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Anisotropic Robin problems with logistic reaction

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Abstract. We consider Robin problems driven by the anisotropic p-Laplace operator and with a logistic reaction. Our analysis covers superdiffusive, subdiffusive and equidiffusive equations. We examine all three cases, and we prove multiplicity properties of positive solutions (superdiffusive case) and uniqueness (subdiffusive and equidiffusive cases). The equidiffusive equation is studied only in the context of isotropic operators. We explain why the more general case cannot be treated.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Robin problem with variable exponent:

$$\left\{ \begin{array}{l} -\Delta_{p(z)}u(z) + \xi(z)u(z)^{p(z)-1} = \lambda u(z)^{q(z)-1} - f(z,u(z)) \text{ in } \quad \Omega, \\ \dfrac{\partial u}{\partial n_{p(z)}} + \beta(z)u^{p(z)-1} = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ u > 0. \end{array} \right\} \tag{P_λ}$$

A feature of the present paper is that in this problem, the exponent of the differential operator is variable, namely $p : \overline{\Omega} \mapsto \mathbb{R}$ is log-Hölder continuous and $1 < \min_{\overline{\Omega}} p$. We point out that this regularity assumption is necessary for related Sobolev embeddings (see Diening et al. [4, Section 8.3]); otherwise, $p(\cdot)$ can be assumed only continuous. We denote by $\Delta_{p(z)}$ the anisotropic p-Laplacian differential operator defined by

$$\Delta_{p(z)}u = \operatorname{div}(|Du|^{p(z)-2}Du) \text{ for all } u \in W^{1,p(z)}(\Omega).$$

This operator is more difficult to deal with since, in contrast to the isotropic (constant exponent) case, it is not homogeneous. In the reaction (right-hand side of problem (P_λ)), there is a parametric term $x \mapsto \lambda x^{q(z)-1}$, $x \geqslant 0$ and a perturbation $-f(z,x)$, with $f(\cdot,\cdot)$ being a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). We assume that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p_+ - 1)$-superlinear as $x \to +\infty$, with $p_+ = \max_{\overline{\Omega}} p$. So, the right-hand side of problem (P_λ) is a generalized logistic reaction. If $f(z,x) = x^{r(z)-1}$ with $r \in C(\overline{\Omega})$ and $p_+ < r_- = \min_{\overline{\Omega}} r$, then we have a usual logistic reaction with variable exponents.

We mention that in the boundary condition, $\frac{\partial u}{\partial n_{p(z)}}$ denotes the variable exponent conormal derivative of u. This directional derivative is interpreted using the nonlinear Green's identity and if $u \in C^1(\overline{\Omega})$,

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then

$$\frac{\partial u}{\partial n_{p(z)}} = |Du|^{p(z)-2}\frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Depending on the relation between the exponents $q(\cdot)$ and $p(\cdot)$, we have three types of logistic equations.

(a) If $p_+ < q_-$, then the equation is "superdiffusive."
(b) If $q_+ < p_-$, then the equation is "subdiffusive."
(c) If $p(z) = q(z)$ for all $z \in \overline{\Omega}$, then the equation is "equidiffusive."

In this paper, we study cases (a) and (b). Case (c) is difficult to deal with in the context of anisotropic equations, because we do not have a satisfactory spectral analysis of the relevant differential operator. The analysis developed in this paper reveals that cases (a) and (b) are different. More precisely, we show that for the superdiffusive equation, we have multiple positive solutions and, in fact, we prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ moves. In contrast, for the subdiffusive equation, we have uniqueness of the positive solution. The equidiffusive equation is treated only for isotropic problems.

The mathematical analysis of nonlinear problems with variable exponent started after the seminal contributions of Zhikov [37,38], in relationship with phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenization. The analysis developed by Zhikov revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon. In particular, Zhikov considered the following model functionals in relationship with the Lavrentiev phenomenon:

$$\begin{aligned} \mathcal{M}(u) &:= \int_\Omega c(z)|\nabla u(z)|^2 \mathrm{d}z, \quad 0 < 1/c(\cdot) \in L^t(\Omega),\ t > 1 \\ \mathcal{V}(u) &:= \int_\Omega |\nabla u(z)|^{p(z)} \mathrm{d}z, \quad 1 < p(z) < \infty. \end{aligned} \tag{1}$$

The functional $\mathcal{M}$ is well known, and there is a loss of ellipticity on the set $\{z \in \Omega;\ c(z) = 0\}$. This functional has been studied in the context of degenerate equations involving Muckenhoupt weights. The functional $\mathcal{V}$ has also been the object of intensive interest nowadays, and a huge literature was developed on it. The energy functional defined by $\mathcal{V}$ was used to build models for strongly anisotropic materials. More precisely, in a material made of different components, the exponent $p(z)$ dictates the geometry of a composite that changes its hardening exponent according to the point.

In the past, nonlinear logistic equations were investigated only in the framework of equations with differential operators which have constant exponents. We mention the works of Cardinali et al. [4], Dong and Chen [7], Filippakis et al. [11], Papageorgiou et al. [19], Papageorgiou et al. [23], Takeuchi [31,32] (superdiffusive problems), El Manouni et al. [8], Winkert [34] (nonhomogeneous Neumann problems), and Ambrosetti and Lupo [2], Ambrosetti and Mancini [3], Kamin and Veron [15], D'Aguì et al. [5], Papageorgiou and Papalini [17], Papageorgiou and Scapellato [22], Papageorgiou and Winkert [24], Papageorgiou and Zhang [25], Rădulescu and Repovš [26], Struwe [28,29] (subdiffusive and equidiffusive equations). Moreover, of the above works only the one by Papageorgiou et al. [23], considers Robin boundary value problems. To the best of our knowledge, there are no works on anisotropic logistic equations.

2. Mathematical background

The analysis of problem (P_λ), uses Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of these spaces can be found in the books of Diening et al. [6] and Rădulescu and Repovš [27].

Let $M(\Omega)$ be the space of all Lebesgue measurable functions $u : \Omega \mapsto \mathbb{R}$. As always we identify two such functions which differ only on a Lebesgue-null subset of Ω. Also, let $E_1 = \{r \in C(\overline{\Omega}) : 1 < r_-\}$. In what follows for any $r \in C(\overline{\Omega})$, $r_- = \min_{\overline{\Omega}} r$, $r_+ = \max_{\overline{\Omega}} r$. Given $r \in E_1$, the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(z)} \mathrm{d}z < +\infty \right\}.$$

This space is equipped with the so-called Luxemburg norm defined by

$$\|u\|_{r(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(z)} \mathrm{d}z \leqslant 1 \right\}.$$

In the sequel, for simplicity we write $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$.

Then, $L^{r(z)}(\Omega)$ is a Banach space, which is separable, reflexive (in fact, uniformly convex). Let $r' \in E_1$, be defined by $r'(z) = \frac{r(z)}{r(z)-1}$ (that is, $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$ for all $z \in \overline{\Omega}$.) Then, we have $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ and we have the following version of Hölder's inequality

$$\int_\Omega |uh| \mathrm{d}z \leqslant \left(\frac{1}{r_-} + \frac{1}{r'_-} \right) \|u\|_{r(z)} \|h\|_{r'(z)}$$

for all $u \in L^{r(z)}(\Omega)$, $h \in L^{r'(z)}(\Omega)$.

If $r_1, r_2 \in E_1$ and $r_1 \leqslant r_2$, then $L^{r_2(z)}(\Omega) \hookrightarrow L^{r_1(z)}(\Omega)$ continuously.

Using the variable exponent Lebesgue spaces, we can define the corresponding variable exponent Sobolev spaces. So, if $r \in E_1$, then the variable exponent Sobolev space $W^{1,r(z)}(\Omega)$ is defined by

$$W^{1,r(z)}(\Omega) = \left\{ u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega) \right\},$$

with Du being the gradient of $u(\cdot)$ in the weak sense. The space $W^{1,r(z)}(\Omega)$ is equipped with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)}$$

for all $u \in W^{1,r(z)}(\Omega)$.

In the sequel, for simplicity we write $\|Du\|_{r(z)} = \||Du|\|_{r(z)}$.

The space $W^{1,r(z)}(\Omega)$ is a separable, reflexive (in fact, uniformly convex) Banach space.

Given $r \in E_1$, we introduce the following critical exponents:

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if } r(z) < N \\ +\infty & \text{if } N \leqslant r(z) \end{cases} \text{ for all } z \in \overline{\Omega},$$

$$r^\partial(z) = \begin{cases} \frac{(N-1)r(z)}{N-r(z)} & \text{if } r(z) < N \\ +\infty & \text{if } N \leqslant r(z) \end{cases} \text{ for all } z \in \partial\Omega.$$

Also, let $\sigma(\cdot)$ denote the $(N-1)$-dimensional Hausdorff (surface) measure on Ω. If $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $q \in C(\overline{\Omega})$ with $1 \leqslant q_-$, then

$$W^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \text{ continuously if } q(z) \leqslant r^*(z), z \in \overline{\Omega},$$
$$W^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \text{ compactly if } q(z) < r^*(z), z \in \overline{\Omega}.$$

Similarly, if $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $q \in C(\partial\Omega)$ with $1 \leqslant \min_{\overline{\Omega}} q$, then using the anisotropic trace theory (see [6, Section 12.1]), we have

$$W^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\partial\Omega) \text{ continuously if } q(z) \leqslant r^{\partial}(z), z \in \overline{\Omega},$$
$$W^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\partial\Omega) \text{ compactly if } q(z) < r^{\partial}(z), z \in \overline{\Omega}.$$

The following modular function is very useful in the study of the variable exponent spaces

$$\rho_r(u) = \int_\Omega |u|^{r(z)} \mathrm{d}z \text{ for all } u \in L^{r(z)}(\Omega).$$

Also, for every $u \in W^{1,r(z)}(\Omega)$ we write $\rho_r(Du) = \rho_r(|Du|)$.
This modular function is closely related to the Luxemburg norm.

Proposition 1. *If $r \in E_1$ and $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then*
(a) $\|u\|_{r(z)} = \theta \Longleftrightarrow \rho_r\left(\frac{u}{\theta}\right) = 1$;
(b) $\|u\|_{r(z)} < 1$ (*resp.* $= 1, > 1$) $\Longleftrightarrow \rho_r(u) < 1$ (*resp.* $= 1, > 1$);
(c) $\|u\|_{r(z)} < 1 \Rightarrow \|u\|_{r(z)}^{r_+} \leqslant \rho_r(u) \leqslant \|u\|_{r(z)}^{r_-}$
$\|u\|_{r(z)} > 1 \Rightarrow \|u\|_{r(z)}^{r_-} \leqslant \rho_r(u) \leqslant \|u\|_{r(z)}^{r_+}$;
(d) $\|u_n\|_{r(z)} \to 0 \Longleftrightarrow \rho_r(u_n) \to 0$;
(e) $\|u_n\|_{r(z)} \to +\infty \Longleftrightarrow \rho_r(u_n) \to +\infty$.

Let $A_{r(z)} : W^{1,r(z)}(\Omega) \mapsto W^{1,r(z)}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A_{r(z)}(u), h\rangle = \int_\Omega |Du|^{p(z)-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z$$

for all $u, h \in W^{1,r(z)}(\Omega)$.

The next proposition summarizes the main properties of this map (see Gasiński and Papageorgiou [14] and Rădulescu and Repovš [27, p. 40]).

Proposition 2. *The operator $A_{r(z)} : W^{1,r(z)}(\Omega) \mapsto W^{1,r(z)}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone, too) and of type $(S)_+$, that is, it has the following property:*

$$\text{“If } u_n \xrightarrow{w} u \text{ in } W^{1,r(z)}(\Omega),\ \limsup_{n\to\infty} \langle A_{r(z)}(u_n), u_n - u\rangle \leqslant 0, \text{ then } u_n \to u \text{ in } W^{1,r(z)}(\Omega).\text{”}$$

We will also use the space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive (order) cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

We will also use the following open cone in $C^1(\overline{\Omega})$:

$$D_+ = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{u^{-1}(0)\cap\partial\Omega} < 0\right\}.$$

Also, if $u,\, v \in \operatorname{int} C_+$, we set

$$R(u,v)(z) = |Du(z)|^p - |Dv(z)|^{p-2}\left(Dv(z), D\left(\frac{u^p}{v^{p-1}}\right)(z)\right)_{\mathbb{R}^N}.$$

From Allegretto and Huang [1], we know that

$$0 \leqslant R(u,v)(z) \text{ for all } z \in \overline{\Omega}.$$

Suppose that X is a Banach space and $\varphi \in C^1(X)$. We set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi).$$

Also, if $u \in W^{1,p(z)}(\Omega)$, then $[u) = \{h \in W^{1,p(z)}(\Omega) :\ u \leqslant h\}$.

We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}}$ is bounded,
$(1 + \|u_n\|_X)\,\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence.″

For every $u \in W^{1,r(z)}(\Omega)$, we define

$$u^+(\cdot) = \max\{u(\cdot), 0\},\ u^-(\cdot) = \max\{-u(\cdot), 0\}.$$

We have $u^+,\, u^- \in W^{1,r(z)}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$.

Now let $\xi \in L^\infty(\Omega)$, $\beta \in L^\infty(\partial\Omega)$, $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$. By $\gamma_p : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma_p(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\mathrm{d}z + \int_\Omega \frac{\xi(z)}{p(z)}|u|^{p(z)}\mathrm{d}z + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}\mathrm{d}\sigma$$

for all $u \in W^{1,p(z)}(\Omega)$.

We have

$$\langle \gamma_p'(u), h\rangle = \langle A_{p(z)}(u), h\rangle + \int_\Omega \xi(z)|u|^{p(z)-2}uh\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^{p(z)-2}uh\mathrm{d}\sigma$$

for all $u \in W^{1,p(z)}(\Omega)$.

Also, let $\rho_0 : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ be the C^1-functional defined by

$$\rho_0(u) = \rho_p(Du) + \int_\Omega \xi(z)|u|^{p(z)}\mathrm{d}z + \int_{\partial\Omega} \beta(z)|u|^{p(z)}\mathrm{d}\sigma$$

for all $u \in W^{1,p(z)}(\Omega)$.

Finally, for notational simplicity, throughout the work, by $\|\cdot\|$ we denote the norm of the anisotropic Sobolev space $W^{1,p(z)}(\Omega)$. Recall that

$$\|u\| = \|u\|_{p(z)} + \|Du\|_{p(z)}$$

for all $u \in W^{1,p(z)}(\Omega)$.

Proposition 3. *There exist* $\hat{c}_0,\, \hat{c} > 0$ *such that*

$$\hat{c}\|u\|^{p_+} \leqslant \gamma_p(u) \leqslant \hat{c}_0\|u\|^{p_-} \text{ if } \|u\| \leqslant 1,$$
$$\hat{c}\|u\|^{p_-} \leqslant \gamma_p(u) \leqslant \hat{c}_0\|u\|^{p_+} \text{ if } \|u\| \geqslant 1.$$

Proof. Recall that we have assumed that $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

We first suppose that $\beta \not\equiv 0$. We define

$$\|u\|_\partial = \inf\left\{\lambda > 0 : \int\limits_{\partial\Omega} \beta(z)\left|\frac{u(z)}{\lambda}\right|^{p(z)} \mathrm{d}\sigma \leqslant 1\right\}$$

and then, we introduce

$$|u| = \|u\|_\partial + \|Du\|_{p(z)} \text{ for all } u \in W^{1,p(z)}(\Omega).$$

Evidently, $|\cdot|$ is a norm on $W^{1,p(z)}(\Omega)$. We will show that $|\cdot|$ and $\|\cdot\|$ are equivalent norms on $W^{1,p(z)}(\Omega)$.

Since $W^{1,p(z)}(\Omega) \hookrightarrow L^{p(z)}(\partial\Omega)$ continuously, we can find $c_1 > 0$ such that

$$\begin{aligned} &\|u\|_\partial \leqslant c_1\|u\| \text{ for all } u \in W^{1,p(z)}(\Omega),\\ &\Rightarrow |u| \leqslant c_2\|u\| \text{ for some } c_2 > 0, \text{ all } u \in W^{1,p(z)}(\Omega). \end{aligned} \tag{2}$$

Next, we show that we can find $c_3 > 0$ such that

$$\|u\|_{p(z)} \leqslant c_3|u| \text{ for all } u \in W^{1,p(z)}(\Omega). \tag{3}$$

Arguing by contradiction, suppose that (3) is not true. We can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ such that

$$|u_n| \leqslant \frac{1}{n}\|u_n\|_{p(z)} \text{ for all } n \in \mathbb{N}. \tag{4}$$

We can always assume that

$$\|u_n\|_{p(z)} = 1 \text{ for all } n \in \mathbb{N}. \tag{5}$$

Then, from (4), we have

$$\begin{aligned} &|u_n| \to 0,\\ &\Rightarrow \|u_n\|_\partial \to 0, \ \|Du_n\|_{p(z)} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{6}$$

From (5) and (6), it follows that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded. So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p(z)}(\Omega), \ u_n \to u \text{ in } L^{p(z)}(\Omega). \tag{7}$$

From (6) and (7), it follows that $u = 0$. Hence, we have

$$u_n \to 0 \text{ in } W^{1,p(z)}(\Omega),$$

which contradicts (5). Therefore, (3) is true and so

$$\|u\| \leqslant c_4|u| \text{ for some } c_4 > 0, \text{ all } u \in W^{1,p(z)}(\Omega). \tag{8}$$

From (2) and (8), we infer that

$$\|\cdot\| \text{ and } |\cdot| \text{ are equivalent norms on } W^{1,p(z)}(\Omega). \tag{9}$$

Now let $\xi \not\equiv 0$ and define

$$\|u\|_* = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \xi(z)\left|\frac{u(z)}{\lambda}\right|^{p(z)} \mathrm{d}z \leqslant 1\right\}.$$

We set

$$|u|_* = \|u\|_* + \|Du\|_{p(z)} \text{ for all } u \in W^{1,p(z)}(\Omega).$$

This is also a norm of $W^{1,p(z)}(\Omega)$ and, as above, we show that

$$\|\cdot\| \text{ and } |\cdot|_* \text{ are equivalent norms on } W^{1,p(z)}(\Omega). \tag{10}$$

Finally, from (9) and (10), we see that we can find $\hat{c}$, $\hat{c}_0 > 0$ such that

$$\hat{c}\|u\|^{p_+} \leqslant \frac{1}{p_+}\rho_0(u) \leqslant \gamma_p(u) \leqslant \frac{1}{p_-}\rho_0(u) \leqslant \hat{c}_0\|u\|^{p_-} \text{ if } \|u\| \leqslant 1$$
$$\hat{c}\|u\|^{p_-} \leqslant \frac{1}{p_+}\rho_0(u) \leqslant \gamma_p(u) \leqslant \frac{1}{p_-}\rho_0(u) \leqslant \hat{c}_0\|u\|^{p_+} \text{ if } 1 < \|u\|.$$

This proof is now complete. □

3. Superdiffusive equation

In this section, we examine superdiffusive anisotropic logistic equations. As we already mentioned in Introduction, in this case we have multiplicity of positive solutions.

The hypotheses on the data of problem (P_λ) are the following.

H_0^a : p, $q \in C^{0,1}(\overline{\Omega})$, $1 < p_- \leqslant p_+ < q_- < q(z) < p^*(z)$ for all $z \in \overline{\Omega}$, $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Remark 1. With these hypotheses, we incorporate in our framework Neumann problems. Just assume $\beta \equiv 0$ (in which case, $\xi \not\equiv 0$).

The hypotheses on the perturbation $f(z,x)$ are the following.

H_1^a : $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $f(z,x) \leqslant a(z)\left(1 + x^{r(z)-1}\right)$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $r \in C(\overline{\Omega})$, $q(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;
(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{q(z)-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q(z)-1}} = 0$ uniformly for a.a. $z \in \Omega$; moreover, for every $\rho > 0$ there exists $m_\rho > 0$ such that $f(z,x) \geqslant m_\rho$ for a.a. $z \in \Omega$, all $x \geqslant \rho$;
(iv) for every $\rho > 0$ and every $\theta > 0$, we can find $\hat{\xi}_\rho^\theta > 0$ such that for a.a. $z \in \Omega$ and every $0 < \lambda \leqslant \theta$, the function $x \mapsto \lambda x^{q(z)-1} - f(z,x) + \hat{\xi}_\lambda^\theta x^{p(z)-1}$ is nondecreasing on $[0,\rho]$.

Remark 2. From hypotheses H_1^a(iii), it is clear that $f(z,x) \geqslant 0$ for a.a. $z \in \Omega$, all $x \geqslant 0$. Also, since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, we may assume without any loss of generality that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. If $r \in C(\overline{\Omega})$ with $q(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$ and $f(z,x) = (x^+)^{r(z)-1}$ for all $z \in \overline{\Omega}$, all $x \in \mathbb{R}$, then hypotheses H_1^a are satisfied. This choice of $f(z,x)$ corresponds to the classical superdiffusive reaction.

We introduce the following two sets:

$$\mathfrak{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has a positive solution}\}$$
$$\text{(this is the set of admissible parameters)},$$
$$S_\lambda = \text{ set of positive solutions of problem } (P_\lambda)$$
$$\text{(this is the solution set of problem } (P_\lambda)).$$

First we show the non-emptiness of the set $\mathfrak{L}$ and determine the regularity properties of the elements of S_λ. In what follows, $F(z,x) = \int_0^x f(z,s)ds$.

Proposition 4. *If hypotheses H_0^a, H_1^a hold, then $\mathfrak{L} \neq \emptyset$ and for all $\lambda > 0$, $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. For every $\lambda > 0$, let $\varphi_\lambda : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \gamma_p(u) + \int_\Omega F(z,u^+)\mathrm{d}z - \int_\Omega \frac{\lambda}{q(z)}(u^+)^{q(z)}\mathrm{d}z$$

for all $u \in W^{1,p(z)}(\Omega)$.

Then, $\varphi_\lambda \in C^1(W^{1,p(z)}(\Omega))$. Hypotheses $\mathrm{H}_1^{\mathrm{a}}$(i), (ii) imply that given $\eta > 0$ we can find $c_5 = c_5(\eta) > 0$ such that

$$\begin{aligned} & f(z,x) \geqslant \eta x^{q(z)-1} - c_5 x^{r(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0, \\ \Rightarrow & F(z,x) \geqslant \frac{\eta}{q(z)}x^{q(z)} - \frac{c_5}{r(z)}x^{r(z)} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \end{aligned} \tag{11}$$

Therefore, if $u \in W^{1,p(z)}(\Omega)$, $\|u\| \geqslant 1$, then

$$\begin{aligned} \varphi_\lambda(u) &\geqslant \gamma_p(u) + \left(\frac{\eta}{q_+} - \frac{\lambda}{q_-}\right)\rho_q(u) \text{ (see (11))} \\ &\geqslant \hat{c}\|u\|^{p_-} \text{ (choosing } \eta > \frac{\lambda q_+}{q_-} \text{ and using Proposition 3)}, \\ \Rightarrow &\ \varphi_\lambda(\cdot) \text{ is coercive.} \end{aligned}$$

In addition, using the anisotropic Sobolev embedding theorem, we see that

$$\varphi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass–Tonelli theorem, we can find $u_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \min\left\{\varphi_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{12}$$

Fix $u \in C_+ \setminus \{0\}$. Then, from hypotheses $\mathrm{H}_0^{\mathrm{a}}$, we see that

$$\varphi_\lambda(u) \leqslant c_6 - \lambda c_7 \text{ for some } c_6 = c_6(u) > 0,\ c_7 = c_7(u) > 0.$$

Hence, for $\lambda > 0$ big we have

$$\begin{aligned} & \varphi_\lambda(u) < 0, \\ \Rightarrow & \varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (12))}, \\ \Rightarrow & u_\lambda \neq 0. \end{aligned}$$

From (12), we have

$$\begin{aligned} & \varphi_\lambda'(u_\lambda) = 0, \\ \Rightarrow &\ \langle \gamma_p'(u_\lambda), h\rangle = \int_\Omega \left(\lambda(u_\lambda^+)^{q(z)-1} - f(z,u_\lambda^+)\right) h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{13}$$

In (13), we use the test function $h = -u_\lambda^- \in W^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned} & \rho_0(u_\lambda^-) = 0, \\ \Rightarrow &\ \rho_p(Du_\lambda^-) \leqslant 0 \text{ and so } u_\lambda^- \equiv c \geqslant 0. \end{aligned}$$

We have

$$\min\left\{c^{p_+}, c^{p_-}\right\}\left(\int_\Omega \xi(z)\mathrm{d}z + \int_{\partial\Omega} \beta(z)\mathrm{d}\sigma\right) \leqslant 0.$$

On account of hypotheses H_0^a, we see that

$$\int_\Omega \xi(z)\mathrm{d}z + \int_{\partial\Omega} \beta(z)\mathrm{d}\sigma > 0.$$

So, it follows that $c = 0$ and we infer that $u_\lambda \geqslant 0$, $u_\lambda \neq 0$.

From Winkert and Zacher [35] (see also Proposition 3.1 of Gasiński and Papageorgiou [14]), we have that $u_\lambda \in L^\infty(\Omega)$ and then Theorem 1.3 of Fan [9] (see also Corollary 3.1 of Tan and Fang [33]), implies that $u_\lambda \in C_+ \setminus \{0\}$.

Hypotheses H_0^a(i), (iii) imply that we can find $c_8 > 0$ such that

$$f(z,x) \geqslant -x^{p(z)-1} - c_8 x^{r(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0.$$

By (13), it follows that

$$\Delta_{p(z)} u_\lambda \leqslant (\|\xi\|_\infty + c_9)\, u_\lambda^{p(z)-1} \text{ in } \Omega \text{ for some } c_9 = c_9(\|u_\lambda\|) > 0.$$

Then, from the anisotropic maximum principle of Zhang [36, Theorem 1.2], we have $u_\lambda \in \operatorname{int} C_+$.

We have proved that for $\lambda > 0$ big enough we have $\lambda \in \mathfrak{L}$, hence $\mathfrak{L} \neq \emptyset$. Moreover, the arguments in the last part of the proof show that $S_\lambda \subseteq \operatorname{int} C_+$.

This proof is now complete. $\square$

Let $\lambda_* = \inf \mathfrak{L}$.

Proposition 5. *If hypotheses* H_0^a, H_1^a *hold, then* $\lambda_* > 0$.

Proof. Suppose $\lambda_* = 0$ and let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{L}$ such that $\lambda_n \downarrow 0$. We can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ for all $n \in \mathbb{N}$. On account of hypothesis H_1^a(ii), the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W^{1,p(z)}(\Omega),\ u_n \to \hat{u} \text{ in } L^{r(z)}(\Omega). \tag{14}$$

We have

$$\langle \gamma_p'(u_n), h\rangle = \int_\Omega \left[\lambda_n u_n^{q(z)-1} - f(z,u_n)\right] h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega),\ n \in \mathbb{N}. \tag{15}$$

We choose $h = u_n - \hat{u} \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (14). We obtain

$$\begin{aligned}&\lim_{n\to\infty} \langle A_{p(z)}(u_n), u_n - \hat{u}\rangle = 0,\\ \Rightarrow\ & u_n \to \hat{u} \text{ in } W^{1,p(z)}(\Omega) \text{ (see Proposition 2)}.\end{aligned}$$

Suppose that $\hat{u} = 0$. Then, we may assume that $\|u_n\| \leqslant 1$ and $\|u_n\|_{q(z)} \leqslant 1$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}&\rho_0(u_n) \leqslant \int_\Omega \left[\lambda_n u_n^{q(z)} - f(z,u_n)u_n\right] \mathrm{d}z,\\ \Rightarrow\ & \hat{c}p_+ \|u_n\|^{p_+} \leqslant \lambda_n \rho_q(u_n) \leqslant \lambda_n c^* \|u_n\|^{q_-}\\ & \quad \text{for some } c^* > 0, \text{all } n \in \mathbb{N} \text{ (recall that } W^{1,p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)),\\ \Rightarrow\ & \frac{\hat{c}p_+}{c^*} \leqslant \lambda_n \|u_n\|^{q_- - p_+},\end{aligned}$$

a contradiction since $p_+ < q_-$. So, $\hat{u} \neq 0$ and taking the limit as $n \to \infty$ in (15), we have

$$\begin{aligned}&\langle \gamma_p'(\hat{u}), h\rangle = -\int_\Omega f(z,\hat{u})h dz,\\ \Rightarrow\ & 0 \leqslant \rho_0(\hat{u}) = -\int_\Omega f(z,\hat{u})\hat{u}\mathrm{d}z < 0 \text{ (see hypothesis } H_1^a\text{(iii))},\end{aligned}$$

a contradiction. Therefore, $\lambda_* > 0$. $\square$

Next, we show that $\mathfrak{L}$ is connected (an upper half line).

Proposition 6. *If hypotheses* H_0^a, H_1^a *hold,* $\lambda \in \mathfrak{L}$ *and* $\eta \in (\lambda, +\infty)$, *then* $\eta \in \mathfrak{L}$.

Proof. Since $\lambda \in \mathfrak{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We have

$$\begin{aligned} -\Delta_{p(z)} u_\lambda + \xi(z) u_\lambda^{p(z)-1} &= \lambda u_\lambda^{q(z)-1} - f(z, u_\lambda) \\ &< \eta u_\lambda^{q(z)-1} - f(z, u_\lambda) \text{ in } \Omega. \end{aligned} \tag{16}$$

We introduce the Carathéodory function $g_\eta : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$g_\eta(z, x) = \begin{cases} \eta u_\lambda(z)^{q(z)-1} - f(z, u_\lambda(z)) & \text{if } x \leqslant u_\lambda(z) \\ \eta x^{q(z)-1} - f(z, x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{17}$$

We set $G_\eta(z, x) = \int_0^x g_\eta(z, s) ds$ and consider the C^1-functional $\psi_\eta : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ defined by

$$\psi_\eta(u) = \gamma_p(u) - \int_\Omega G_\eta(z, u) \mathrm{d}z \text{ for all } u \in W^{1,p(z)}(\Omega).$$

Using (17) and Proposition 3, we see that

- $\psi_\eta(\cdot)$ is coercive.
- $\psi_\eta(\cdot)$ is sequentially weakly lower semicontinuous.

Therefore, we can find $u_\eta \in W^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &\psi_\eta(u_\eta) = \min \left\{ \psi_\eta(u) : u \in W^{1,p(z)}(\Omega) \right\}, \\ &\Rightarrow \langle \psi_\eta'(u_\eta), h \rangle = 0 \text{ for all } h \in W^{1,p(z)}(\Omega), \\ &\Rightarrow \langle \gamma_p'(u_\eta), h \rangle = \int_\Omega g_\eta(z, u_\eta) h \text{ for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{18}$$

In (18) we choose $h = (u_\lambda - u_\eta)^+ \in W^{1,p(z)}(\Omega)$. Then,

$$\begin{aligned} \langle \gamma_p'(u_\eta), (u_\lambda - u_\eta)^+ \rangle &= \int_\Omega \left(\eta u_\lambda^{q(z)-1} - f(z, u_\lambda) \right) (u_\lambda - u_\eta)^+ \mathrm{d}z \text{ (see (17))} \\ &> \int_\Omega \left(\lambda u_\lambda^{q(z)-1} - f(z, u_\lambda) \right) (u_\lambda - u_\eta)^+ \mathrm{d}z \text{ (see (16))} \\ &= \langle \gamma_p'(u_\lambda), (u_\lambda - u_\eta)^+ \rangle \text{ (since } u_\lambda \in S_\lambda), \\ \Rightarrow\ & u_\lambda \leqslant u_\eta. \end{aligned} \tag{19}$$

From (19), (17) and (18), it follows that $u_\eta \in S_\eta$ and so $\eta \in \mathfrak{L}$.
This proof is now complete. □

As a by-product of the above proof, we have the following corollary.

Corollary 7. *If hypotheses* H_0^a, H_1^a *hold,* $\lambda \in \mathfrak{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\eta \in (\lambda, +\infty)$, *then* $\eta \in \mathfrak{L}$ *and we can find* $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$ *such that* $u_\lambda \leqslant u_\eta$.

We can improve this corollary as follows.

Proposition 8. *If hypotheses* H_0^a, H_1^a *hold,* $\lambda \in \mathfrak{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ *and* $\eta \in (\lambda, +\infty)$, *then* $\eta \in \mathfrak{L}$ *and we can find* $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$ *such that* $u_\lambda - u_\eta \in D_+$.

Proof. From Corollary 7, we have that $\eta \in \mathfrak{L}$ and there exists $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$ such that

$$u_\lambda \leqslant u_\eta. \tag{20}$$

Let $\rho = \|u_\eta\|_\infty$ and $\hat{\xi}_\rho^\eta > 0$ be as postulated by hypothesis $\mathrm{H}_1^{\mathrm{a}}$(iv). We have

$$\begin{aligned}
& -\Delta_{p(z)} u_\lambda + \left(\xi(z) + \hat{\xi}_\rho^\eta\right) u_\lambda^{p(z)-1} \\
&= \lambda u_\lambda^{q(z)-1} - f(z, u_\lambda) + \hat{\xi}_\rho^\eta u_\lambda^{p(z)-1} \\
&\leqslant \lambda u_\eta^{q(z)-1} - f(z, u_\eta) + \hat{\xi}_\rho^\eta u_\eta^{p(z)-1} \text{ (see (20) and hypothesis } \mathrm{H}_1^{\mathrm{a}}\text{(iv))} \\
&< \eta u_\eta^{q(z)-1} - f(z, u_\eta) + \hat{\xi}_\rho^\eta u_\eta^{p(z)-1} \text{ (since } \eta > \lambda) \\
&= -\Delta_{p(z)} u_\eta + \left(\xi(z) + \hat{\xi}_\rho^\eta\right) u_\eta^{p(z)-1}.
\end{aligned} \tag{21}$$

Since $u_\eta \in \operatorname{int} C_+$, we have

$$0 < c^* \leqslant (\eta - \lambda) u_\eta(z) \text{ for all } z \in \overline{\Omega}.$$

So, using Proposition 2.5 of Papageorgiou et al. [20], we conclude that $u_\eta - u_\lambda \in D_+$.
This proof is now complete. □

Next, we show that for $\lambda > \lambda_*$, we have multiple positive solutions. More precisely, we will show that for $\lambda > \lambda_*$ problem (P_λ) has at least a pair of positive solutions.

Proposition 9. *If hypotheses* $\mathrm{H}_0^{\mathrm{a}}$, $\mathrm{H}_1^{\mathrm{a}}$ *hold and* $\lambda > \lambda_*$, *then problem* (P_λ) *has at least two positive solutions*

$$u_0,\ \hat{u} \in \operatorname{int} C_+.$$

Proof. Since $\lambda > \lambda_*$, we can find $\mu \in (\lambda_*, \lambda) \cap \mathfrak{L}$. Then, let $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ and consider the Carathéodory function $k_\lambda : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$k_\lambda(z, x) = \begin{cases} \lambda u_\mu(z)^{q(z)-1} - f(z, u_\mu(z)) & \text{if } x \leqslant u_\mu(z) \\ \lambda x^{q(z)-1} - f(z, x) & \text{if } u_\mu < x. \end{cases} \tag{22}$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s) ds$ and consider the C^1-functional $\Upsilon_\lambda : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ defined by

$$\Upsilon_\lambda(u) = \gamma_p(u) - \int_\Omega K_\lambda(z, u) \mathrm{d}z \text{ for all } u \in W^{1,p(z)}(\Omega).$$

As before (see the proof of Proposition 6), via the direct method of the calculus of variations, we can find $u_0 \in W^{1,p(z)}(\Omega)$ such that

$$\Upsilon_\lambda(u_0) = \min\left\{\Upsilon_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\}, \tag{23}$$

$$\Rightarrow \langle \Upsilon_\lambda'(u_0), h\rangle = 0 \text{ for all } h \in W^{1,p(z)}(\Omega). \tag{24}$$

From (24), using $h = (u_\mu - u_0)^+ \in W^{1,p(z)}(\Omega)$ and (22), we infer that $u_\mu \leqslant u_0$ (see the proof of Proposition 6). In fact arguing as in the proof Proposition 8 and using Proposition 2.5 of [20], we obtain

$$u_0 - u_\mu \in D_+. \tag{25}$$

From (22), we see that

$$\varphi\big|_{[u_\mu)} = \Upsilon_\lambda\big|_{[u_\mu)} + d_0 \text{ with } d_0 \in \mathbb{R}. \tag{26}$$

From (26), (25) and (23), it follows that

$$\begin{aligned}&u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda,\\ \Rightarrow\, &u_0 \text{ is a local } W^{1,p(z)}(\Omega)\text{-minimizer of } \varphi_\lambda\\ &\text{(see Gasiński and Papageorgiou [14] and Tan \textit{and} Fang [33]).}\end{aligned} \tag{27}$$

On account of hypothesis H_1^a(iii), we can find $\delta > 0$ such that

$$\begin{aligned}&f(z,x) \geqslant -x^{q(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta,\\ \Rightarrow\, &F(z.x) \geqslant -\frac{1}{q(z)}x^{q(z)-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta.\end{aligned} \tag{28}$$

Let $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \leqslant \delta$. We can always choose $\delta \in (0,1)$ small so that we also have that $\|u\| \leqslant 1$. Then,

$$\begin{aligned}\varphi_\lambda(u) &\geqslant \gamma_p(u) - \frac{1}{q_-}(1+\lambda)\rho_q(u^+) \text{ (see (28))}\\ &\geqslant \hat{c}\|u\|^{p^+} - c_{10}\|u\|^{q_-} \text{ for some } c_{10} = c_{10}(\lambda) > 0.\end{aligned}$$

Since $p_+ < q_-$, we see that by taking $\delta > 0$ even smaller if necessary we have

$$\begin{aligned}&\varphi_\lambda(u) > 0 \text{ for all } u \in C^1(\overline{\Omega}) \text{ with } 0 < \|u\|_{C^1(\overline{\Omega})} \leqslant \delta,\\ \Rightarrow\, &u = 0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda(\cdot),\\ \Rightarrow\, &u = 0 \text{ is a local } W^{1,p(z)}(\Omega)\text{-minimizer of } \varphi_\lambda(\cdot) \text{ (see [14, 21]).}\end{aligned} \tag{29}$$

We may assume that

$$0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u_0).$$

The analysis is similar if the opposite inequality holds, using (29) instead of (27).

It is easy to see that $K_{\varphi_\lambda} \subseteq C_+$. Hence, we may assume that K_{φ_λ} is finite (otherwise we already have whole sequence of distinct positive solutions in $\operatorname{int} C_+$ and so we are done). Then, from (27) and Theorem 5.7.6 of Papageorgiou et al. [21, p. 449], we can find $\rho \in (0,1)$ small such that

$$0 = \varphi_\lambda(0) \leqslant \varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda. \tag{30}$$

Recall that $\varphi_\lambda(\cdot)$ is coercive (see the proof of Proposition 4). Hence, $\varphi_\lambda(\cdot)$ satisfies the C-condition (see [21, p. 369]). Then, this fact and (30) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p(z)}(\Omega)$ such that

$$\begin{aligned}&\hat{u} \in K_{\varphi_\lambda} \subseteq \operatorname{int} C_+ \cup \{0\},\ 0 = \varphi_\lambda(u_0) \leqslant \varphi_\lambda(u_0) < m_\lambda \leqslant \varphi(\hat{u}) \text{ (see (30))},\\ \Rightarrow\, &\hat{u} \in \operatorname{int} C_+ \text{ is a second positive solution of problem } (P_\lambda), \hat{u} \neq 0.\end{aligned}$$

This proof is now complete. □

Next, we check the admissibility of the critical parameter λ_*.

Proposition 10. *If hypotheses* H_0^a, H_1^a *hold, then* $\lambda_* \in \mathfrak{L}$.

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq (\lambda_*, +\infty)$ and assume that $\lambda_n \downarrow \lambda_*$. We can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$, $n \in \mathbb{N}$. On account of hypotheses H_1^a(ii), we have that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega)$ is bounded. Then, we can find $c_{11} > 0$ such that $\|u_n\|_\infty \leqslant c_{11}$ for all $n \in \mathbb{N}$ (see Fan [9, Theorem 1.3] and Fukagai and Narukawa [13, Lemma 3.3.]), we can find $\theta \in (0,1)$ and $c_{12} > 0$ such that

$$u_n \in C^{1,\theta}(\overline{\Omega}),\ \|u_n\|_{C^{1,\theta}(\overline{\Omega})} \leqslant c_{12} \text{ for all } n \in \mathbb{N}.$$

We know that $C^{1,\theta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly. So, by passing to a subsequence if necessary, we may assume that

$$u_n \to u_* \text{ in } C^1(\overline{\Omega}). \tag{31}$$

Suppose that $u_* = 0$. We have

$$\langle \gamma_p'(u_n), h\rangle = \int_\Omega \left(\lambda_n u_n^{q(z)-1} - f(z,u_n)\right) h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega). \tag{32}$$

In (32), we choose $h = u_n \in W^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned} &\rho_0(u_n) = \lambda_n \rho_q(u_n) - \int_\Omega f(z,u_n)u_n\mathrm{d}z,\\ \Rightarrow\, & p_-\gamma_p(u_n) \leqslant \lambda_n\|u_n\|^{q_-} - \int_\Omega f(z,u_n)u_n\mathrm{d}z\\ &\quad \text{(see Proposition 3 and its proof, Proposition 1 and recall that } u_* = 0),\\ \Rightarrow\, & p_-\hat{c}\|u_n\|^{p_+} \leqslant \lambda_n\|u_n\|^{q_-} \ (\text{since } f \geqslant 0),\\ \Rightarrow\, & p_-\hat{c} \leqslant \lambda_n\|u_n\|^{q_- - p_+} \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Passing to the limit as $n \to \infty$ and since $p_+ < q_-$, we have a contradiction (see (31) and recall that we have assumed that $u_* = 0$). This proves that $u_* \neq 0$. We pass to the limit as $n \to \infty$ in (32) and using (31) we obtain

$$\begin{aligned} &\langle \gamma_p'(u_*), h\rangle = \int_\Omega \left(\lambda_* u_*^{q(z)-1} - f(z,u_*)\right) h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega),\\ \Rightarrow\, & u_* \in S_\lambda \subseteq \operatorname{int} C_+ \text{ and } \lambda_* \in \mathfrak{L}. \end{aligned}$$

This proof is now complete. □

Therefore, we have proved that

$$\mathfrak{L} = [\lambda_*, +\infty).$$

So, summarizing the situation for the superdiffusive anisotropic logistic equation, we can state the following bifurcation-type result, which describes the changes in the set of positive solutions as the parameter $\lambda > 0$ varies.

Theorem 11. *If hypotheses* $\mathrm{H}_0^{\mathrm{a}}$, $\mathrm{H}_1^{\mathrm{a}}$ *hold, then there exists* $\lambda_* > 0$ *such that*

(a) *for every* $\lambda > \lambda_*$, *problem* (P_λ) *has at least two positive solutions* u_0, $\hat{u} \in \operatorname{int} C_+$, $u_0 \neq \hat{u}$;
(b) *for* $\lambda = \lambda_*$, *problem* (P_λ) *has at least one positive solution* $u_* \in \operatorname{int} C_+$;
(c) *for every* $\lambda \in (0, \lambda_*)$, *problem* ($P_\lambda$) *has no positive solution.*

4. Subdiffusive equation

In this section, we examine the subdiffusive equation. As we already mentioned in Introduction, the situation is different from the superdiffusive case and now we have uniqueness of the positive solution.

The hypotheses on the data of problem (P_λ) are the following:

$\mathrm{H}_0^{\mathrm{b}}$: p, $q \in C^{0,1}(\overline{\Omega})$, $1 < q_- \leqslant q_+ < p_-$, $\xi \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

$\mathrm{H}_1^{\mathrm{b}}$: $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leqslant f(z,x) \leqslant a(z)\left(1+x^{r(z)-1}\right)$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$ and $p(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;
(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p(z)-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q(z)-1}} = 0$ uniformly for a.a. $z \in \Omega$;
(iv) for a.a. $z \in \Omega$, the function $x \mapsto \frac{f(z,x)}{x^{q_+-1}}$ is nondecreasing on $\mathring{\mathbb{R}}_+ = (0,+\infty)$.

Remark 3. As in the superdiffusive case, we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. The classical subdiffusive perturbation $(x^+)^{r(z)-1}$ with $r \in C(\overline{\Omega})$, $p(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$ satisfies the above hypotheses.

The next theorem provides a complete picture for the positive solutions of the subdiffusive equation.

Theorem 12. *If hypotheses* $\mathrm{H}_0^{\mathrm{b}}$, $\mathrm{H}_1^{\mathrm{b}}$ *hold, then for every* $\lambda > 0$ *problem* (P_λ) *admits a unique positive solution* $u_\lambda \in \operatorname{int} C_+$ *and* $u_\lambda \to 0$ *in* $C^1(\overline{\Omega})$ *as* $\lambda \to 0^+$.

Proof. Let $\varphi_\lambda : W^{1,p(z)}(\Omega) \mapsto \mathbb{R}$ be the energy functional of problem (P_λ) introduced in the proof of Proposition 4. We know that $\varphi_\lambda \in C^1(W^{1,p(z)}(\Omega))$. Since we deal with the subdiffusive case, we have $q_+ < p_-$ (see hypotheses $\mathrm{H}_0^{\mathrm{b}}$). This fact in conjunction with hypothesis $\mathrm{H}_1^{\mathrm{b}}$(ii) and Proposition 3, implies that

$$\varphi_\lambda(\cdot) \text{ is coercive.}$$

Also, the anisotropic Sobolev embedding theorem, implies that

$$\varphi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

Therefore, we can find $u_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \min\left\{\varphi_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{33}$$

On account of hypothesis $\mathrm{H}_1^{\mathrm{b}}$(iii), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{q(z)} x^{q(z)} \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leqslant x \leqslant \delta. \tag{34}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that

$$tu(z) \in (0,\delta] \text{ for all } z \in \overline{\Omega}. \tag{35}$$

Using (34) and (35), we have that

$$\varphi_\lambda(tu) \leqslant \frac{t^{p_-}}{p_-}\rho_0(u) - \frac{t^{q_+}}{q_+}(\lambda-\varepsilon)\rho_q(u) \text{ (recall } t \in (0,1)).$$

Let $\varepsilon \in (0,\lambda)$. Then,

$$\varphi_\lambda(tu) \leqslant c_{13}t^{p_-} - c_{14}t^{q_+} \text{ for some } c_{13} > 0 \text{ and } c_{14} = c_{14}(\lambda) > 0.$$

Since $q_+ < p_-$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\varphi_\lambda(tu) < 0, \\ \Rightarrow\, &\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (33))}, \\ \Rightarrow\, &u_\lambda \neq 0. \end{aligned}$$

From (33) we have

$$\begin{aligned} &\varphi_\lambda'(u_\lambda) = 0, \\ \Rightarrow\, &\langle \gamma_p'(u_\lambda), h\rangle = \int_\Omega \left(\lambda (u_\lambda^+)^{q(z)-1} - f(z,u_\lambda^+)\right) h dz \text{ for all } h \in W^{1,p(z)}(\Omega). \end{aligned} \tag{36}$$

In (36) we use test function $h = -u_\lambda^- \in W^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned} &\rho_0(u_\lambda^-) = 0, \\ \Rightarrow\, &p_+\hat{c}\|u_\lambda^-\|^{p_+} \leqslant 0 \text{ if } \|u_\lambda^-\| \leqslant 1 \\ &p_+\hat{c}\|u_\lambda^-\|^{p_-} \leqslant 0 \text{ if } \|u_\lambda^-\| \geqslant 1 \\ &\text{(see the proof of Proposition 3).} \end{aligned}$$

Hence, we have $u_\lambda \geqslant 0$, $u_\lambda \neq 0$. Moreover, as before using the anisotropic regularity theory (see Fan [9]) and the anisotropic maximum principle (see Zhang [36]), we have that $u_\lambda \in \operatorname{int} C_+$.

Next, we show the uniqueness of this positive solution. To this end, we introduce the integral functional $j : L^1(\Omega) \mapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \gamma_p(u^{1/q_+}) & \text{if } u \geqslant 0,\ u^{1/q_+} \in W^{1,p(z)}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \left\{u \in L^1(\Omega) : j(u) < +\infty\right\}$ (the effective domain of $j(\cdot)$). From Theorem 2.2 of Takáč and Giacomoni [30], we know that $j(\cdot)$ is convex.

Suppose that $v_\lambda \in W^{1,p(z)}(\Omega)$ is another positive solution of problem (P_λ). Again we show that $v_\lambda \in \operatorname{int} C_+$. Then, Proposition 4.1.22 of Papageorgiou et al. [21, p. 274], implies that

$$\frac{u_\lambda}{v_\lambda} \in L^\infty(\Omega) \text{ and } \frac{v_\lambda}{u_\lambda} \in L^\infty(\Omega).$$

We set $h = u_\lambda^{q_+} - v_\lambda^{q_+} \in W^{1,p(z)}(\Omega)$. Then, for $|t| < 1$ small we have

$$u_\lambda^{q_+} + th \in \operatorname{dom} j \text{ and } v_\lambda^{q_+} + th \in \operatorname{dom} j.$$

This fact and the convexity of $j(\cdot)$, imply that $j(\cdot)$ is Gateaux differentiable at $u_\lambda^{q_+}$ and at $v_\lambda^{q_+}$ in the direction h. A direct calculation using Green's identity gives

$$\begin{aligned} j_\lambda'(u_\lambda^{q_+})(h) &= \frac{1}{q_+}\int_\Omega \frac{-\Delta_{p(z)}u_\lambda + \xi(z)u_\lambda^{p(z)-1}}{u_\lambda^{q_+-1}} h\mathrm{d}z \\ &= \frac{1}{q_+}\int_\Omega \frac{\lambda u_\lambda^{q(z)-1} - f(z,u_\lambda)}{u_\lambda^{q_+-1}} h\mathrm{d}z, \end{aligned}$$

and

$$\begin{aligned} j_\lambda'(v_\lambda^{q_+})(h) &= \frac{1}{q_+}\int_\Omega \frac{-\Delta_{p(z)}v_\lambda + \xi(z)v_\lambda^{p(z)-1}}{u_\lambda^{q_+-1}} h\mathrm{d}z \\ &= \frac{1}{q_+}\int_\Omega \frac{\lambda v_\lambda^{q(z)-1} - f(z,v_\lambda)}{v_\lambda^{q_+-1}} h\mathrm{d}z. \end{aligned}$$

The convexity of $j(\cdot)$ implies that $j'(\cdot)$ is monotone. Therefore,

$$\begin{aligned} 0 \leqslant\ &\frac{1}{q_+}\int_\Omega \lambda\left(\frac{1}{u_\lambda^{q_+-q(z)}} - \frac{1}{v_\lambda^{q_+-q(z)}}\right) h\mathrm{d}z \\ &+ \frac{1}{q_+}\int_\Omega \left(\frac{f(z,u_\lambda)}{u_\lambda^{q_+-1}} - \frac{f(z,v_\lambda)}{v_\lambda^{q_+-1}}\right) h\mathrm{d}z \leqslant 0, \\ \Rightarrow\ &u_\lambda = v_\lambda \text{ (see hypothesis } \mathrm{H}_1^{\mathrm{b}}\text{(iv))}. \end{aligned}$$

This proves the uniqueness of the positive solution $u_\lambda \in \operatorname{int} C_+$.

Finally, we determine the asymptotic behavior of u_λ as $\lambda \to 0^+$. So, let $\lambda_n \downarrow 0$ and set $u_n = u_{\lambda_n} \in \operatorname{int} C_+$, $n \in \mathbb{N}$ be the uniqueness positive solution of problem (p_{λ_n}). We have

$$\langle \gamma_p'(u_n), h\rangle = \int_\Omega \left(\lambda_n u_n^{q(z)-1} - f(z, u_n)\right) h\mathrm{d}z \tag{37}$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

In (37), we choose $h = u_n \in W^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned} &\rho_0(u_n) \leqslant \lambda_1 \rho_q(u_n) \text{ for all } n \in \mathbb{N} \text{ (since } f \geqslant 0, \text{see } \mathrm{H}_1^{\mathrm{b}}(\mathrm{i})), \\ \Rightarrow\, &p_+ \hat{c}\|u_n\|^{p_-} \leqslant \lambda_1 \rho_q(u_n) \text{ for all } n \in \mathbb{N} \text{ and for } \|u_n\| \geqslant 1. \end{aligned}$$

Since $W^{1,p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$ and $q_+ < p_-$, it follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p(z)}(\Omega) \text{ is bounded.}$$

Then as before, from the anisotropic regularity (see the proof of Proposition 10), we can find $\theta \in (0, 1)$ and $c_{15} > 0$ such that

$$u_n \in C^{1,\theta}(\overline{\Omega}),\ \|u_n\|_{C^{1,\theta}(\overline{\Omega})} \leqslant c_{15} \text{ for all } n \in \mathbb{N}. \tag{38}$$

From (38) and the compact embedding of $C^{1,\theta}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ we see that of at least for a subsequence, we have

$$u_n \to \hat{u} \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{39}$$

Passing to the limit as $n \to \infty$ in (37) and using (39), we obtain

$$\langle \gamma_p'(\hat{u}), h\rangle = -\int_\Omega f(z, \hat{u}) h\mathrm{d}z \text{ for all } h \in W^{1,p(z)}(\Omega).$$

Let $h = \hat{u} \in W^{1,p(z)}(\Omega)$. Then,

$$\begin{aligned} &\rho_0(\hat{u}) \leqslant 0 \text{ (since } f \geqslant 0,\ \hat{u} \geqslant 0), \\ \Rightarrow\, &\hat{u} = 0 \text{ (see Proposition 3).} \end{aligned}$$

So, from (39) we conclude that

$$u_\lambda \to 0^+ \text{ as } \lambda \to 0^+.$$

This proof is now complete. □

5. Equidiffusive equation

In the equidiffusive case, we can only deal with the isotropic equation. The reason for this is that in the anisotropic case, there is no satisfactory spectral analysis of the differential operator. More precisely, if we set

$$\hat{\lambda}_1 = \inf\left\{\frac{\gamma_p(u)}{\int_\Omega \frac{1}{p(z)}|u|^{p(z)}\mathrm{d}z} : u \in W^{1,p(z)}(\Omega),\ u \neq 0\right\},$$

then it can happen that $\hat{\lambda}_1 = 0$ even if $\xi \not\equiv 0$ or $\beta \not\equiv 0$ (see Fan [10]). We are not aware of any reasonable conditions on the exponent $p(\cdot)$ (aside from being constant), which will guarantee that $\hat{\lambda}_1 > 0$. This prevents us from dealing with the anisotropic equidiffusive equation.

In contrast, in the isotropic case, if we have that $\xi \not\equiv 0$ or $\beta \not\equiv 0$ (as we have done throughout this work), then $\hat{\lambda}_1 > 0$ and the analysis of the equidiffusive equation can proceed without problems. We show that the situation is similar to the subdiffusive case and we have uniqueness of the positive solution.

The hypotheses on the data of problem (P_λ) are the following:

H_0^c : $q(z) = p(z) = p > 1$ for all $z \in \overline{\Omega}$ (isotropic problem), $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi \geqslant 0$, $\beta \geqslant 0$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

H_1^c : $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leqslant f(z, x) \leqslant a(z)\left(1 + x^{r-1}\right)$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for a.a. $z \in \Omega$, the function $x \mapsto \frac{f(z,x)}{x^{p-1}}$ is increasing on $\mathring{\mathbb{R}}_+ = (0, +\infty)$ and for a.a. $z \in \Omega$ and all $x > 0$, we have $0 < f(z, x)$.

Remark 4. Again we can set $f(z, x) = 0$ for a.a. $z \in \Omega$. Moreover, the classical perturbation $f(z, x) = f(x) = (x^+)^{r-1}$ ($p < r < p^*$) satisfies the above hypotheses.

Consider the following nonlinear eigenvalue problem.

$$\left\{\begin{array}{ll} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{40}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is "eigenvalue", if for this $\hat{\lambda}$ problem (40) admits a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$, known as an "eigenfunction" corresponding to $\hat{\lambda}$. We know that under hypotheses H_0^c, problem (40) admits a smallest eigenvalue $\hat{\lambda}_1 > 0$, which is simple, isolated and admits the following variational characterization

$$\hat{\lambda}_1 = \inf\left\{\frac{\hat{\gamma}_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0\right\} > 0, \tag{41}$$

with $\hat{\gamma}_p(u) = \|Du\|_p^p + \int\limits_\Omega \xi(z)|u|^p \mathrm{d}z + \int\limits_{\partial\Omega} \beta(z)|u|^p \mathrm{d}\sigma$ for all $u \in W^{1,p}(\Omega)$ (see Fragnelli et al. [12]). The infimum in (41) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. By $\hat{u}_1$ we denote the positive, L^p-normalized eigenfunction corresponding to $\hat{\lambda}_1$. The nonlinear regularity theory and the nonlinear maximum principle, imply that $\hat{u}_1 \in \operatorname{int} C_+$. We mention that for every eigenvalue $\hat{\lambda} > \hat{\lambda}_1$, the corresponding eigenfunctions are nodal (sign-changing).

The isotropic equidiffusive case is very similar to the subdiffusive case, except that now the infimum of the admissible parameter λ is $\hat{\lambda}_1 > 0$. The existence and uniqueness theorem for the isotropic equidiffusive equation is the following:

Theorem 13. *If hypotheses* H_0^c, H_1^c *hold, then*

(a) *for every* $\lambda > \hat{\lambda}_1$ *problem* (P_λ) *has a unique positive solution* $u_\lambda \in \operatorname{int} C_+$ *and* $u_\lambda \to 0$ *in* $C^1(\overline{\Omega})$ *as* $\lambda \to 0^+$;

(b) *for every* $\lambda \in (0, \hat{\lambda}_1]$ *problem* (P_λ) *has no positive solution.*

Proof. (a) Let $\lambda > \hat{\lambda}_1$ and let $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$ be the energy functional for problem (P_λ) (see the proof of Proposition 4). On account of hypothesis H_1^c(ii), we see that $\varphi_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \min\left\{\varphi_\lambda(u) : u \in W^{1,p}(\Omega)\right\}. \tag{42}$$

On account of hypothesis H_1^c(iii) given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{p} x^p \text{ for a.a.} z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta. \tag{43}$$

Choose $t \in (0,1)$ small so that $0 < t\hat{u}_1(z) \leqslant \delta$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). Then, using (43) we have

$$\varphi_\lambda(t\hat{u}_1) \leqslant \frac{t^p}{p}\left(\hat{\lambda}_1 + \varepsilon - \lambda\right) \text{ (recall } \|\hat{u}_1\|_p = 1).$$

Since $\lambda > \hat{\lambda}_1$, we choose $\varepsilon \in (0, \lambda - \hat{\lambda}_1)$ and have

$$\begin{aligned} &\varphi_\lambda(t\hat{u}_1) < 0, \\ \Rightarrow &\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (42))}, \\ \Rightarrow &u_\lambda \neq 0. \end{aligned}$$

From (42) we have $\langle \varphi_\lambda'(u_\lambda), h\rangle = 0$ for all $h \in W^{1,p}(\Omega)$ and by choosing $h = -u_\lambda^- \in W^{1,p}(\Omega)$, we see that $u_\lambda \geqslant 0$, $u_\lambda \neq 0$. From Proposition 2.10 of Papageorgiou and Rădulescu [18], we know that $u_\lambda \in L^\infty(\Omega)$. Then, Theorem 2 of Lieberman [16] implies that $u_\lambda \in C_+ \setminus \{0\}$. Finally, the nonlinear maximum principle implies that $u_\lambda \in \operatorname{int} C_+$.

Next we check the uniqueness of u_λ. So, suppose that $v_\lambda \in W^{1,p}(\Omega)$ is another positive solution of problem (P_λ). Again, we have $v_\lambda \in \operatorname{int} C_+$. We have

$$\begin{aligned} &\int_\Omega \left(\lambda - \frac{f(z,u_\lambda)}{u_\lambda^{p-1}}\right)(u_\lambda^p - v_\lambda^p)\mathrm{d}z \\ =&\int_\Omega \left(\lambda u_\lambda^{p-1} - f(z,u_\lambda)\right)(u_\lambda - \frac{v_\lambda^p}{u_\lambda^{p-1}})\mathrm{d}z \\ =&\int_\Omega \left(-\Delta_p u_\lambda + \xi(z)u_\lambda^{p-1}\right)(u_\lambda - \frac{v_\lambda^p}{u_\lambda^{p-1}})\mathrm{d}z \\ =&\int_\Omega |Du_\lambda|^{p-2}\left(Du_\lambda, D\left(u_\lambda - \frac{v_\lambda^p}{u_\lambda^{p-1}}\right)\right)_{\mathbb{R}^N}\mathrm{d}z + \int_{\partial\Omega}\beta(z)(u_\lambda^p - v_\lambda^p)\mathrm{d}\sigma \\ +&\int_\Omega \xi(z)(u_\lambda^p - v_\lambda^p)\mathrm{d}z \text{ (using Green's identity)} \\ =&\|Du_\lambda\|_p^p - \|Dv_\lambda\|_p^p + \int_\Omega R(v_\lambda,u_\lambda)\mathrm{d}z + \int_\Omega \xi(z)(u_\lambda^p - v_\lambda^p)\mathrm{d}z \\ +&\int_{\partial\Omega}\beta(z)(u_\lambda^p - v_\lambda^p)\mathrm{d}\sigma \end{aligned} \tag{44}$$

(using the nonlinear Picone's identity, see Allegretto and Huang [18]).

Interchanging the roles of u_λ and v_λ in the above argument, we obtain

$$\begin{aligned}&\int_\Omega \left(\lambda - \frac{f(z,v_\lambda)}{v_\lambda^{p-1}}\right)(v_\lambda^p - u_\lambda^p)\mathrm{d}z \\ &= \|Dv_\lambda|_p^p - \|Du_\lambda\|_p^p + \int_\Omega R(u_\lambda, v_\lambda)\mathrm{d}z + \int_\Omega \xi(z)(v_\lambda^p - u_\lambda^p)\mathrm{d}z \\ &+ \int_{\partial\Omega} \beta(z)(v_\lambda^p - u_\lambda^p)\mathrm{d}\sigma. \qquad (45)\end{aligned}$$

Adding (44) and (45) and using hypothesis H_1^c(iv) and the fact that $R \geqslant 0$, we obtain $u_\lambda = v_\lambda$, which proves the uniqueness of the positive solution $u_\lambda \in \operatorname{int} C_+$ of problem (P_λ).

Now let $\lambda_n \downarrow \hat{\lambda}_1$ and let $u_n = u_{\lambda_n} \in \operatorname{int} C_+$ be the unique solution of problem (p_{λ_n}). As in the proof of Theorem 12, we have

$$u_n \to \hat{u} \text{ in } C^1(\overline{\Omega}).$$

Then, in the limit as $n \to \infty$ we have

$$\left\{\begin{array}{ll} -\Delta_p \hat{u} + \xi(z)\hat{u}^{p-1} = \hat{\lambda}_1 \hat{u}^{p-1} - f(z,\hat{u}) & \text{in } \Omega, \\ \dfrac{\partial \hat{u}}{\partial n_p} + \beta(z)\hat{u}^{p-1} = 0 & \text{on } \partial\Omega,\ \hat{u} \geqslant 0. \end{array}\right\} \qquad (46)$$

If $\hat{u} \neq 0$, then from (46) we have

$$\gamma_p(\hat{u}) = \hat{\lambda}_1 \|\hat{u}\|_p^p - \int_\Omega f(z,\hat{u})\hat{u}\mathrm{d}z < \hat{\lambda}_1 \|\hat{u}\|_p^p \text{ (see hypothesis } H_1^c\text{(iv))}.$$

This contradicts (41). Hence, $\hat{u} = 0$ and we have

$$u_\lambda \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to 0^+.$$

(b) Suppose $0 < \lambda \leqslant \hat{\lambda}_1$. If λ is admissible, we can find $u_\lambda \in \operatorname{int} C_+$ such that

$$\langle \frac{1}{p}\hat{\gamma}'_p(u_\lambda), h\rangle = \int_\Omega \left(\lambda u_\lambda^{p-1} - f(z,u_\lambda)\right) h\mathrm{d}z \text{ for all } h \in W^{1,p}(\Omega).$$

Let $h = u_\lambda \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned}\hat{\gamma}_p(u) &= \lambda\|u_\lambda\|_p^p + \int_\Omega f(z,u_\lambda)u_\lambda\mathrm{d}z \\ &\Rightarrow 0 > \left(\hat{\lambda}_1 - \lambda\right)\|u_\lambda\|_p^p \text{ (see hypothesis } H_1^c\text{(iv))},\end{aligned}$$

a contradiction. So $\lambda \in (0, \hat{\lambda}_1]$ is not admissible.

This proof is now complete. □

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