$; see condition (6) in Theorem 4.1. Further, we assume that condition (C^2_M) holds true for every $t\geq t_0$, where t_0 can be different from zero.

The second part of this work is devoted to a multiplicity result. In presence of symmetries (f is assumed to be odd), the existence of infinitely many weak solutions to problem $(D^p_{M,f})$ is achieved exploiting the $\mathbf{Z}_2$-symmetric version of the MPT; see Theorem 2.2.

The plan of the paper is as follows. Section 2 and 3 are devoted to our abstract framework and preliminaries. Successively, in Sections 4 and 5 we give the main results; see Theorems 4.1 and 5.1. Finally, some concrete examples of applications are presented in the last section (see Example 6.1 and 6.3).

2. Abstract framework

Let $W^{1,p}_0(\Omega)$, $p>1$, be the usual Sobolev space, equipped with the norm

$$\|u\|:=\Big(\int_\Omega |\nabla u(x)|^p\,dx\Big)^{1/p}.$$

Further, let $W^{-1,p'}(\Omega)$, with $1/p+1/p'=1$, be its topological dual and denote the duality brackets for the pair $(W^{-1,p'}(\Omega),W^{1,p}_0(\Omega))$ by $\langle\cdot,\cdot\rangle$. We denote by p^* the critical exponent of the Sobolev embedding $W^{1,p}_0(\Omega)\hookrightarrow L^q(\Omega)$. Recall that if $p<N$ then $p^*=Np/(N-p)$ and for every $q\in[1,p^*]$ there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)}\leq c_q\|u\|, \tag{3}$$

for every $u\in W^{1,p}_0(\Omega)$. Moreover, when $p\geq N$, this inequality holds for any $q\in[1,+\infty[$, since $p^*=+\infty$.

For the sake of completeness, we recall that a C^1-functional $\varphi\colon X\to\mathbf{R}$, where X is a real Banach space with topological dual X^*, satisfies the *Palais–Smale condition at level* $\mu\in\mathbf{R}$ (briefly $(\mathrm{PS})_\mu$) when

$(\mathrm{PS})_\mu$ every sequence $\{u_n\}$ in X such that

$$\varphi(u_n) \to \mu \quad \text{and} \quad \|\varphi'(u_n)\|_{X^*} \to 0,$$

as $n \to \infty$, possesses a convergent subsequence.

We say that φ satisfies the *Palais–Smale condition* (in short (PS)) if $(\mathrm{PS})_\mu$ holds for every $\mu \in \mathbf{R}$.

With the above notations, our main tools are the classical MPT and its $\mathbf{Z}_2$-symmetric version recalled respectively in the next Theorems 2.1 and 2.2. As pointed out by Brezis and Browder [15], the mountain pass theorem *"extends ideas already present in Poincaré and Birkhoff"*.

Theorem 2.1. [4] and [26, Theorem 2.2, p. 7] *Let $(X, \|\cdot\|)$ be a real Banach space and let $\varphi \in C^1(X;\mathbf{R})$ such that $\varphi(0_X) = 0$ and satisfying the* (PS) *condition. Suppose that*

(I_1) *there exist constant $\rho, \alpha > 0$ such that $\varphi(u) \geq \alpha$ if $\|u\| = \rho$,*

(I_2) *there exists $e \in X$ with $\|e\| > \rho$ such that $\varphi(e) \leq 0$.*

Then φ possesses a critical value $c \geq \alpha$, which can be characterized as

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} \varphi(u),$$

where

$$\Gamma := \{\gamma \in C([0,1];X) \colon \gamma(0) = 0 \wedge \gamma(1) = e\}.$$

Theorem 2.2. [26, p. 5] *Let X be an infinite dimensional real Banach space and let $\varphi \in C^1(X;\mathbf{R})$ be even, satisfying the* (PS) *condition, and $\varphi(0_X) = 0$. Suppose that condition (I_1) holds in addition to the following:*

(I_2') *For each finite dimensional subspace $X_1 \subset X$, the set*

$$\{u \in X_1 \colon \varphi(u) \geq 0\}$$

is bounded in X.

Then φ has an unbounded sequence of critical values.

We cite the monograph by Kristály, Rădulescu and Varga [22] as general reference on variational setting adopted in this paper.

3. Notations and preliminaries

Set $\Phi \colon W_0^{1,p}(\Omega) \to \mathbf{R}$ the smooth functional defined by

$$\Phi(u) := \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u(x)|^p\,dx\right), \tag{4}$$

for all $u \in W_0^{1,p}(\Omega)$, where we recall that

$$\widehat{M}(t) := \int_0^t [M(s)]^{p-1}\,ds,$$

for every $t \in [0,+\infty)$. Then the Fréchet derivative $\Phi' \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\langle \Phi'(u), v\rangle = \left[M\left(\int_\Omega |\nabla u(x)|^p\,dx\right)\right]^{p-1} \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\,dx,$$

for every $u, v \in W_0^{1,p}(\Omega)$.

Moreover, it is well-known that the operator $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot \nabla v(x)\, dx,$$

for every $u, v \in W_0^{1,p}(\Omega)$, satisfies the (S_+) property. This means that for every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ (weakly) in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0, \tag{5}$$

then $u_n \to u$ (strongly) in $W_0^{1,p}(\Omega)$.

4. Existence result

Put

$$\lambda_{1,p}(\Omega) := \inf\left\{ \frac{\int_\Omega |\nabla u(x)|^p\, dx}{\int_\Omega |u(x)|^p\, dx} : u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}.$$

Our first result extends a pioneering property established by Ambrosetti and Rabinowitz [4] under similar assumptions, provided that $M \equiv 1$ and $p = 2$.

Theorem 4.1. *Let us assume that $M\colon [0,+\infty) \to [0,+\infty)$ is a continuous map such that condition (C_M^1) and (C_M^2) hold. Further, require that $f\colon \bar{\Omega}\times \mathbf{R} \to \mathbf{R}$ is a continuous function that verifies:*

h$_1$) *The subcritical growth condition:*

$$|f(x,t)| \leq c(1+|t|^{q-1}), \quad \forall\, x \in \bar{\Omega},\, \forall\, t \in \mathbf{R},$$

where $c > 0$ and $p < q < p^*$;

h$_2$) *The Ambrosetti–Rabinowitz (briefly (AR)) condition: the functional $F(x,\xi) := \int_0^\xi f(x,t)\, dt$ is θ-superhomogeneous at infinity, that is, there exists $t^\star > 0$ such that*

$$0 < \theta F(x,\xi) \leq f(x,\xi)\xi \quad \forall\, x \in \bar{\Omega},\, \forall\, |\xi| \geq t_\star,$$

where $\theta > p$.

We assume that

$$\limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \leq \lambda, \tag{6}$$

uniformly for $x \in \bar{\Omega}$, where

$$\lambda < m_0^{p-1}\lambda_{1,p}(\Omega).$$

Then the Dirichlet problem

$$\begin{cases} -\Big[M\Big(\displaystyle\int_\Omega |\nabla u(x)|^p dx\Big)\Big]^{p-1}\Delta_p u = f(x,u) & \text{in } \Omega,\\ u|_{\partial\Omega} = 0, \end{cases} \tag{$D^p_{M,f}$}$$

has at least one nontrivial weak solution in $W_0^{1,p}(\Omega)$.

For the sake of completeness we recall that a *weak solution* of problem $(D^p_{M,f})$ is a function $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ such that

$$\langle \Phi'(u), v\rangle = \int_\Omega f(x,u(x))v(x)\, dx,$$

for every $v \in W_0^{1,p}(\Omega)$. Further we note that, if $f\colon \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is locally Lipschitz, then every weak solution of problem $(D^p_{M,f})$ is classical.

4.1. Some remarks on our assumptions. The validity of the next lemma will be crucial in the sequel.

Lemma 4.2. *Suppose that conditions* (C^1_M) *and* (C^2_M) *hold. Then there are constants* $m_1 > 0$ *and* $m_2 \geq 0$ *such that*

$$\widehat{M}(t) \leq m_1 t + m_2, \tag{7}$$

for every $t \in [0, +\infty)$.

Proof. Let $t_1 > t_0$, where t_0 appears in hypothesis (C^2_M). By our assumptions we easily have

$$\frac{[M(t)]^{p-1}}{\widehat{M}(t)} \leq \frac{1}{t},$$

for every $t \in \,]t_1, +\infty)$. Integrating the above relation, we obtain

$$\int_{t_1}^{t} \frac{[M(s)]^{p-1}}{\widehat{M}(s)}\, ds = \log \frac{\widehat{M}(t)}{\widehat{M}(t_1)} \leq \log \frac{t}{t_1},$$

for every $t \in \,]t_1, +\infty)$. Thus

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_1)}{t_1} t,$$

for every $t \in \,]t_1, +\infty)$. Hence the growth condition (7) holds taking, for instance, $m_1 := \dfrac{\widehat{M}(t_1)}{t_1}$ and $m_2 := \max\limits_{t \in [0,t_1]} \widehat{M}(t)$. □

Owing to conditions (C^1_M) and (C^2_M), by Lemma 4.2 one has the following inequalities:

$$(\widehat{C}_M) \quad \frac{m_0^{p-1}}{p}\|u\|^p \leq \Phi(u) \leq \frac{m_1}{p}\|u\|^p + \frac{m_2}{p},$$

for every $u \in W_0^{1,p}(\Omega)$.

Remark 4.3. We observe that the (AR) condition implies that

$$F(x, \tau\xi) \geq F(x, \xi)\tau^\theta,$$

for every $x \in \bar{\Omega}$, $|\xi| \geq t_\star$ and $\tau \geq 1$. Indeed, for $\tau = 1$, clearly the equality holds. Otherwise, fix $|\xi| \geq t_\star$ and define $g(x, \tau) := F(x, \tau\xi)$, for every $x \in \bar{\Omega}$ and $\tau \in]1, +\infty)$. By (AR) condition it follows that

$$\frac{g'(x, \tau)}{g(x, \tau)} \geq \frac{\theta}{\tau},$$

for every $x \in \bar{\Omega}$ and $\tau > 1$. Integrating in $]1, \tau]$ it follows that

$$\int_1^\tau \frac{g'(x, s)}{g(x, s)}\, ds = \log \frac{g(x, \tau)}{g(x, 1)} \geq \log \tau^\theta.$$

In conclusion, since for every $x \in \bar{\Omega}$, $|\xi| \geq t_\star$ and $\tau > 1$ one has

$$F(x, \tau\xi) =: g(x, \tau) \geq g(x, 1)\tau^\theta = F(x, \xi)\tau^\theta.$$

The claim is verified.

4.2. Proof of Theorem 4.1. Under the assumptions of Theorem 4.1, we define the C^1-functional

$$\varphi(u) := \Phi(u) - \int_\Omega F(x,u(x))\,dx, \quad u \in W_0^{1,p}(\Omega),$$

whose critical points are the weak solutions of problem $(D^p_{M,f})$. In order to prove our result, we apply Theorem 2.1 to this functional. In the next three lemmas we shall verify the mountain pass theorem conditions.

Lemma 4.4. *Every Palais–Smale sequence for the functional φ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a Palais–Smale sequence, that is,

$$\varphi(u_n) \to \mu, \tag{8}$$

for $\mu \in \mathbf{R}$ and

$$\|\varphi'(u_n)\|_{W^{-1,p'}} \to 0. \tag{9}$$

We argue by contradiction. So, suppose that the conclusion is not true. Passing to a subsequence if necessary, we may assume that

$$\|u_n\| \to +\infty.$$

By conditions (C_M^1) and (C_M^2), it follows that there exists $n_0 \in \mathbf{N}$ such that

$$\begin{aligned}\varphi(u_n) - \frac{\langle \varphi'(u_n), u_n\rangle}{\theta} &= [M(\|u_n\|^p)]^{p-1}\left[\frac{\widehat{M}(\|u_n\|^p)}{p\,[M(\|u_n\|^p)]^{p-1}} - \frac{\|u_n\|^p}{\theta}\right] \\ &\quad + \int_\Omega \left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]dx, \\ &\geq m_0^{p-1}\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p \\ &\quad + \int_\Omega \left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]dx,\end{aligned} \tag{10}$$

for every $n \geq n_0$. Thus

$$\begin{aligned}m_0^{p-1}\left(\frac{\theta - p}{\theta p}\right)\|u_n\|^p &\leq \varphi(u_n) - \frac{\langle \varphi'(u_n), u_n\rangle}{\theta} \\ &\quad - \int_{|u_n(x)|>t_\star}\left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]dx, \\ &\quad + M\,\mathrm{meas}\,(\Omega), \quad \forall\, n \geq n_0,\end{aligned}$$

where "meas (Ω)" denotes the standard Lebesgue measure of Ω and

$$M := \sup\left\{\left|\frac{f(x,t)t}{\theta} - F(x,t)\right| : x \in \bar{\Omega}, |t| \leq t_\star\right\}.$$

Now, we observe that, the (AR) condition yields

$$\int_{|u_n(x)|>t_\star}\left[\frac{f(x,u_n(x))u_n(x)}{\theta} - F(x,u_n(x))\right]dx \geq 0.$$

So, we deduce that

$$m_0^{p-1}\left(\frac{\theta-p}{\theta p}\right)\|u_n\|^p \leq \varphi(u_n) - \frac{\langle\varphi'(u_n),u_n\rangle}{\theta} + M\,\mathrm{meas}\,(\Omega),$$

for every $n \geq n_0$. Then, for every $n \geq n_0$ one has

$$C\|u_n\|^p \leq \left\{\varphi(u_n) + \|\varphi'(u_n)\|_{W^{-1,p'}}\frac{\|u_n\|}{\theta} + M\,\mathrm{meas}\,(\Omega)\right\},$$

where $C := m_0^{p-1}\left(\dfrac{\theta-p}{\theta p}\right) > 0$. In conclusion, dividing by $\|u_n\|$ and letting $n \to \infty$, we obtain a contradiction. $\square$

The above Lemma implies that the C^1-functional φ verifies the Palais–Smale condition as proved in the next result.

Lemma 4.5. *The functional φ satisfies the compactness* (PS) *condition.*

Proof. Take $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a Palais–Smale sequence. Thus, by Lemma 4.4, the sequence $\{u_n\}$ is necessarily bounded in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega)$ is reflexive, we may extract a subsequence that for simplicity we call again $\{u_n\}$, such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. We will prove that u_n strongly converges to $u \in W_0^{1,p}(\Omega)$. Exploiting the derivative $\varphi'(u_n)(u_n - u)$, we obtain

$$\langle\Phi'(u_n), u_n - u\rangle = \langle\varphi'(u_n), u_n - u\rangle + \int_\Omega f(x,u_n(x))(u_n-u)(x)\,dx.$$

Since $\|\varphi'(u_n)\|_{W^{-1,p'}} \to 0$ and the sequence $\{u_n - u\}$ is bounded in $W_0^{1,p}(\Omega)$, taking into account that

$$|\langle\varphi'(u_n), u_n - u\rangle| \leq \|\varphi'(u_n)\|_{W^{-1,p'}}\|u_n - u\|,$$

one has

$$\langle\varphi'(u_n), u_n - u\rangle \to 0.$$

Further, by h$_1$) and taking into account that $u_n \to u$ in $L^q(\Omega)$ we obtain

$$\int_\Omega |f(x,u_n(x))||u_n(x) - u(x)|\,dx \to 0.$$

We can conclude (by (C_M^1)) that

$$\langle A(u_n), u_n - u\rangle \to 0,$$

as $n \to \infty$. Since the operator A has the (S_+) property, in conclusion, $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. Hence, as claimed, the functional φ fulfills condition (PS). $\square$

Lemma 4.6. *The functional φ has the geometry of the MPT. More precisely*:

1. *There exists $r > 0$ such that*

$$\inf_{\|u\|=r}\varphi(u) > 0.$$

2. *For some $u_0 \in W_0^{1,p}(\Omega)$ one has*

$$\varphi(\tau u_0) \to -\infty,$$

as $\tau \to +\infty$.

Proof. 1. We choose $\varepsilon > 0$ small enough, verifying

$$m_0^{p-1} > \frac{\lambda + \varepsilon}{\lambda_{1,p}(\Omega)}.$$

By condition (6) there exists $\delta_\varepsilon > 0$ such that

$$\frac{f(x,t)}{|t|^{p-2}t} \leq \lambda + \varepsilon,$$

for every $x \in \bar{\Omega}$ and $|t| \leq \delta_\varepsilon$. Hence, one has

$$F(x,\xi) \leq \frac{\lambda + \varepsilon}{p}|\xi|^p,$$

for every $|\xi| \leq \delta_\varepsilon$. As a consequence of the above inequality, using hypotheses h_1), the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ and $(\widehat{C}_M)$, we can write

$$\begin{aligned}
\varphi(u) &\geq \frac{m_0^{p-1}}{p}\|u\|^p - \int_{|u(x)|\leq\delta_\varepsilon} \frac{\lambda+\varepsilon}{p}|u(x)|^p\,dx - C\int_{|u(x)|>\delta_\varepsilon} |u(x)|^q\,dx \\
&\geq \frac{m_0^{p-1}}{p}\|u\|^p - \frac{\lambda+\varepsilon}{p\lambda_{1,p}(\Omega)}\|u\|^p - C\|u\|^q,
\end{aligned}$$

for a suitable positive constant C.

Now, set $r := \|u\|^p$ and observe that for $r > 0$ small enough, we have

$$\frac{1}{p}\left(m_0^{p-1} - \frac{\lambda+\varepsilon}{\lambda_{1,p}(\Omega)}\right) r - Cr^{q/p} > 0,$$

bearing in mind that $q > p$. Hence

$$\inf_{\|u\|=r} \varphi(u) > 0.$$

2. Let us choose an element $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\mathrm{meas}\,(\{x \in \Omega\colon u_0(x) \geq t_\star\}) > 0.$$

Being $F(x,\xi)$ a θ-superhomogeneous function if $|\xi| \geq t_\star$, for $\tau > 1$, we obtain

$$\begin{aligned}
\varphi(\tau u_0) &\leq \frac{m_1}{p}\|\tau u_0\|^p + \frac{m_2}{p} - \int_\Omega F(x,\tau u_0(x))\,dx \\
&\leq \tau^p \frac{m_1\|u_0\|^p}{p} - \tau^\theta \int_{|u_0(x)|\geq t_\star} F(x,u_0(x))\,dx + \frac{m_2}{p} + M\,\mathrm{meas}\,(\Omega),
\end{aligned}$$

where

$$M := \sup\left\{|F(x,\xi)| : x \in \bar{\Omega}, |\xi| \leq t_\star\right\}.$$

Thus, the (AR) condition implies that

$$\varphi(\tau u_0) \to -\infty,$$

as $\tau \to +\infty$. This concludes the proof. $\square$

5. Infinitely many solutions

In this section, we prove the existence of infinitely many weak solutions for problem $(D^p_{M,f})$ exploiting the $\mathbf{Z}_2$-symmetric version of the MPT recalled in Theorem 2.2. Our main result is the following.

Theorem 5.1. *Assume that* $f\colon \bar\Omega \times \mathbf{R} \to \mathbf{R}$ *odd with respect to the second argument and that the conditions of Theorem 4.1 are satisfied. Then problem* $(D^p_{M,f})$ *has infinitely many weak solutions.*

Proof. Since the function f is odd in the second argument the functional φ is even. Moreover φ verifies that $\varphi(0_{W_0^{1,p}}) = 0$ and by Lemmas 4.4 and 4.5, it satisfies also the (PS) condition. On the other hand by Lemma 4.6, there exist positive constants ρ and α such that $\varphi(u) \geq \alpha$ if $\|u\| = \rho$. Hence, we need to prove just that for each finite dimensional subspace $X_1 \subset W_0^{1,p}(\Omega)$, the set

$$\{u \in X_1 \colon \varphi(u) \geq 0\}$$

is bounded in $W_0^{1,p}(\Omega)$.

Indeed, let $u \in W_0^{1,p}(\Omega)$ arbitrary and denote

$$\Omega_< := \{x \in \Omega \colon |u(x)| < t_\star\},$$

as well as

$$\Omega_\geq := \{x \in \Omega \colon |u(x)| \geq t_\star\}.$$

We shall prove that φ satisfies the following estimate

$$\varphi(u) \leq \frac{m_1}{p}\|u\|^p + \frac{m_2}{p} - \int_\Omega \gamma(x)|u(x)|^\theta\, dx + \kappa, \tag{11}$$

where κ is a suitable positive constant and $\gamma \in L^\infty(\Omega)$, with $\gamma > 0$ in Ω.

To show the above inequality, let us start observing that by h_1), the function F satisfies

$$|F(x,\xi)| \leq C_1(1 + |\xi|^q), \quad \forall\, x \in \bar\Omega,\ \forall\, \xi \in \mathbf{R}. \tag{12}$$

We claim that there exists $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ in Ω, such that

$$F(x,\xi) \geq \gamma(x)|\xi|^\theta, \quad \forall\, x \in \bar\Omega,\ \forall\, |\xi| \geq t_\star. \tag{13}$$

Indeed, since F is θ-superhomogeneous, we have that

$$F(x,\xi) \geq \gamma_1(x)|\xi|^\theta, \quad \forall\, x \in \bar\Omega,\ \forall\, \xi \geq t_\star,$$

where $\gamma_1(x) := F(x,t_\star)/t_\star^\theta$. It is easy to see that $\gamma_1 \in L^\infty(\Omega)$ and $\gamma_1 > 0$ in Ω.

In a similar way, it follows that

$$F(x,\xi) \geq \gamma_2(x)|\xi|^\theta, \quad \forall\, x \in \bar\Omega,\ \forall\, \xi \leq -t_\star,$$

with $\gamma_2(x) := F(x,-t_\star)/t_\star^\theta$. Also in this case $\gamma_2 \in L^\infty(\Omega)$ and $\gamma_2 > 0$ in Ω. Then (13) holds with

$$\gamma(x) := \max\{\gamma_1(x), \gamma_2(x)\},$$

for every $x \in \bar\Omega$.

Now, by condition (12) we conclude that

$$\int_{\Omega_<} F(x,u(x))\, dx \geq -C_1(t_\star^q + 1)\, \mathrm{meas}\,(\Omega).$$

Further, inequality (13) yields

$$\int_{\Omega_{\geq}} F(x, u(x))\, dx \geq \int_{\Omega_{\geq}} \gamma(x)|u(x)|^{\theta}\, dx.$$

Then

$$\begin{aligned}\varphi(u) &\leq \frac{m_1}{p}\|u\|^p + \frac{m_2}{p} - \left(\int_{\Omega_<} F(x, u(x))\, dx + \int_{\Omega_{\geq}} F(x, u(x))\, dx\right)\\ &\leq \frac{m_1}{p}\|u\|^p + \frac{m_2}{p} - \int_{\Omega_{\geq}} \gamma(x)|u(x)|^{\theta}\, dx + C_1(t_\star^q + 1)\,\mathrm{meas}\,(\Omega)\\ &\leq \frac{m_1}{p}\|u\|^p + \frac{m_2}{p} - \int_{\Omega} \gamma(x)|u(x)|^{\theta}\, dx + \kappa,\end{aligned}$$

where

$$\kappa := (\|\gamma\|_\infty t_\star^{\theta} + C_1(t_\star^q + 1))\,\mathrm{meas}\,(\Omega).$$

Hence, inequality (11) is proved.

At this point, observe that the functional $\|\cdot\|_\gamma\colon W_0^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\|u\|_\gamma := \left(\int_\Omega \gamma(x)|u(x)|^{\theta}\, dx\right)^{1/\theta},$$

is a norm in $W_0^{1,p}(\Omega)$. Since in X_1 the norms $\|\cdot\|$ and $\|\cdot\|_\gamma$ are equivalent (X_1 is finite dimensional), there exists a positive constant κ_{X_1} such that

$$\|u\| \leq \kappa_{X_1}\|u\|_\gamma,$$

for every $u \in W_0^{1,p}(\Omega)$. Consequently, we have that

$$\varphi(u) \leq \frac{m_1\kappa_{X_1}^p}{p}\|u\|_\gamma^p - \|u\|_\gamma^{\theta} + \kappa + \frac{m_2}{p},$$

for every $u \in X_1$. Hence

$$\frac{m_1\kappa_{X_1}^p}{p}\|u\|_\gamma^p - \|u\|_\gamma^{\theta} + \kappa + \frac{m_2}{p} \geq 0,$$

for every

$$u \in \{u \in X_1 \colon \varphi(u) \geq 0\}.$$

Since $\theta > p$ we conclude that the above set is bounded. The proof is complete. □

Remark 5.2. Note that condition (6) in Theorem 5.1 can be removed if, instead of Theorem 2.2, we apply the more general Theorem 9.12 contained in Rabinowitz [26, p. 55].

6. Some examples

In this section Ω is an open and bounded subset of $\mathbf{R}^3$ with smooth boundary $\partial\Omega$. We present two simple applications of our main results taking $p = 2$. The first one is a direct consequence of Theorem 4.1.

Example 6.1. Consider the following nonlocal equation with the Dirichlet boundary condition:

$$(E_1)\qquad \begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = u^3+u^4 \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{cases}$$

where

$$M\left(\int_\Omega |\nabla u(x)|^2\,dx\right) := 2+\frac{\sin\left(\int_\Omega |\nabla u(x)|^2\,dx\right)}{1+\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^2}.$$

Owing to Theorem 4.1, problem (E_1) admits one nontrivial classical solution. Indeed, a direct computation ensures that the continuous function

$$M(t) := 2+\frac{\sin t}{1+t^2}, \quad \forall\, t\geq 0,$$

satisfies conditions (C_M^1) and (C_M^2). Further, the locally Lipschitz continuous function $f:\mathbf{R}\to\mathbf{R}$ given by

$$f(t) := t^3+t^4, \quad \forall\, t\in\mathbf{R},$$

verifies all the hypotheses of Theorems 4.1.

Remark 6.2. We note that in Example 6.1 condition (C_M^2) is verified for every $t\geq t_0$, where t_0 is the unique positive solution of the following real equation

$$\int_0^t\left(2+\frac{\sin s}{1+s^2}\right)ds - t\left(2+\frac{\sin t}{1+t^2}\right)=0.$$

In conclusion, the existence of infinitely many solutions for the next elliptic nonlocal problem (namely (E_2)) can be studied by using Theorem 5.1.

Example 6.3. Consider the following problem in presence of an odd nonlinearity:

$$(E_2)\qquad \begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u(x)|^2\,dx\right)\Delta u = u^3+u \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0, \end{cases}$$

where

$$M\left(\int_\Omega |\nabla u(x)|^2\,dx\right) := 1+\frac{\cos\left(\int_\Omega |\nabla u(x)|^2\,dx\right)}{1+\left(\int_\Omega |\nabla u(x)|^2\,dx\right)^2}.$$

Owing to Theorem 5.1, problem (E_2) admits infinitely many classical solutions.

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