

A bifurcation problem associated to a convex, asymptotically linear function

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Abstract - We consider the bifurcation problem associated to a convex, asymptotically linear function and we study the behaviour of the stable solution and the existence and related properties of the unstable solutions.

Un problème de bifurcation associé à une fonction
convexe, asymptotiquement linéaire

Résumé - On considère le problème de bifurcation associé à une fonction convexe, asymptotiquement linéaire et on étudie le comportement des solutions stable et instables, ainsi que l'existence de ces dernières.

Version française abrégée - Dans cette note on considère le problème

$$(1) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega \end{cases}$$

dans les conditions suivantes : Ω est un ouvert borné connexe régulier de \mathbf{R}^N ; $f : \mathbf{R} \rightarrow \mathbf{R}$ est une fonction de classe C^1 , convexe, non négative, telle que $f(0) > 0$ et $f'(0) > 0$. De plus, f est une fonction asymptotiquement linéaire vers ∞ , c'est-à-dire,

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \lim_{t \rightarrow \infty} f'(t) = a \in (0, +\infty)$$

On suppose que λ est un paramètre positif et on cherche u dans $C^2(\Omega) \cap C(\overline{\Omega})$.

Sous ces hypothèses, on sait (voir [1]) qu'il existe $\lambda^* \in (0, \infty)$ tel que pour tout $\lambda < \lambda^*$ (resp. $\lambda > \lambda^*$), le problème (1) admet une solution (n'a aucune solution). Enfin, pour $\lambda < \lambda^*$, il existe une solution minimale $u(\lambda)$. De plus, $u(\lambda)$ est une solution stable et l'application $\lambda \mapsto u(\lambda)$ est convexe et croissante.

On se propose d'étudier les questions suivantes :

- i) l'existence de plusieurs solutions;
- ii) l'existence d'une solution pour $\lambda = \lambda^*$;
- iii) le comportement de la deuxième solution.

Dans ce cadre, nos résultats principaux sont les suivants:

THÉORÈME 1. - Si $f(t) \geq at$ pour tout t , alors

i) $\lambda^* = \frac{\lambda_1}{a}$.

ii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty$, uniformément sur les sous-ensembles compacts de Ω .

iii) $u(\lambda)$ est l'unique solution de (1) pour $\lambda \in (0, \lambda^*)$.

iv) (1) n'a pas de solutions si $\lambda = \lambda^*$.

THÉORÈME 2. - S'il existe $t_0 \in \mathbf{R}$ tel que $f(t_0) < at_0$, alors

i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$, où $\lambda_0 = \min_{t>0} \frac{f(t)}{t}$.

ii) (1) admet une seule solution, u^* , pour $\lambda = \lambda^*$.

iii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = u^*$, uniformément sur Ω .

iv) Si $\lambda \in (0, \frac{\lambda_1}{a}]$, $u(\lambda)$ est l'unique solution de (1).

v) Si $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, le problème (1) a au moins une solution instable $v(\lambda)$.

De plus, pour tout choix de $v(\lambda)$ on a

vi) $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} v(\lambda) = \infty$, uniformément sur les sous-ensembles compacts de Ω .

vii) $\lim_{\lambda \rightarrow \lambda^*} v(\lambda) = u^*$, uniformément sur Ω .

On utilise les notations suivantes: si $\alpha \in L^\infty(\Omega)$, alors $\lambda_j(-\Delta - \alpha)$ et $\varphi_j(-\Delta - \alpha)$ sont la j -ème valeur propre (resp. la j -ème fonction propre) de l'opérateur $-\Delta - \alpha$. Si $\alpha = 0$, on les note λ_j et φ_j . On suppose toujours que $\varphi_1 > 0$ et que $\|\varphi_j\|_{L^2(\Omega)} = 1$.

Pour la démonstration de ces deux résultats un argument essentiel est le Lemme 3, qui montre que $u(\lambda)$ vérifie l'alternative suivante: ou bien $u(\lambda)$ converge vers ∞ uniformément sur les compacts de Ω , ou bien $u(\lambda)$ converge vers une solution du problème (1). L'existence d'une solution instable est obtenue via le théorème de Ambrosetti-Rabinowitz.

INTRODUCTION - We study the problem

$$(1) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where: Ω is a smooth bounded connected open set in \mathbf{R}^N , $u \in C^2(\Omega) \cap C(\overline{\Omega})$, λ is a positive parameter, $f \in C^1(\mathbf{R}, \mathbf{R})$ is convex, nonnegative, with $f(0) > 0$ and $f'(0) > 0$. Moreover, we suppose that f is asymptotically linear in the sense that

$$(2) \quad \lim_{t \rightarrow \infty} \frac{f(t)}{t} = a \in (0, \infty)$$

Under these hypotheses it is known (see [1]) that there exists $\lambda^* \in (0, \infty)$ such that

i) (1) has no solution for $\lambda > \lambda^*$.

ii) (1) has solution for every $\lambda \in (0, \lambda^*)$.

iii) when $\lambda \in (0, \lambda^*)$ there exists a minimal solution, $u(\lambda)$, which can also be described as the unique solution u such that

$$(3) \quad \lambda_1(-\Delta - \lambda f'(u)) > 0$$

(Such solutions are called *stable*).

iv) $u(\lambda)$ increases with λ .

Here and in what follows, if $\alpha \in L^\infty(\Omega)$, then $\lambda_j(-\Delta - \alpha)$ and $\varphi_j(-\Delta - \alpha)$ denote the j -th eigenvalue (eigenfunction, respectively) of $-\Delta - \alpha$. We always suppose $\varphi_1 > 0$ and $\int_{\Omega} \varphi_j^2 = 1$. If $\alpha = 0$ we write λ_j and φ_j .

In this paper we are concerned with the following questions:

- i) what happens when $\lambda = \lambda^*$,
- ii) the behaviour of $u(\lambda)$ for λ near λ^* ,
- iii) under what circumstances (1) has solutions different from $u(\lambda)$.

The main results are the following:

THEOREM 1.- *If $f(t) \geq at$ for each t , then:*

- i) $\lambda^* = \frac{\lambda_1}{a}$
- ii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty$, *u.c.s.* Ω .
- iii) $u(\lambda)$ is the only solution of (1) when $\lambda \in (0, \lambda^*)$.
- iv) (1) has no solution when $\lambda = \lambda^*$.

THEOREM 2.- *If there exists t_0 such that $f(t_0) < at_0$, then:*

- i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$
- ii) (1) has exactly one solution, say u^* , when $\lambda = \lambda^*$.
- iii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = u^*$ *u.* Ω .
- iv) when $\lambda \in (0, \frac{\lambda_1}{a}]$, (1) has no solution but $u(\lambda)$.
- v) when $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, (1) has at least an unstable solution, say $v(\lambda)$.

Moreover, for each choice of $v(\lambda)$ we have

- vi) $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} v(\lambda) = \infty$ *u.c.s.* Ω .
- vii) $\lim_{\lambda \rightarrow \lambda^*} v(\lambda) = u^*$ *u.* Ω .

Here, $\lambda_0 = \min_{t>0} \frac{f(t)}{t}$, a solution u is called *unstable* if $\lambda_1(-\Delta - \lambda f'(u)) \leq 0$, while *u.c.s.* and *u.* mean uniformly on compact subsets (uniformly, resp.).

After the sketches of the proofs, we discuss the problem of the order of convergence to ∞ in the Theorems 1 and 2. As all integrals are taken over Ω , we shall omit this in our writing.

I. Proofs of the Theorems 1 and 2

We mention first some auxiliary results:

LEMMA 1. Let $\alpha \in L^\infty(\Omega)$, $w \in H_0^1(\Omega) \setminus \{0\}$, $w \geq 0$, be such that $\lambda_1(-\Delta - \alpha) \leq 0$ and

$$(4) \quad -\Delta w \geq \alpha w$$

Then $\lambda_1(-\Delta - \alpha) = 0$, $-\Delta w = \alpha w$ and $w > 0$ in Ω .

This follows multiplying (4) by $\varphi_1(-\Delta - \alpha)$ and integrating by parts.

LEMMA 2. i) $\lambda^* \geq \frac{\lambda_1}{a}$.

ii) if $f(t) = at + b$, $b > 0$, then $\lambda^* = \frac{\lambda_1}{a}$ and (1) has no solution when $\lambda = \lambda^*$.

iii) if (1) has solution when $\lambda = \lambda^*$, it is necessarily unstable.

iv) (1) has at most one solution when $\lambda = \lambda^*$.

v) $u(\lambda)$ is the only solution of (1) such that $\lambda_1(-\Delta - \lambda f'(u)) \geq 0$.

Proof. - i) 0 and the solution $u \in H_0^1(\Omega)$ of $-\Delta u = \lambda(au + f(0))$ are sub and supersolution for (1) when $\lambda \in (0, \frac{\lambda_1}{a})$.

ii) It suffices to prove the second part, which follows, by contradiction, multiplying by φ_1 and integrating.

iii) Otherwise, in view of the implicit function theorem, λ^* would not be maximal.

iv) If v_1 is such a solution, then $v_2 = \lim_{\lambda \rightarrow \lambda^*} u(\lambda)$ is also a solution and $v_2 \leq v_1$. With $w = v_1 - v_2 \geq 0$, we have $-\Delta w \geq f'(v_2)w$. Hence, either $w = 0$ or $w > 0$, but then $f(v_1) = av_1 + f(0)$. The last possibility contradicts ii).

v) As in iv), if v were such a solution different from $u(\lambda)$, then $f'(v) = f'(u(\lambda))$.

LEMMA 3. The following assertions are equivalent:

i) $\lambda^* = \frac{\lambda_1}{a}$.

ii) (1) has no solution in $C^2(\Omega) \cap C(\overline{\Omega})$ when $\lambda = \lambda^*$.

iii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty$ u.c.s. Ω .

Proof. - i) \implies ii) Any such solution u is a priori unstable. But this forces f to be linear in $[0, \max_{\Omega} u]$, which contradicts Lemma 2.

ii) \implies iii) It is enough to show that $u(\lambda)$ is bounded in $L^2(\Omega)$. Suppose the contrary. Then, by the Theorem 4.1.9., p. 94, of [2], $u(\lambda)$ converges in $L_{loc}^1(\Omega)$ to some u^* . If $u(\lambda) = k(\lambda)w(\lambda)$, $k(\lambda) > 0$, $\int w^2(\lambda) = 1$, we get the existence of some w , weak \star cluster point of $(w(\lambda))$ in $H_0^1(\Omega)$ such that $w \geq 0$, $\int w^2 = 1$ and $-\Delta w = 0$.

Obviously, iii) \implies ii). It remains to see that [iii) and ii)] \implies i). If w is obtained as above, this time it verifies $-\Delta w = \lambda^* a w$. Hence, $\lambda^* a = \lambda_1$ and $w = \varphi_1$.

COROLLARY 1. *Under the hypotheses of the Lemma 3,*

$$\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \quad u.\overline{\Omega}$$

Via a bootstrap argument and the Theorem 8.34, p. 211 in [3], we get that $w(\lambda)$ is bounded in $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$. We apply afterwards the Arzela-Ascoli Theorem.

Proof of the Theorem 1.- i), ii), iv). Suppose (1) has a solution u when $\lambda = \frac{\lambda_1}{a}$. Then, $-\Delta u \geq \lambda_1 u$. Lemma 1 implies $f(u) = au + f(0)$, but this contradicts Lemma 2.

iii) If u is a solution, then $\lambda_1(-\Delta - \lambda f'(u)) > \lambda_1(-\Delta - \lambda_1) = 0$.

Proof of the Theorem 2.- i) Suppose $\lambda^* = \frac{\lambda_1}{a}$. Then

$$0 = \lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \int \varphi_1 [(\lambda_1 - a\lambda)u(\lambda) + \lambda(au(\lambda) - f(u(\lambda)))] \geq \frac{-l\lambda_1}{a} \int \varphi_1 > 0,$$

where $l = \lim_{t \rightarrow \infty} [f(t) - at] < 0$. If we suppose $\lambda^* \geq \frac{\lambda_1}{\lambda_0}$, we obtain a similar contradiction.

ii), iii), iv) are obvious.

v) is a consequence of the Ambrosetti-Rabinowitz Theorem. Let

$$\lambda \in (\frac{\lambda_1}{a}, \lambda^*), \epsilon_0 = \frac{a\lambda - \lambda_1}{2\lambda_1}, u_0 = u(\lambda), F(t) = \lambda \int_0^t f(s) ds, X = H_0^1(\Omega),$$

$$J_\epsilon(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(u) + \frac{\epsilon}{2} \int |\nabla(u - u_0)|^2, u \in X, \epsilon \in [0, \epsilon_0]$$

Then it is known (see [1]) that $J_0 \in C^1(X, \mathbf{R})$ and u_0 is a local minimum for J_0 . Hence, u_0 is a local strict minimum for J_ϵ , $\epsilon \in (0, \epsilon_0]$. Since $\lim_{t \rightarrow \infty} \sup_{\epsilon \in [0, \epsilon_0]} J_\epsilon(t\varphi_1) = -\infty$, there exists $v_0 \in X$ with $J_\epsilon(v_0) \leq J_\epsilon(u_0)$, for each ϵ . If

$$\wp = \{p \in C([0, 1], X) : p(0) = u_0, p(1) = v_0\}$$

and $c_\epsilon = \inf_{\wp} \max_{[0, 1]} J_\epsilon \circ p$, then $c_0 \leq c_\epsilon \leq \max_{[u_0, v_0]} J_0 + \frac{\epsilon}{2} \int |\nabla(v_0 - u_0)|^2$.

The variational problem satisfies a Palais-Smale type condition, in the sense that if

$$(5) \quad (J_{\epsilon_n}(u_n)) \text{ is bounded}$$

and

$$(6) \quad J'_{\epsilon_n}(u_n) \longrightarrow 0$$

then (u_n) contains a convergent subsequence. By standard arguments, it is enough to find a subsequence bounded in L^2 . Suppose the contrary: let $u_n = k_n w_n$, $\int w_n^2 = 1$, $k_n > 0$, $\lim_{n \rightarrow \infty} k_n = \infty$, $\epsilon_n \rightarrow \epsilon \in [0, \epsilon_0]$. Then, by (5), if we modify f such that $\lim_{t \rightarrow -\infty} f(t) = 0$, we get (up to a subsequence) $w_n \rightarrow w$, both in weak $\star H_0^1$ and L^2 sense, with $-(1+\epsilon)\Delta w = \lambda a w^+$. Hence $w^+ = w$, which contradicts the choice of ϵ_0 . Hence there exists $(v_\epsilon)_{\epsilon \in (0, \epsilon_0]}$ pre-compact in $H_0^1(\Omega)$ such that

$$(7) \quad \begin{cases} -\Delta v_\epsilon = \lambda f(v_\epsilon) + \epsilon(u_0 - v_\epsilon) \\ J_\epsilon(v_\epsilon) = c_\epsilon > J_\epsilon(u_0) \end{cases}$$

(Note that this implies $v_\epsilon \neq u_0$ and v_ϵ unstable). Let v be a limit point of v_ϵ when $\epsilon \rightarrow 0$. Then v is the desired solution. Indeed, v is unstable as limit of unstable solutions.

vi) follows immediately if we show that $(v(\lambda))$ is bounded in $L^2(\Omega)$ when λ is near λ^* . The contrary would give as in Lemma 3 that $\lambda^* = \frac{\lambda_1}{a}$.

vii) If we suppose the contrary we obtain the same contradiction as in the proof of ii) \implies iii) in Lemma 3.

Further results: 1) If $\lambda_0 \geq \frac{a\lambda_1}{\lambda_2}$ (or, more generally, if $\lambda^* \leq \frac{\lambda_2}{a}$) then $v(\lambda)$ is unique. This follows from [4], p. 838. This implies that in this case v depends C^1 on λ .

2) If $\lambda^* > \frac{\lambda_2}{a}$ then there exists $\epsilon > 0$ such that $v(\lambda)$ is unique in $(\frac{\lambda_1}{a}, \frac{\lambda_2}{a} + \epsilon)$. Indeed, for $\lambda = \frac{\lambda_2}{a}$ we have, if v is an unstable solution of (1), that $\lambda_2(-\Delta - \lambda f'(v)) \geq 0$. The equality would imply that f is linear in $[0, \max_\Omega v]$, which is contradictory. Since $\lambda_1(-\Delta - \lambda f'(v)) < 0$, we get the uniqueness when $\lambda = \frac{\lambda_2}{a}$ via the previous remark and the implicit function theorem. A routine argument shows that the uniqueness remains true in a neighborhood of $\frac{\lambda_2}{a}$.

A natural question is to estimate the speed of convergence to ∞ of $u(\lambda)$ in Theorem 1. Regarding the equality

$$(8) \quad \int \varphi_1[(\lambda_1 - a\lambda)u(\lambda) + \lambda(au(\lambda) - f(u(\lambda)))] = 0$$

one can obtain the following results:

3) If $l = \lim_{t \rightarrow \infty} [f(t) - at] \in (0, \infty)$, then

$$(9) \quad \frac{a(\lambda_1 - a\lambda)}{\lambda_1 l \int \varphi_1} u(\lambda) \rightarrow \varphi_1 \quad u.\overline{\Omega}$$

4) If $l = 0$ then

$$(\lambda_1 - a\lambda)u(\lambda) \rightarrow 0 \quad u.\overline{\Omega}$$

In this case the answer depends heavily on f . For example:

- i) if $f(t) = t + \frac{1}{t+2}$ then $u(\lambda) \sim \frac{c}{\sqrt{\lambda_1 - \lambda}} \varphi_1$;
- ii) if $f(t) = t + \frac{1}{(t+1)^2}$ then $u(\lambda) \rightarrow \infty$ like no power of $(\lambda_1 - \lambda)$.

Similarly,

- 5) If $l \in (-\infty, 0)$ then (9) is true with $v(\lambda)$ instead of $u(\lambda)$.
- 6) If $l = -\infty$ then

$$(\lambda_1 - a\lambda)v(\lambda) \rightarrow \infty \quad u.c.s.\Omega$$

- 7) In the above statements we can allow $f'(0) = 0$ if f is strictly convex near 0.

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THE STUDY OF A BIFURCATION PROBLEM ASSOCIATED TO AN ASYMPTOTICALLY LINEAR FUNCTION

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Introduction

In this paper we consider the problem

$$(1) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where: Ω is a smooth connected bounded open set in \mathbf{R}^N , $f : \mathbf{R} \rightarrow \mathbf{R}$ is a C^1 convex nonnegative function such that $f(0) > 0$, $f'(0) > 0$ and f is asymptotically linear, that is

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} = a \in (0, +\infty)$$

In what follows we suppose that λ is a positive parameter and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

We point out some well known facts about the problem (1) (see [5] for details):

- i) there exists $\lambda^* \in (0, +\infty)$ such that (1) has (has no) solution when $\lambda \in (0, \lambda^*)$ ($\lambda \in (\lambda^*, +\infty)$, resp.).
- ii) for $\lambda \in (0, \lambda^*)$, among the solutions of (1) there exists a minimal one, say $u(\lambda)$.
- iii) $\lambda \mapsto u(\lambda)$ is a C^1 convex increasing function.
- iv) $u(\lambda)$ can be characterized as the only solution u of (1) such that the operator $-\Delta - \lambda f'(u)$ is coercive.

In what follows, we discuss some natural problems raised by (1):

- i) what can be said when $\lambda = \lambda^*$?
- ii) which is the behaviour of $u(\lambda)$ when λ approaches λ^* ?
- iii) are there other solutions of (1) excepting $u(\lambda)$?
- iv) if so, which is their behaviour?

Before mentioning our main results, we give some definitions and notations:

- i) let $\lim_{t \rightarrow \infty} (f(t) - at) = l \in [-\infty, \infty)$. We say that f obeys *the monotone case* (*the non-monotone case*) if $l \geq 0$ ($l < 0$, resp.).

ii) if $\alpha \in L^\infty(\Omega)$ we shall denote by $\varphi_j(\alpha)$ and $\lambda_j(\alpha)$ the j th eigenfunction (eigenvalue, resp.) of $-\Delta - \alpha$. We consider that $\int_{\Omega} \varphi_j(\alpha) \varphi_k(\alpha) = \delta_{jk}$ and $\varphi_1(\alpha) > 0$. If $\alpha = 0$ we shall write φ_j (λ_j , resp.).

iii) a solution u of (1) is said to be *stable* if $\lambda_1(\lambda f'(u)) > 0$ and *unstable* otherwise.

iv) $u.c.s.\Omega$ and $u.\overline{\Omega}$ will mean “uniformly on compact subsets of Ω ” (“uniformly on Ω ”, resp.).

All the integrals considered are over Ω , so that we shall omit Ω in writing.

Now we can state the main results:

THEOREM A.- *If f obeys the monotone case, then:*

- i) $\lambda^* = \frac{\lambda_1}{a}$
- ii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty$, $u.c.s. \Omega$.
- iii) $u(\lambda)$ is the only solution of (1) when $\lambda \in (0, \lambda^*)$.
- iv) (1) has no solution when $\lambda = \lambda^*$.

THEOREM B.- *If f obeys the non-monotone case, then:*

- i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$, where $\lambda_0 = \min_{t>0} \frac{f(t)}{t}$
- ii) (1) has exactly one solution, say u^* , when $\lambda = \lambda^*$.
- iii) $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = u^*$ $u.\overline{\Omega}$.
- iv) when $\lambda \in (0, \frac{\lambda_1}{a}]$, (1) has no solution but $u(\lambda)$.
- v) when $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, (1) has at least an unstable solution, say $v(\lambda)$.

For each choice of $v(\lambda)$ we have

- vi) $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} v(\lambda) = \infty$ $u.c.s. \overline{\Omega}$.
- vii) $\lim_{\lambda \rightarrow \lambda^*} v(\lambda) = u^*$ $u.\overline{\Omega}$.

After we establish these results, we discuss the problem of the order of convergence to ∞ in the theorems A and B.

1. Proof of Theorem A

LEMMA 1. Let $\alpha \in L^\infty(\Omega)$, $w \in H_0^1(\Omega) - \{0\}$, $w \geq 0$, be such that $\lambda_1(\alpha) \leq 0$ and

$$(2) \quad -\Delta w \geq \alpha w$$

Then:

- i) $\lambda_1(\alpha) = 0$
- ii) $-\Delta w = \alpha w$
- iii) $w > 0$ in Ω .

Proof: If we multiply (2) by $\varphi_1(\alpha)$ and integrate by parts, we obtain

$$\int \alpha \varphi_1(\alpha) w + \lambda_1(\alpha) \int \varphi_1(\alpha) w \geq \int \alpha \varphi_1(\alpha) w$$

Now this means that $\lambda_1(\alpha) = 0$ and $-\Delta w = \alpha w$. Since $w \geq 0$ and $w \not\equiv 0$, we get $w = c\varphi_1(\alpha)$ for some $c > 0$, which concludes the proof. \square

LEMMA 2. (the linear case) If $f(t) = at + b$ when $t \geq 0$, with $a, b > 0$, then

- i) $\lambda^* = \frac{\lambda_1}{a}$.
- ii) (1) has no solution when $\lambda = \lambda^*$.

Proof: i), ii) If $\lambda \in (0, \frac{\lambda_1}{a})$ then the problem

$$(3) \quad \begin{cases} -\Delta u - \lambda au = \lambda b & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution in $H_0^1(\Omega)$ which is positive in view of Stampacchia maximum principle (see [5]). Now Ω smooth and $-\Delta u = \lambda au + \lambda b \in H_0^1(\Omega)$ mean $u \in H^3(\Omega)$ and so on. We get $u \in H^\infty(\Omega)$ and therefore $u \in C^\infty(\overline{\Omega})$. We have thus exhibited a smooth solution of (1) when $\lambda \in (0, \frac{\lambda_1}{a})$

We claim that (1) has no solution if $\lambda^* = \frac{\lambda_1}{a}$. For if u were such a solution, multiplying (1) by φ_1 and integrating by parts, we get $\int \varphi_1 = 0$, which contradicts $\varphi_1 > 0$. \square

LEMMA 3. i) $\lambda^* \geq \frac{\lambda_1}{a}$.

- ii) if (1) has solution when $\lambda = \lambda^*$, it is necessarily unstable.
- iii) (1) has at most a solution when $\lambda = \lambda^*$.
- iv) $u(\lambda)$ is the only solution of (1) such that $\lambda_1(\lambda f'(u)) \geq 0$.

Proof: i) It is enough to exhibit a super and sub solution for $\lambda \in (0, \frac{\lambda_1}{a})$, that is: $\underline{U}, \overline{U} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\underline{U} \leq \overline{U}$,

$$\begin{cases} -\Delta \overline{U} \geq \lambda f(\overline{U}) & \text{in } \Omega \\ \overline{U} \geq 0 & \text{on } \partial\Omega \end{cases}$$

and that the reversed inequalities hold for \underline{U} (see [5] for the method of super and subsolutions).

Take some $b > 0$ such that $f(t) \leq at + b$ for nonnegative t . Let \overline{U} be the solution of (3) with $b = f(0)$ and $\underline{U} \equiv 0$. We have $f(t) \leq at + b$ for $t > 0$ and this implies $f(\overline{U}) \leq a\overline{U} + b$ in view of the positivity of \overline{U} . The remaining part is trivial.

ii) Suppose that (1) with $\lambda = \lambda^*$ has a solution u^* with $\lambda_1(\lambda^* f'(u^*)) > 0$. Then by the implicit function theorem applied to

$$G : \{u \in C^{2, \frac{1}{2}}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\} \times \mathbf{R} \rightarrow C^{0, \frac{1}{2}}(\overline{\Omega}), \quad G(u, \lambda) = -\Delta u - \lambda f(u)$$

it follows that (1) has solution for λ in a neighbourhood of λ^* , contradicting by this the definition of λ^* .

iii) Let u be such a solution. Then u is a supersolution for (1) when $\lambda \in (0, \lambda^*)$ and therefore $u \geq u(\lambda)$ for such λ . This shows that $u(\lambda)$ (which increases with λ) tends in $L^1(\Omega)$ sense to a limit $u^* \leq u$. Since $-\Delta u(\lambda) = \lambda f(u(\lambda))$ we get $-\Delta u^* = \lambda^* f(u^*)$. In order to conclude that u^* is a solution of (1), it is enough to prove that $u^* \in H_0^1(\Omega)$ and to deduce from this first that either $-\Delta u^* \in L^{2^*}(\Omega)$ and hence $u^* \in W^{2, 2^*}(\Omega)$ when $N > 2$, or $-\Delta u^* \in L^4(\Omega)$ and hence $u^* \in C^{0, \frac{1}{2}}(\overline{\Omega})$ if $N = 1, 2$ (using theorems 8.34 and 9.15 in [7]). The first case is then concluded via a bootstrap argument, while the second one using the theorem 4.3 in [7] (here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent).

Now we claim that $u(\lambda)$ is bounded in $H_0^1(\Omega)$. Indeed, if we multiply (1) by $u(\lambda)$ and integrate by parts we get

$$\int |\nabla u(\lambda)|^2 = \lambda \int f(u(\lambda))u(\lambda) \leq \lambda^* \int u f(u)$$

Thus, $u(\lambda) \rightharpoonup u^*$ in $H_0^1(\Omega)$ if $\lambda \rightarrow \lambda^*$. Indeed, if v is a weak- \star cluster point of $u(\lambda)$ when $\lambda \rightarrow \lambda^*$, then, up to a subsequence, $u(\lambda) \rightarrow v$ a.e. But $u(\lambda) \rightarrow u$ a.e. We have hence obtained that $u^* \in H_0^1(\Omega)$. The proof will be concluded if we show that $u = u^*$. Let $w = u - u^* \geq 0$. Then

$$(4) \quad -\Delta w = \lambda^*(f(u) - f(u^*)) \geq \lambda^* f'(u^*)w$$

We also have $\lambda_1(\lambda^* f'(u^*)) \leq 0$, so that lemma 1 implies that either $w = 0$ or $w > 0$, $\lambda_1(\lambda^* f'(u^*)) = 0$ and $-\Delta w = \lambda^* f'(u^*)w$. If we take (4) into account the last equality implies that f is linear in all the intervals $[u^*(x), u(x)]$, $x \in \Omega$. It is easy to see that this forces f to be linear in $[0, \max_{\Omega} u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta$ and $f(u^*) = \alpha u^* + \beta$. We have

$$0 = \lambda_1(\lambda^* f'(u^*)) = \lambda_1(\lambda^* \alpha) = \lambda_1 - \lambda^* \alpha,$$

that is $\lambda^* = \frac{\lambda_1}{\alpha}$. The last conclusion contradicts Lemma 2.

iv) Suppose (1) has a solution $u \neq u(\lambda)$ with $\lambda_1(\lambda f'(u)) \geq 0$. Then $u > u(\lambda)$ by the strong maximum principle (see the theorem 3.5. in [7]). Let $w = u - u(\lambda) > 0$. Then

$$(5) \quad -\Delta w = \lambda(f(u) - f(u(\lambda))) \leq \lambda f'(u)w$$

If we multiply (5) by $\varphi = \varphi_1(\lambda f'(u))$ and integrate by parts we get

$$\lambda \int f'(u) \varphi w + \lambda_1(\lambda f'(u)) \int \varphi w \leq \lambda \int f'(u) \varphi w$$

Thus, $\lambda_1(\lambda f'(u)) = 0$ and in (5) we have equality, that is f is linear in $[0, \max_{\Omega} u]$. Let $\alpha, \beta > 0$ be such that $f(u) = \alpha u + \beta$, $f(u(\lambda)) = \alpha u(\lambda) + \beta$. Then

$$0 = \lambda_1(\lambda f'(u)) = \lambda_1(\lambda f'(u(\lambda))),$$

contradiction. □

The following result is a reformulation of the theorem 4.1.9. in [9].

LEMMA 4. *Let (u_n) be a sequence of nonnegative superharmonic functions in Ω . Then*

either

$$i) \lim_{n \rightarrow \infty} u_n = \infty \quad \text{u.c.s. } \Omega$$

or

$$ii) (u_n) \text{ contains a subsequence which converges in } L^1_{loc}(\Omega) \text{ to some } u^*.$$

LEMMA 5. *The following conditions are equivalent:*

$$i) \lambda^* = \frac{\lambda_1}{a}$$

$$ii) (1) \text{ has no solution when } \lambda = \lambda^*$$

$$iii) \lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty \quad \text{u.c.s. } \Omega$$

Proof: $i) \implies ii)$ Suppose the contrary. Let u be such a solution. As we have already seen, $\lambda_1(\lambda^* f'(u)) \leq 0$. But $\lambda_1(\lambda^* f'(u)) \geq \lambda_1(\lambda^* a) = 0$.

Hence $\lambda_1(\lambda^* f'(u)) = 0$, that is $f'(u) = a$. As already happened, this contradicts lemma 2.

ii) \implies iii) Suppose the contrary. We prove first that $u(\lambda)$ are uniformly bounded in $L^2(\Omega)$. Suppose again the contrary. Then, up to a subsequence, $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda) \rightarrow \infty$ and $\int w^2(\lambda) = 1$.

Suppose, using again a subsequence if necessary, that $u(\lambda) \rightarrow u^*$ in $L^1_{loc}(\Omega)$. Then $\frac{\lambda}{k(\lambda)}f(u(\lambda)) \rightarrow 0$ in $L^1_{loc}(\Omega)$, that is

$$(6) \quad -\Delta w(\lambda) \rightarrow 0 \quad \text{in } L^1_{loc}(\Omega)$$

It is easy to see that $(w(\lambda))$ is bounded in $H^1_0(\Omega)$. Indeed,

$$\begin{aligned} \int |\nabla w(\lambda)|^2 &= \int -\Delta w(\lambda)w(\lambda) = \int \frac{\lambda}{k(\lambda)}f(u(\lambda))w(\lambda) \leq \\ &\leq \lambda^* \int (aw^2(\lambda) + \frac{f(0)}{k(\lambda)}w(\lambda)) \leq \lambda^*a + c \int w(\lambda) \leq \\ &\leq \lambda^*a + c \int \sqrt{|\Omega|} \quad (\text{for a suitable } c > 0) \end{aligned}$$

Let $w \in H^1_0(\Omega)$ be such that, up to a subsequence,

$$(7) \quad w(\lambda) \rightarrow w \quad \text{weakly in } H^1_0(\Omega) \text{ and strongly in } L^2(\Omega)$$

Then, by (6), $-\Delta w = 0$, and by (7), $w \in H^1_0(\Omega)$ and $\int w^2 = 1$. We have obtained the desired contradiction. Hence $(u(\lambda))$ is bounded in $L^2(\Omega)$. As above, $u(\lambda)$ is bounded in $H^1_0(\Omega)$. Let $u \in H^1_0(\Omega)$ be such that, up to a subsequence, $u(\lambda) \rightarrow u$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then by (1) we get that u is a $H^1_0(\Omega)$ solution of $-\Delta u = \lambda^* f(u)$. As we have already done, we get that in fact u is a solution of (1) when $\lambda = \lambda^*$. This contradiction concludes the proof.

iii) \implies ii). As we have seen, if (1) has a solution when $\lambda = \lambda^*$, it is necessarily equal to $\lim_{\lambda \rightarrow \lambda^*} u(\lambda)$, which cannot happen in the given context.

[iii) and ii)] \implies i) Let $u(\lambda) = k(\lambda)w(\lambda)$ with $k(\lambda)$ and $w(\lambda)$ as above. This time $\lim_{\lambda \rightarrow \lambda^*} k(\lambda) = \infty$. As above we get a uniform bound for $(w(\lambda))$ in $H^1_0(\Omega)$. Let $w \in H^1_0(\Omega)$ be such that, up to a subsequence, $w(\lambda) \rightarrow w$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$. Then $-\Delta w(\lambda) \rightarrow -\Delta w$ in $\mathcal{D}'(\Omega)$ and $\frac{\lambda}{k(\lambda)}f(u(\lambda)) \rightarrow \lambda^*aw$ in $L^2(\Omega)$. (The last statement will be shown out in the proof of Lemma 9). So we obtain

$$-\Delta w = \lambda^*aw, \quad w \in H^1_0(\Omega), \quad w \geq 0, \quad \int w^2 = 1$$

But this means exactly that $\lambda^* = \frac{\lambda_1}{a}$ (and $w = \varphi_1$). \square

LEMMA 6. *The following conditions are equivalent:*

i) $\lambda^* > \frac{\lambda_1}{a}$

ii) (1) has exactly a solution, say u^* , when $\lambda = \lambda^*$.

iii) $u(\lambda)$ is converging to u^* which is the unique solution of (1) when $\lambda = \lambda^*$.

Proof: We have already seen that $\lambda^* \geq \frac{\lambda_1}{a}$. This makes this lemma a reformulation of the preceding one apart the fact that the limit in iii) is u^* . Since we know that $u(\lambda) \rightarrow u^*$ a.e., it is enough to prove that $u(\lambda)$ has a limit in $C(\overline{\Omega})$ when $\lambda \rightarrow \lambda^*$. Even less, it is enough to prove that $u(\lambda)$ is relatively compact in $C(\overline{\Omega})$. This will be done via the Arzela-Ascoli Theorem if we show that $(u(\lambda))$ is bounded in $C^{0, \frac{1}{2}}(\overline{\Omega})$. Now $0 < u(\lambda) < u^*$ implies $0 < f(u(\lambda)) < f(u^*)$, which offers a uniform bound for $-\Delta u(\lambda)$ in $L^{2N}(\Omega)$. The desired bound is now a consequence of the theorem 8.34 in [7] (see also the remark from the page 212) and of the closed graph theorem. \square

Proof of Theorem A:

i), ii) and iv) will follow together if we prove one of them. We shall prove that $\lambda^* = \frac{\lambda_1}{a}$ by showing that (1) has no solution when $\lambda = \frac{\lambda_1}{a}$. For suppose u were such a solution. Then

$$(8) \quad -\Delta u = \lambda f(u) \geq \lambda_1 u$$

If we multiply (8) by φ_1 and integrate by parts we get $\lambda f(u) = \lambda_1 u$, contradicting the fact that $f(0) > 0$.

iii) taking into account the lemma 3 iv), it is enough to prove that for $\lambda \in (0, \frac{\lambda_1}{a})$ any solution u verifies $\lambda_1(\lambda f'(u)) \geq 0$. But

$$-\Delta - \lambda f'(u) \geq -\Delta - \lambda a$$

which shows that

$$\lambda_1(\lambda f'(u)) \geq \lambda_1(\lambda a) = \lambda_1 - \lambda a > 0$$

\square

2. Proof of Theorem B

i) We prove first that $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$. For this aim, we shall see that (1) has no solution when $\lambda = \frac{\lambda_1}{\lambda_0}$. Suppose the contrary and let u be such a solution. Then multiplying (1) by φ_1 and integrating by parts we get

$$(9) \quad \lambda_1 \int \varphi_1 u = \lambda \int \varphi_1 f(u)$$

In our case, (9) becomes

$$\lambda_1 \int \varphi_1 u = \frac{\lambda_1}{\lambda_0} \int \varphi_1 f(u) \geq \lambda_1 \int \varphi_1 u$$

which forces $f(u) = \lambda_0 u$ and, as above, this contradicts $f(0) > 0$.

The remaining part of *i*), *ii*) and *iii*) are equivalent in view of the lemmas 3 *iii*) and 6. We shall prove that $\lambda^* > \frac{\lambda_1}{a}$ supposing the contrary. Then $\lim_{\lambda \rightarrow \lambda^*} u(\lambda) = \infty$ *u.c.s.* Ω and $\lambda^* = \frac{\lambda_1}{a}$. If we examine (9) rewritten as

$$(10) \quad \begin{aligned} 0 &= \int \varphi_1 [\lambda_1 u(\lambda) - \lambda f(u(\lambda))] = \\ &= \int \varphi_1 [(\lambda_1 - a\lambda)u(\lambda) - \lambda(f(u(\lambda)) - au(\lambda))] \geq -\lambda \int \varphi_1 [f(u(\lambda)) - au(\lambda)] \end{aligned}$$

we see that the righthand side integrand converges monotonously to $l\varphi_1$ when $\lambda \rightarrow \lambda^*$. Here $l = \lim_{t \rightarrow \infty} (f(t) - at) < 0$. Passing to the limit in (10) we obtain the contradictory inequality

$$0 \geq -l\lambda \int \varphi_1 > 0$$

We have seen that $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$ and we know that (1) has solution when $\lambda = \lambda^*$. This shows that $\lambda^* < \frac{\lambda_1}{\lambda_0}$.

iv) can be proved exactly in the same way as *iii*) in the theorem A.

Since all the solutions of (1) are positive, we may modify $f(t)$ as we wish for negative t . In what follows we shall suppose, additionally, that f is increasing.

For the proof of *v*) we shall use some known results that we point out in what follows:

THE AMBROSETTI-RABINOWITZ THEOREM: *Let E be a Banach space, $J \in C^1(E, \mathbf{R})$, $u_0 \in E$. Suppose that there exist $R, \rho > 0, v_0 \in E$ such that*

$$(11) \quad J(u) \geq J(u_0) + \rho \quad \text{if} \quad \|u - u_0\| = R$$

$$(12) \quad J(v_0) \leq J(u_0)$$

Suppose that the following condition is satisfied:

(PS) every sequence (u_n) in E such that $(J(u_n))$ is bounded in \mathbf{R} and $J'(u_n) \rightarrow 0$ in E^ is relatively compact in E .*

Let

$$\mathcal{P} = \{p \in C([0, 1], E) : p(0) = u_0, p(1) = v_0\}$$

and

$$c = \inf_{\mathcal{P}} \max_{[0,1]} F \circ p$$

Then there exists $u \in E$ such that $J(u) = c$ and $J'(u) = 0$.

Note that $c > J(u_0)$ and that is why $u \neq u_0$ (see [5] for details).

We want to find out solutions of (1) different from $u(\lambda)$, that is critical points, others than $u(\lambda)$, of

$$J : E \rightarrow \mathbf{R}, \quad J(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(u)$$

where $E = H_0^1(\Omega)$ and $F(t) = \lambda \int_0^t f(s)ds$. We take $u(\lambda)$ as u_0 for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

We have

LEMMA 7. i) $J \in C^1(E, \mathbf{R})$

ii) For $u, v \in E$ we have $J'(u)v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v$

iii) u_0 is a local minimum for J .

The proof can be found in [5].

In order to apply the Ambrosetti-Rabinowitz Theorem we transform u_0 into a local strict minimum by modifying J . Let

$$J_\epsilon : E \rightarrow \mathbf{R}, \quad J_\epsilon(u) = J(u) + \frac{\epsilon}{2} \int |\nabla(u - u_0)|^2$$

In view of the preceding lemma we obviously have

i) $J \in C^1(E, \mathbf{R})$

ii) $J'_\epsilon(u) \cdot v = \int \nabla u \cdot \nabla v - \lambda \int f(u)v + \epsilon \int \nabla(u - u_0) \cdot \nabla v$

iii) u_0 is a local strict minimum for J_ϵ if $\epsilon > 0$ (so that (11) is verified).

We prove first the existence of a v_0 good for all ϵ near 0.

LEMMA 8. Let $\epsilon_0 = \frac{\lambda_a - \lambda_1}{2\lambda_1}$. Then there exists $v_0 \in E$ such that $J_\epsilon(v_0) < J_\epsilon(u_0)$ for $\epsilon \in [0, \epsilon_0]$.

Proof: Note that $J_\epsilon(u)$ is bounded by $J_0(u)$ and $J_{\epsilon_0}(u)$. It suffices to prove that

$$\lim_{t \rightarrow \infty} J_{\epsilon_0}(t\varphi_1) = -\infty$$

But

$$(13) \quad J_\epsilon(t\varphi_1) = \frac{\lambda_1}{2}t^2 + \frac{\epsilon_0}{2}\lambda_1 t^2 -$$

$$-\epsilon_0 \lambda_1 t \int \varphi_1 u_0 + \frac{\epsilon_0}{2} \int |\nabla u_0|^2 - \int F(t\varphi_1)$$

Let $\alpha = \frac{3a\lambda + \lambda_1}{4\lambda}$. Since $\alpha < a$, there exists $\beta \in \mathbf{R}$ such that $f(s) \geq \alpha s + \beta$ for all s , which implies that $F(s) \geq \frac{\alpha\lambda}{2} s^2 + \beta\lambda s$ when $s \geq 0$. Then (13) shows that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^2} J_{\epsilon_0}(t\varphi_1) \leq \frac{\lambda_1 + \epsilon_0 \lambda_1 - \lambda\alpha}{2} < 0$$

because of the choice of α . □

LEMMA 9. *The condition (PS) is satisfied uniformly in ϵ , that is if*

$$(14) \quad (J_{\epsilon_n}(u_n)) \text{ is bounded in } \mathbf{R}, \quad \epsilon_n \in [0, \epsilon_0]$$

and

$$(15) \quad J'_{\epsilon_n}(u_n) \rightarrow 0 \text{ in } E^*$$

then (u_n) is relatively compact in E .

Proof: Since any subsequence of (u_n) verifies (14) and (15), it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E . Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \rightarrow u$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and *a.e.*, and $\epsilon_n \rightarrow \epsilon$. Now (15) gives that

$$-\Delta u_n - \lambda f(u_n) - \epsilon_n \Delta(u_n - u_0) \rightarrow 0 \quad \text{in } \mathcal{D}'(\Omega)$$

Note that $f(u_n) \rightarrow f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \leq a|u_n - u|$. This shows that

$$-(1 + \epsilon_n) \Delta u_n \rightarrow \lambda f(u) - \epsilon \Delta u_0 \quad \text{in } \mathcal{D}'(\Omega),$$

that is

$$(16) \quad -\Delta u - \lambda f(u) - \epsilon \Delta(u - u_0) = 0$$

The above equality multiplied by u gives

$$(17) \quad (1 + \epsilon) \int |\nabla u|^2 - \lambda \int u f(u) - \epsilon \lambda \int u f(u_0) = 0$$

Now (15) multiplied by (u_n) gives

$$(18) \quad (1 + \epsilon_n) \int |\nabla u_n|^2 - \lambda \int u_n f(u_n) - \epsilon_n \lambda \int u_n f(u_0) \rightarrow 0$$

in view of the boundedness of (u_n) . The middle term in (18) tends to $-\lambda \int u f(u)$ and the last one to $-\epsilon \lambda \int u f(u_0)$ in view of the $L^2(\Omega)$ -convergence of u_n and $f(u_n)$. Hence, if we compare the first terms in (17) and (18) we get that $\int |\nabla u_n|^2 \rightarrow \int |\nabla u|^2$, which insures us that $u_n \rightarrow u$ in $H_0^1(\Omega)$. Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$ -boundedness of (u_n) implies the $H_0^1(\Omega)$ -boundedness of (u_n) as it can be seen by examining (14).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_{L^2(\Omega)} \rightarrow \infty$. Let $u_n = k_n w_n$ with $k_n > 0$, $\int w_n^2 = 1$ and $k_n \rightarrow \infty$. We may suppose $\epsilon_n \rightarrow \epsilon$. Then

$$(19) \quad 0 = \lim_{n \rightarrow \infty} \frac{J_{\epsilon_n}(u_n)}{k_n^2} = \lim_{n \rightarrow \infty} \left[\frac{1}{2} \int |\nabla w_n|^2 - \frac{1}{k_n^2} \int F(u_n) + \frac{\epsilon_n}{2} \int \left| \nabla \left(w_n - \frac{u_0}{k_n} \right) \right|^2 \right]$$

Now

$$\frac{\epsilon_n}{2} \int \left| \nabla \left(w_n - \frac{u_0}{k_n} \right) \right|^2 = \frac{\epsilon_n}{2} \int |\nabla w_n|^2 + \frac{\epsilon_n}{2k_n^2} \int |\nabla u_0|^2 - \frac{\epsilon_n \lambda}{k_n} \int w_n f(u_0)$$

Thus (19) can be rewritten

$$\lim_{n \rightarrow \infty} \left[\frac{1 + \epsilon_n}{2} \int |\nabla w_n|^2 - \frac{1}{k_n^2} \int F(u_n) \right] = 0$$

But

$$|F(u_n)| = |F(k_n w_n)| \leq \frac{\lambda a}{2} k_n^2 w_n^2 + \lambda b |k_n w_n|$$

because $|f(t)| \leq a|t| + b$. Here $b = f(0)$. This shows that $(\frac{1}{k_n^2} \int F(u_n))$ is bounded and this must also be true for $\|w_n\|_{H_0^1(\Omega)}$. Now let $w \in H_0^1(\Omega)$ be such that (up to a subsequence) $w_n \rightarrow w$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and *a.e.*. Note that $\int w^2 = 1$. We claim that

$$(20) \quad -(1 + \epsilon) \Delta w = \lambda a w^+$$

Indeed, (15) divided by k_n gives

$$(21) \quad (1 + \epsilon_n) \int \nabla w_n \cdot \nabla v - \lambda \int \frac{f(u_n)}{k_n} v - \frac{\epsilon_n \lambda}{k_n} \int f(u_0) v \rightarrow 0$$

for each $v \in H_0^1(\Omega)$. Now

$$(1 + \epsilon_n) \int \nabla w_n \cdot \nabla v \rightarrow (1 + \epsilon) \int \nabla w \cdot \nabla v$$

Hence (20) can be concluded from (21) if we show that $\frac{1}{k_n}f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $\frac{1}{k_n}f(u_n) = \frac{1}{k_n}f(k_n w_n)$ and it is easy to see that the required limit is equal to aw^+ in the set

$$\{x \in \Omega : w_n(x) \rightarrow w(x) \neq 0\}$$

If $w(x) = 0$ and $w_n(x) \rightarrow w(x)$, let $\epsilon > 0$ and n_0 be such that $|w_n(x)| < \epsilon$ for $n \geq n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \leq \epsilon a + \frac{b}{k_n} \text{ for such } n,$$

that is the required limit is 0. Thus, $\frac{f(u_n)}{k_n} \rightarrow aw^+$ a.e. Here $b = f(0)$. Now $w_n \rightarrow w$ in $L^2(\Omega)$ and thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see theorem IV.9 in [4]).

Since $\frac{1}{k_n}f(u_n) \leq a|w_n| + \frac{1}{k_n}b$, it follows that $\frac{1}{k_n}f(u_n)$ is also dominated. Hence (20) is now obtained. Now (20) and the maximum principle imply $w \geq 0$ and (20) becomes

$$(22) \quad \begin{cases} -\Delta w = \frac{\lambda a}{1+\epsilon} w \\ w \geq 0 \\ \int w^2 = 1 \end{cases}$$

Thus $\frac{\lambda a}{1+\epsilon} = \lambda_1$ (and $w = \varphi_1$), which contradicts the fact that $\epsilon \in [0, \epsilon_0]$ and the choice of ϵ_0 . This contradiction finishes the proof of the lemma 9. \square

LEMMA 10. c_ϵ is uniformly bounded.

Proof: The fact that J_ϵ increases with ϵ implies $c_\epsilon \in [c_0, c_{\epsilon_0}]$. \square

Now we continue the proof of the theorem B v): for $\epsilon \in (0, \epsilon_0]$, let $v_\epsilon \in H_0^1(\Omega)$ be such that

$$(23) \quad -\Delta v_\epsilon = \frac{\lambda}{1+\epsilon} f(v_\epsilon) + \frac{\lambda \epsilon}{1+\epsilon} f(u_0)$$

and

$$(24) \quad J_\epsilon(v_\epsilon) = c_\epsilon$$

The relation (24) and the lemmas 9 and 10 show that there exists $v \in H_0^1(\Omega)$ such that $v_\epsilon \rightarrow v$ in $H_0^1(\Omega)$ as $\epsilon \rightarrow 0$. Now (23) implies

$$-\Delta v = \lambda f(v)$$

The last assertions to be proved are that $v \neq u_0 = u(\lambda)$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$. Note that v_ϵ is a solution of (23) different from u_0 and hence unstable, in the sense that

$$\lambda_1\left(\frac{\lambda}{1+\epsilon}f'(v_\epsilon)\right) \leq 0$$

Indeed (23) is an equation of the form

$$-\Delta u = g(u) + h(x)$$

where g is convex and positive and h is positive. Then, if it has solutions, it has a minimal one, say u , with $\lambda_1(g'(u)) \geq 0$ (see [5]). Now the proof of the lemma 3 *iv*) shows that for all other solutions v we have $\lambda_1(g'(v)) < 0$. In our case, u_0 stands for u and v_ϵ for v . All we have to prove now is that the limit of a sequence of unstable solutions is also unstable, which will be done in

LEMMA 11. *Let $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $\mu_n \rightarrow \mu$ be such that $\lambda_1(\mu_n f'(u_n)) \leq 0$.*

Then $\lambda_1(\mu f'(u)) \leq 0$.

Proof: The fact that $\lambda_1(\alpha) \leq 0$ is equivalent to the existence of a $\varphi \in H_0^1(\Omega)$ such that

$$\int |\nabla \varphi|^2 \leq \int \alpha \varphi^2 \text{ and } \int \varphi^2 = 1$$

follows from the Hilbert-Courant min-max principle.

Let $\varphi_n \in H_0^1(\Omega)$ be such that

$$(25) \quad \int |\nabla \varphi_n|^2 \leq \int \mu_n f'(u_n) \varphi_n^2$$

and

$$(26) \quad \int \varphi_n^2 = 1$$

Since $f' \leq a$, (25) shows that (φ_n) is bounded in $H_0^1(\Omega)$. Let $\varphi \in H_0^1(\Omega)$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H_0^1(\Omega)$. Then the righthand side of (25) converges, up to a subsequence, to $\mu \int f'(u) \varphi^2$. This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega)$ as in the theorem IV.9 in [4]. Since

$$\int \varphi^2 = 1 \text{ and } \int |\nabla \varphi|^2 \leq \liminf \int |\nabla \varphi_n|^2,$$

we get the desired result.

The fact that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ follows via a bootstrap argument:

$$v \in H_0^1(\Omega) \Rightarrow f(v) \in L^{2^*}(\Omega) \Rightarrow v \in W^{2,2^*}(\Omega) \Rightarrow \dots$$

The key facts are:

- a) if $v \in L^p(\Omega)$ then $f(v) \in L^p(\Omega)$
- b) an elliptic regularity result (theorem 9.15 in [7]).
- c) the Sobolev embeddings.

vi) Suppose the contrary. Then there are $\mu_n \rightarrow \frac{\lambda_1}{a}$, v_n an unstable solution of (1) with $\lambda = \mu_n$, and $v \in L_{loc}^1(\Omega)$ such that $v_n \rightarrow v$ in $L_{loc}^1(\Omega)$

We claim first that (v_n) cannot be bounded in $H_0^1(\Omega)$. Otherwise, let $w \in H_0^1(\Omega)$ be such that, up to a subsequence, $v_n \rightarrow w$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Then

$$-\Delta v_n \rightarrow -\Delta w \text{ in } \mathcal{D}'(\Omega) \text{ and } f(v_n) \rightarrow f(w) \text{ in } L^2(\Omega),$$

which shows that $-\Delta w = \frac{\lambda_1}{a} f(w)$.

It follows that $w \in C^2(\Omega) \cap C(\overline{\Omega})$, that is w is a solution of (1). From Lemma 11 it follows that

$$(27) \quad \lambda_1 \left(\frac{\lambda_1}{a} f'(w) \right) \leq 0$$

Now (27) shows that $w \neq u(\frac{\lambda_1}{a})$, which contradicts *iv)* of the Theorem.

The fact that (v_n) is not bounded in $H_0^1(\Omega)$ implies that (v_n) is not bounded in $L^2(\Omega)$. Indeed, we have seen that the $L^2(\Omega)$ -boundedness implies the $H_0^1(\Omega)$ one. So, let $v_n = k_n w_n$, where $k_n > 0$, $\int w_n^2 = 1$ and up to a subsequence $k_n \rightarrow \infty$.

We have

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n) \rightarrow 0 \text{ in } L_{loc}^1(\Omega)$$

(and hence we have convergence also in the distribution sense) and (w_n) is seen to be bounded in $H_0^1(\Omega)$ with an already provided argument. If w is a \star -cluster point of (w_n) in $H_0^1(\Omega)$, we obtain $-\Delta w = 0$ and $\int w^2 = 1$, the desired contradiction.

vii) As before, it is enough to prove the $L^2(\Omega)$ -boundedness of $v(\lambda)$ near λ^* and to use the uniqueness property of u^* . Suppose the contrary. Let $\mu_n \rightarrow \lambda^*$, $\|v_n\|_{L^2(\Omega)} \rightarrow \infty$, where v_n are the corresponding solutions of (1). If we write again $v_n = k_n w_n$, then

$$(28) \quad -\Delta w_n = \frac{\mu_n}{k_n} f(u_n)$$

The fact that the righthand side of (28) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in $H_0^1(\Omega)$. Let w be such that up to a subsequence $w_n \rightarrow w$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. A computation already done shows that

$$-\Delta w = \lambda^* a w, \quad w \geq 0 \text{ and } \int w^2 = 1,$$

which forces λ^* to be $\frac{\lambda_1}{a}$. This contradiction concludes the proof. \square

3. Some further remarks

As we have seen in the proofs of the Theorems 1 and 2, we have that

i) in the monotone case, $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1$ in $H_0^1(\Omega)$.

ii) in the non-monotone case, $\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1$ in $H_0^1(\Omega)$.

It is natural to try to find out:

i) if the above limits continue to exist in a more restrictive sense, say in $C(\overline{\Omega})$.

ii) which is the asymptotic behaviour of $\|u(\lambda)\|_{L^2(\Omega)}$ and $\|v(\lambda)\|_{L^2(\Omega)}$ when λ is near $\frac{\lambda_1}{a}$.

It is easy to answer the first question. We have

PROPOSITION 1. *i) in the monotone case,*

$$\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda) = \varphi_1 \quad \text{in } C^1(\overline{\Omega})$$

ii) in the non-monotone case,

$$\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} \frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda) = \varphi_1 \quad \text{in } C^1(\overline{\Omega})$$

Proof: i) The proof is essentially the same as for the Lemma 6: it is enough to prove that $(\frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda))$ is relatively compact in $C^1(\overline{\Omega})$ (when λ is near $\frac{\lambda_1}{a}$), which can be done by showing that it is bounded in $C^{1, \frac{1}{2}}(\overline{\Omega})$. But this follows from the fact that the above set is bounded in $H_0^1(\Omega)$ and a bootstrap argument (note that a uniform bound for $w(\lambda) = \frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda)$ in some $L^p(\Omega)$, $1 < p < \infty$ provides a uniform bound for $-\Delta w(\lambda)$ in $L^p(\Omega)$ for the same p).

ii) is identical with i). \square

Moreover, we have

PROPOSITION 2. *If $w(\lambda)$ is either $\frac{1}{\|u(\lambda)\|_{L^2(\Omega)}} u(\lambda)$ or $\frac{1}{\|v(\lambda)\|_{L^2(\Omega)}} v(\lambda)$, then $\frac{\varphi_1}{w(\lambda)}$ is uniformly bounded when λ is near $\frac{\lambda_1}{a}$.*

Proof: Note that the strong maximum principle implies that $\frac{\partial w(\lambda)}{\partial \nu} < 0$ on $\partial\Omega$ and hence $\frac{\varphi_1}{w(\lambda)}$ can be extended to a continuous function on $\overline{\Omega}$ by setting

$$\frac{\varphi_1}{w(\lambda)}(x) = \frac{\frac{\partial \varphi_1}{\partial \nu}(x)}{\frac{\partial w(\lambda)}{\partial \nu}(x)} \quad \text{for } x \in \partial\Omega.$$

LEMMA 12. *There exists $\epsilon_0 > 0$ such that if*

$$\omega_0 = \{x \in \mathbf{R}^N : d(x, \partial\Omega) < \epsilon_0\}$$

then

- i) *for each $x \in \omega_0$ there is a unique $x_0 \in \partial\Omega$ such that $d(x, \partial\Omega) = |x - x_0|$.*
- ii) *if $\Pi(x) = x_0$, then $\Pi \in C^1(\omega_0)$ (x, x_0 are as above).*
- iii) *if $|x - \Pi(x)| = \epsilon$ then $x = \Pi(x) - \epsilon\nu(\Pi(x))$ or $x = \Pi(x) + \epsilon\nu(\Pi(x))$, according to the case $x \in \Omega$ or $x \notin \Omega$.*
- iv) *if $x \in \Omega$ then $[x, \Pi(x)) \subset \Omega$.*

The proof can be found in [10]. □

Let $\omega = \omega_0 \cap \Omega$ and $K = \Omega \setminus \omega$. Since $w(\lambda) \rightarrow \varphi_1$ $u.\overline{\Omega}$, for λ close enough to $\frac{\lambda_1}{a}$ we have $w(\lambda)|_K > \frac{1}{2} \min_K \varphi_1$, that is $\frac{\varphi_1}{w(\lambda)} < c$ in K for such λ and a suitable c . If $x \in \omega$, let $x_0 = \Pi(x)$. Then

$$(29) \quad \frac{\varphi_1(x)}{w(\lambda, x)} = \frac{\varphi_1(x) - \varphi_1(x_0)}{w(\lambda, x) - w(\lambda, x_0)} = \frac{-\epsilon \frac{\partial \varphi_1}{\partial \nu(x_0)}(x_0 + \tau(x - x_0))}{-\epsilon \frac{\partial w}{\partial \nu(x_0)}(\lambda, x_0 + \tau(x - x_0))}$$

for some $\tau \in (0, 1)$. Taking a smaller ϵ_0 , if necessary, we may suppose that $\frac{\partial w}{\partial \nu(\Pi(x))}(x) < 0$ on $\overline{\omega}$. Then, as above, the quotient in (29) is smaller than some $c_1 > 0$ for λ near $\frac{\lambda_1}{a}$. □

For the second question the answer is delicate. For example we have

PROPOSITION 3. *Suppose f to obey the monotone case, that is $f(t) \geq at$ for all t , and let*

$$l = \lim_{t \rightarrow \infty} [f(t) - at] \geq 0.$$

Then

$$\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} (\lambda_1 - a\lambda) \|u(\lambda)\|_{L^2(\Omega)} = \frac{\lambda_1}{a} l \int \varphi_1.$$

Proof: Let L_0 be a limit point of $(\lambda_1 - a\lambda) \|u(\lambda)\|_{L^2(\Omega)}$ when $\lambda \rightarrow \frac{\lambda_1}{a}$. If we rewrite

$$(10) \quad \int \varphi_1 [(\lambda_1 - a\lambda)u(\lambda) - \lambda(f(u(\lambda)) - au(\lambda))] = 0$$

in the form

$$(30) \quad \int \varphi_1(\lambda_1 - a\lambda) \|u(\lambda)\|_{L^2(\Omega)} w(\lambda) = \int \lambda \varphi_1(f(u(\lambda)) - au(\lambda))$$

and we note that the righthand side integrand converges dominated to $\frac{\lambda_1}{a} l \varphi_1$ when $\lambda \rightarrow \frac{\lambda_1}{a}$, and that the lefthand side integrand tends to $L_0 \varphi_1^2 u \cdot \bar{\Omega}$ if $L_0 < \infty$ and to ∞ uniformly in Ω if $L_0 = \infty$ (on an appropriate sequence of λ), we get that

$$L_0 = \frac{\lambda_1}{a} l \int \varphi_1$$

□

It is obvious that the answer is good only when $l > 0$. If $l = 0$ then it shows only that $\|u(\lambda)\|_{L^2(\Omega)}$ grows slower than $\frac{1}{\lambda_1 - a\lambda}$. As we shall see below, in this case the answer depends heavily on f .

EXAMPLE 1. Let $f(t) = t + \frac{1}{t+2}$ when $t \geq 0$ (defined no matter how for negative t). Then

$$\lim_{\lambda \rightarrow \lambda_1} \sqrt{\lambda_1 - \lambda} \|u(\lambda)\|_{L^2(\Omega)} = \sqrt{\lambda_1 |\Omega|}$$

Proof: With the usual decomposition $u(\lambda) = k(\lambda)w(\lambda)$, if we divide (10) by $\sqrt{\lambda_1 - \lambda}$ we get

$$(31) \quad \int \varphi_1 \sqrt{\lambda_1 - \lambda} k(\lambda) w(\lambda) = \int \frac{\lambda \varphi_1}{\sqrt{\lambda_1 - \lambda} k(\lambda) w(\lambda) + 2\sqrt{\lambda_1 - \lambda}}$$

We claim first that $\liminf_{\lambda \rightarrow \lambda_1} \sqrt{\lambda_1 - \lambda} k(\lambda) > 0$. Otherwise, let $\mu_n \rightarrow \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n} k(\mu_n) \rightarrow 0$. Then

$$\sqrt{\lambda_1 - \mu_n} k(\mu_n) w(\mu_n) \varphi_1 \rightarrow 0 \quad u \cdot \bar{\Omega}$$

and

$$\sqrt{\lambda_1 - \mu_n} k(\mu_n) w(\mu_n) + 2\sqrt{\lambda_1 - \mu_n} \rightarrow 0 \quad u \cdot \bar{\Omega},$$

which contradicts (31) for large n .

We shall also prove that $\limsup_{\lambda \rightarrow \lambda_1} \sqrt{\lambda_1 - \lambda} k(\lambda) < \infty$. Suppose the contrary. Let $\mu_n \rightarrow \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n} k(\mu_n) \rightarrow \infty$. Then the lefthand side of (31) tends to ∞ with n . We shall show that the righthand side remains bounded and the contradiction will conclude the proof. Now $\frac{\varphi_1}{w(\mu_n)}$ is uniformly bounded by some $M > 0$, so that the righthand side integrand is less than $\frac{\lambda_1 M}{\sqrt{\lambda_1 - \mu_n} k(\mu_n)}$, which is bounded.

Let $c \in (0, +\infty)$ be a limit point of $\sqrt{\lambda_1 - \lambda}k(\lambda)$ when $\lambda \rightarrow \lambda_1$. Let $\mu_n \rightarrow \lambda_1$ be such that $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \rightarrow c$ and $\sqrt{\lambda_1 - \mu_n}k(\mu_n) \geq \frac{c}{2}$. Then the lefthand side of (31) tends to c , while the righthand side integrand is dominated by $\frac{2\lambda_1 M}{c}$ and converges *a.e.* to $\frac{\lambda_1}{c}$. Hence $c = \frac{\lambda_1}{c}|\Omega|$ which finishes the proof. \square

Note that a similar computation can be made if $f(t) = \sqrt{t^2 + 1}$.

If $f(t) - at$ decays to ∞ faster than $\frac{1}{t}$ then the behaviour becomes more complicated, as shows

EXAMPLE 2. Let $f(t) = t + \frac{1}{(t+1)^2}$. Then $\|u(\lambda)\|_{L^2(\Omega)}$ tends to ∞ like no power of $(\lambda_1 - \lambda)$. More precisely,

- i) $\lim_{\lambda \rightarrow \lambda_1} (\lambda_1 - \lambda)^\alpha \|u(\lambda)\|_{L^2(\Omega)} = \infty$ if $\alpha \leq \frac{1}{3}$.
- ii) $\lim_{\lambda \rightarrow \lambda_1} (\lambda_1 - \lambda)^\alpha \|u(\lambda)\|_{L^2(\Omega)} = 0$ if $\alpha > \frac{1}{3}$.

Proof: We shall need first some estimations for $\int \frac{1}{\varphi_1}$ and $\int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1}$.

LEMMA 13. i) There exist positive constants K_1, K_2 and ϵ_1 such that

$$K_1 |\ln \epsilon| \leq \int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} \leq K_2 |\ln \epsilon| \quad \text{for } \epsilon \in (0, \epsilon_1).$$

$$\text{ii) } \int \frac{1}{\varphi_1} = \infty.$$

Proof: ii) follows obviously from i).

i) Let ϵ_0 and ω_0 as in Lemma 12. Let

$$\Phi : \omega_0 \rightarrow \partial\Omega \times (-\epsilon_0, \epsilon_0) \quad \text{and} \quad \Psi : \partial\Omega \times (-\epsilon_0, \epsilon_0) \rightarrow \omega_0$$

be defined by

$$\Phi(x) = (\Pi(x), \langle x - \Pi(x), \nu(x) \rangle) \quad \text{and} \quad \Psi(x_0, \epsilon) = x_0 + \epsilon \nu(x_0).$$

Then Φ, Ψ are smooth and $\Psi = \Phi^{-1}$, so that if we replace if necessary ϵ_0 with a smaller number, we may suppose that there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq |J(\Psi)| \leq C_2$ on ω_0 .

We claim that there exist $C_3, C_4 > 0$ such that

$$C_3 d(x, \partial\Omega) \leq \varphi_1(x) \leq C_4 d(x, \partial\Omega)$$

when $x \in \omega$, if we replace, eventually, ϵ_0 with a smaller number. Indeed, as $\max_{\partial\Omega} \frac{\partial \varphi_1}{\partial \nu} < 0$, we obtain that

$$-C_3 = \sup_{x \in \omega} \frac{\partial \varphi_1(x)}{\partial \nu(\Pi(x))} < 0$$

if ϵ_0 is small enough.

Let $C_4 = \max_{\bar{\Omega}} |\varphi'|$. Then if $x \in \omega$ we get

$$\varphi_1(x) = \varphi_1(x) - \varphi_1(\Pi(x)) = -d(x, \Pi(x)) \frac{\partial \varphi_1(y)}{\partial \nu(\Pi(x))}$$

for some $y \in [x, \Pi(x)]$ and also the desired result.

Take $\epsilon_1 < \min(\inf_{\Omega \setminus \omega} \varphi_1, C_3 \epsilon_0)$. Now if $\epsilon < \epsilon_1$ then:

$$\int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} = \int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \int \mathbf{1}_{\{\epsilon < \varphi_1 < \epsilon_1\}} \frac{1}{\varphi_1}$$

Note that

$$\left\{ \frac{\epsilon}{C_3} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_4} \right\} \subset \{\epsilon < \varphi_1 < \epsilon_1\} \subset \left\{ \frac{\epsilon}{C_4} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_3} \right\}$$

and

$$\frac{1}{C_4 d(x, \partial\Omega)} \leq \frac{1}{\varphi_1(x)} \leq \frac{1}{C_3 d(x, \partial\Omega)}$$

there. Then

$$\begin{aligned} & \int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_4} \int \mathbf{1}_{\{\frac{\epsilon}{C_3} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_4}\}} \frac{1}{d(x, \partial\Omega)} \leq \\ & \leq \int \mathbf{1}_{\{\varphi_1 > \epsilon\}} \frac{1}{\varphi_1} \leq \int \mathbf{1}_{\{\varphi_1 \geq \epsilon_1\}} \frac{1}{\varphi_1} + \frac{1}{C_3} \int \mathbf{1}_{\{\frac{\epsilon}{C_4} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_3}\}} \frac{1}{d(x, \partial\Omega)} \end{aligned}$$

It remains to find, for example, $C_5, C_6 > 0$ such that

$$C_5 |\ln \epsilon| \leq I = \int \mathbf{1}_{\{\frac{\epsilon}{C_4} < d(x, \partial\Omega) < \frac{\epsilon_1}{C_3}\}} \frac{1}{d(x, \partial\Omega)} \leq C_6 (|\ln \epsilon| + 1)$$

Now with the changement of coordinates $x = \Psi(x_0, \delta)$ we get

$$I = \int_{\partial\Omega \times (\frac{\epsilon}{C_4}, \frac{\epsilon_1}{C_3})} \frac{1}{\delta} |J(\Psi)| ds(x_0) d\delta,$$

so that

$$C_1 |\partial\Omega| \ln \frac{C_4 \epsilon_1}{C_3 \epsilon} \leq I \leq C_2 |\partial\Omega| \ln \frac{C_4 \epsilon_1}{C_3 \epsilon}$$

and the desired estimation follows easily. The proof of the Lemma is completed. \square

Now in order to prove i) of the Example 2 it is enough to show that

$$\lim_{\lambda \rightarrow \lambda_1} (\lambda_1 - \lambda)^{\frac{1}{3}} \|u(\lambda)\|_{L^2(\Omega)} = \infty$$

Suppose that there exist $\mu_n \rightarrow \lambda_1$ and $c < \infty$ such that

$$(\lambda_1 - \mu_n)^{\frac{1}{3}} k_n \rightarrow c, \quad \text{where } k_n = \|u(\mu_n)\|_{L^2(\Omega)}$$

If we divide (10) written with $\lambda = \mu_n$ by $(\lambda_1 - \mu_n)^{\frac{2}{3}}$ we get

$$(32) \quad \int \varphi_1 (\lambda_1 - \mu_n)^{\frac{1}{3}} k_n w_n = \lambda \int \frac{\varphi_1}{(\lambda_1 - \mu_n)^{\frac{2}{3}} (k_n w_n + 1)^2}$$

where $w_n = \frac{1}{k_n} u(\mu_n)$.

If $c = 0$ then the lefthand side in (32) tends to 0, while the second one to ∞ . Hence $c \in (0, \infty)$. The fact that $k_n \rightarrow \infty$ implies that for each $\epsilon > 0$, $2k_n w_n + 1 < \epsilon k_n^2$, for large n , so that the righthand side of (32) is larger than

$$\frac{\lambda}{2c^2} \int \frac{\varphi_1}{\varphi_1^2 + \epsilon}$$

for n big enough to have $(\lambda_1 - \mu_1)^{\frac{2}{3}} k_n^2 < 2c^2$. Since the limit of the lefthand side is c , we get that

$$c \geq \frac{\lambda_1}{2c^2} \int \frac{\varphi_1}{\varphi_1^2 + \epsilon}$$

for all $\epsilon > 0$. Letting $\epsilon \rightarrow 0$ we obtain $c = \infty$, the desired contradiction.

ii) Suppose the contrary. Then there exist $\alpha > \frac{1}{3}$, $\mu_n \rightarrow \lambda_1$, $c \in (0, +\infty]$ such that $(\lambda_1 - \mu_n)^{\alpha} k_n \rightarrow c$, where $k_n = \|u(\mu_n)\|_{L^2(\Omega)}$.

Let $\beta = 3\alpha - 1 > 0$. Then (10) with $\lambda = \mu_n$ divided by $(\lambda_1 - \mu_n)^{1-\alpha}$ gives

$$(33) \quad \int \varphi_1 (\lambda_1 - \mu_n)^{\alpha} k_n w_n = \lambda \int \frac{\varphi_1}{(\lambda_1 - \mu_n)^{2\alpha-\beta} (k_n w_n + 1)^2} (= I_n)$$

The limit of the lefthand side is $c \in (0, +\infty]$. I_n can be estimated as follows:

$$I_n = \int \dots = \int \mathbf{1}_{\{\varphi_1 < \lambda_1 - \mu_n\}} \dots + \int \mathbf{1}_{\{\varphi_1 \geq \lambda_1 - \mu_n\}} \dots = J_n + K_n$$

Now

$$0 < J_n \leq \int \frac{\lambda_1 - \mu_n}{(\lambda_1 - \mu_n)^{2\alpha-\beta}} = (\lambda_1 - \mu_n)^{\alpha} |\Omega| \rightarrow 0$$

while

$$0 < K_n \leq \frac{M(\lambda_1 - \mu_n)^{\beta}}{c^2} \int \mathbf{1}_{\{\varphi_1 \geq \lambda_1 - \mu_n\}} \frac{1}{\varphi_1},$$

where $M = \sup_n \max \frac{w_n^2}{\varphi_1^2} < \infty$ (as shows the proof of the Proposition 2).

Lemma 13 shows that the last expression is $O((\lambda_1 - \mu_n)^\beta |\ln(\lambda_1 - \mu_n)|)$, that is it tends to zero with n .

In the non-monotone case $\|v(\lambda)\|_{L^2(\Omega)}$ grows faster to ∞ . We have

PROPOSITION 4. *Let f obey the non-monotone case and let*

$$\lim_{t \rightarrow \infty} [f(t) - at] = l \in [-\infty, 0).$$

Then

$$\lim_{\lambda \rightarrow \frac{\lambda_1}{a}} (\lambda_1 - a\lambda) \|v(\lambda)\|_{L^2(\Omega)} = l$$

The proof is identical to that of the preceding Proposition.

The result is good only when $l \in \mathbf{R}$. When $l = -\infty$, we give an example.

EXAMPLE 3. *If $f(t) = t + 2 - \sqrt{t+1}$, then*

$$\lim_{\lambda \rightarrow \lambda_1} (\lambda - \lambda_1)^2 \|v(\lambda)\|_{L^2(\Omega)} = \left(\int \varphi_1 \sqrt{\varphi_1} \right)^2$$

Proof: If we multiply (10) by $\lambda - \lambda_1$ we get

$$\begin{aligned} (34) \quad & \int \varphi_1 (\lambda - \lambda_1) \sqrt{k(\lambda)} \sqrt{w(\lambda)} [\lambda - (\lambda - \lambda_1) \sqrt{k(\lambda)} \sqrt{w(\lambda)}] = \\ & = 2\lambda(\lambda - \lambda_1) \int \varphi_1 - \lambda \int \varphi_1 [\sqrt{(\lambda - \lambda_1)^2 k(\lambda) w(\lambda) + (\lambda - \lambda_1)^2} - \sqrt{(\lambda - \lambda_1)^2 k(\lambda) w(\lambda)}] \end{aligned}$$

where $k(\lambda), w(\lambda)$ are as usual. We prove first that $\limsup_{\lambda \rightarrow \lambda_1} (\lambda - \lambda_1)^2 k(\lambda) < \infty$. Suppose there exist $\mu_n \rightarrow \lambda_1$ such that $(\mu_n - \lambda_1)^2 k(\mu_n) \rightarrow \infty$. Then the righthand side of (34) tends to 0, while the lefthand side is, for a suitable choice of $C_1, C_2 > 0$, less than

$$C_1(\lambda - \lambda_1) \sqrt{k(\lambda)} - C_2(\lambda - \lambda_1)^2 k(\lambda)$$

so it tends to $-\infty$.

Suppose now that

$$(35) \quad \liminf_{\lambda \rightarrow \lambda_1} (\lambda - \lambda_1)^2 k(\lambda) = 0.$$

The last integral in (34) is positive, so that (34) gives

$$(36) \quad \int \varphi_1 \sqrt{k(\lambda)} \sqrt{w(\lambda)} [\lambda - (\lambda - \lambda_1) \sqrt{k(\lambda)} \sqrt{w(\lambda)}] \leq 2\lambda \int \varphi_1$$

But the assumption (35) makes the lefthand side of (36) to tend to ∞ for a suitable λ . The contradiction shows that (35) is false.

Now let $c \in (0, +\infty)$ be any limit point of $(\lambda - \lambda_1)^2 k(\lambda)$ when $\lambda \rightarrow \lambda_1$. Then (34) shows that $c = (\int \varphi_1 \sqrt{\varphi_1})^2$.

All other functions we have tested behaved well in the sense that $\|v(\lambda)\|_{L^2(\Omega)} \sim Cg(\frac{1}{\lambda - \lambda_1})$ where g is the inverse of the antiderivative of

$$[0, +\infty) \ni t \longmapsto \frac{1}{at + f(0) + 1 - f(t)}$$

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PERIODIC SOLUTIONS OF THE EQUATION

$$-\Delta v = v(1 - |v|^2) \text{ IN } \mathbf{R} \text{ AND } \mathbf{R}^2$$

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1. Introduction

We study in this paper the existence of periodic functions $v : \mathbf{R} \rightarrow \mathbf{C}$ which satisfy the equation

$$(1) \quad -v'' = v(1 - |v|^2).$$

As observed in [BMR], the functions

$$(2) \quad Ae^{ikx} \quad , \text{ where } k \in \mathbf{R}, A \in \mathbf{C}, |A|^2 + k^2 = 1,$$

are such solutions.

For fixed T , we also study the number of solutions of (1) with principal period T . The problem is that (1) has too many solutions, that is, if v is a solution, then

$$(3) \quad x \longmapsto \alpha v(x_0 \pm x)$$

is also a solution if $|\alpha| = 1$ and $x_0 \in \mathbf{R}$. In order to avoid such a redundancy, we shall first obtain a “canonical form” of solutions of (1). Namely, let V be a periodic solution of (1). We may suppose that $x = 0$ is a maximum point for $|V|^2$. Then one can find $\epsilon \in \{-1, +1\}$ and $\alpha \in \mathbf{C}$, $|\alpha| = 1$ such that

$$x \xrightarrow{v} \alpha V(\epsilon x)$$

satisfies, apart (1), the conditions

$$(4) \quad \begin{cases} v_1(0) = a > 0 \\ v_1'(0) = 0 \\ v_2(0) = 0 \\ v_2'(0) = b \geq 0 \end{cases} ,$$

where $v = v_1 + iv_2$ and $a = \max |v|$. It is obvious that the system (1)+(4) gives all the geometrically distinct solutions of (1), that is solutions that cannot be obtained one from another by the procedure (3).

In what follows, we shall simply write “ T -periodic solutions” instead of “solutions of principal period T ”. Our first result concerns the existence and the multiplicity of “ T -periodic solutions”.

2. The main result

Our main result is the following

Theorem. i) If $T \leq 2\pi$, there are no T -periodic solutions.

ii) If $T > 2\pi$, there is exactly one real solution v of (1)+(4), that is a solution for which $v_2 \equiv 0$. Moreover, v depends analytically on T .

iii) There is some $T_1 > 2\pi$ such that, for $2\pi < T \leq T_1$, (1)+(4) has no other T -periodic solutions apart those given by ii) above and (2), for $k = \frac{2\pi}{T}$, $A = \sqrt{1 - k^2}$.

iv) For $T > T_1$, (1)+(4) has other T -periodic solutions apart these two.

v) For each T , the number of T -periodic solutions is finite.

vi) For large T , (1)+(4) has at least

$$\frac{5}{8}T^2 + O(T \log T)$$

T -periodic solutions.

Remark: In fact, we shall find all the solutions of (1)+(4). More precisely, we shall exhibit a set $\Omega = \overline{\Omega} \subset \mathbf{R}^2$ such that, roughly speaking,

i) if $(a, b) \notin \Omega$, then the solution of (1)+(4) has a finite life time for positive or negative x .

ii) if $(a, b) \in \partial\Omega$, we obtain the solutions given by (2) or ii) of the Theorem.

iii) if $(a, b) \in \text{Int}\Omega$, then $v \neq 0$, v has a global existence, $|v|$ and $\frac{d}{dx} \frac{v}{|v|}$ are periodic functions. For such (a, b) , if T_0 is the principal period of $|v|$ and φ is (globally) defined such that $v = e^{i\varphi}|v|$, then v is periodic if and only if $\varphi(T_0) - \varphi(0) \in \pi$. Given $q = \frac{m}{n} \in \mathbb{Q}$, $q > 0$, $(m, n) = 1$, the set

$$\{(a, b) \in \text{Int } \Omega ; \quad \varphi(T_0) - \varphi(0) = \pi q\}$$

is a smooth curve, which for example can be parametrized as $(a, b(a))$, $a \in (a_0, 1)$, where a_0 is depending on q . If $T_0(a)$ denotes the principal period of $|v|$ for the initial datae $(a, b(a))$,

then $\lim_{a \nearrow 1} T_0(a) = \infty$ and this curve raises a smooth curve of periodic solutions of (1)+(4), with principal periods $T(a) = nT_0(a)$ (if m is even) or $T(a) = 2nT_0(a)$ (if m is odd).

Actually, the diagram of bifurcation of the distinguished solutions is given by Picture 1.

For the instant, we do not know whether the curves $q = \text{const.}$ are like a) or like b) in Picture 1. In other words, we do not know whether T increases or not along these curves. If the first possibility holds, the minimum number of solutions given by (38) is the exact one. After the proof of the theorem, we shall give a sufficient condition for this happens (see the Remarks following the proof).

Finally, the last paragraph is devoted to the existence, in the whole \mathbf{R}^2 , of 2-periodic solutions which are geometrically distinct to the real ones. Some existence and non-existence results are obtained.

3. Proof of Theorem

Let us note first that

$$(5) \quad a \leq 1.$$

Suppose the contrary. Let $M > 1$ be such that

$$\min |v| < M < \max |v|.$$

Let I be an interval such that $|v| > M$ in I and $|v| = M$ on ∂I . (Note that such an interval is necessarily finite). Since

$$(|v|^2)'' \geq 2|v|^2(|v|^2 - 1) > 0$$

in I , it follows that $|v| \leq M$ in I , which contradicts our choice of I .

Next we shall prove that

$$(6) \quad b^2 \leq a^2(1 - a^2).$$

Indeed, for small x we have

$$v_1(x) = a - \frac{a(1 - a^2)}{2}x^2 + O(x^3),$$

$$v_2(x) = bx + O(x^3),$$

so that (6) follows from the fact that $x = 0$ is a local maximum.

Now let

$$\Omega = \{(a, b) \in (0, 1] \times [0, 1]; \quad b^2 \leq a^2(1 - a^2)\}.$$

We have obtained that if (1)+(4) raises a non-null periodic solution such that $x = 0$ is a local maximum, then necessarily $(a, b) \in \Omega$.

We shall first study the case $(a, b) \in \partial\Omega$.

Case 1 If $b = a\sqrt{1 - a^2}$, it follows that

$$v(x) = ae^{ikx} \quad , \text{ where } k = \sqrt{1 - a^2}.$$

Indeed, (2) provides a solution for (1)+(4) in this case.

Case 2 If $b = 0$, one gets easily that $v_2 = 0$. If $a = 1$, we get the trivial solution $v(x) \equiv 1$, so that in what follows we shall assume that $a \in (0, 1)$.

Note first that v_1 cannot be positive (negative) into an infinite interval if v is periodic. For, otherwise, v_1 would be a periodic concave (convex) function, that is a constant function. This is impossible for our choice of a and b .

Let x_1, x_2 be two consecutive zeros of v_1 . We may suppose that $v(x) > 0$ if $x_1 < x < x_2$, so that $v'(x_1) > 0$, $v'(x_2) < 0$. If x_3 is the smallest $x > x_2$ such that $v(x_3) = 0$, it follows that $v(x) < 0$ if $x_2 < x < x_3$.

If we prove the fact that $x_2 - x_1 > \pi$, it will also follow that $x_3 - x_1 > 2\pi$ and that there is no $x \in (x_1, x_3)$ such that $v(x) = 0$ and $v'(x) > 0$. We will get that the principal period of v must be $> 2\pi$. This will be done in

Lemma 1. *Let $f : \mathbf{R} \rightarrow [0, 1]$ be such that the set $\{x; f(x) = 0 \text{ or } f(x) = 1\}$ contains only isolated points. Let v be a real function such that $v(x_1) = v(x_2) = 0$, and $v(x) > 0$ in (x_1, x_2) . If, for $x \in [x_1, x_2]$,*

$$(7) \quad -v'' = vf,$$

then $x_2 - x_1 > \pi$.

Proof. We may assume that $x_1 = 0$. Multiplying (7) by $\varphi(x) := \sin \frac{\pi x}{x_2}$ and integrating by parts, we obtain that

$$\int_0^{x_2} v\varphi > \int_0^{x_2} vf\varphi = \left(\frac{\pi}{x_2}\right)^2 \int_0^{x_2} v\varphi,$$

that is $x_2 > \pi$. □

Incidentally, this proves i) of the Theorem.

Returning to the Case 2, we shall explicitly integrate (1)+(4) as one usually does for the Weierstrass Elliptic Functions. Multiplying (1) by v'_1 , we find

$$(8) \quad v_1'^2 = -v_1^2 + \frac{1}{2}v_1^4 + a^2 - \frac{1}{2}a^4.$$

It follows that, as far as the solution of (1)+(4) exists, we have $|v_1| \leq a$ and $|v'_1| \leq \sqrt{a^2 - \frac{1}{2}a^4}$. Hence the solution of (1)+(4) is globally defined.

Note that $v'_1(0) = 0$, $v''_1(0) < 0$, so that v_1 decreases for small $x > 0$. Moreover, $v'_1(x) < 0$ for $0 < x < \tau$, where

$$\tau = \sup\{x > 0; \ v_1(y) > 0 \text{ for all } 0 < y < x\}.$$

Indeed, suppose the contrary. Then, taking (8) into account, we obtain the existence of some $\tau_0 > 0$ such that $v_1(\tau_0) = a$, $\tau_0 < \tau$. If we consider the smallest $\tau_0 > 0$ such that the above equality occurs, we have $v_1(x) < a$ if $0 < x < \tau_0$. Since $v_1(0) = v_1(\tau_0) = a$, it follows that there exists some $0 < \tau_1 < \tau_0$ such that $v'_1(\tau_1) = 0$, which is the desired contradiction. Hence we have

$$(9) \quad v'_1 = -\sqrt{a^2 - \frac{1}{2}a^4 - v_1^2 + \frac{1}{2}v_1^4} < 0 \quad \text{in } (0, \tau).$$

It follows that, if $0 < x < \tau$, then

$$\int_{v(x)}^a \frac{1}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} dt = x,$$

which gives

$$(10) \quad \tau = \int_0^a \frac{dt}{\sqrt{\frac{1}{2}t^4 - t^2 + a^2 - \frac{1}{2}a^4}} := \tau(a).$$

From (1), we obtain $v_1(\tau + x) = -v_1(\tau - x)$, $v_1(2\tau - x) = -v_1(x)$, $v_1(4\tau + x) = v_1(x)$, so it is easy to see that v is periodic of principal period $T(a) = 4\tau(a)$.

Now (10) can be rewritten as

$$(11) \quad \tau(a) = \int_0^1 \frac{1}{\sqrt{(1 - \xi^2)[1 - \frac{a^2}{2}(1 + \xi^2)]}} d\xi,$$

so that τ increases with a and

$$\lim_{a \searrow 0} \tau(a) = \frac{\pi}{2}, \quad \lim_{a \nearrow 1} \tau(a) = +\infty.$$

Since $\tau'(a) > 0$, it follows that the mapping

$$T(a) \longmapsto a := a(T)$$

is analytic, so that ii) is completely proved. Moreover,

$$\lim_{T \searrow 2\pi} a(T) = 0 \quad \text{and} \quad \lim_{T \nearrow \infty} a(T) = 1,$$

so that the diagram of “real” solutions is that depicted in Picture 1.

Next we return to the points (a, b) which are interior to Ω .

Case 3 Let $(a, b) \in \text{Int}\Omega$.

Write, for small x ,

$$(12) \quad v(x) = e^{i\varphi(x)} w(x)$$

with $\varphi(0) = 0$ and $w > 0$.

One can easily see that w satisfies

$$(13) \quad -w'' = w(1 - w^2) - \frac{a^2 b^2}{w^3}$$

and

$$(14) \quad \begin{cases} w(0) = a \\ w'(0) = 0, \end{cases}$$

while φ is given by

$$(15) \quad \varphi' = \frac{ab}{w^2}, \quad \varphi(0) = 0.$$

Hence, if the system (13)+(14) has a global positive solution, it follows that (12) is global. Moreover, if w is periodic of period T_0 , then

$$(16) \quad v(nT_0 + x) = e^{in\varphi(T_0)} e^{i\varphi(x)} w(x) \quad \text{for } 0 \leq x < T_0, \quad n = 0, 1, \dots$$

so that (1)+(4) gives a periodic solution if and only if $\varphi(T_0) \in \pi$.

We shall prove the global existence in

Lemma 2. *If $(a, b) \in \text{Int } \Omega$, then (13)+(14) have a global positive periodic solution.*

Proof. Note that the assumption made on (a, b) implies that $w''(0) < 0$, so that, multiplying as above (13) by w' , we obtain , for small $x > 0$,

$$(17) \quad w'^2 = -w^2 + \frac{1}{2}w^4 - \frac{a^2b^2}{w^2} + a^2 - \frac{1}{2}a^4 + b^2$$

and

$$(18) \quad w' = -\sqrt{-w^2 + \frac{1}{2}w^4 - \frac{a^2b^2}{w^2} + a^2 - \frac{1}{2}a^4 + b^2}.$$

Now (17) implies that w and w' are bounded as far as the solution exists and, moreover, that

$$\inf\{w(x); \ w \text{ exists}\} > 0.$$

It follows that w is a global solution. Let

$$\tau = \sup\{x > 0; \ w'(y) < 0 \text{ for all } 0 < y < x\}.$$

Note that (18) is valid if $0 < x < \tau$.

Let c be the only root of

$$f(x) := -x^2 + \frac{1}{2}x^4 - \frac{a^2b^2}{x^2} + a^2 - \frac{1}{2}a^4 + b^2 = 0$$

which is positive and inferior to a .

Since $f(x) < 0$ if $0 < x < c$ or $x > a$, x close to a , it follows from (17) that

$$(19) \quad c \leq w(x) \leq a \quad \text{for all } x \in \mathbf{R}.$$

Claim 1. $\lim_{x \nearrow \tau} w(x) = c$.

Proof of Claim 1. If $\tau < \infty$, it follows that $w'(\tau) = 0$. Now (17) together with the definitions of τ and c show that $w(\tau) = c$. If $\tau = \infty$, then we have $\lim_{x \rightarrow \infty} w(x) \geq c$. If we would have $\lim_{x \rightarrow \infty} w(x) > c$, there would exist a constant $M > 0$ such that $w'(x) \leq -M$ for each $x > 0$. The latest inequality contradicts (19) for large x .

As we did before, for $0 < x < \tau$, (18) gives

$$(20) \quad x = \int_{w(x)}^a \frac{1}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}} dt,$$

so that

$$(21) \quad \tau = \int_c^a \frac{1}{\sqrt{-t^2 + \frac{1}{2}t^4 - \frac{a^2b^2}{t^2} + a^2 - \frac{1}{2}a^4 + b^2}} dt < \infty.$$

It follows by a reflection argument that $w(2\tau) = w(0) = a$, $w'(2\tau) = w'(0) = 0$, so that w is (2τ) -periodic. \square

Next, in order to make simpler the computations that follow, it is useful to replace the (a, b) -coordinates into other ones, by associating to (a, b) the point (A, C) , where $A = a^2$, $C = c^2$ with a, c as above. This changement of coordinates maps Ω analytically into

$$\omega := \{(A, C); \ 0 < C < A, \ 2A + C < 2\}$$

(see Picture 2).

It follows from the above discussion that to each $(A, C) \in \omega$ it corresponds a solution (w, φ) of (13)-(15) such that w and φ' are periodic of period given by (after a suitable change of variables)

$$(22) \quad T_0 = T_0(A, C) = 2\sqrt{2} \int_0^\infty \frac{1}{\sqrt{(y^2 + 1)[(2 - 2A - C)y^2 + (2 - A - 2C)]}} dy.$$

Moreover, $\varphi(0) = 0$ and

$$(23) \quad \varphi(T_0) = \sqrt{2AC(2 - A - C)} \int_0^{\tau(A, C)} \frac{1}{w^2(y)} dy,$$

where $\tau(A, C) = \frac{1}{2}T_0(A, C)$.

Now the change of variables $w(y) = t$ yields, with $\varphi(A, C) := \varphi(T_0(A, C))$,

$$(24) \quad \varphi(A, C) = \sqrt{2AC(2 - A - C)} \int_0^\infty \sqrt{\frac{y^2 + 1}{(2 - 2A - C)y^2 + (2 - A - 2C)}} \cdot \frac{1}{Ay^2 + C} dy,$$

and (22), (24) show that $(A, C) \mapsto (T_0, \varphi)$ is an analytic map. Moreover, (22) gives that

$$(25) \quad T_0 > \pi, \quad \lim_{(A, C) \rightarrow (0, 0)} T_0(A, C) = \pi, \quad \inf_{|(A, C)| \geq \varepsilon > 0} T_0(A, C) > \pi.$$

A lower estimate for φ will be given in

Lemma 3. $\varphi > \frac{\pi}{\sqrt{2}}$ and $\lim_{(A, C) \rightarrow (0, 0)} \varphi(A, C) = \frac{\pi}{\sqrt{2}}.$

Proof. If we put $y = \sqrt{\frac{C}{A}}z$ in (24), we obtain

$$(26) \quad \varphi(A, C) = \sqrt{2(2 - A - C)} \int_0^\infty \sqrt{\frac{Cz^2 + A}{C(2 - 2A - C)z^2 + A(2 - A - 2C)}} \frac{1}{z^2 + 1} dz,$$

so that the second assertion follows from the Lebesgue Dominated Convergence Theorem.

For the first one, it is enough to show that for given $0 < k < 1$, the function

$$(0, \frac{2}{k+2}) \ni A \xrightarrow{\psi} \varphi(A, kA)$$

is increasing.

After a short computation, we find that

$$(27) \quad \psi'(A) = \frac{2k}{\sqrt{2 - (k+1)A}} \int_0^\infty \sqrt{\frac{ky^2 + 1}{[k(2 - (k+2)A)y^2 + (2 - (2k+1)A)]^3}} dy > 0.$$

□

Incidentally, this shows that φ has no critical points and that the level curves $\varphi = \text{const.}$ are analytic and can be parametrized as

$$(28) \quad (A(k), kA(k)).$$

Lemma 4. $\lim_{A \nearrow \frac{2}{k+2}} \psi(A) = \infty.$

Proof. It follows from (26) that

$$\psi(A) > \sqrt{2(2 - \frac{2(k+1)}{k+2})} \int_0^\infty \sqrt{\frac{kx^2 + 1}{k(2 - (k+2)A)x^2 + 2 - (2k+1)A}} \frac{dx}{x^2 + 1},$$

and the last integral tends monotonically to $+\infty$ by the Beppo Levi Theorem. □

From the above Lemma, we obtain that the parametrization (28) is valid for $k \in (0, 1)$. Moreover, (27) shows that the mapping

$$(29) \quad k \longmapsto A(k)$$

is analytic. Of course, the level line $\varphi = \text{const.}$ is non-void if and only if $\text{const.} > \frac{\pi}{\sqrt{2}}$. This will be assumed in the sequel. We shall prove that (29) provides a decreasing mapping.

Indeed, if we consider now ψ as $\psi(A, k)$, then it follows from (27) that $\frac{\partial \psi}{\partial A}$ increases with k . Hence, if $k_1 < k_2$, then

$$\psi(A, k_1) < \psi(A, k_2),$$

that is $A(k)$ decreases with k .

We obtain the existence of

$$\lim_{k \nearrow 1} A(k) := A_0 \quad \text{and} \quad \lim_{k \searrow 0} A(k) := A_1 > A_0,$$

From Lemma 3, $A_0 > 0$.

Claim 2. $A_1 = 1$.

Proof of Claim 2. It follows from (26) and the Lebesgue Dominated Convergence Theorem that

$$\lim_{(A,C) \rightarrow (A_2,0)} \varphi(A, C) = \frac{\pi}{\sqrt{2}} \quad \text{if } 0 < A_2 < 1,$$

so that, taking Lemma 3 into account, we obtain that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varphi(A, C) < \frac{\pi}{\sqrt{2}} + \varepsilon$$

if $0 < A < 1 - \delta$, $0 < C < \delta$. This completes the proof of the claim. \square

At this stage of the proof, we know that the level lines $\varphi = \text{const.}$ are analytic, all of them “end” at $(1,0)$ and “begin” at (A_0, A_0) for some suitable $0 < A_0 < 1$, A_0 depending on the constant. Moreover, if $q_1 < q_2$, the line $\varphi = q_1$ lies below the line $\varphi = q_2$ (see Picture 3).

Now A_0 can be found implicitly, because φ can be extended by continuity on the line segment MN . This shows that

$$q = \varphi(A_0, A_0) = \frac{\pi}{2} \sqrt{\frac{1 - A_0}{2 - 3A_0}},$$

that is

$$(30) \quad A_0 = A_0(q) = \frac{8q^2 - \pi^2}{12q^2 - 3\pi^2}.$$

Returning to the proof of the theorem, note that iii) and iv) follow easily from the above calculation. Indeed, for small A and C , if $\varphi(A, C) = \pi \frac{m}{n}$ is a rational multiple of π , then $n \geq 4$, so that, taking into account the fact that $T_0(A, C) \geq \pi$, it follows that for small A the period of v is at least 4π . Now the existence of T_1 follows from (25).

In order to prove v), note that the level line $\varphi = q$ contains a T -periodic solution if and only if

$$(31) \quad \begin{cases} q = \pi \frac{m}{n}, (m, n) = 1 & \text{and there exists } (A, C) \text{ on the level line} \\ \text{such that} & T_0(A, C) = \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd} \end{cases} \end{cases}.$$

We shall prove that

$$(32) \quad \lim_{2A+C \nearrow 2} T_0(A, C) = \infty.$$

Suppose (32) proved for the moment. Obviously, if $\varphi(A_n, C_n) \rightarrow \infty$, then $2A_n + C_n \rightarrow 2$. It follows from (32) that, for q large enough, $T_0(A, C) > T$ if (A, C) is on the level line $\varphi = q$, so that (31) cannot hold for such q . Hence, in order to prove v) it remains to show that, for given q, T_0 , the set

$$\mathcal{M} = \{(A, C); \varphi(A, C) = q, T_0(A, C) = T_0\}$$

is finite.

Let

$$\mathcal{C}_1 = \{(A, C); \varphi(A, C) = q\}.$$

Since \mathcal{C}_1 is an analytic curve, \mathcal{M} is finite provided that $(1, 0)$ and $(A_0(q), A_0(q))$ are not cluster points of \mathcal{M} . For $(1, 0)$, this follows from the fact that, according to (32), $T_0(A, C)$ approaches $+\infty$ as A approaches 1 along \mathcal{C}_1 . In particular, $T_0(A, C)$ is not constant along \mathcal{C}_1 . In order to see what happens in $(A_0(q), A_0(q))$, we perform the following trick: let

$$\omega_1 = \omega \cup \{(C, A); (A, C) \in \omega\} \cup \{(A, A); 0 < A < 1\}$$

(see Picture 4).

Obviously, (24) extends φ to an analytic function φ_1 in ω_1 . The change of variables $z = \frac{1}{y}$ in (24) shows that $\varphi(A, C) = \varphi(C, A)$. Note also that (27) continues to hold for $k = 1$. This shows that φ_1 has no critical points and that $T_0(A, C)$ tends to $+\infty$ at the both ends of $\varphi_1 = \text{const}$. Hence, φ can assume the same value only a finite number of times.

All it remains to do is

Proof of (32). Let $A_n < 1, 0 < C_n < 1$ be such that $2A_n + C_n \nearrow 2$. Then

$$(33) \quad T_0(A_n, C_n) > 2\sqrt{2} \int_0^\infty \frac{dy}{\sqrt{(y^2 + 1)[(2 - 2A_n - C_n)y^2 + 2]}},$$

and the right hand side of (33) tends to $+\infty$ from the Beppo Levi Theorem. \square

The proof of v) is completed.

Next we return to the proof of vi). Take $q = \pi \frac{m}{n}$, $(m, n) = 1$, $\frac{m}{n} > \frac{1}{\sqrt{2}}$. Then the level line $\varphi = q$ is nonempty and smooth. If we put

$$(34) \quad T_0(q) = 2\sqrt{2} \int_0^\infty \frac{dy}{\sqrt{(y^2 + 1)[(2 - 3A_0(q))y^2 + (2 - 3A_0(q))]} = \pi \sqrt{\frac{24q^2 - 6\pi^2}{16q^2 - 5\pi^2}},$$

it follows from (32) that, along $\varphi = q$, T_0 assumes all the values between $T_0(q)$ and $+\infty$. We obtain that, for fixed T , (1)+(4) has at least one T -periodic solution corresponding to each q such that

$$(35) \quad q = \pi \frac{m}{n}, \quad (m, n) = 1, \quad \frac{m}{n} > \frac{1}{\sqrt{2}}, \quad T_0(q) < \begin{cases} \frac{T}{n}, & \text{if } m \text{ is even} \\ \frac{T}{2n}, & \text{if } m \text{ is odd} \end{cases}.$$

Hence it suffices to count, for large T , the number of elements of $A \cup B$, where

$$(36) \quad A = \{(m, n); (m, n) = 1, m \text{ is even}, \frac{m}{n} > \frac{1}{\sqrt{2}}, 24m^2n^2 - 6n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\}$$

and

$$(37) \quad B = \{(m, n); (m, n) = 1, m \text{ is odd}, \frac{m}{n} > \frac{1}{\sqrt{2}}, 96m^2n^2 - 24n^4 < (16m^2n^2 - 5n^2)\pi^2T^2\}.$$

Note that

$$A \cup B \supset \{(m, n); (m, n) = 1, m \geq n, m \leq \sqrt{\frac{5}{24}}\pi T\}.$$

It follows that there are at least

$$(38) \quad \sum_{1 \leq m \leq \sqrt{\frac{5}{24}}\pi T} \Phi(m)$$

solutions, where Φ is the Euler's Function. Now a Theorem of Mertens (see [Ch]) asserts that the sum in (38) is

$$(39) \quad \frac{5}{8}T^2 + O(T \log T).$$

The proof of the Theorem is completed. \square

Remarks: 1) It is obvious that (38) does not provide an accurate estimate. On the other hand, one may see that the number of elements of $A \cup B$ is $O(T^2)$.

2) (35) counts all the T -periodic solutions if and only if T_0 increases along $\varphi=q=\text{const.}$ as far as A increases from $A_0(q)$ to 1. A sufficient condition is that $(A, C) \mapsto (T_0(A, C), \varphi(A, C))$ is a local diffeomorphism. This relies on the following fact: let ω be an open connected set of \mathbf{R}^2 and $f : \omega \rightarrow \mathbf{R}^2$ a local diffeomorphism. If the level lines $f_2=\text{const.}$ are connected, then $f : \omega \rightarrow f(\omega)$ is a global diffeomorphism.

3) It follows from the proof that the diagram of bifurcation is, indeed, as in Picture 1. For example, the level line $\varphi = q$, $q = \pi \frac{m}{n}$, raises a branch of periodic solutions which starts from a solution of the form (2). Note that, on a level line, the solutions oscillate more and more as $A \nearrow 1$, in the sense that $\max |v|$ and $\min |v|$ approach 1 and 0 as A approaches 1. It is also easy to see that, in Picture 1, the points T_1, T_2, T_3, \dots are isolated.

4) One may prove that, if $a = \max |v|$ for a T -periodic solution, then

- i) $a^2 + (\frac{2\pi}{T})^2 = 1$ if v is given by (b);
- ii) $a^2 + (\frac{2\pi}{T})^2 > 1$ if v is a real solution;
- iii) $a^2 + (\frac{2\pi}{T})^2 < 1$ if v is a “complex” solution.

5) We have seen that the solution of (1)+(4) is globally existent if $(A, C) \in \bar{\omega}$. The same happens if $(A, C) \in \bar{\omega}_1$. There is nothing surprising in this, because starting with some $(A, C) \in \omega_1 \setminus \omega$ means considering the “canonical form” of (1) with $x = 0$ a local minimum, this time.

Let Ω_1 be the inverse image of ω_1 with respect to the mapping $(a, b) \mapsto (A, C)$. Considering some point (a, b) , $a \geq 0$, $b \geq 0$ such that $(a, b) \notin \bar{\Omega}_1$, it is easy to carry out once again (13)-(21) in order to prove that this time v has a finite left or right life time.

4. Existence of non-trivial periodic solutions in \mathbf{R}^2

We are concerned with the existence of double periodic solutions, that is of functions $u : \mathbf{R}^2 \rightarrow \mathbf{C}$ solutions of

$$(40) \quad -\Delta u = u(1 - |u|^2), \quad u \in L_{loc}^2(\mathbf{R}^2),$$

such that there exist $\omega_1, \omega_2 \in \mathbf{R}^2$ linearly independent with

$$(41) \quad u(x + \omega_j) = u(x), \quad j = 1, 2.$$

Of course, we have already obtained such solutions: take $\omega_1 = (2T, 0)$ with $T > \pi$, ω_2 arbitrary and u a $2T$ -periodic real solution. Even simpler, one may take $u = \text{const.}$, $|u| = 0$ or 1 .

Therefore, we shall look for non-trivial solutions, that is solutions enjoying the property

$$(42) \quad \begin{cases} \text{there is no } v : \mathbf{R} \rightarrow \mathbf{C} \text{ solution of (1) such that} \\ u(x) = v(\alpha_1 x_1 + \alpha_2 x_2) \text{ for some } \alpha \in \mathbf{C}, |\alpha| = 1. \end{cases}$$

We start with a non-existence result.

Proposition 1. *If $|\omega_1|, |\omega_2|$ are small enough, all the solutions of (40)-(41) are constant.*

We shall use in the proof

Lemma 5. *Let u be a solution of (40)-(41). Then $|u| \leq 1$ (so that u is smooth).*

Proof of Lemma 5. We follow an idea from [BMR]. It follows easily from (40) that $u \in H_{loc}^1(\mathbf{R}^2)$. Let

$$P = \{\lambda\omega_1 + \mu\omega_2; \quad 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1\}.$$

Let φ be a $C_0^\infty(\mathbf{R}^2)$ -function such that $\varphi \geq 0$, $\varphi = 1$ in a neighborhood of 0, and $\varphi_n(x) = \frac{1}{n^2} \varphi(\frac{x}{n})$ for $n = 1, 2, \dots$

Multiplying (40) with $u(|u|^2 - 1)^+ \varphi_n$ and integrating by parts, we get, as $n \rightarrow \infty$,

$$\int_{P \cap [|u| \geq 1]} |\nabla u|^2 (|u|^2 - 1) + \int_{P \cap [|u| \geq 1]} |\nabla |u|^2|^2 \leq - \int_{P \cap [|u| \geq 1]} |u|^2 (|u|^2 - 1)^2,$$

that is $|u| \leq 1$ a.e. It follows that $u \in L^\infty$, so that u may be supposed smooth. \square

Proof of Proposition 1. Let $(\varphi_n)_{n \geq 0}$ be an orthonormal basis of eigenfunctions of $-\Delta$ in $H_p^1(P)$ (here “ p ” means periodic conditions on ∂P) with corresponding eigenvalues $(\lambda_n)_{n \geq 0}$. We may suppose $\varphi_0 = 1$, so that $\lambda_n > 0$ for all $n \geq 1$. If $|\omega_1|, |\omega_2|$ are small enough, then $\lambda_n > 2$ if $n \geq 1$.

Let u be a solution of (40)-(41) and write

$$u = \sum c_n \varphi_n, \quad u|u|^2 = \sum d_n \varphi_n.$$

Integrating (40) over P , we find that $c_0 = d_0$. Multiplying (40) by φ_n , $n \geq 1$ and integrating we obtain, if $d_n \neq 0$,

$$|d_n| = (\lambda_n - 1)|c_n| > |c_n|.$$

Since $|u| \leq 1$, we have

$$\int_P |u|^2 \geq \int_P |u|^6,$$

that is

$$\sum |c_n|^2 \geq \sum |d_n|^2.$$

Examining these formulae, we see that $c_n = d_n = 0$ if $n \geq 1$, that is u is constant. \square

Concerning the existence of solutions of (40)-(42), we have been able to prove it if P is a rectangle large enough.

Proposition 2. *Let P be large enough such that the first eigenvalue of $-\Delta$ in $H_0^1(R)$ is inferior to 1, where $R = \frac{1}{2}P$.*

Then (40)-(42) has solutions.

Proof. Let

$$J : H_0^1(R) \rightarrow \mathbf{R}, \quad J(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{4} \int (1 - |u|^2)^2$$

Then J is a C^1 -function (see [BN]), even, bounded from below. It is not difficult to see that it satisfies the (PS)-condition:

(PS) if $(u_n) \subset H_0^1(R)$ is such that $(J(u_n))$ is bounded and $J'(u_n) \rightarrow 0$ in $H^{-1}(R)$, then (u_n) is relatively compact in $H_0^1(R)$.

Now $J(0) = \frac{|R|}{4}$ and, if φ_1 is the first eigenfunction of $-\Delta$ in $H_0^1(R)$, then $J(\varepsilon \varphi_1) < J(0)$ for small ε .

More generally, if the k -th eigenvalue is inferior to 1, one can easily see that there is some $R > 0$ such that $J(u) < J(0)$ if $u \in \text{Sp}\{\varphi_1, \dots, \varphi_k\}$ and $\|u\| = R$. Here φ_j denotes the eigenfunction corresponding to the k -th eigenvalue.

It follows from Theorem 8.10 in [R] that J has at least k pairs $(u_j, -u_j)$ of critical points which are different from 0. Let u_0 be a critical point of J in R . Suppose $R = (0, a) \times (0, b)$. Define $u : P \rightarrow \mathbf{C}$ by

$$u(x') = u(x'') = -u_0(x), \quad u(x''') = u_0(x),$$

where $x = (x_1, x_2)$, $x' = (2a - x_1, x_2)$, $x'' = (x_1, 2b - x_2)$, $x''' = (2a - x_1, 2b - x_2)$.

It is obvious that u satisfies (41). It is not hard to see that u_0 is regular (see [G]). It follows then by a simple calculation that u satisfies (40).

Finally, suppose (42) does not hold. Let $\vec{\beta} = (\alpha_2, -\alpha_1)$ where $\alpha = \alpha_1 + i\alpha_2$ is as in (42). Then u must be constant along each parallel to $\vec{\beta}$. Since any such line intersects the grid generated by P , it follows that $u \equiv 0$, which is not the case. \square

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ON THE GINZBURG-LANDAU ENERGY WITH WEIGHT

Cătălin LEFTER and Vicențiu D. RĂDULESCU

Abstract. We study the behavior as $\varepsilon \rightarrow 0$ of minimizers (u_ε) of the Ginzburg-Landau energy E_ε^w with the weight w . We prove the convergence (up to a subsequence) to a harmonic map whose singularities have degree $+1$. We also find the expression of the renormalized energy and deduce that the configuration of singularities is a minimum point of this functional. Our work is motivated by a problem raised by F. Bethuel, H. Brezis and F. Hélein in [4].

Sur l'énergie de Ginzburg-Landau avec poids

Résumé. On étudie le comportement quand $\varepsilon \rightarrow 0$ des minimiseurs (u_ε) de l'énergie de Ginzburg-Landau E_ε^w avec le poids w . On montre la convergence (à une sous-suite près) vers une application harmonique dont les singularités ont les degrés $+1$. On trouve aussi l'expression de l'énergie renormalisée et on déduit que la configuration des singularités est un point de minimum de cette fonctionnelle. Notre travail est motivé par un problème posé par F. Bethuel, H. Brezis et F. Hélein dans [4].

Version française abrégée. Soit G un ouvert borné, régulier et simplement connexe dans \mathbf{R}^2 . On fixe une condition aux limites $g : \partial G \rightarrow S^1$ telle que $d = \deg(g, \partial G) > 0$. Soit $w \in C^1(\overline{G}, \mathbf{R})$, $w > 0$ dans \overline{G} . On considère l'énergie de Ginzburg-Landau avec le poids w :

$$E_\varepsilon^w(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2 w, \quad \varepsilon > 0,$$

définie pour tout $u \in H^1(G; \mathbf{R}^2)$. Soit u_ε un minimiseur de E_ε^w dans la classe

$$H_g^1(G; \mathbf{R}^2) = \{u \in H^1(G; \mathbf{R}^2); u = g \text{ sur } \partial G\}.$$

Pour caractériser le comportement des minimiseurs dans le cas $w \equiv 1$, ainsi que la configuration limite, F. Bethuel, H. Brezis et F. Hélein ont défini (voir [2],[4]) l'énergie renormalisée par

$$W(b, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |b_i - b_j| + \frac{1}{2} \int_{\partial G} \Phi_0(g \wedge g_\tau) - \pi \sum_{j=1}^k d_j R_0(b_j) ,$$

où $b = (b_1, \dots, b_k)$ est une configuration de k points distincts dans G de degrés $\bar{d} = (d_1, \dots, d_k)$, avec $d = d_1 + \dots + d_k$. Les applications Φ_0 et R_0 sont définies de manière unique par

$$(1) \quad \begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^k d_j \delta_{b_j} , & \text{dans } G \\ \frac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau , & \text{sur } \partial G \\ \int_{\partial G} \Phi_0 = 0 \end{cases}$$

et

$$(2) \quad R_0(x) = \Phi_0(x) - \sum_{j=1}^k d_j \log |x - b_j| .$$

On désigne par $W(b)$ l'énergie renormalisée quand tous les degrés sont égaux à $+1$.

Théorème 1. *Il existe une suite $\varepsilon_n \rightarrow 0$ et exactement d points a_1, \dots, a_d dans G tels que*

$$u_{\varepsilon_n} \rightarrow u_\star \quad \text{dans } H_{\text{loc}}^1(\bar{G} \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2) ,$$

où u_\star est l'application harmonique canonique associée aux singularités a_1, \dots, a_d de degrés $+1$ et à la donnée au bord g .

De plus, $a = (a_1, \dots, a_d)$ minimise la fonctionnelle

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^d \log w(b_j)$$

parmi toutes les configurations $b = (b_1, \dots, b_d)$ de d points distincts dans G .

On a

$$\lim_{n \rightarrow \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d |\log \varepsilon_n|\} = W(a) + \frac{\pi}{2} \sum_{j=1}^d \log w(a_j) + d\gamma ,$$

où γ est une constante universelle.

Théorème 2. *Soit*

$$W_n = \frac{1}{4\varepsilon_n^2} (1 - |u_{\varepsilon_n}|^2)^2 w.$$

Alors la suite (W_n) converge dans la topologie faible \star de $C(\overline{G})$ vers

$$W_\star = \frac{\pi}{2} \sum_{j=1}^d \delta_{a_j}.$$

Let G be a smooth, simply connected domain in \mathbf{R}^2 and $w \in C^1(\overline{G}, \mathbf{R})$, $w > 0$ in \overline{G} . We consider the Ginzburg-Landau energy with the weight w

$$E_\varepsilon^w(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2 w,$$

where:

- a) $\varepsilon > 0$ is a (small) parameter.
- b) $g : \partial G \rightarrow S^1$ is a smooth data with a topological degree $d > 0$.

Studying the behavior of minimizers u_ε of E_ε^w in the case $w \equiv 1$, F. Bethuel, H. Brezis and F. Hélein have proved (see [2], [4]) that there exists d points a_1, \dots, a_d in G such that (up to a subsequence) $u_{\varepsilon_n} \rightarrow u_\star$ in $C_{\text{loc}}^k(\overline{G} \setminus \{a_1, \dots, a_d\})$, where u_\star is the canonical harmonic map associated to g and $a = (a_1, \dots, a_d)$. In order to locate the singularities at the limit, they have defined the renormalized energy associated to a given configuration $b = (b_1, \dots, b_k)$ of distinct points in G with associated degrees $\bar{d} = (d_1, \dots, d_k)$, $d_1 + \dots + d_k = d$ by

$$W(b, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |b_i - b_j| + \frac{1}{2} \int_{\partial G} \Phi_0(g \wedge g_\tau) - \pi \sum_{j=1}^k d_j R_0(b_j),$$

where Φ_0 is the unique solution of

$$(1) \quad \begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^k d_j \delta_{b_j}, & \text{in } G \\ \frac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau, & \text{on } \partial G \\ \int_{\partial G} \Phi_0 = 0 \end{cases}$$

and

$$(2) \quad R_0(x) = \Phi_0(x) - \sum_{j=1}^k d_j \log |x - b_j| \quad .$$

We shall denote by $W(a)$ the renormalized energy when $k = d$ and all degrees equal $+1$. F. Bethuel, H. Brezis and F. Hélein have proved in [4] that the functional W is related to the asymptotic behavior of minimizers u_ε as follows:

$$(3) \quad \lim_{\varepsilon \rightarrow 0} \{E_\varepsilon(u_\varepsilon) - \pi d |\log \varepsilon|\} = W(a, \bar{d}, g) + d\gamma,$$

where γ is an universal constant, $d_i = 1$ for all i and the configuration (a_1, \dots, a_d) achieves the minimum of W .

This work is motivated by the Open Problem 2, p. 137 in [4]. We are concerned with the study of the convergence of minimizers of E_ε^w , as well as with the corresponding expression of the renormalized energy. We prove that the behavior of minimizers is of the same type as in the case $w \equiv 1$, the change appearing in the expression of the renormalized energy and, consequently, in the location of singularities of the limit u_\star of u_ε . Our Theorem 2 generalizes another result of F. Bethuel, H. Brezis and F. Hélein concerning the behavior of u_ε . We then prove in Theorem 3 a vanishing gradient property for the configuration of singularities obtained at the limit. The last theorem is devoted to a description of the renormalized energy by the “shrinking holes” method which was developed in [4], Chapter I.

Theorem 1. *There is a sequence $\varepsilon_n \rightarrow 0$ and exactly d points a_1, \dots, a_d in G such that*

$$u_{\varepsilon_n} \rightarrow u_\star \quad \text{in } H_{\text{loc}}^1(\bar{G} \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2),$$

where u_\star is the canonical harmonic map associated to the singularities a_1, \dots, a_d of degrees $+1$ and to the boundary data g .

Moreover, $a = (a_1, \dots, a_d)$ minimizes the functional

$$\widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^d \log w(b_j)$$

among all configurations $b = (b_1, \dots, b_d)$ of d distinct points in G .

In addition we have

$$\lim_{n \rightarrow \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d |\log \varepsilon_n|\} = W(a) + \frac{\pi}{2} \sum_{j=1}^d \log w(a_j) + d\gamma,$$

where γ is some universal constant, the same as in (3).

Remark. The functional \widetilde{W} may be regarded as the renormalized energy corresponding to the energy E_ε^w .

If $c, \varepsilon, \eta > 0$ are constant, let

$$I(\varepsilon, \eta) = \min\{E_\varepsilon(u); u \in H^1(B_\eta(0); \mathbf{R}^2) \text{ and } u(x) = \frac{x}{\eta} \text{ on } \partial B_\eta(0)\}.$$

For $x \in G$, denote

$$M_\eta(x) = \sup_{B(x, \eta) \cap \overline{G}} w \quad \text{and} \quad m_\eta(x) = \inf_{B(x, \eta) \cap \overline{G}} w.$$

Sketch of the proof. The first part of the conclusion may be obtained by adapting the techniques developed in [1], [2], [3], [4] taking into account the estimate

$$(4) \quad \frac{1}{\varepsilon_n^2} \int_G (1 - |u_{\varepsilon_n}|^2)^2 w \leq C,$$

which is deduced by using the ideas in [6].

The proof of the second part of the theorem is divided into 3 steps:

Step 1. *An upper bound for $E_\varepsilon^w(u_\varepsilon)$.* If $b = (b_j)$ is an arbitrary configuration of d distinct points in G , then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$(5) \quad E_\varepsilon^w(u_\varepsilon) \leq \sum_{j=1}^d I\left(\frac{\varepsilon}{\eta \sqrt{M_\eta(b_j)}}, 1\right) + W(b) + \pi d \log \frac{1}{\eta} + O(\eta) \quad \text{as } \eta \rightarrow 0,$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ is a quantity which is bounded by $C\eta$, with C independent of $\eta > 0$ small enough.

Step 2. *A lower bound for $E_{\varepsilon_n}^w(u_{\varepsilon_n})$.* If a_1, \dots, a_d are the singularities of u_\star and $\eta > 0$, then there is $N_0 = N_0(\eta) \in \mathbb{N}$ such that, for each $n \geq N_0$,

$$(6) \quad E_{\varepsilon_n}^w(u_{\varepsilon_n}) \geq \sum_{j=1}^d I\left(\frac{\varepsilon_n}{\alpha \eta \sqrt{m_{\alpha \eta}(a_j)}}, 1\right) + \pi d \log \frac{1}{\eta} + W(a) + O(\eta).$$

Here $\alpha = 1 + \eta$ and $O(\eta)$ is a quantity with the same behavior as in (5).

Step 3. *The final conclusion.* From (5), (6) and the asymptotic expression of $I(\varepsilon, \eta)$ as $\frac{\varepsilon}{\eta} \rightarrow 0$ (see [4]), we obtain

$$(7) \quad W(b) + \frac{\pi}{2} \sum_{j=1}^d \log M_\eta(b_j) - \pi d \log \varepsilon_n + d\gamma + o(1) \geq$$

$$\geq W(a) + \frac{\pi}{2} \sum_{i=1}^d \log m_\eta(a_i) - \pi d \log \varepsilon_n + \pi d \log \frac{1}{\eta} - \pi d \log \frac{1}{\eta} + d\gamma + o(1),$$

where $o(1)$ stands for a quantity which goes to 0 as $\varepsilon_n \rightarrow 0$ for fixed η . Adding $\pi d \log \varepsilon_n$ and passing to the limit firstly as $n \rightarrow \infty$ and then as $\eta \rightarrow 0$, we obtain that $a = (a_1, \dots, a_d)$ is a global minimum point of \widetilde{W} . We also deduce that

$$\lim_{n \rightarrow \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d |\log \varepsilon_n|\} = W(a) + \frac{\pi}{2} \sum_{j=1}^d \log w(a_j) + d\gamma.$$

Theorem 2. *Set*

$$W_n = \frac{1}{4\varepsilon_n^2} (1 - |u_{\varepsilon_n}|^2)^2 w.$$

Then (W_n) converges in the weak \star topology of $C(\overline{G})$ to

$$W_\star = \frac{\pi}{2} \sum_{j=1}^d \delta_{a_j}.$$

The expression of the renormalized energy \widetilde{W} allows us, by using the results obtained in [4], to give the analogue of the vanishing gradient property obtained in [4], Chapter VIII.2.

Taking into account Theorem 1 and using the expression of DW (see Theorem VIII.3 in [4]) we obtain

Theorem 3. (“Vanishing gradient property”) *If $a = (a_1, \dots, a_d)$ is as in Theorem 1, then*

$$\nabla R_0(a_j) + \sum_{i \neq j} \frac{a_j - a_i}{|a_j - a_i|^2} = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}, \quad \text{for each } j.$$

As in [4], Chapter I.4, we may define the renormalized energy by considering a suitable variational problem in a domain with shrinking holes.

Let b_1, \dots, b_k be distinct points in G . Fix $d_1, \dots, d_k \in \mathbb{N}$ and a smooth data $g : \partial G \rightarrow S^1$ of degree $d = d_1 + \dots + d_k$. For each $\eta > 0$ small enough, define

$$G_\eta^w = G \setminus \bigcup_{j=1}^k \overline{\omega_{j,\eta}},$$

where

$$\omega_{j,\eta} = B\left(b_j, \frac{\eta}{\sqrt{w(b_j)}}\right).$$

Set

$$\mathcal{E}_\eta^w = \{v \in H^1(G_\eta^w; S^1); \deg(v, \partial\omega_{j,\eta}) = d_j \text{ and } v = g \text{ on } \partial G\}.$$

Let u_η be a solution of

$$(8) \quad \min_{u \in \mathcal{E}_\eta^w} \int_{G_\eta^w} |\nabla u|^2.$$

The following result shows that the renormalized energy \widetilde{W} is what remains in the energy after the singular “core energy” $\pi d |\log \eta|$ has been removed.

Theorem 4. *We have the following asymptotic estimate:*

$$\frac{1}{2} \int_{G_\eta^w} |\nabla u_\eta|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) |\log \eta| + \widetilde{W}(b, \bar{d}, g) + O(\eta), \quad \text{as } \eta \rightarrow 0,$$

where

$$\widetilde{W}(b, \bar{d}, g) = W(b, \bar{d}, g) + \frac{\pi}{2} \left(\sum_{j=1}^k d_j^2 \log w(b_j) \right).$$

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ON THE GINZBURG-LANDAU ENERGY WITH WEIGHT

(Sur l'énergie de Ginzburg-Landau avec poids)

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Abstract. This paper gives a solution to an open problem raised by Bethuel, Brezis and Hélein. We study the Ginzburg-Landau energy with weight. We find the expression of the renormalized energy and we show that the finite configuration of singularities of the limit is a minimum point of this functional. We find a vanishing gradient type property and then we obtain the renormalized energy by Bethuel, Brezis and Hélein's shrinking holes method.

Résumé. Ce travail donne la solution d'un problème ouvert de Bethuel, Brezis and Hélein. On étudie l'énergie de Ginzburg-Landau avec poids. Nous trouvons l'expression de l'énergie renormalisée et on prouve que la configuration finie des singularités de la limite est un point de minimum pour cette fonctionnelle. Nous montrons une propriété du type "vanishing gradient" et on obtient ensuite l'énergie renormalisée avec la méthode "shrinking holes" de Bethuel, Brezis et Hélein.

Keywords: Ginzburg-Landau energy with weight, renormalized energy.

Classification A.M.S. : 35 J 60, 35 Q 99.

1. Introduction

In a recent book [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the vortices related to the Ginzburg-Landau functional. Similar functionals appear in the study of problems occuring in superconductivity or the theory of superfluids.

In [BBH4], F. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\varepsilon \rightarrow 0$ of minimizers u_ε of the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2$$

in the class of functions

$$H_g^1(G) = \{u \in H^1(G; \mathbf{R}^2); u = g \text{ on } \partial G\},$$

where:

- a) $\varepsilon > 0$ is a (small) parameter.
- b) G is a smooth, simply connected, starshaped domain in \mathbf{R}^2 .
- c) $g : \partial G \rightarrow S^1$ is a smooth data with a topological degree $d > 0$.

They obtained the convergence of (u_{ε_n}) in certain topologies to u_\star . The function u_\star is a *harmonic map* from $G \setminus \{a_1, \dots, a_d\}$ to S^1 , and is *canonical*, in the sense that

$$\frac{\partial}{\partial x_1} \left(u_\star \wedge \frac{\partial u_\star}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(u_\star \wedge \frac{\partial u_\star}{\partial x_2} \right) = 0 \quad \text{in } \mathcal{D}'(G).$$

Recall (see [BBH4]) that a canonical harmonic map u_\star with values in S^1 and singularities b_1, \dots, b_k of degrees d_1, \dots, d_k may be expressed as

$$u_\star(x) = \left(\frac{x - b_1}{|x - b_1|} \right)^{d_1} \cdots \left(\frac{x - b_k}{|x - b_k|} \right)^{d_k} e^{i\varphi_0(x)},$$

with

$$\Delta \varphi_0 = 0 \quad \text{in } G.$$

They also defined the notion of renormalized energy $W(b, \bar{d}, g)$ associated to a given configuration $b = (b_1, \dots, b_k)$ of distinct points with associated degrees $\bar{d} = (d_1, \dots, d_k)$. For simplicity we set $W(b) = W(b, \bar{d}, g)$ when $k = d$ and all the degrees equal +1. The expression of the renormalized energy W is given by

$$W(b, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |b_i - b_j| + \frac{1}{2} \int_{\partial G} \Phi_0(g \wedge g_\tau) - \pi \sum_{j=1}^k d_j R_0(b_j),$$

where Φ_0 is the unique solution of

$$(1) \quad \begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^k d_j \delta_{b_j}, & \text{in } G \\ \frac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau, & \text{on } \partial G \\ \int_{\partial G} \Phi_0 = 0 \end{cases}$$

and

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^k d_j \log |x - b_j|.$$

The functional W is also related to the asymptotic behavior of minimizers u_ε as follows:

$$(2) \quad \lim_{\varepsilon \rightarrow 0} \{E_\varepsilon(u_\varepsilon) - \pi d \mid \log \varepsilon \mid\} = \min_{b \in G^d} W(b) + d\gamma,$$

where γ is an universal constant, $k = d$, $d_i = +1$ for all i and the configuration $a = (a_1, \dots, a_d)$ achieves the minimum of W .

We study in this paper a similar problem, related to the Ginzburg-Landau energy with the weight w , that is

$$E_\varepsilon^w(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2 w,$$

with $w \in C^1(\overline{G})$, $w > 0$ in \overline{G} . Throughout, u_ε will denote a minimizer of E_ε^w . We mention that u_ε verifies the Ginzburg-Landau equation with weight

$$(3) \quad \begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) w & \text{in } G \\ u_\varepsilon = g & \text{on } \partial G. \end{cases}$$

Our work is motivated by the Open Problem 2, p. 137 in [BBH4]. We are concerned in this paper with the study of the convergence of minimizers, as well as with the corresponding expression of the renormalized energy. We prove that the behavior of minimizers is of the same type as in the case $w \equiv 1$, the change appearing in the expression of the renormalized energy and, consequently, in the location of singularities of the limit u_\star of u_{ε_n} . In our proof we borrow some of the ideas from Chapter VIII in [BBH4], without relying on the vanishing gradient property that is used there. We then prove a corresponding vanishing gradient property for the configuration of singularities obtained at the limit. In the last section we obtain the new renormalized energy by a variant of the “shrinking holes” method which was developed in [BBH4], Chapter I.

2. The renormalized energy

Theorem 1. *There is a sequence $\varepsilon_n \rightarrow 0$ and exactly d points a_1, \dots, a_d in G such that*

$$u_{\varepsilon_n} \rightarrow u_\star \quad \text{in } H_{\text{loc}}^1(\overline{G} \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2),$$

where u_\star is the canonical harmonic map associated to the singularities a_1, \dots, a_d of degrees $+1$ and to the boundary data g .

Moreover, $a = (a_1, \dots, a_d)$ minimizes the functional

$$(4) \quad \widetilde{W}(b) = W(b) + \frac{\pi}{2} \sum_{j=1}^d \log w(b_j)$$

among all configurations $b = (b_1, \dots, b_d)$ of d distinct points in G .

In addition, the following holds:

$$(5) \quad \lim_{n \rightarrow \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d |\log \varepsilon_n|\} = W(a) + \frac{\pi}{2} \sum_{j=1}^d \log w(a_j) + d\gamma,$$

where γ is some universal constant, the same as in (2).

Remark. The functional \widetilde{W} may be regarded as the renormalized energy corresponding to the energy E_ε^w .

Before giving the proof, we shall make some useful notations: given the constants $c, \varepsilon, \eta > 0$, set

$$I^c(\varepsilon, \eta) = \min\{E_\varepsilon^c(u); u \in H^1(B_\eta; \mathbf{R}^2) \text{ and } u(x) = \frac{x}{\eta} \text{ on } \partial B_\eta\}.$$

Here $B_\eta = B(0, \eta) \subset \mathbf{R}^2$.

For $x \in G$, denote

$$M_\eta(x) = \sup_{B(x, \eta) \cap \overline{G}} w \quad \text{and} \quad m_\eta(x) = \inf_{B(x, \eta) \cap \overline{G}} w.$$

Note that

$$I^c(\varepsilon, \eta) = I^c\left(\frac{\varepsilon}{\eta}, 1\right) = I^1\left(\frac{\varepsilon}{\eta\sqrt{c}}, 1\right)$$

and

$$I^{c_1}(\varepsilon, \eta) \leq I^{c_2}(\varepsilon, \eta),$$

provided $c_1 \leq c_2$.

We shall drop the superscript c if it equals 1.

Proof of Theorem 1. The first part of the conclusion may be obtained by adapting the techniques developed in [BBH1], [BBH2], [BBH3], [BBH4] (see also [S]). We shall point out only the main steps that are necessary to prove the convergence:

a) Using the techniques from [S] we find a sequence $\varepsilon_n \rightarrow 0$ such that, for each n ,

$$(6) \quad \frac{1}{\varepsilon_n^2} \int_G (1 - |u_{\varepsilon_n}|^2)^2 w \leq C.$$

b) Using the methods developed in [BBH4], Chapters 3-5, we determine the “bad” disks, as well as the fact that their number is uniformly bounded. These techniques allow us to prove the convergence of (u_{ε_n}) weakly in $H_{\text{loc}}^1(G \setminus \{a_1, \dots, a_k\}; \mathbf{R}^2)$ to u_\star , which is the canonical harmonic map associated to a_1, \dots, a_k with some degrees d_1, \dots, d_k and to the given boundary data.

c) The strong convergence of (u_{ε_n}) in $H_{\text{loc}}^1(\overline{G} \setminus \{a_1, \dots, a_k\}; \mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1. Now the local convergence of (u_{ε_n}) in $G \setminus \{a_1, \dots, a_k\}$ in stronger topologies, say C^2 , may be easily obtained by a bootstrap argument in (3). This implies that

$$(7) \quad \frac{1 - |u_{\varepsilon_n}|^2}{\varepsilon_n^2} w \rightarrow |\nabla u_\star|^2,$$

uniformly on every compact subset of $G \setminus \{a_1, \dots, a_k\}$.

d) For each $1 \leq j \leq k$, $\deg(u_\star, a_j) \neq 0$. Indeed, if not, then as in Step 1 of Theorem 2 [BBH3], the H^1 -convergence is extended up to a_j , which becomes a “removable singularity”.

e) The fact that all degrees equal +1 may be deduced as in Theorem VI.2, [BBH4].

f) The points a_1, \dots, a_d lie in G . The proof of this fact is similar to the corresponding result in [BBH4].

The proof of the second part of the theorem is divided into 3 steps:

Step 1. *An upper bound for $E_\varepsilon^w(u_\varepsilon)$.*

We shall prove that if $b = (b_j)$ is an arbitrary configuration of d distinct points in G , then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$(8) \quad E_\varepsilon^w(u_\varepsilon) \leq \sum_{j=1}^d I\left(\frac{\varepsilon}{\eta \sqrt{M_\eta(b_j)}}, 1\right) + W(b) + \pi d \log \frac{1}{\eta} + O(\eta) \quad \text{as } \eta \rightarrow 0,$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ is a quantity which is bounded by $C\eta$, with C independent of $\eta > 0$ small enough.

The idea is to construct a suitable comparison function v_ε . Let $\eta < \eta_0$, where $\eta_0 = \min_{j,k} \{\text{dist}(b_j, \partial G), |b_j - b_k|\}$. Applying Theorem I.9 in [BBH4] to the configuration

b , we find $\tilde{u} : G_\eta := G \setminus \bigcup_{j=1}^d \overline{B(b_j, \eta)} \rightarrow S^1$ with $\tilde{u} = g$ on ∂G and $\alpha_j \in \mathbb{C}$, $|\alpha_j| = 1$ such that

$$\tilde{u} = \alpha_j \frac{z - b_j}{|z - b_j|} \quad \text{on } \partial B(b_j, \eta)$$

and

$$(9) \quad \frac{1}{2} \int_{G_\eta} |\nabla \tilde{u}|^2 = \pi d \log \frac{1}{\eta} + W(b) + O(\eta), \quad \text{as } \eta \rightarrow 0.$$

We define v_ε as follows: let $v_\varepsilon = \tilde{u}$ on G_η and, in $B(b_j, \eta)$, let v_ε be a minimizer of E_ε^w on $H_h^1(B(b_j, \eta); \mathbf{R}^2)$, where $h = \tilde{u}|_{\partial B(b_j, \eta)}$. We have the following estimate

$$(10) \quad E_\varepsilon^w(v_\varepsilon|_{B(b_j, \eta)}) \leq I^{M_\eta(b_j)}(\varepsilon, \eta) = I\left(\frac{\varepsilon}{\eta \sqrt{M_\eta(b_j)}}, 1\right).$$

The desired conclusion follows from (9), (10) and $E_\varepsilon^w(u_\varepsilon) \leq E_\varepsilon^w(v_\varepsilon)$.

Step 2. A lower bound for $E_{\varepsilon_n}^w(u_{\varepsilon_n})$.

We shall prove that, if a_1, \dots, a_d are the singularities of u_\star , then given any $\eta > 0$, there is $N_0 = N_0(\eta) \in \mathbb{N}$ such that, for each $n \geq N_0$,

$$(11) \quad E_{\varepsilon_n}^w(u_{\varepsilon_n}) \geq \sum_{j=1}^d I\left(\frac{\varepsilon_n}{\alpha \eta \sqrt{m_{\alpha \eta}(a_j)}}, 1\right) + \pi d \log \frac{1}{\eta} + W(a) + O(\eta).$$

Here $\alpha = 1 + \eta$ and $O(\eta)$ is a quantity with the same behavior as in (8).

Indeed, for a fixed a_j , supposed to be 0, u_\star may be written

$$u_\star = e^{i(\psi + \theta)},$$

where ψ is a smooth harmonic function in a neighbourhood of 0. We may assume, without loss of generality, that $\psi(0) = 0$.

In the annulus $A_{\eta, \alpha \eta} = \{x \in \mathbf{R}^2; \eta \leq |x| \leq \alpha \eta\}$ the function u_{ε_n} may be written, for n large enough, as

$$u_{\varepsilon_n} = \rho_n e^{i(\psi_n + \theta)},$$

where ψ_n is a smooth function and $0 < \rho_n \leq 1$. Define, for $\eta \leq r \leq \alpha \eta$, the interpolation function

$$v_n(r, \theta) = \frac{r - \eta + \rho_n(\eta, \theta)(\alpha \eta - r)}{\eta(\alpha - 1)} \cdot e^{i\left[\frac{\alpha \eta - r}{\eta(\alpha - 1)} \psi_n(\eta, \theta) + \theta\right]}.$$

We have

$$\begin{aligned} \frac{1}{\varepsilon_n^2} \int_{A_{\eta, \alpha\eta}} (1 - |v_n|^2)^2 w &\leq \frac{\|w\|_{L^\infty}}{\varepsilon_n^2} \cdot \int_{\eta}^{\alpha\eta} \frac{r}{\eta} \left(\int_{\partial B_\eta} (1 - |u_n|^2)^2 d\sigma \right) dr = \\ &= \|w\|_{L^\infty} \cdot \frac{\alpha + 1}{2} \eta^2 \int_{\partial B_\eta} \frac{(1 - |u_n|^2)^2}{\varepsilon_n^2} d\sigma \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This convergence is motivated by (7). We also observe that the convergence of (u_{ε_n}) in $H_{\text{loc}}^1(G \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2)$ implies

$$(12) \quad \int_{A_{\eta, \alpha\eta}} |\nabla v_n|^2 \rightarrow \int_{A_{\eta, \alpha\eta}} |\nabla v|^2, \quad \text{as } \eta \rightarrow 0,$$

where

$$v(\eta, \theta) = e^{i \left[\frac{\alpha\eta - r}{\eta(\alpha - 1)} \psi(\eta, \theta) + \theta \right]}.$$

Thus, we may write, for $n \geq N_1$,

$$E_{\varepsilon_n}^w(v_n|_{A_{\eta, \alpha\eta}}) = \frac{1}{2} \int_{A_{\eta, \alpha\eta}} |\nabla v|^2 + o(1).$$

We prove in what follows that

$$(13) \quad \int_{A_{\eta, \alpha\eta}} |\nabla v|^2 = O(\eta).$$

Indeed, since

$$|\nabla v|^2 = \frac{\psi^2(\eta, \theta)}{\eta^2(\alpha - 1)^2} + \frac{1}{r^2} \left[\frac{\alpha\eta - r}{\eta(\alpha - 1)} \psi_\theta(\eta, \theta) + 1 \right]^2$$

and

$$\psi(r, \theta) \leq Cr, \quad |\psi_r(r, \theta)| \leq C, \quad |\psi_\theta(r, \theta)| \leq Cr,$$

the desired conclusion follows by a straightforward calculation.

We obtain

$$(14) \quad E_{\varepsilon_n}^w(v_{\varepsilon_n}|_{B(a_j, \eta)}) \geq I^{m_{\alpha\eta}(a_j)}(\varepsilon_n, \alpha\eta) + O(\eta).$$

On the other hand, by the convergence of (u_{ε_n}) in $H_{\text{loc}}^1(G \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2)$ it follows that

$$(15) \quad E_{\varepsilon_n}^w(u_{\varepsilon_n}|_{G_\eta}) = \int_{G_\eta} |\nabla u_\star|^2 + O(\eta),$$

for ε_n sufficiently small.

Taking into account (12)-(15) we obtain the desired result.

Step 3. *The final conclusion.*

It follows from [BBH4], Chapter IX that

$$(16) \quad I(\varepsilon, \eta) = \pi \left| \log \frac{\varepsilon}{\eta} \right| + \gamma + o(1) \quad \text{as } \frac{\varepsilon}{\eta} \rightarrow 0,$$

where the constant γ represents the minimum of the renormalized energy corresponding to the boundary data x in B_1 .

From (8) and (11) we obtain

$$(17) \quad \begin{aligned} W(b) + \frac{\pi}{2} \sum_{j=1}^d \log M_\eta(b_j) - \pi d \log \varepsilon_n + d\gamma + o(1) &\geq \\ &\geq W(a) + \frac{\pi}{2} \sum_{i=1}^d \log m_\eta(a_i) - \pi d \log \varepsilon_n + \pi d \log \frac{1}{\eta} - \pi d \log \frac{1}{\eta} + d\gamma + o(1), \end{aligned}$$

where $o(1)$ stands for a quantity which goes to 0 as $\varepsilon_n \rightarrow 0$ for fixed η . Adding $\pi d \log \varepsilon_n$ and passing to the limit firstly as $n \rightarrow \infty$ and then as $\eta \rightarrow 0$, we obtain that $a = (a_1, \dots, a_d)$ is a global minimum point of \widetilde{W} . We also deduce that

$$\lim_{n \rightarrow \infty} \{E_{\varepsilon_n}^w(u_{\varepsilon_n}) - \pi d \left| \log \varepsilon_n \right| \} = W(a) + \frac{\pi}{2} \sum_{j=1}^d \log w(a_j) + d\gamma.$$

We now generalize another result from [BBH4] concerning the behavior of u_ε .

Theorem 2. *Set*

$$W_n = \frac{1}{4\varepsilon_n^2} (1 - |u_{\varepsilon_n}|^2)^2 w.$$

Then (W_n) converges in the weak \star topology of $C(\overline{G})$ to

$$W_\star = \frac{\pi}{2} \sum_{j=1}^d \delta_{a_j}.$$

Proof. The boundedness of (W_n) in $L^1(G)$ follows directly from (6). Hence (up to a subsequence), W_n converges in the sense of measures of \overline{G} to some W_\star . With the same

techniques as those developed in [BBH3] (Theorem 2) or [BBH4] (Theorem X.3) we can obtain that, for any compact subset K of $\overline{G} \setminus \bigcup_{j=1}^d \{a_j\}$,

$$\frac{1}{\varepsilon_n^2} \|1 - |u_{\varepsilon_n}|^2\|_{L^\infty(K)} \leq C_K .$$

Hence

$$\text{supp } W_\star \subset \bigcup_{j=1}^d \{a_j\} .$$

Therefore

$$W_\star = \sum_{j=1}^d m_j \delta_{a_j} \quad \text{with } m_j \in \mathbf{R} .$$

We now determine m_j using the same methods as in [BBH4]. Fix one of the points a_j (supposed to be 0) and consider $B_R = B(0, R)$ for R small enough so that B_R contains no other point a_i ($i \neq j$). As in the proof of the Pohozaev identity, multiplying the Ginzburg-Landau equation (3) by $x \cdot \nabla u_\varepsilon$ and integrating on B_R we obtain

$$\begin{aligned} (18) \quad & \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 + \frac{1}{2\varepsilon^2} \int_{B_R} (1 - |u_\varepsilon|^2)^2 w + \frac{1}{4\varepsilon^2} \int_{B_R} (1 - |u_\varepsilon|^2)^2 (\nabla w \cdot x) = \\ & = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\varepsilon}{\partial \tau} \right|^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1 - |u_\varepsilon|^2)^2 w . \end{aligned}$$

Passing to the limit in (18) as $\varepsilon \rightarrow 0$ and using the convergence of W_n we find

$$(19) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\star}{\partial \nu} \right|^2 + 2m_j = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial u_\star}{\partial \tau} \right|^2 .$$

Using now the expression of u_\star around a singularity we deduce that, on ∂B_R ,

$$(20) \quad \left| \frac{\partial u_\star}{\partial \nu} \right|^2 = \left| \frac{\partial \theta}{\partial \nu} + \frac{\partial \psi}{\partial \nu} \right|^2 = \left| \frac{\partial \psi}{\partial \nu} \right|^2 .$$

$$(21) \quad \left| \frac{\partial u_\star}{\partial \tau} \right|^2 = \left| \frac{\partial \theta}{\partial \tau} + \frac{\partial \psi}{\partial \tau} \right|^2 = \frac{1}{R^2} + \frac{2}{R} \frac{\partial \psi}{\partial \tau} + \left| \frac{\partial \psi}{\partial \tau} \right|^2 .$$

Inserting (20) and (21) into (19) we obtain

$$(22) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial \psi}{\partial \nu} \right|^2 + 2m_j = \pi + \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial \psi}{\partial \tau} \right|^2 .$$

On the other hand, multiplying $\Delta\psi = 0$ by $x \cdot \nabla\psi$ and integrating on B_R we find

$$(23) \quad \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial\psi}{\partial\nu} \right|^2 = \frac{R}{2} \int_{\partial B_R} \left| \frac{\partial\psi}{\partial\tau} \right|^2 .$$

Thus, from (17) and (18) we obtain

$$m_j = \frac{\pi}{2} .$$

□

3. The vanishing gradient property of the renormalized energy with weight

The expression of the renormalized energy \widetilde{W} allows us, by using the results obtained in [BBH4], to give an expression of the vanishing gradient property in the case of a weight.

From (4) it follows that

$$(24) \quad D\widetilde{W}(b_1, \dots, b_d) = DW(b_1, \dots, b_d) + \frac{\pi}{2} \left(\frac{\nabla w(b_1)}{w(b_1)}, \dots, \frac{\nabla w(b_d)}{w(b_d)} \right) ,$$

for each configuration $b = (b_1, \dots, b_d) \in G^d$.

Recall now Theorem VIII.3 in [BBH4], which gives the expression of the differential of W in an arbitrary configuration of distinct points $b = (b_1, \dots, b_d) \in G^d$:

$$(25) \quad \begin{aligned} DW(b) &= -2\pi \left[\left(\frac{\partial S_1}{\partial x_1}(b_1), \frac{\partial S_1}{\partial x_2}(b_1) \right), \dots, \left(\frac{\partial S_d}{\partial x_1}(b_d), \frac{\partial S_d}{\partial x_2}(b_d) \right) \right] = \\ &= 2\pi \left[\left(-\frac{\partial H_1}{\partial x_2}(b_1), \frac{\partial H_1}{\partial x_1}(b_1) \right), \dots, \left(-\frac{\partial H_d}{\partial x_2}(b_d), \frac{\partial H_d}{\partial x_1}(b_d) \right) \right] . \end{aligned}$$

Here $S_j(x) = \Phi_0(x) - \log |x - b_j|$ in G and Φ_0 the unique solution of

$$\begin{cases} \Delta\Phi_0 = 2\pi \sum_{j=1}^d \delta_{b_j} , \text{ in } G \\ \frac{\partial\Phi_0}{\partial\nu} = g \wedge g_\tau , \text{ on } \partial G \\ \int_{\partial G} \Phi_0 = 0 . \end{cases}$$

The function H_j is harmonic around b_j and is related to u_\star by

$$u_\star(x) = \frac{x - b_j}{|x - b_j|} e^{iH_j(x)}, \quad \text{near } b_j.$$

Let

$$R_0(x) = S_j(x) - \sum_{i \neq j} \log |x - b_i|.$$

Our variant of the vanishing gradient property in [BBH4] (Corollary VIII.1) is:

Theorem 3. *The following properties are equivalent:*

- i) $a = (a_1, \dots, a_d)$ is a critical point of the renormalized energy \widetilde{W} .
- ii) $\nabla S_j(a_j) = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, for each j .
- iii) $\nabla H_j(a_j) = \frac{1}{4w(a_j)} \left(-\frac{\partial w}{\partial x_2}(a_j), \frac{\partial w}{\partial x_1}(a_j) \right)$, for each j .
- iv) $\nabla R_0(a_j) + \sum_{i \neq j} \frac{a_j - a_i}{|a_j - a_i|^2} = \frac{1}{4} \frac{\nabla w(a_j)}{w(a_j)}$, for each j .

The proof follows by the above considerations and the fact that, for each j ,

$$\nabla R_0(x) = \nabla S_j(x) - \sum_{i \neq j} \frac{x - a_i}{|x - a_i|^2}.$$

4. Shrinking holes and the renormalized energy with weight

As in [BBH4], Chapter I.4, we may define the renormalized energy by considering a suitable variational problem in a domain with “shrinking holes”.

Let, as above, G be a smooth, bounded and simply connected domain in \mathbf{R}^2 and let b_1, \dots, b_k be distinct points in G . Fix $d_1, \dots, d_k \in \mathbb{N}$ and a smooth data $g : \partial G \rightarrow S^1$ of degree $d = d_1 + \dots + d_k$. For each $\eta > 0$ small enough, define

$$G_\eta^w = G \setminus \bigcup_{j=1}^k \overline{\omega_{j,\eta}},$$

where

$$\omega_{j,\eta} = B\left(b_j, \frac{\eta}{\sqrt{w(b_j)}}\right).$$

Set

$$\mathcal{E}_\eta^w = \{v \in H^1(G_\eta^w; S^1); \deg(v, \partial\omega_{j,\eta}) = d_j \text{ and } v = g \text{ on } \partial G\}.$$

We consider the minimization problem

$$(26) \quad \min_{u \in \mathcal{E}_\eta^w} \int_{G_\eta^w} |\nabla u|^2.$$

The following result shows that the renormalized energy \widetilde{W} is what remains in the energy after the singular “core energy” $\pi d |\log \eta|$ has been removed.

Theorem 4. *We have the following asymptotic estimate:*

$$\frac{1}{2} \int_{G_\eta^w} |\nabla u_\eta|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) |\log \eta| + \widetilde{W}(b, \bar{d}, g) + O(\eta), \quad \text{as } \eta \rightarrow 0,$$

where

$$\widetilde{W}(b, \bar{d}, g) = W(b, \bar{d}, g) + \frac{\pi}{2} \left(\sum_{j=1}^k d_j^2 \log w(b_j) \right).$$

Proof. As in [BBH4], Chapter I we associate to (26) the linear problem:

$$(27) \quad \begin{cases} \Delta \Phi_\eta = 0, & \text{in } G_\eta^w \\ \Phi_\eta = C_j = \text{Const.}, & \text{on each } \partial\omega_{j,\eta} \\ \int_{\partial\omega_{j,\eta}} \frac{\partial \Phi_\eta}{\partial \nu} = 2\pi d_j, & \text{for each } j = 1, \dots, k \\ \frac{\partial \Phi_\eta}{\partial \nu} = g \wedge g_\tau, & \text{on } \partial G \\ \int_{\partial G} \Phi_\eta = 0. \end{cases}$$

With the same techniques as in [BBH4] (see Lemma I.2), one may prove that

$$\|\Phi_\eta - \Phi_0\|_{L^\infty(G_\eta^w)} = O(\eta),$$

where Φ_0 is the unique solution of (1).

Note that the link between Φ_η and an arbitrary solution u_η of (26) is

$$(28) \quad \begin{cases} u_\eta \wedge \frac{\partial u_\eta}{\partial x_1} = -\frac{\partial \Phi_\eta}{\partial x_2} & \text{in } G_\eta^w \\ u_\eta \wedge \frac{\partial u_\eta}{\partial x_2} = \frac{\partial \Phi_\eta}{\partial x_1} & \text{in } G_\eta^w \end{cases}$$

From now on the proof follows the same lines as of Theorem I.7 in [BBH4]. □

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MINIMIZATION PROBLEMS AND CORRESPONDING RENORMALIZED ENERGIES

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1. Introduction

Let G be a smooth bounded simply connected domain in \mathbf{R}^2 . Let $a = (a_1, \dots, a_k)$ be a configuration of distinct points in G and $\bar{d} = (d_1, \dots, d_k) \in \mathbf{Z}^k$. Consider a smooth boundary data $g : \partial G \rightarrow S^1$ whose topological degree is $d = d_1 + \dots + d_k$. Let also $\rho > 0$ be sufficiently small and denote

$$\Omega_\rho = G \setminus \bigcup_{i=1}^k \overline{B(a_i, \rho)}, \quad \Omega = G \setminus \{a_1, \dots, a_k\}.$$

In [BBH4], B. Bethuel, H. Brezis and F. Hélein have studied the behavior as $\rho \rightarrow 0$ of solutions of the minimization problem

$$(1) \quad E_{\rho, g} = \min_{u \in \mathcal{E}_{\rho, g}} \int_{\Omega_\rho} |\nabla v|^2,$$

where

$$\mathcal{E}_{\rho, g} = \{v \in H^1(\Omega_\rho; S^1); v = g \text{ on } \partial G \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i, \text{ for } i = 1, \dots, k\}.$$

They proved that (1) has a unique solution, say u_ρ . By analysing the behavior of u_ρ as $\rho \rightarrow 0$, they obtained the renormalized energy $W(a, \bar{d}, g)$ through the following asymptotic expansion:

$$(2) \quad \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\rho|^2 = \pi \left(\sum_{i=1}^k d_i^2 \right) \log \frac{1}{\rho} + W(a, \bar{d}, g) + O(\rho), \quad \text{as } \rho \rightarrow 0.$$

If $G = B_1$ and $g(\theta) = e^{di\theta}$ we give an explicit formula for $W(a, \bar{d}, g)$:

$$(3) \quad W(a, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| - \pi \sum_{i,j} d_i d_j \log |1 - a_i \bar{a}_j| .$$

It is natural to ask what happens if we try to minimize the Dirichlet energy $\int_{\Omega_\rho} |\nabla v|^2$ with respect to other classes of test functions. Let

$$\mathcal{F}_\rho = \{v \in H^1(\Omega_\rho; S^1); \deg(v, \partial G) = d \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i, \text{ for } i = 1, \dots, k\} .$$

In [BBH4] it is proved that the problem

$$(4) \quad F_\rho = \min_{u \in \mathcal{F}_\rho} \int_{\Omega_\rho} |\nabla v|^2 ,$$

has a unique solution v_ρ . We find an analogous asymptotic estimate of (2) for the problem (4). More precisely, we prove that

$$(5) \quad \frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 = \pi \left(\sum_{i=1}^k d_i^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \bar{d}) + O(\rho) , \quad \text{as } \rho \rightarrow 0 .$$

The connection between the renormalized energy $W(a, \bar{d}, g)$ from [BBH4] and the new renormalized energy $\widetilde{W}(a, \bar{d})$ is

$$(6) \quad \widetilde{W}(a, \bar{d}) = \inf_{\substack{g: \partial G \rightarrow S^1 \\ \deg(g, \partial G) = d}} W(a, \bar{d}, g) .$$

Moreover the infimum in (6) is achieved. In the case $G = B_1$ we prove that

$$(7) \quad \widetilde{W}(a, \bar{d}) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \pi \sum_{i,j} d_i d_j \log |1 - a_i \bar{a}_j| .$$

We also study the behavior as $\rho \rightarrow 0$ of solutions of the minimization problem

$$(8) \quad F_{\rho, A} = \min_{v \in \mathcal{F}_{\rho, A}} \int_{\Omega_\rho} |\nabla v|^2 ,$$

where

$$\mathcal{F}_{\rho, A} = \{v \in \mathcal{F}_\rho; \int_{\partial G} \left| \frac{\partial v}{\partial \tau} \right|^2 \leq A\} .$$

We find an analogue of (5): if w_ρ is a solution of (8) then

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla w_\rho|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}_A(a, \bar{d}) + o(1) , \quad \text{as } \rho \rightarrow 0 ,$$

where

$$\widetilde{W}_A(a, \bar{d}) = \inf \{ W(a, \bar{d}, g); \deg(g; \partial G) = d \text{ and } \int_{\partial G} \left| \frac{\partial g}{\partial \tau} \right|^2 \leq A \}$$

and the infimum is attained.

In the last section we minimize the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u|^2)^2$$

in the class

$$\mathcal{H}_{d,A} = \{ u \in H^1(G; \mathbf{R}^2); |u| = 1 \text{ on } \partial G, \deg(u, \partial G) = d \text{ and } \int_{\partial G} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq A \}.$$

We prove that $\mathcal{H}_{d,A}$ is non-empty if A is sufficiently large and that the infimum of E_ε is achieved. If u_ε is a minimizer, we prove the convergence as $\varepsilon \rightarrow 0$ of u_ε to u_\star , which is a canonical harmonic map with values in S^1 and d singularities, say a_1, \dots, a_d . Moreover, the configuration $a = (a_1, \dots, a_d)$ minimizes the renormalized energy \widetilde{W}_A .

2. The renormalized energy for prescribed singularities and degrees

We recall that in [BBH4] the study of the minimization problems (1) and (4) is related to the unique solutions Φ_ρ , respectively $\hat{\Phi}_\rho$, of the following linear problems:

$$(9) \quad \begin{cases} \Delta \Phi_\rho = 0 & \text{in } \Omega_\rho \\ \Phi_\rho = C_i = \text{Const.} & \text{on each } \partial\omega_i \text{ with } \omega_i = B(a_i, \rho) \\ \int_{\partial\omega_i} \frac{\partial \Phi_\rho}{\partial \nu} = 2\pi d_i & i = 1, \dots, k \\ \frac{\partial \Phi_\rho}{\partial \nu} = g \wedge g_\tau & \text{on } \partial G \\ \int_{\partial G} \Phi_\rho = 0 \end{cases}$$

and

$$(10) \quad \begin{cases} \Delta \hat{\Phi}_\rho = 0 & \text{in } \Omega \\ \hat{\Phi}_\rho = C_i = \text{Const.} & \text{on } \partial\omega_i \text{ } i = 1, \dots, k \\ \hat{\Phi}_\rho = 0 & \text{on } \partial G \\ \int_{\partial\omega_i} \frac{\partial \hat{\Phi}_\rho}{\partial \nu} = 2\pi d_i & i = 1, \dots, k. \end{cases}$$

We also recall that Φ_ρ converges uniformly as $\rho \rightarrow 0$ to Φ_0 , which is the unique solution of

$$(11) \quad \begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^k d_j \delta_{a_j} & \text{in } G \\ \frac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau & \text{on } \partial G \\ \int_{\partial G} \Phi_0 = 0 . \end{cases}$$

The explicit formula for $W(a, \bar{d}, g)$ found in [BBH4] is

$$(12) \quad W(a, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \frac{1}{2} \int_{\partial G} \Phi_0(g \wedge g_\tau) - \pi \sum_{i=1}^k d_i R_0(a_i) ,$$

where

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^k d_j \log |x - a_j| .$$

We recall (see [BBH4]) that v is a canonical harmonic map with values in S^1 and boundary data g if it is harmonic and satisfies

$$\begin{cases} v \wedge \frac{\partial v}{\partial x_1} = -\frac{\partial \Phi_0}{\partial x_2} & \text{in } \Omega \\ v \wedge \frac{\partial v}{\partial x_2} = \frac{\partial \Phi_0}{\partial x_1} & \text{in } \Omega , \end{cases}$$

or, equivalently,

$$\frac{\partial}{\partial x_1} \left(v \wedge \frac{\partial v}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(v \wedge \frac{\partial v}{\partial x_2} \right) = 0 \quad \text{in } \mathcal{D}'(G) .$$

If v is canonical and has singularities $a_1, \dots, a_k \in G$ with topological degrees d_1, \dots, d_k then v has the form

$$v(z) = \left(\frac{z - a_1}{|z - a_1|} \right)^{d_1} \cdots \left(\frac{z - a_k}{|z - a_k|} \right)^{d_k} e^{i\varphi(z)} ,$$

where φ is a uniquely determined smooth harmonic function in G .

We know from Chapter I in [BBH4] that

$$(13) \quad \begin{cases} v_\rho \wedge \frac{\partial v_\rho}{\partial x_1} = -\frac{\partial \hat{\Phi}_\rho}{\partial x_2} & \text{in } \Omega_\rho \\ v_\rho \wedge \frac{\partial v_\rho}{\partial x_2} = \frac{\partial \hat{\Phi}_\rho}{\partial x_1} & \text{in } \Omega_\rho . \end{cases}$$

So

$$(14) \quad |\nabla v_\rho| = |\nabla \hat{\Phi}_\rho| \quad \text{in } \Omega_\rho .$$

Lemma 1. $\hat{\Phi}_\rho$ converges to $\hat{\Phi}_0$ in $L^\infty(\Omega_\rho)$ as $\rho \rightarrow 0$. More precisely, there exists $C > 0$ such that

$$(15) \quad \|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \leq C\rho.$$

For the proof of Lemma 1 we need the following result of Bethuel, Brezis and Hélein (see [BBH4], Lemma I.4):

Lemma 2. *Let v be a solution of*

$$(16) \quad \begin{cases} \Delta v = 0 & \text{in } \Omega_\rho \\ v = 0 & \text{on } \partial G \\ \int_{\partial\omega_j} \frac{\partial v}{\partial \nu} = 0 & \text{for each } j . \end{cases}$$

Then

$$\sup_{\Omega_\rho} v - \inf_{\Omega_\rho} v \leq \sum_{j=1}^k (\sup_{\omega_j} v - \inf_{\omega_j} v) .$$

Proof of Lemma 1. We apply Lemma 2 to the function $v = \hat{\Phi}_\rho - \hat{\Phi}_0$. Since $\hat{\Phi}_\rho = \text{Const.}$ on each $\partial B(a_j, \rho)$, it follows that

$$\sup_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) - \inf_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) \leq \sum_{j=1}^k \left(\sup_{\partial B(a_j, \rho)} \hat{\Phi}_0 - \inf_{\partial B(a_j, \rho)} \hat{\Phi}_0 \right) \leq C\rho .$$

Using now the fact that $\hat{\Phi}_\rho - \hat{\Phi}_0 = 0$ on ∂G we obtain

$$(17) \quad \|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \leq C\rho .$$

□

Remark. By Lemma 1 and standard elliptic estimates it follows that $\hat{\Phi}_\rho$ converges in $C_{\text{loc}}^k(\Omega \cup \partial G)$ as $\rho \rightarrow 0$, for each $k \geq 0$.

Theorem 1. As $\rho \rightarrow 0$ then (up to a subsequence) v_ρ converges in $C_{\text{loc}}^k(\Omega \cup \partial G)$ to v_0 , which is a canonical harmonic map.

Moreover, the limits of two such sequences differ by a multiplicative constant of modulus 1.

Proof. We may write, locally on $\Omega_\rho \cup \partial G$, $v_\rho = e^{i\varphi_\rho}$ with $0 \leq \varphi_\rho \leq 2\pi$. Thus, by (13),

$$(18) \quad \begin{cases} \frac{\partial \varphi_\rho}{\partial x_1} = -\frac{\partial \hat{\Phi}_\rho}{\partial x_2} & \text{in } \Omega_\rho \\ \frac{\partial \varphi_\rho}{\partial x_2} = \frac{\partial \hat{\Phi}_\rho}{\partial x_1} & \text{in } \Omega_\rho . \end{cases}$$

Hence, up to a subsequence, φ_ρ converges in $C_{\text{loc}}^k(\Omega \cup \partial G)$. This means that v_ρ converges (up to a subsequence) in $C_{\text{loc}}^k(\Omega \cup \partial G)$ to some v_0 . Denote by $g_\rho = v_\rho|_{\partial G}$. It is clear that g_ρ converges to some g_0 and v_0 satisfies

$$(19) \quad \begin{cases} v_0 \wedge \frac{\partial v_0}{\partial x_1} = -\frac{\partial \hat{\Phi}_0}{\partial x_2} & \text{in } \Omega \\ v_0 \wedge \frac{\partial v_0}{\partial x_2} = \frac{\partial \hat{\Phi}_0}{\partial x_1} & \text{in } \Omega \\ v_0 = g_0 & \text{on } \partial G , \end{cases}$$

which means that v_0 is a canonical harmonic map.

We now consider two sequences v_{ρ_n} and v_{ν_n} which converge to v_1 and v_2 . Locally,

$$\varphi_{\rho_n} \rightarrow \varphi_1 \quad \text{and} \quad \varphi_{\nu_n} \rightarrow \varphi_2 .$$

Thus, $\nabla \varphi_1 = \nabla \varphi_2$, so φ_1 and φ_2 differ locally by an additive constant, which means that v_1 and v_2 differ locally by a multiplicative constant of modulus 1. By the connectedness of Ω , this constant is global. \square

Let

$$\hat{R}_0(x) = \hat{\Phi}_0(x) - \sum_{j=1}^k d_j \log |x - a_j| .$$

We observe that \hat{R}_0 is a smooth harmonic function in G .

Theorem 2. We have the following asymptotic estimate:

$$(20) \quad \frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \bar{d}) + O(\rho) , \quad \text{as } \rho \rightarrow 0 ,$$

where

$$(21) \quad \widetilde{W}(a, \bar{d}) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| - \pi \sum_{j=1}^k d_j \hat{R}_0(a_j) .$$

Proof. We follow the ideas of the proof of Theorem I.7 in [BBH4].

Since $\hat{\Phi}_\rho$ is harmonic in Ω_ρ and $\hat{\Phi}_\rho = 0$ on ∂G we may write

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 = \frac{1}{2} \int_{\Omega_\rho} |\nabla \hat{\Phi}_\rho|^2 = -\frac{1}{2} \sum_{j=1}^k \int_{\partial B(a_j, \rho)} \frac{\partial \hat{\Phi}_\rho}{\partial \nu} \hat{\Phi}_\rho = -\pi \sum_{j=1}^k d_j \hat{\Phi}_\rho \left(\partial B(a_j, \rho) \right) .$$

By Lemma 1 and the expression of \hat{R}_0 we easily deduce (20). \square

Theorem 3. *The following equality holds:*

$$(22) \quad \widetilde{W}(a, \bar{d}) = \inf_{\deg(g; \partial G) = d} W(a, \bar{d}, g)$$

and the infimum is achieved.

Proof. *Step 1.* $\widetilde{W}(a, \bar{d}) \leq \inf_{\deg(g; \partial G) = d} W(a, \bar{d}, g)$.

Suppose not, then there exist $\varepsilon > 0$ and $g : \partial G \rightarrow S^1$ with $\deg(g; \partial G) = d$ such that

$$(23) \quad W(a, \bar{d}, g) + \varepsilon \leq \widetilde{W}(a, \bar{d}) .$$

Thus, if u_ρ is a solution of (1), then

$$(24) \quad \begin{aligned} \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\rho|^2 &= \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + W(a, \bar{d}, g) + O(\rho) \geq \\ &\geq \frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \bar{d}) + O(\rho) , \quad \text{as } \rho \rightarrow 0 . \end{aligned}$$

We obtain a contradiction by (23) and (24).

Step 2. If g_ρ and g_0 are as in the proof of Theorem 1, then

$$\widetilde{W}(a, \bar{d}) = W(a, \bar{d}, g_0) .$$

For $r > 0$ let $u_{\rho, r}$ be a solution of the minimization problem

$$(25) \quad \min_{u \in \mathcal{E}_{r, g_\rho}} \int_{\Omega_r} |\nabla u|^2 .$$

Denote $u_{\rho,\rho} = u_\rho$ and $\Phi_{\rho,r}$ the solution of the associated linear problem (see (9)). Let $\Phi_{\rho,0}$ be the solution of (11) for g replaced by g_ρ .

We recall (see Theorem I.6 in [BBH4]) that

$$(26) \quad \Phi_{\rho,r} \rightarrow \Phi_{\rho,0} \quad \text{in } C_{\text{loc}}^k(\Omega \cup \partial G) \quad \text{as } r \rightarrow 0$$

and

$$(27) \quad \left| \frac{1}{2} \int_{\Omega_r} |\nabla u_{\rho,r}|^2 - \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{r} - W(a, \bar{d}, g_\rho) \right| \leq C_{g_\rho} r,$$

where $C_g = C(g) > 0$ is a constant which depends on the boundary data g .

Our aim is to prove that C_{g_ρ} is uniformly bounded for $\rho > 0$. Indeed, analysing the proof of Theorem I.7 in [BBH4] we observe that C_{g_ρ} depends on \tilde{C}_{g_ρ} , which appears in

$$(28) \quad \|\Phi_{\rho,r} - \Phi_{\rho,0}\|_{L^\infty(\Omega_r)} \leq \sum_{j=1}^k \left[\sup_{\partial B(a_j,r)} \Phi_{\rho,0} - \inf_{\partial B(a_j,r)} \Phi_{\rho,0} \right] \leq \tilde{C}_{g_\rho} r.$$

It is clear at this stage, by the convergence of g_ρ and elliptic estimates, that \tilde{C}_{g_ρ} is uniformly bounded.

Observe now that the map $C^1(\partial G; S^1) \ni g \mapsto W(a, \bar{d}, g)$ is continuous. We have

$$\begin{aligned} & |W(a, \bar{d}, g_0) - \widetilde{W}(a, \bar{d})| \leq \left| \frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 - \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} - \widetilde{W}(a, \bar{d}) \right| + \\ & + \left| \frac{1}{2} \int_{\Omega_\rho} |\nabla v_\rho|^2 - \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} - W(a, \bar{d}, g_\rho) \right| + |W(a, \bar{d}, g_\rho) - W(a, \bar{d}, g_0)| \leq \\ & \leq O(\rho) + C\rho + |W(a, \bar{d}, g_\rho) - W(a, \bar{d}, g_0)| \rightarrow 0 \quad \text{as } \rho \rightarrow 0. \end{aligned}$$

Thus

$$\widetilde{W}(a, \bar{d}) = W(a, \bar{d}, g_0),$$

which concludes the proof of Step 2. \square

Theorem 4. *For fixed A , if w_ρ is a solution of the minimization problem (8) then the following holds:*

$$(29) \quad \frac{1}{2} \int_{\Omega_\rho} |\nabla w_\rho|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}_A(a, \bar{d}) + o(1), \quad \text{as } \rho \rightarrow 0,$$

where

$$(30) \quad \widetilde{W}_A(a, \bar{d}) = \inf \{ W(a, \bar{d}, g); \deg(g; \partial G) = d \text{ and } \int_{\partial G} \left| \frac{\partial g}{\partial \tau} \right|^2 \leq A \} ,$$

and the infimum is achieved.

Moreover, w_ρ converges in $C_{\text{loc}}^{0,\alpha}(\Omega \cup \partial G)$ to the canonical harmonic map associated to g_0, a, \bar{d} .

Proof. The existence of w_ρ is obvious. Let $g_\rho = w_\rho|_{\partial G}$. It follows from Chapter I in [BBH4] that

$$(31) \quad \left| \frac{1}{2} \int_{\Omega_\rho} |\nabla w_\rho|^2 - \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + W(a, \bar{d}, g_\rho) \right| \leq C_{g_\rho} \cdot \rho , \quad \text{as } \rho \rightarrow 0 ,$$

where C_g depends only on g, a and \bar{d} .

By the boundedness of g_ρ in $H^1(\partial G)$ we may suppose that (up to a subsequence)

$$g_\rho \rightharpoonup g_0 \quad \text{weakly in } H^1(\partial G), \text{ as } \rho \rightarrow 0 .$$

As in the proof of Theorem 3 (see (28)) we deduce that C_{g_ρ} is uniformly bounded.

We now prove that the map $g \mapsto W(a, \bar{d}, g)$ is continuous in the weak topology of $H^1(\partial G)$. Taking into account the weak convergence of g_ρ to g_0 and the Sobolev embedding Theorem we obtain

$$g_\rho \wedge \frac{\partial g_\rho}{\partial \tau} \rightharpoonup g_0 \wedge \frac{\partial g_0}{\partial \tau} \quad \text{weakly in } L^2(\partial G), \text{ as } \rho \rightarrow 0 .$$

Using (11), it follows that

$$\Phi_{\rho,0} \rightharpoonup \Phi_0 \quad \text{weakly in } H^1(G), \text{ as } \rho \rightarrow 0 .$$

So, by the Rellich Theorem,

$$\Phi_{\rho,0} \rightarrow \Phi_0 \quad \text{strongly in } L^2(G), \text{ as } \rho \rightarrow 0 .$$

Therefore,

$$\int_{\partial G} \Phi_{\rho,0} \left(g_\rho \wedge \frac{\partial g_\rho}{\partial \tau} \right) \rightarrow \int_{\partial G} \Phi_0 \left(g_0 \wedge \frac{\partial g_0}{\partial \tau} \right) \quad \text{as } \rho \rightarrow 0 .$$

We also deduce, using elliptic estimates, that for each i ,

$$R_{\rho,0}(a_i) \rightarrow R_0(a_i) \quad \text{as } \rho \rightarrow 0 .$$

Thus, by (12), we obtain the continuity of the map $g \mapsto W(a, \bar{d}, g)$. Hence, by (31), we easily deduce (29).

The fact that the infimum in (30) is achieved may be deduced with similar arguments as in the proof of Theorem 3.

The convergence of w_ρ to a canonical harmonic map follows easily from the convergence of g_ρ . \square

3. Renormalized energies in a particular case

We shall calculate in the first part of this section the expressions of $\widetilde{W}(a, \bar{d})$ and $\widetilde{W}(a, \bar{d}, g)$ when $G = B(0; 1)$ and $g(\theta) = e^{id\theta}$, for an arbitrary configuration $a = (a_1, \dots, a_k)$.

Proposition 1. *The expression of the renormalized energy $\widetilde{W}(a, \bar{d})$ is given by*

$$\widetilde{W}(a, \bar{d}) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \pi \sum_{i,j} d_i d_j \log |1 - a_i \bar{a}_j| .$$

Proof. Let \hat{R}_0 be defined as in the preceding section. Then

$$\begin{cases} \Delta \hat{R}_0 = 0 & \text{in } B_1 \\ \hat{R}_0(x) = -\sum_{j=1}^k d_j \log |x - a_j| & \text{if } x \in \partial B_1 . \end{cases}$$

It follows from the linearity of this problem that it is sufficient to compute \hat{R}_0 when the configuration of points consists of one point, say a . Hence, by the Poisson formula, for each $x \in B_1$,

$$(32) \quad \hat{R}_0(x) = -\frac{d}{2\pi} (1 - |x|^2) \int_{\partial B_1} \frac{\log |z - a|}{|z - x|^2} dz .$$

We first observe that

$$(33) \quad \hat{R}_0(x) = 0 \quad \text{if } a = 0 .$$

If $a \neq 0$ and $a^* = \frac{a}{|a|^2}$, then

$$(34) \quad \begin{aligned} \hat{R}_0(x) &= -\frac{d}{2\pi} (1 - |x|^2) \int_{\partial B_1} \frac{\log |z - a^*| + \log |a|}{|z - x|^2} dz = \\ &= -d \log |x - a^*| - d \log |a| . \end{aligned}$$

Hence, by (33) and (34)

$$(35) \quad \hat{R}_0(x) = \begin{cases} 0 & \text{if } a = 0 \\ -d \log |x - a^*| - d \log |a| & \text{if } a \neq 0 . \end{cases}$$

In the case of a general configuration $a = (a_1, \dots, a_k)$ one has

$$(36) \quad \hat{R}_0(x) = -\sum_{j=1}^k d_j \log |x - a_j^*| - \sum_{j=1}^k d_j \log |a_j| .$$

Applying now Theorem 2 we obtain

$$\widetilde{W}(a, \bar{d}) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \pi \sum_{i,j} d_i d_j \log |1 - a_i \bar{a}_j| .$$

□

Proposition 2. *The expression of $W(a, \bar{d}, g)$ if $G = B_1$ and $g(\theta) = e^{id\theta}$ is given by*

$$(37) \quad W(a, \bar{d}, g) = -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| - \pi \sum_{i,j} d_i d_j \log |1 - a_i \bar{a}_j| .$$

Proof. We shall use the expression (12) for the renormalized energy $W(a, \bar{d}, g)$. As above, we observe that it suffices to compute R_0 for one point, say a .

We define on $B(0; 1) \setminus \{a\}$ the function \mathcal{G} by

$$(38) \quad \mathcal{G}(x) = \begin{cases} \frac{d}{2\pi} \log |x - a| + \frac{d}{2\pi} \log |x - a^\star| - \frac{d}{4\pi} |x|^2 + \mathcal{C} & \text{if } a \neq 0 \\ \frac{d}{2\pi} \log |x| - \frac{d}{4\pi} |x|^2 + \mathcal{C} & \text{if } a = 0 \end{cases}$$

and we choose the constant \mathcal{C} such that

$$\int_{\partial B_1} \mathcal{G} = 0 .$$

It follows that, for every $a \in B_1$,

$$(39) \quad \mathcal{C} = \frac{d}{4\pi} + \frac{d}{2\pi} \log |a| .$$

The function \mathcal{G} satisfies

$$(40) \quad \begin{cases} \Delta \mathcal{G} = d\delta_a - \frac{d}{\pi} & \text{in } B_1 \\ \frac{\partial \mathcal{G}}{\partial \nu} = 0 & \text{on } \partial B_1 \\ \int_{\partial B_1} \mathcal{G} = 0 . \end{cases}$$

It follows now from (11) that

$$\begin{cases} \Delta \left(\frac{\Phi_0}{2\pi} \right) = d\delta_a & \text{in } B_1 \\ \frac{\partial}{\partial \nu} \left(\frac{\Phi_0}{2\pi} \right) = \frac{d}{2\pi} & \text{on } \partial B_1 \\ \int_{\partial B_1} \frac{\Phi_0}{2\pi} = 0 . \end{cases}$$

Thus the function $\Psi = \frac{\Phi_0}{2\pi} - \frac{d}{4\pi} (|x|^2 - 1)$ satisfies

$$(41) \quad \begin{cases} \Delta \Psi = d\delta_a - \frac{d}{\pi} & \text{in } B_1 \\ \frac{\partial \Psi}{\partial \nu} = 0 & \text{on } \partial B_1 \\ \int_{\partial B_1} \Psi = 0 . \end{cases}$$

By uniqueness, it follows from (40) and (41) that

$$(42) \quad \frac{\Phi_0}{2\pi} - \frac{d}{4\pi} (|x|^2 - 1) = \frac{d}{2\pi} \log |x - a| + \frac{d}{2\pi} \log |x - a^*| - \frac{d}{4\pi} |x|^2 + \mathcal{C} .$$

Taking into account the expression of \mathcal{C} given in (39), as well as the link between Φ_0 and R_0 we obtain (37). \square

Remark. It follows by Theorem 3 and Propositions 1 and 2 that

$$\sum_{i \neq j} d_i d_j \log |a_i - a_j| + \sum_{j=1}^k d_j^2 \log(1 - |a_j|^2) \leq 0 .$$

A very interesting problem is the study of configurations which minimize $W(a, \bar{d}, g)$ with \bar{d} and g prescribed. This relies on the behavior of minimizers of the Ginzburg-Landau energy (see [BBH4] for further details).

Proposition 3. *If $k = 2$ and $d_1 = d_2 = 1$, then the minimal configuration for W is unique (up to a rotation) and consists of two points which are symmetric with respect to the origin.*

Proof. Let a and b be two distinct points in B_1 . Then

$$\begin{aligned} -\frac{W}{\pi} &= \log(|a|^2 + |b|^2 - 2|a| \cdot |b| \cdot \cos \varphi) + \log(1 + |a|^2|b|^2 - 2|a| \cdot |b| \cdot \cos \varphi) + \\ &\quad + \log(1 - |a|^2) + \log(1 - |b|^2) , \end{aligned}$$

where φ denotes the angle between the vectors \overrightarrow{Oa} and \overrightarrow{Ob} . So, a necessary condition for the minimum of W is $\cos \varphi = -1$, that is the points a , O and b are colinear, with O between a and b . Hence one may suppose that the points a and b lie on the real axis and $-1 < b < 0 < a < 1$. Denote

$$f(a, b) = 2 \log(a - b) + 2 \log(1 - ab) + \log(1 - a^2) + \log(1 - b^2) .$$

A straightforward calculation, based on the Jensen inequality and the symmetry of f , shows that $a = -b = 5^{-1/4}$. \square

4. The behavior of minimizers of the Ginzburg-Landau energy

We assume throughout this section that G is strictly starshaped about the origin.

In [BBH2] and [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the behavior of minimizers of the Ginzburg-Landau energy E_ε in

$$H_g^1(G; \mathbf{R}^2) = \{u \in H^1(G; \mathbf{R}^2); u = g \text{ on } \partial G\},$$

for some smooth fixed $g : \partial G \rightarrow S^1$, $\deg(g; \partial G) = d > 0$. Our aim is to study a similar problem, that is the behavior of minimizers u_ε of E_ε in the class

$$(43) \quad \mathcal{H}_{d,A} = \{u \in H^1(G; \mathbf{R}^2); |u| = 1 \text{ on } \partial G, \deg(u, \partial G) = d \text{ and } \int_{\partial G} \left| \frac{\partial u}{\partial \tau} \right|^2 \leq A\}.$$

It would have seemed more natural to minimize E_ε in the class

$$\mathcal{H}_d = \{u \in H^1(G; \mathbf{R}^2); |u| = 1 \text{ on } \partial G, \deg(u, \partial G) = d\}$$

but, as observed by F. Bethuel, H. Brezis and F. Hélein, the infimum of E_ε is not attained. To show this, they consider the particular case when $G = B_1$, $d = 1$ and $g(x) = x$. This is the reason why we take the infimum of E_ε on the class $\mathcal{H}_{d,A}$, that was also considered by F. Bethuel, H. Brezis and F. Hélein.

Theorem 5. *For each sequence $\varepsilon_n \rightarrow 0$, there is a subsequence (also denoted by ε_n) and exactly d points a_1, \dots, a_d in G such that*

$$u_{\varepsilon_n} \rightarrow u_\star \quad \text{in } H_{\text{loc}}^1(\overline{G} \setminus \{a_1, \dots, a_d\}; \mathbf{R}^2),$$

where u_\star is a canonical harmonic map with values in S^1 and singularities a_1, \dots, a_d of degrees $+1$.

Moreover, the configuration $a = (a_1, \dots, a_d)$ is a minimum point of

$$\widetilde{W}_A(a, \bar{d}) := \min \{W(a, \bar{d}, g); \deg(g; \partial G) = d \text{ and } \int_{\partial G} \left| \frac{\partial g}{\partial \tau} \right|^2 \leq A\}.$$

Proof. *Step 1.* The existence of u_ε .

For fixed ε , let u_ε^n be a minimizing sequence for E_ε in $\mathcal{H}_{d,A}$. It follows that (up to a subsequence)

$$u_\varepsilon^n \rightharpoonup u_\varepsilon \quad \text{weakly in } H^1$$

and, by the boundedness of $u_\varepsilon^n|_{\partial G}$ in $H^1(\partial G)$, we obtain that

$$u_{\varepsilon_n}|_{\partial G} \rightarrow u_\varepsilon|_{\partial G} \quad \text{strongly in } H^{\frac{1}{2}}(\partial G) .$$

This means that, if $g_\varepsilon = u_\varepsilon|_{\partial G}$, then

$$\deg(g_\varepsilon; \partial G) = d .$$

By the lower semi-continuity of E_ε , u_ε is a minimizer of E_ε . Moreover, this u_ε satisfies the Ginzburg-Landau equation

$$(44) \quad -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{in } G .$$

Step 2. A fundamental estimate.

As in the proof of Theorem III.2 in [BBH4], multiplying (47) by $x \cdot \nabla u_\varepsilon$ and integrating on G , we find

$$(45) \quad \begin{aligned} & \frac{1}{2} \int_{\partial G} (x \cdot \nu) \left(\frac{\partial u_\varepsilon}{\partial \nu} \right)^2 + \frac{1}{2\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 = \\ & = \frac{1}{2} \int_{\partial G} (x \cdot \nu) \left(\frac{\partial g_\varepsilon}{\partial \tau} \right)^2 - \int_{\partial G} (x \cdot \tau) \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial g_\varepsilon}{\partial \tau} . \end{aligned}$$

Using now the boundedness of g_ε in $H^1(\partial G)$ and the fact that G is strictly starshaped we easily obtain

$$(46) \quad \int_{\partial G} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 + \frac{1}{\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 \leq C ,$$

where C depends only on A and d .

Step 3. A fundamental Lemma.

The following result is an adapted version of Theorem III.3 in [BBH4] which is essential towards locating the singularities at the limit.

Lemma 3. *There exist positive constants λ_0 and μ_0 (which depend only on G , d and A) such that if*

$$\frac{1}{\varepsilon^2} \int_{G \cap B_{2\ell}} (1 - |u_\varepsilon|^2)^2 \leq \mu_0 ,$$

where $B_{2\ell}$ is some disc of radius 2ℓ in \mathbf{R}^2 with

$$\frac{\ell}{\varepsilon} \geq \lambda_0 \quad \text{and} \quad \ell \leq 1 ,$$

then

$$(47) \quad |u_\varepsilon(x)| \geq \frac{1}{2} \quad \text{if } x \in G \cap B_\ell .$$

The proof of Lemma is essentially the same as of the cited theorem, after observing that

$$\|\nabla u_\varepsilon\|_{L^\infty(G)} \leq \frac{C}{\varepsilon} ,$$

where C depends only on G , d and A .

Step 4. The convergence of u_ε .

Using Lemma 1 and the estimate (46), we may apply the methods developed in Chapters III-V in [BBH4] to determine the “bad” discs, as well as the fact that their number is uniformly bounded. The same techniques allow us to prove the weak convergence in $H_{\text{loc}}^1(G \setminus \{a_1, \dots, a_k\}; \mathbf{R}^2)$ of a subsequence, also denoted by u_{ε_n} , to some u_\star .

As in [BBH4], Chapter X (see also [S]) one may prove that, for each $p < 2$,

$$u_{\varepsilon_n} \rightarrow u_\star \quad \text{in } W^{1,p}(G) .$$

This allows us to pass at the limit in

$$\frac{\partial}{\partial x_1} \left(u_{\varepsilon_n} \wedge \frac{\partial u_{\varepsilon_n}}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(u_{\varepsilon_n} \wedge \frac{\partial u_{\varepsilon_n}}{\partial x_2} \right) = 0 \quad \text{in } \mathcal{D}'(G)$$

and to deduce that u_\star is a canonical harmonic map.

The strong convergence of (u_{ε_n}) in $H_{\text{loc}}^1(\overline{G} \setminus \{a_1, \dots, a_k\}; \mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1.

We then observe that for all j , $\deg(u_\star, a_j) \neq 0$. Indeed, if not, then as in Step 1 of Theorem 2 in [BBH3], the H^1 -convergence is extended up to a_j , which becomes a “removable singularity”. The fact that all these degrees equal $+1$ and the points a_1, \dots, a_d are not on the boundary may be deduced as in Theorem VI.2 [BBH4].

The following steps are devoted to characterize the limiting configuration as a minimum point of the renormalized energy \widetilde{W}_A .

Step 5. An upper bound for $E_\varepsilon(u_\varepsilon)$.

For $R > 0$, let $I(R)$ be the infimum of E_ε on $H_g^1(G)$ with $G = B(0; \frac{\varepsilon}{R})$ and $g(x) = \frac{x}{|x|}$ on ∂G . Following the ideas of the proof of Lemma VIII.1 in [BBH4] one may show that if $b = (b_j)$ is an arbitrary configuration of d distinct points in G and g is such that $\deg(g, \partial G) = d$ and $\int_{\partial G} |\frac{\partial g}{\partial \tau}|^2 \leq A$, then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$(48) \quad E_\varepsilon(u_\varepsilon) \leq dI\left(\frac{\varepsilon}{\eta}\right) + W(b, g) + \pi d \log \frac{1}{\eta} + O(\eta) , \quad \text{as } \eta \rightarrow 0$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ stands for a quantity which is bounded by $C\eta$, where C is a constant depending only on g .

Step 6. A lower bound for $E_{\varepsilon_n}(u_{\varepsilon_n})$.

With the same proof as of Step 2 of Theorem 1 in [LR] one may show that if a_1, \dots, a_d are the singularities of u_\star and $\eta > 0$, then there is $N_0 = N_0(\eta) \in \mathbf{N}$ such that, for each $n \geq N_0$,

$$(49) \quad E_{\varepsilon_n}(u_{\varepsilon_n}) \geq dI\left(\frac{\varepsilon_n}{\eta(1+\eta)}\right) + \pi d \log \frac{1}{\eta} + W(a, g_0) + O(\eta) ,$$

where $O(\eta)$ is a quantity bounded by $C\eta$, where C depends only on g_0 .

Step 7. The limiting configuration is a minimum point for \widetilde{W}_A .

Taking into account that (see [BBH4], Chapter III)

$$I(\varepsilon) = \pi |\log \varepsilon| + \gamma + O(\varepsilon) ,$$

we obtain by (48) and (49)

$$(50) \quad \begin{aligned} W(b, g) - \pi d \log \varepsilon_n + d\gamma + O\left(\frac{\varepsilon_n}{\eta}\right) &\geq \\ &\geq W(a, g_0) - \pi d \log \varepsilon_n + d\gamma + O(\eta) . \end{aligned}$$

Adding $\pi d \log \varepsilon_n$ in (50) and passing to the limit firstly as $n \rightarrow \infty$ and then as $\eta \rightarrow 0$, we find

$$(51) \quad W(a, g_0) \leq W(b, g) .$$

As b and g are arbitrary chosen it follows that $a = (a_1, \dots, a_d)$ is a global minimum point of

$$(52) \quad \widetilde{W}_A(b) = \min \{W(b, g); \deg(g, \partial G) = d \text{ and } \int_{\partial G} |\frac{\partial g}{\partial \tau}|^2 \leq A\} .$$

□

Remark. The infimum in (52) is achieved because of the continuity of the mapping $\mathcal{H}_{d,A} \ni g \longmapsto W(b, g)$ with respect to the weak topology of $H^1(\partial G)$.

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THE RENORMALIZED ENERGY ASSOCIATED TO A HARMONIC MAP

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Introduction

In [BBH2], F. Bethuel, H. Brezis and F. Hélein have studied several problems which occur in superconductivity and superfluids and they have introduced the notion of renormalized energy. We recall the essential facts: Let $G \subset \mathbf{R}^2$ be a smooth simply connected bounded domain and let $g : \partial G \rightarrow S^1$ be a smooth map of topological degree $d > 0$. Consider a configuration $a = (a_1, \dots, a_k)$ of distinct points in G and $\bar{d} = (d_1, \dots, d_k) \in \mathbf{Z}^k$ such that $d_1 + \dots + d_k = d$. The *canonical harmonic map* $u_0 : \Omega = G \setminus \{a_1, \dots, a_k\} \rightarrow S^1$ associated to (a, \bar{d}, g) is defined by

$$(1) \quad u_0(z) = \left(\frac{z - a_1}{|z - a_1|} \right)^{d_1} \cdot \dots \cdot \left(\frac{z - a_k}{|z - a_k|} \right)^{d_k} \cdot e^{i\varphi_0(z)} \quad \text{if } z \in G,$$

where

$$\begin{cases} \Delta \varphi_0 = 0 & \text{in } G \\ u_0 = g & \text{on } \partial G. \end{cases}$$

For each $\rho > 0$ sufficiently small we define

$$G_\rho = G \setminus \bigcup_{j=1}^k \overline{B(a_j, \rho)}.$$

The renormalized energy $W(a, \bar{d}, g)$ appears in Chapter I of [BBH2] as

$$(2) \quad W(a, \bar{d}, g) = \lim_{\rho \rightarrow 0} \left\{ \frac{1}{2} \int_{G_\rho} |\nabla u_0|^2 - \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} \right\}.$$

We also recall that any *harmonic map* $u : \Omega \rightarrow S^1$, $u = g$ on ∂G with $\deg(u, a_j) = d_j$ has the form

$$(3) \quad u = e^{i\psi} u_0 \quad \text{on } \Omega$$

where

$$(4) \quad \begin{cases} \psi(x) = \sum_{j=1}^k c_j \log |x - a_j| + \phi(x) \\ \psi = 0 \quad \text{on } \partial G \\ \Delta \phi = 0 \quad \text{on } G. \end{cases}$$

In the first section we define a notion of renormalized energy associated to a harmonic map u , which coincides with $W(a, \bar{d}, g)$ when $u = u_0$. In the second part of this paper we give an explicit formula for our notion of renormalized energy.

1. The main result

Theorem 1. *For any harmonic map $u : \Omega \rightarrow S^1$ of the form (3) the following limit exists and is finite*

$$(5) \quad \lim_{p \nearrow 2} \left\{ \frac{1}{2} \int_G |\nabla u|^p - \frac{\pi}{2-p} \sum_{j=1}^k (c_j^2 + d_j^2) \right\} + \frac{\pi}{2} \sum_{j=1}^k (c_j^2 + d_j^2) \cdot \log \left(\sum_{j=1}^k (c_j^2 + d_j^2) \right) =: W(u)$$

Moreover

$$(6) \quad W(u) = \lim_{\rho \rightarrow 0} \left\{ \frac{1}{2} \int_{G_\rho} |\nabla u|^2 - \pi \left(\sum_{j=1}^k (c_j^2 + d_j^2) \right) \log \frac{1}{\rho} \right\}.$$

Proof. Fix $\rho > 0$ such that the closed balls $\overline{B(a_j, \rho)}$ are mutually disjoint and included in G .

We shall estimate ∇u in the neighbourhood of a singularity a_j , supposed to be 0. There exists a smooth harmonic function ζ such that, if $0 < |x| \leq \rho$,

$$u(x) = e^{i(c_j \log |x| + d_j \theta + \zeta(x))}.$$

Hence

$$(7) \quad |\nabla u| = |\nabla(c_j \log |x| + d_j \theta + \zeta)| =$$

$$= \left| \frac{c_j}{|x|} (\cos \theta, \sin \theta) + \frac{d_j}{|x|} (-\sin \theta, \cos \theta) + \nabla \zeta \right| = \left| \frac{\sqrt{c_j^2 + d_j^2}}{|x|} e^{i(\theta + \theta_0)} + \nabla \zeta \right| ,$$

where $\theta_0 \in [0, 2\pi)$ depends only on c_j and d_j .

We observe that the term $\nabla \zeta$ is negligible in $\int_{B(0, \rho)} |\nabla u|^p$, in the sense that

$$(8) \quad \int_{B(0, \rho)} \left| |\nabla u|^p - \left| \frac{\sqrt{c_j^2 + d_j^2}}{|x|} e^{i(\theta + \theta_0)} \right|^p \right| \leq (\text{The Mean Value Theorem})$$

$$\leq C \int_{B(0, \rho)} \frac{1}{r^{p-1}} dx = O(\rho) \quad \text{as } p \nearrow 2 .$$

Therefore

$$\begin{aligned} & \frac{1}{2} \int_G |\nabla u|^p - \frac{\pi}{2-p} \sum_{j=1}^k (c_j^2 + d_j^2)^{\frac{p}{2}} = \\ &= \frac{1}{2} \int_{G_\rho} |\nabla u|^p + \sum_{j=1}^k \left[\frac{1}{2} \int_{B(a_j, \rho)} |\nabla u|^p - \frac{\pi}{2-p} (c_j^2 + d_j^2)^{\frac{p}{2}} \right] \leq \\ &\leq \frac{1}{2} \int_{G_\rho} |\nabla u|^p + \sum_{j=1}^k \left[\frac{\pi}{2-p} (c_j^2 + d_j^2)^{\frac{p}{2}} \rho^{2-p} - \frac{\pi}{2-p} (c_j^2 + d_j^2)^{\frac{p}{2}} \right] + C_1 \rho , \quad \text{as } p \nearrow 2 , \end{aligned}$$

for some fixed constant C_1 .

It follows that

$$(9) \quad \limsup_{p \nearrow 2} \left\{ \frac{1}{2} \int_G |\nabla u|^p - \frac{\pi}{2-p} \sum_{j=1}^k (c_j^2 + d_j^2)^{\frac{p}{2}} \right\} \leq$$

$$\leq \frac{1}{2} \int_{G_\rho} |\nabla u|^2 - \pi \left(\sum_{j=1}^k (c_j^2 + d_j^2) \right) \log \frac{1}{\rho} + C_1 \rho .$$

At the same manner we can find a constant C_2 such that

$$(10) \quad \liminf_{p \nearrow 2} \left\{ \frac{1}{2} \int_G |\nabla u|^p - \frac{\pi}{2-p} \sum_{j=1}^k (c_j^2 + d_j^2)^{\frac{p}{2}} \right\} \geq$$

$$\geq \frac{1}{2} \int_{G_\rho} |\nabla u|^2 - \pi \left(\sum_{j=1}^k (c_j^2 + d_j^2) \right) \log \frac{1}{\rho} - C_2 \rho .$$

The relations (9) and (10) show that the two limits are finite and their difference is $O(\rho)$. Since ρ is arbitrary, it follows that the limit in (5) exists and is finite.

Now we can also deduce from (9) and (10) that (6) holds. □

Corollary 1. *For each u as in Theorem 1,*

$$\lim_{p \nearrow 2} (2-p) \int_G |\nabla u|^p = 2\pi \sum_{j=1}^k (c_j^2 + d_j^2) .$$

The proof of this equality follows obviously from Theorem 1.

Corollary 2. *If u_0 is the canonical harmonic map associated to (a, \bar{d}, g) then*

$$W(u_0) = W(a, \bar{d}, g) .$$

The proof follows immediately from (6).

2. An explicit formula for the renormalized energy

Our purpose in what follows is to give an explicit formula for the renormalized energy $W(u)$, for any harmonic map $u : \Omega \rightarrow S^1$. To do this, we shall use the asymptotic evaluate given by (6).

It follows by (3) that

$$(11) \quad \begin{cases} u \wedge \frac{\partial u}{\partial x_1} = u_0 \wedge \frac{\partial u_0}{\partial x_1} + \frac{\partial \psi}{\partial x_1} & \text{in } \Omega \\ u \wedge \frac{\partial u}{\partial x_2} = u_0 \wedge \frac{\partial u_0}{\partial x_2} + \frac{\partial \psi}{\partial x_2} & \text{in } \Omega . \end{cases}$$

We recall (see Chapter 1 in [BBH2]) that

$$(12) \quad \begin{cases} u_0 \wedge \frac{\partial u_0}{\partial x_1} = -\frac{\partial \Phi_0}{\partial x_2} & \text{in } \Omega \\ u_0 \wedge \frac{\partial u_0}{\partial x_2} = \frac{\partial \Phi_0}{\partial x_1} & \text{in } \Omega , \end{cases}$$

where Φ_0 is the (unique) solution of

$$\begin{cases} \Delta \Phi_0 = 2\pi \sum_{j=1}^k d_j \delta_{a_j} & \text{in } G \\ \frac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau & \text{on } \partial G \\ \int_{\partial G} \Phi_0 = 0 . \end{cases}$$

Inserting (12) into (11) we obtain

$$(13) \quad \begin{cases} u \wedge \frac{\partial u}{\partial x_1} = -\frac{\partial \Phi_0}{\partial x_2} + \frac{\partial \psi}{\partial x_1} & \text{in } \Omega \\ u \wedge \frac{\partial u}{\partial x_2} = \frac{\partial \Phi_0}{\partial x_1} + \frac{\partial \psi}{\partial x_2} & \text{in } \Omega . \end{cases}$$

We have by (11) and (12)

$$(14) \quad \begin{aligned} \frac{1}{2} \int_{G_\rho} |\nabla u|^2 &= \frac{1}{2} \int_{G_\rho} |\nabla \Phi_0|^2 + \frac{1}{2} \int_{G_\rho} |\nabla \psi|^2 + \\ &+ \int_{G_\rho} \left[\frac{\partial \psi}{\partial x_1} \left(u_0 \wedge \frac{\partial u_0}{\partial x_1} \right) + \frac{\partial \psi}{\partial x_2} \left(u_0 \wedge \frac{\partial u_0}{\partial x_2} \right) \right] . \end{aligned}$$

In Chapter I from [BBH2] it is proved that

$$(15) \quad \frac{1}{2} \int_{G_\rho} |\nabla \Phi_0|^2 = \pi \left(\sum_{j=1}^k d_j^2 \right) \log \frac{1}{\rho} + W(a, \bar{d}, g) + O(\rho) \quad \text{as } \rho \rightarrow 0 .$$

We show now that the third term in the right side of (14) is $O(\rho)$ as $\rho \rightarrow 0$. Indeed, since u_0 is an harmonic map and $\psi = 0$ on ∂G , we have

$$(16) \quad \begin{aligned} \int_{G_\rho} \left[\frac{\partial \psi}{\partial x_1} \left(u_0 \wedge \frac{\partial u_0}{\partial x_1} \right) + \frac{\partial \psi}{\partial x_2} \left(u_0 \wedge \frac{\partial u_0}{\partial x_2} \right) \right] &= \\ &= \int_{G_\rho} \operatorname{div} \left(\psi(u_0 \wedge \frac{\partial u_0}{\partial x_1}), \psi(u_0 \wedge \frac{\partial u_0}{\partial x_2}) \right) = - \sum_{j=1}^k \int_{\partial B(a_j, \rho)} \psi \frac{\partial \Phi_0}{\partial \tau} . \end{aligned}$$

Around each a_j one may write

$$(17) \quad \psi = c_j \log |x - a_j| + \phi_j , \quad \Delta \phi_j = 0$$

$$(18) \quad \Phi_0 = d_j \log |x - a_j| + S_j , \quad \Delta S_j = 0 .$$

Thus, by (17) and (18),

$$(19) \quad \begin{aligned} \int_{\partial B(a_j, \rho)} \psi \frac{\partial \Phi_0}{\partial \tau} &= \int_{\partial B(a_j, \rho)} \frac{\partial S_j}{\partial \tau} \left(c_j \log |x - a_j| + \phi_j \right) = \\ &= \int_{\partial B(a_j, \rho)} \frac{\partial S_j}{\partial \tau} \phi_j = O(\rho) \quad \text{as } \rho \rightarrow 0 . \end{aligned}$$

All it remains to do now is to estimate $\int_{G_\rho} |\nabla \psi|^2$. We have

$$\begin{aligned}
(20) \quad \int_{G_\rho} |\nabla \psi|^2 &= - \sum_{j=1}^k \int_{\partial B(a_j, \rho)} \psi \frac{\partial \psi}{\partial \nu} = \\
&= - \sum_{j=1}^k \int_{\partial B(a_j, \rho)} (c_j \log |x - a_j| + \phi_j) \frac{\partial}{\partial \nu} \left(c_j \log |x - a_j| + \phi_j \right) = \\
&= - \sum_{j=1}^k \int_{\partial B(a_j, \rho)} (c_j \log \rho + \phi_j) \left(\frac{c_j}{\rho} + \frac{\partial \phi_j}{\partial \nu} \right) = \\
&= 2\pi \left(\sum_{j=1}^k c_j^2 \right) \log \frac{1}{\rho} - 2\pi \sum_{j=1}^k c_j \phi_j(a_j) - \\
&\quad - \left(\sum_{j=1}^k c_j \int_{\partial B(a_j, \rho)} \frac{\partial \phi_j}{\partial \nu} \right) \log \rho - \sum_{j=1}^k c_j \int_{\partial B(a_j, \rho)} \phi_j \frac{\partial \phi_j}{\partial \nu} = \\
&= 2\pi \left(\sum_{j=1}^k c_j^2 \right) \log \frac{1}{\rho} - 2\pi \sum_{j=1}^k c_j \phi_j(a_j) + O(\rho) \quad \text{as } \rho \rightarrow 0.
\end{aligned}$$

So, by (6), (14), (15), (16), (19) and (20) we have obtained

Theorem 2. *For any harmonic map u ,*

$$\begin{aligned}
W(u) &= W(a, \bar{d}, g) - \pi \sum_{j=1}^k c_j \phi_j(a_j) = \\
&= W(u_0) - \pi \sum_{i \neq j} c_i c_j \log |a_i - a_j| - \pi \sum_{j=1}^k c_j \phi(a_j),
\end{aligned}$$

where ϕ was defined in (4).

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