Analysis and Applications, Vol. 13, No. 5 (2015) 481–506

DOI: 10.1142/S0219530514500134

Resonant $(p, 2)$-equations with asymmetric reaction

Nikolaos S. Papageorgiou

National Technical University
Department of Mathematics
Zografou Campus, Athens 15780, Greece
npapg@math.ntua.gr

Vicenţiu D. Rădulescu

Department of Mathematics, Faculty of Science
King Abdulaziz University, Jeddah, Saudi Arabia

Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764, 014700 Bucharest, Romania
vicentiu.radulescu@imar.ro

Received 8 November 2013
Accepted 26 November 2013
Published 21 April 2014

We consider a nonlinear, nonhomogeneous Dirichlet problem driven by the sum of a p-Laplacian and a Laplacian, $2 < p < \infty$ ($(p, 2)$-equation) and with a reaction which exhibits asymmetric behavior at $+\infty$ and at $-\infty$ and is resonant. Using variational methods together with Morse theoretic arguments, we prove the existence of two and three nontrivial solutions.

Keywords: $(p, 2)$-Equation; resonance; nonlinear regularity; critical groups; Picone's identity.

Mathematics Subject Classification 2010: 35B34, 35J20, 35J60, 58E05

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $p > 2$ be a real number.

In this paper we study the following nonlinear nonhomogeneous elliptic equation ($(p, 2)$-equation):

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{1.1}$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(\|Du\|^{p-2} Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory reaction (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous). The aim of this work is to prove a multiplicity theorem (in particular, a three solutions theorem), when the reaction $f(z,\cdot)$ is $(p-1)$-linear near $\pm\infty$, but exhibits asymmetric behavior at $+\infty$ and at $-\infty$. More precisely, we assume that the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ crosses the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ as $x \in \mathbb{R}$ moves from $-\infty$ to $+\infty$. Another interesting feature of our framework is that we allow for resonance to occur at both $+\infty$ and $-\infty$. At $+\infty$ the resonance can occur with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$, while at $-\infty$ with respect to the second eigenvalue $\hat{\lambda}_2(p) > \hat{\lambda}_1(p)$.

Problems with an asymmetric nonlinearity were studied by Chabrowski and Yang [5], Chang [6], de Paiva and Massa [11], de Paiva and Presoto [12], Motreanu, Motreanu and Papageorgiou [18], Perera [25] (for semilinear Dirichlet problems), by Motreanu, Motreanu and the first author [19] (for nonlinear equations driven by the Dirichlet p-Laplacian) and by the authors [21] (for semilinear Neumann problems with an indefinite and unbounded potential). None of the aforementioned works permits resonance.

We mention that $(p,2)$-equations (that is, equations driven by the sum of a p-Laplacian and a Laplacian, with $2 < p < \infty$) arise in mathematical physics (see [4] (quantum physics) and [7] (plasma physics)). Recently some existence and multiplicity results for such equations were proved by Cingolani and Degiovanni [9], Cingolani and Vannella [10], the authors [22, 23], and Sun [28]. However, none of the aforementioned works treats the asymmetric resonant case. We also refer to the recent book by Ciarlet [8] for the rigorous qualitative analysis of many models described by nonlinear partial differential equations.

Our approach combines variational methods based on the critical point theory with Morse theoretic arguments (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools that will be used in the sequel.

2. Mathematical Background

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short) if the following property holds:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ, which is more general than the usual Palais–Smale condition. Both conditions compensate for the fact that the ambient space X need not be locally compact. Using the C-condition, we can have the following minimax characterization of certain critical values of φ. The result is know in the literature as the "mountain pass theorem" (see, for example, [13, 27]).

Theorem 2.1. *Let X be a Banach space. Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$ with $\|x_1 - x_0\| > \rho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \eta_\rho,$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = x_0, \gamma(1) = x_1\}$. Then $c \ge \eta_\rho$ and c is a critical value of φ.

In the analysis of problem (1.1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space $C_0^1(\bar\Omega)$ defined by

$$C_0^1(\bar\Omega) = \{u \in C^1(\bar\Omega) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\bar\Omega) : u(z) \ge 0 \text{ for all } z \in \bar\Omega\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth in the $x \in \mathbb{R}$ variable, that is,

$$|f_0(z,x)| \le a(z)(1 + |z|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with

$$a \in L^\infty(\Omega)_+, \quad 1 < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z, u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is a special case of a more general result due to Aizicovici, the first author and Staicu [2] and it relates local $C_0^1(\bar\Omega)$ and $W_0^{1,p}(\Omega)$-minimizers of φ_0. The result is essentially a consequence of the nonlinear regularity theory (see [15, 17]).

Proposition 2.2. *Let $\hat{u} \in W_0^{1,p}(\Omega)$ be a local $C_0^1(\bar{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(\hat{u}) \leq \varphi_0(\hat{u} + h) \quad \text{for all } h \in C_0^1(\bar{\Omega}),\ \|h\|_{C_0^1(\bar{\Omega})} \leq \rho_0.$$

Then $\hat{u} \in C_0^{1,\beta}(\bar{\Omega})$ with $\beta \in (0,1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(\hat{u}) \leq \varphi_0(\hat{u} + h) \quad \text{for all } h \in W_0^{1,p}(\Omega),\ \|h\| \leq \rho_1.$$

In the above result and in the sequel we denote by $\|\cdot\|$ the norm of $W_0^{1,p}(\Omega)$. Using Poincaré's inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Note that by $\|\cdot\|$ we also denote the norm in $\mathbb{R}^N$. However, no confusion is possible since it will always be clear from the context which norm is used.

For every $x \in \mathbb{R}$, we set $x^{\pm} = \max\{0, \pm x\}$. Then for $u \in W_0^{1,p}(\Omega)$ we can define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^- \quad \text{and} \quad |u| = u^+ + u^-.$$

Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

This is the Nemytskii map corresponding to h.

For every $r \in (1, \infty)$, let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* (\frac{1}{r} + \frac{1}{r'} = 1)$ be the nonlinear map defined by

$$\langle A_r(u), v \rangle = \int_\Omega \|Du\|^{r-2} (Du, Dv)_{\mathbb{R}^N}\, dz \quad \text{for all } u, v \in W_0^{1,r}(\Omega). \tag{2.1}$$

If $r = 2$, then we write $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. The next proposition summarizes the basic properties of this map (see, for example, [13]).

Proposition 2.3. *The nonlinear map $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ defined by* (2.1) *is bounded* (*that is, maps bounded sets to bounded sets*), *continuous, strictly monotone* (*strongly monotone if* $r \geq 2$) *hence maximal monotone too and of type* $(S)_+$, *that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and* $\limsup_{n\to\infty} \langle A_r(u_n), u_n - u \rangle \leq 0$, *then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

Next we recall some basic facts about the spectrum of the Dirichlet r-Laplace operator. So, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following nonlinear weighted eigenvalue problem:

$$-\Delta_r u(z) = \hat{\lambda} m(z) |u(z)|^{r-2} u(z) \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{2.2}$$

A number $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue if problem (2.2) admits a nontrivial solution $\hat{u}$, which is an eigenfunction corresponding to $\hat{\lambda}$. Problem (2.2) admits a smallest eigenvalue denoted by $\hat{\lambda}_1(r, m)$. The following facts are known about this first eigenvalue:

- $\hat{\lambda}_1(r, m) > 0$;
- $\hat{\lambda}_1(r, m)$ is isolated, that is, we can find $\epsilon > 0$ such that the interval $(\hat{\lambda}_1(r, m), \hat{\lambda}_1(r, m) + \epsilon)$ contains no eigenvalues;
- $\hat{\lambda}_1(r, m)$ is simple, that is, if $\hat{u}_1, \hat{u}_2$ are eigenfunctions corresponding to $\hat{\lambda}_1(r, m)$ then $\hat{u}_1 = \xi \hat{u}_2$ for some $\xi \neq 0$;
- $\hat{\lambda}_1(r, m) > 0$ admits the following variational characterization:

$$\hat{\lambda}_1(r, m) = \inf \left\{ \frac{\|Du\|_r^r}{\displaystyle\int_\Omega m(z)|u|^r dz} : u \in W_0^{1,r}(\Omega), u \neq 0 \right\}. \tag{2.3}$$

The infimum in (2.3) is attained on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(r, m) > 0$. It is clear from (2.3) that the elements of this one-dimensional eigenspace do not change sign. In what follows by $\hat{u}_1(r, m)$ we denote the positive L^r-normalized (that is, $\|\hat{u}_1(r, m)\|_r = 1$) eigenfunction corresponding to $\hat{\lambda}_1(r, m) > 0$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, [13, pp. 737–738]) imply that $\hat{u}_1(r, m) \in \operatorname{int} C_+$. The first eigenvalue $\hat{\lambda}_1(r, m) > 0$, as a function of the weight m, exhibits the following strict monotonicity property.

Proposition 2.4. *Assume that $m, \hat{m} \in L^\infty(\Omega)_+ \backslash \{0\}, m(z) \leq \hat{m}(z)$ a.e. in $\Omega, m \neq \hat{m}$. Then $\hat{\lambda}_1(r, \hat{m}) < \hat{\lambda}_1(r, m)$.*

If $\sigma(r, m)$ denotes the set of eigenvalues of (2.2), then $\sigma(r, m)$ is closed. This fact and since $\hat{\lambda}_1(r, m) > 0$ is isolated, imply that the second eigenvalue is well-defined by

$$\hat{\lambda}_2^*(r, m) = \inf\{\hat{\lambda} : \hat{\lambda} \in \sigma(r, m), \hat{\lambda} > \hat{\lambda}_1(r, m)\}.$$

The Ljusternik–Schnirelmann minimax scheme produces a whole strictly increasing sequence $\{\hat{\lambda}_k(r, m)\}_{k \geq 1}$ of eigenvalues such that $\hat{\lambda}_k(r, m) \to +\infty$. We know that $\hat{\lambda}_2^*(r, m) = \hat{\lambda}_2(r, m)$, that is, the second eigenvalue and the second Ljusternik–Schnirelmann eigenvalue coincide. However, we do not know if the Ljusternik–Schnirelmann sequence exhausts the spectral set $\sigma(r, m)$. This is the case if $r = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equation). We point out that $\hat{\lambda}_1(r, m) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (that is, sign changing) eigenfunctions.

When $r = 2$, the eigenvalue problem is linear and so we can define the eigenspace for every eigenvalue. By $E(\hat{\lambda}_k(2, m))$ (for all $k \geq 1$) we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2, m)$. The elements of this eigenspace have the "unique continuation property", namely if $u \in E(\hat{\lambda}_k(2, m))$ and u vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. The regularity theory implies that $E(\hat{\lambda}_k(2, m)) \subseteq C_0^1(\bar{\Omega}), k \geq 1$. Also, we have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{\bigoplus_{k\geq 1} E(\hat{\lambda}_k(2,m))}.$$

If $m \equiv 1$, then we write $\hat{\lambda}_1(r,m) = \hat{\lambda}_1(r)$, $\hat{\lambda}_2(r,m) = \hat{\lambda}_2(r)$, and $\hat{u}_1(r,m) = \hat{u}_1(r)$.

As a straightforward consequence of (2.3) and of the fact that $\hat{u}_1(r) \in \operatorname{int} C_+$, we have the following easy lemma (see [20, p. 356]).

Lemma 2.5. *If $\vartheta \in L^\infty(\Omega), \vartheta(z) \leq \hat{\lambda}_1(p)$ a.e. in Ω and $\vartheta \neq \hat{\lambda}_1(p)$, then there exists $c_0 > 0$ such that*

$$\|Du\|_p^p - \int_\Omega \vartheta(z)|u|^p dz \geq c_0\|u\|^p \quad \textit{for all } u \in W_0^{1,p}(\Omega).$$

We conclude this section by recalling some basic facts from Morse theory (critical groups) which we will use in the sequel. So, let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ we denote by $H_k(Y_1, Y_2)$ the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2).

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \quad K_\varphi = \{x \in X : \varphi'(x) = 0\},$$
$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let $x \in X$ be an isolated critical point of φ with $\varphi(x) = c$ (that is, $x \in K_\varphi^c$). Then the critical groups of φ at x are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, \varphi^c \cap U\backslash\{x\}) \quad \text{for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geq 0.$$

The second deformation theorem (see, for example, [13, p. 628]) implies that the above definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We set

$$M(t, x) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, x) t^k \quad \text{for all } t \in \mathbb{R}, \ \text{all } x \in K_\varphi$$

$$P(t, \infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{x\in K_\varphi} M(t,x) = P(t,\infty) + (1+t)Q(t), \tag{2.4}$$

where $Q(t) = \sum_{k\geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

The next proposition is a useful tool in the computation of critical groups at infinity and it extends an earlier result for Hilbert spaces by Liang and Su [16].

Proposition 2.6. *Let X be a Banach space and assume that $(t,x) \mapsto h_t(x)$ is a $C^1([0,1]\times X)$-function which maps bounded sets to bounded sets, the maps $x \mapsto (h_t)'(x)$ and $t \mapsto \partial_t h_t(x)$ are both locally Lipschitz, h_0 and h_1 satisfy the C-condition*

$$|\partial_t h_t(x)| \leq c_1(\|u\|_X^q + \|u\|_X^p) \quad \textit{for all } x \in X$$

with $c_1 > 0$, $1 < q < p < \infty$, and there exist $\gamma_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that

$$h_t(x) \leq \gamma_0 \Rightarrow (1+\|x\|_X)\|(h_t)'(x)\|_{X^*} \geq \delta_0(\|x\|_X^q + \|x\|_X^p) \quad \textit{for all } t \in [0,1].$$

Then $C_k(h_0,\infty) = C_k(h_1,\infty)$ for all $k \geq 0$.

Proof. Since $h \in C^1([0,1]\times X)$, we know that it admits a pseudo-gradient vector field $\hat{v}_t(x)$ (see, for example, [13, p. 616]). From the construction of the pseudo-gradient vector field we deduce that

$$\hat{v}_t(x) = (\partial_t h_t(x), v_t(x)),$$

with $(t,x) \mapsto v_t(x)$ locally Lipschitz and for all $t \in [0,1]$, $v_t(\cdot)$ is the pseudo-gradient vector field corresponding to $h_t(\cdot)$. So, for all $t \in [0,1]$ and all $x \in X$, we have

$$\|(h_t)'(x)\|_{X^*}^2 \leq \langle (h_t)'(x), v_t(x)\rangle \quad \text{and} \quad \|v_t(x)\|_X \leq 2\|(h_t)'(x)\|_{X^*}. \tag{2.5}$$

Given $t \in [0,1]$, we consider the map $w_t : X \to X$ defined by

$$w_t(x) = -\frac{|\partial_t h_t(x)|}{\|(h_t)'(x)\|_{X^*}^2} v_t(x) \quad \text{for all } x \in X.$$

Evidently, $[t,x] \mapsto w_t(x)$ is well-defined and locally Lipschitz. Let $\gamma \leq \gamma_0$ be such that

$$h_0^\gamma \neq \emptyset \quad \text{or} \quad h_1^\gamma \neq \emptyset.$$

If we cannot find such a $\gamma \leq \gamma_0$, then $C_k(h,\infty) = C_k(h,\infty) = \delta_{k,0}\mathbb{Z}$ for all $k \geq 0$.

Assume that $h_0^\gamma \neq \emptyset$ and let $y \in h_0^\gamma$. We consider the following abstract Cauchy problem

$$\frac{d\sigma}{dt} = w_t(\sigma) \text{ on } [0,1], \quad \sigma(0) = y. \tag{2.6}$$

Problem (2.6) admits a local flow $\sigma(t,y)$ (see, for example, [13, p. 618]). In what follows, for notational simplicity, we drop y from the description of σ. Using the chain rule, we have

$$\begin{aligned}\frac{d}{dt}h_t(\sigma) &= \left\langle (h_t)'(\sigma), \frac{d\sigma}{dt}\right\rangle + \partial_t h_t(\sigma)\\ &= \left\langle (h_t)'(\sigma), \frac{-|\partial_t h_t(\sigma)|}{\|(h_t)'(\sigma)\|^2_{X^*}} v_t(\sigma)\right\rangle + \partial_t h_t(\sigma)\\ &\le -|\partial_t h_t(\sigma)| + \partial_t h_t(\sigma) \quad \text{(see (2.5))}\\ &\le 0\\ \Rightarrow t &\mapsto h_t(\sigma) \text{ is nonincreasing.}\end{aligned}$$

Hence for $t \ge 0$ small, we have

$$\begin{aligned}&h_t(\sigma(t)) \le h_0(\sigma(0)) = h_0(y) \le \gamma \le \gamma_0\\ &\Rightarrow (1 + \|\sigma(t)\|_X)\|(h_t)'(\sigma(t))\|_{X^*} \ge \delta_0(\|\sigma(t)\|^q_X + \|\sigma(t)\|^p_X).\end{aligned} \tag{2.7}$$

Then

$$\begin{aligned}|w_t(\sigma(t))| &\le \frac{|\partial_t h_t(\sigma(t))|}{\|(h_t)'(\sigma(t))\|^2_{X^*}}\|v_t(\sigma(t))\|_X\\ &\le \frac{C_1(\|\sigma(t)\|^q_X + \|\sigma(t)\|^p_X)}{\|(h_t)'(\sigma(t))\|^2_{X^*}} 2\|(h_t)'(x)\|_{X^\tau} \quad \text{(see (2.5))}\\ &\le \frac{C_1(\|\sigma(t)\|^q_X + \|\sigma(t)\|^p_X)}{\delta_0(\|\sigma(t)\|^q_X + \|\sigma(t)\|^p_X)}(1 + \|\sigma(t)\|_X) \quad \text{(see (2.7))}\\ &= \frac{C_1}{\delta_0}(1 + \|\sigma(t)\|_X) \quad \text{for all } t \in [0,1] \text{ small.}\end{aligned}$$

This means that the flow in (2.6) is global on $[0, 1]$.

Then $\sigma(t, \cdot)$ is a homeomorphism of h_0^γ onto a subset D_0 of h_1^γ. Also, reversing the time $t \to 1 - t$ and using the corresponding global flow $\sigma_*(\cdot, v)$ (here $v \in h_1^\gamma$), we deduce that h_1^γ is homeomorphic to a subset D_1 of h_0^γ.

Let

$$\eta(t, y) = \sigma_*(t, \sigma(1, y)) \quad \text{for all } (t, y) \in [0, 1] \times h_0^\gamma.$$

Then we have

$$\eta(0, \cdot) \text{ is homotopy equivalent to } \mathrm{id}|_{D_0}(\cdot) \text{ and } \eta(1, \cdot) = (\sigma_*)_1 \circ \sigma_1(\cdot). \tag{2.8}$$

Similarly, if

$$\eta_*(t, v) = \sigma(t, \sigma_*(1, v)) \quad \text{for all } (t, v) \in [0, 1] \times h_1^\gamma,$$

then

$$\eta_*(0, \cdot) \text{ is homotopy equivalent to } \mathrm{id}|_{D_1}(\cdot) \text{ and } \eta_*(1, \cdot) = \sigma_1 \circ (\sigma_*)_1(\cdot). \tag{2.9}$$

Recall that D_0 and H_0^γ are homeomorphic. Similarly D_1 and h_1^γ are homeomorphic. Combining these facts with (2.8) and (2.9), we infer that the level sets h_0^γ and h_1^γ

are homotopy equivalent. Therefore

$$\begin{aligned} &H_k(X, h_0^\gamma) = H_k(X, h_1^\gamma) \quad \text{for all } k \geq 0 \quad \text{(see [14, p. 387])} \\ \Rightarrow\ &C_k(h_0, \infty) = C_k(h, \infty) \quad \text{for all } k \geq 0. \end{aligned}$$

This completes the proof. □

3. Two Nontrivial Solutions

In this section we establish the existence of two nontrivial solutions for problem (1.1) without imposing any differentiability condition on $f(z, \cdot)$.

First we produce a positive solution. To this end, we impose the following conditions on the reaction $f(z, x)$:

$\underline{H_1}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + |x|^{p-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \hat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$ and if $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x\to+\infty} [f(z, x)x - pF(z, x)] = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exists a function $\eta \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_1(p) \leq \eta(z) \text{ for a.a. } z \in \Omega, \quad \hat{\lambda}_1(p) \neq \eta \quad \text{and}$$

$$\eta(z) \leq \liminf_{x\to-\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x\to-\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \hat{\lambda}_2(p) \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exist functions $\beta, \hat{\beta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_1(2) \leq \beta(z) \text{ for a.a. } z \in \Omega, \quad \hat{\lambda}_1(2) \neq \beta \quad \text{and}$$

$$\beta(z) \leq \liminf_{x\to 0} \frac{f(z, x)}{x} \leq \limsup_{x\to 0} \frac{f(z, x)}{x} \leq \hat{\beta}(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) $f(z, x)x - pF(z, x) \geq 0$ for a.a. $z \in \Omega$, all $x \leq 0$, and $f(z, \cdot)$ is lower locally Lipschitz on $[0, +\infty)$.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.1. *Assume that hypotheses H_1 hold. Then the functional φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for all } n \geq 1, \text{ some } M_1 > 0 \tag{3.1}$$

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$\left|\langle A_p(u_n), h\rangle - \int_\Omega f(z, u_n) h dz\right| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+. \tag{3.3}$$

In (3.3) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$\left|\|Du_n^+\|_p^p + \|Du_n^+\|_2^2 - \int_\Omega f(z, u_n^+) u_n^+ dz\right| \leq \epsilon_n \quad \text{for all } n \geq 1. \tag{3.4}$$

Using (3.4) we will show that the sequence $\{u_n^+\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ is bounded. Arguing indirectly, suppose that the sequence is not bounded in $W_0^{1,p}(\Omega)$. Then by passing to a subsequence if necessary, we may assume that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \geq 1$. Then $\|y_n\| = 1$ and $y_n \geq 0$ for all $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega). \tag{3.5}$$

From (3.4), we have

$$\|Dy_n\|_p^p \leq \frac{\epsilon_n}{\|u_n^+\|^p} + \int_\Omega \frac{f(z, u_n^+)}{\|u_n^+\|^{p-1}} y_n dz \quad \text{for all } n \geq 1. \tag{3.6}$$

From hypothesis H_1(i) it is clear that

$$\left\{\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^{p'}(\Omega) \text{ as } n \to \infty. \tag{3.7}$$

Using hypothesis H_1(ii), as in [1], we show that

$$\begin{aligned} &g(z) = \vartheta(z) y(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ with } \vartheta \in L^\infty(\Omega), \\ &\vartheta(z) \leq \hat{\lambda}_1(p) \text{ a.e. in } \Omega. \end{aligned} \tag{3.8}$$

Hence, if in (3.6) we pass to the limit as $n \to \infty$ and use (3.5), (3.7), (3.8) we obtain

$$\|Dy\| - \int_\Omega \vartheta(z) y^p dz \leq 0. \tag{3.9}$$

If $\vartheta \neq \hat{\lambda}_1(p)$, then from (3.9) and Lemma 2.5, we have

$$c_0 \|y\|^p \leq 0, \quad \text{hence } y = 0.$$

Then from (3.6) it follows that $Dy_n \to 0$ in $L^p(\Omega, \mathbb{R}^N)$ and so $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$.

Next suppose that $\vartheta(z) = \hat{\lambda}_1(p)$ a.e. in Ω. Then from (3.9) and (2.3) we have

$$\|Dy\|_p^p = \hat{\lambda}_1(p)\|y\|_p^p$$
$$\Rightarrow y = \xi \hat{u}_1(p) \quad \text{for some } \xi > 0 \quad (\text{see } (2.3)).$$

Since $y \in \operatorname{int} C_+$, we have $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$ and so by virtue of hypothesis H_1(ii) we have

$$f(z, u_n^+(z))u_n^+(z) - pF(z, u_n^+(z)) \to \infty \quad \text{for a.a. } z \in \Omega$$
$$\Rightarrow \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \to +\infty \quad (\text{by Fatou's lemma}). \tag{3.10}$$

On the other hand, from (3.1) we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \int_\Omega pF(z, u_n)dz \leq p\ M_1 \quad \text{for all } n \geq 1. \tag{3.11}$$

Also from (3.3) with $h = u_n \in W_0^{1,p}(\Omega)$, we obtain

$$-\|Du_n\|_p^p - \|Du_n\|_2^2 + \int_\Omega f(z, u_n)u_n dz \leq \epsilon_n \quad \text{for all } n \geq 1. \tag{3.12}$$

Adding (3.11) and (3.12) we have

$$\left(\frac{p}{2} - 1\right)\|Du_n\|_2^2 + \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz$$
$$\leq M_2 \quad \text{for some } M_2 > 0, \ \text{all } n \geq 1$$
$$\Rightarrow \int_\Omega [f(z, u_n)u_n - pF(z, u_n)]dz \leq M_2 \quad \text{for all } n \geq 1 \ (\text{recall } p > 2)$$
$$\Rightarrow \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz + \int_\Omega [f(z, -u_n^-)(-u_n^-) - pF(z, -u_n^-)]dz$$
$$\leq M_2 \quad \text{for all } n \geq 1$$
$$\Rightarrow \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq M_2 \quad \text{for all } n \geq 1 \ (\text{see } H_1(v)). \tag{3.13}$$

Comparing (3.10) and (3.13), we reach a contradiction which proves that

$$\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.14}$$

Next we show that $\{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Again we argue by contradiction. So, assume that $\|u_n^-\| \to \infty$ and let $v_n = \frac{u_n^-}{\|u_n^-\|}$ $n \geq 1$. Then $\|v_n\| = 1$, $v_n \geq 0$

for all $n \geq 1$ and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad v_n \to v \text{ in } L^p(\Omega), \;\; v \geq 0. \tag{3.15}$$

From (3.3) and (3.14), we have

$$\begin{aligned} &\left| \langle A_p(-u_n^-), h\rangle + \langle A(-u_n^-), h\rangle - \int_\Omega f(z, -u_n^-) h dz \right| \\ &\qquad \leq M_3 \|h\| \quad \text{for some } M_3 > 0, \;\; \text{all } n \geq 1 \\ \Rightarrow &\left| \langle A_p(-v_n), h\rangle + \frac{1}{\|u_n^-\|^{p-2}} \langle A(-v_n), h\rangle - \int_\Omega \frac{f(z, -u_n^-)}{\|u_n^-\|^{p-1}} h dz \right| \\ &\qquad \leq \frac{M_3 \|h\|}{\|u_n^-\|^{p-1}} \quad \text{for all } n \geq 1. \end{aligned} \tag{3.16}$$

Hypothesis H_1(i) implies that

$$\left\{ \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \right\}_{n \geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

Passing to a subsequence if necessary and using hypothesis H_1(iii) we have

$$\begin{aligned} \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \xrightarrow{w} -\hat{\eta} v^{p-1} \text{ in } L^{p'}(\Omega) \quad \text{with } \eta(z) \leq \hat{\eta}(z) \leq \hat{\lambda}_2(p) \\ \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.17}$$

In (3.16) we choose $h = v - v_n \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.15) and (3.17). Since $p > 2$, we obtain

$$\begin{aligned} &\lim_{n \to \infty} \langle A_p(-v_n), v - v_n\rangle = 0 \\ &\Rightarrow v_n \to v \text{ in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.3)}, \quad \text{hence } \|v\| = 1, \;\; v \geq 0. \end{aligned} \tag{3.18}$$

Therefore, if in (3.16) we pass to the limit as $n \to \infty$ and use (3.17) and (3.18) and the fact that $p > 2$, we deduce that

$$\begin{aligned} &\langle A_p(-v), h\rangle = \int_\Omega -\hat{\eta} v^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \\ \Rightarrow\; &A_p(v) = \hat{\eta} v^{p-1} \\ \Rightarrow\; &-\Delta_p v(z) = \hat{\eta}(z) v(z)^{p-1} \quad \text{a.e. in } \Omega, \;\; v|_{\partial\Omega} = 0. \end{aligned} \tag{3.19}$$

From Proposition 2.4, we have

$$\hat{\lambda}_1(p, \hat{\eta}) < \hat{\lambda}_1(p, \hat{\lambda}_1(p)) = 1.$$

So, from (3.19) it follows that v must be nodal, which contradicts (3.18). This proves that

$$\begin{aligned} &\{u_n^-\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded} \\ \Rightarrow\; &\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded} \quad \text{(see (3.14))}. \end{aligned}$$

Hence, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{3.20}$$

In (3.3) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.20). Then

$$\begin{aligned}
&\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0 \\
\Rightarrow &\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \le 0 \quad \text{(since } A \text{ is monotone)} \\
\Rightarrow &\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \le 0 \\
\Rightarrow &u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 2.3).}
\end{aligned}$$

This proves that φ satisfies the C-condition. □

We consider the positive truncation of $f(z,\cdot)$ defined by

$$f_+(z,x) = f(z,x^+).$$

This is a Carathéodory function. We set $F_+(z,x) = \int_0^x f_+(z,s)ds$ and consider the C^1-functional $\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_+(z,u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.2. *Assume that hypotheses H_1 hold. Then the functional φ_+ is coercive.*

Proof. We argue indirectly. So, suppose that φ_+ is not coercive. Then we can find $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ and $M_4 > 0$ such that

$$\|u_n\| \to \infty \text{ as } n \to \infty \quad \text{and} \quad \varphi_+(u_n) \le M_4 \text{ for all } n \ge 1.$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \ge 1$. Then $\|y_n\| = 1$ for all $n \ge 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega). \tag{3.21}$$

We have

$$\begin{aligned}
&\frac{1}{p}\|Du_n\|_p^p + \frac{1}{2}\|Du_n\|_2^2 - \int_\Omega F_+(z,u_n)dz \le M_4 \quad \text{for all } n \ge 1 \\
\Rightarrow &\frac{1}{p}\|Dy_n\|_p^p + \frac{1}{2\|u_n\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \frac{F_+(z,u_n)}{\|u_n\|^p}dz \le \frac{M_4}{\|u_n\|^p} \quad \text{for all } n \ge 1.
\end{aligned} \tag{3.22}$$

Hypothesis H_1(ii) implies that given $\epsilon > 0$, we can find $M_5 = M_5(\epsilon) > 0$ such that

$$\begin{aligned} & f(z,x) \le (\hat{\lambda}_1(p) + \epsilon)x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M_5 \\ \Rightarrow\ & F(z,x) \le \frac{1}{p}(\hat{\lambda}_1(p) + \epsilon)x^p \quad \text{for a.a. } x \in \Omega, \text{ all } x \ge M_5 \\ \Rightarrow\ & \frac{pF(z,x)}{x^p} \le \hat{\lambda}_1(p) + \epsilon \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M_5 \\ \Rightarrow\ & \limsup_{x\to+\infty} \frac{pF(z,x)}{x^p} \le \hat{\lambda}_1(p) + \epsilon \quad \text{uniformly for a.a. } z \in \Omega. \end{aligned}$$

Since $\epsilon > 0$, is arbitrary, we let $\epsilon \downarrow 0$ to conclude that

$$\limsup_{x\to+\infty} \frac{pF_+(z,x)}{x^p} \le \hat{\lambda}_1(p) \quad \text{uniformly for a.a. } z \in \Omega. \tag{3.23}$$

Hypothesis H_1(i) implies that

$$\Rightarrow \left\{ \frac{F_+(\cdot, u_n(\cdot))}{\|u_n\|^p} \right\}_{n\ge1} \subseteq L^1(\Omega) \quad \text{uniformly integrable.}$$

Then from the Dunford–Pettis theorem and using (3.23), at least for a subsequence, we have

$$\begin{aligned} \frac{F_+(\cdot, u_n(\cdot))}{\|u_n\|^p} \xrightarrow{w} \frac{1}{p}\vartheta(y^+)^p \quad \text{in } L^1(\Omega) \text{ with } \vartheta \in L^\infty(\Omega), \\ \vartheta(z) \le \hat{\lambda}_1(p) \text{ a.a. in } \Omega. \end{aligned} \tag{3.24}$$

We return to (3.22), pass to the limit as $n \to \infty$ and use (3.21) and (3.24). Since $2 < p$, we obtain

$$\frac{1}{p}\|Dy\|_p^p \le \frac{1}{p}\int_\Omega \vartheta(y^+)^p dz \tag{3.25}$$

$$\Rightarrow \|Dy^+\|_p^p \le \int_\Omega \vartheta(y^+)^p dz. \tag{3.26}$$

If $\vartheta \ne \hat{\lambda}_1(p)$, then from (3.26) and Lemma 2.5, we have

$$c_0\|y^+\|^p \le 0, \quad \text{hence } y \le 0.$$

Then from (3.25) it follows that $y = 0$ and this from (3.22) implies that

$$\begin{aligned} & Dy_n \to 0 \quad \text{in } L^p(\Omega, \mathbb{R}^N) \\ \Rightarrow\ & y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega). \end{aligned}$$

This contradicts the fact that $\|y_n\| = 1$ for all $n \ge 1$.

Next we assume that $\vartheta(z) = \hat{\lambda}_1(p)$ a.e. in Ω. In this case from (3.26), we have $y^+ = \xi\hat{u}_1(p)$ with $\xi \ge 0$. If $\xi = 0$, then as above we reach a contradiction. So, we assume that $\xi > 0$. We have

$$u_n^+(z) \to +\infty \quad \text{for a.a. } z \in \Omega. \tag{3.27}$$

By virtue of hypothesis H_1(ii), given $\xi > 0$, we can find $M_6 = M_6(\xi) > 0$ such that

$$f(z,x)x - pF(z,x) \geq \xi \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq M_6. \tag{3.28}$$

For a.a. $z \in \Omega$ and all $x \geq M_6$, we have

$$\begin{aligned} \frac{d}{dx}\frac{F(z,x)}{x^p} &= \frac{f(z,x)x^p - pF(z,x)x^{p-1}}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\ &\geq \frac{\xi}{x^{p+1}} \quad \text{(see (3.28))} \\ \Rightarrow \frac{F(z,x)}{v^p} - \frac{F(z,x)}{x^p} &\geq -\frac{\xi}{p}\left[\frac{1}{v^p} - \frac{1}{x^p}\right] \quad \text{for a.a. } z \in \Omega, \ \text{all } v \geq x \geq M_6. \end{aligned} \tag{3.29}$$

So, if in (3.29) we let $v \to +\infty$ and use (3.23), we obtain

$$\begin{aligned} &\frac{\hat{\lambda}_1(p)}{p} - \frac{F(z,x)}{x^p} \geq \frac{\xi}{p}\frac{1}{x^p} \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq M_6 \\ \Rightarrow &\frac{\hat{\lambda}_1(p)}{p}\, x^p - F(z,x) \geq \frac{\xi}{p} \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq M_6 \\ \Rightarrow &\lim_{x \to +\infty}\left[\frac{\hat{\lambda}_1(p)}{p}x^p - F(z,x)\right] \geq \frac{\xi}{p} \quad \text{uniformly for a.a. } z \in \Omega. \end{aligned}$$

But $\xi > 0$ is arbitrary. So, we conclude that

$$\lim_{x \to +\infty}\left[\frac{\hat{\lambda}_1(p)}{p}x^p - F(z,x)\right] = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Thus, using (3.27), we have

$$\begin{aligned} &\frac{\hat{\lambda}_1(p)}{p}u_n^+(z)^p - F(z,u_n^+(z)) \to +\infty \quad \text{for a.a. } z \in \Omega \\ \Rightarrow &\int_\Omega \left[\frac{\hat{\lambda}_1(p)}{p}u_n^+(z)^p - F_+(z,u_n^+(z))\right] dz \to +\infty \quad \text{(by Fatou's Lemma).} \end{aligned} \tag{3.30}$$

Recall that

$$\begin{aligned} &\frac{1}{p}\|Du_n\|_p^p + \frac{1}{2}\|Du_n\|_2^2 - \int_\Omega F_+(z,u_n^+)dz \leq M_4 \quad \text{for all } n \geq 1, \\ &\int_\Omega \left[\frac{\hat{\lambda}_1(p)}{p}(u_n^+)^p - F_+(z,u_n^+)\right] dz \leq M_4 \quad \text{for all } n \geq 1 \ \text{(see (2.3)).} \end{aligned} \tag{3.31}$$

Comparing (3.30) and (3.31), we have a contradiction which proves that φ_+ is coercive. $\square$

Now we can produce a first solution for problem (1.1) which is positive.

Proposition 3.3. *Assume that hypotheses H_1 hold. Then problem* (1.1) *admits a positive solution $u_0 \in \operatorname{int} C_+$ which is a local minimizer of the energy functional φ.*

Proof. From Proposition 3.2, we know that φ_+ is coercive. Also, using the Sobolev embedding theorem, we see that φ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_+(u_0) = \inf\{\varphi_+(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.32}$$

Hypothesis H_1(iv) implies that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{2}(\beta(z) - \epsilon)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{3.33}$$

Since $\hat{u}_1(2) \in \operatorname{int} C_+$, for $\lambda \in (0,1)$ small, we have $\lambda\hat{u}_1(2)(z) \in [0,\delta]$ for all $z \in \bar{\Omega}$. Then

$$\begin{aligned}\varphi_+(\lambda\hat{u}_1(2)) &= \frac{\lambda^p}{p}\|D\hat{u}_1(2)\|_p^p + \frac{\lambda^2}{2}\hat{\lambda}_1(2) - \int_\Omega F_+(z, \lambda\hat{u}_1(2))dz \\ &\leq \frac{\lambda^p}{p}\|D\hat{u}_1(2)\|_p^p \\ &\quad - \frac{\lambda^2}{2}\left[\int_\Omega (\beta(z) - \hat{\lambda}_1(2))\hat{u}_1(2)^2 dz + \epsilon\hat{\lambda}_1(2)\right] \quad \text{(see (3.33))}.\end{aligned}$$

The hypothesis on $\beta(\cdot)$ (see H_1(iv)) and since $\hat{u}_1(2) \in \operatorname{int} C_+$, imply that

$$\xi_* = \int_\Omega (\beta(z) - \hat{\lambda}_1(2))\hat{u}_1(2)^2 dz > 0.$$

So, if we choose $\epsilon \in (0, \xi_*/\hat{\lambda}_1(2))$, then

$$\begin{aligned}&\varphi_+(\lambda\hat{u}_1(2)) < 0 \\ \Rightarrow\ &\varphi_+(u_0) < 0 = \varphi_+(0) \ \text{(see (3.32))}, \quad \text{hence } u_0 \neq 0.\end{aligned}$$

From (3.32) we have

$$\begin{aligned}&\varphi_+'(u_0) = 0 \\ \Rightarrow\ &A_p(u_n) + A(u_n) = N_f(u_0).\end{aligned} \tag{3.34}$$

On (3.34) we act with $-u_n^- \in W_0^{1,p}(\Omega)$. We obtain

$$\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 = 0, \quad \text{hence } u_0 \geq 0, \ u_0 \neq 0.$$

Then from (3.34) we have

$$-\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \ \text{a.e. in } \Omega, \quad u_0|_{\partial\Omega} = 0.$$

From [15, p. 286], we know that $u_0 \in L^\infty(\Omega)$. Then Theorem 2.1 of [17] implies that $u_0 \in C_+\backslash\{0\}$.

Evidently hypotheses H_1(i), (iv) imply that for every $\rho > 0$, we can find $\hat{\xi}_\rho > 0$ such that $f(z,x)x + \xi_\rho |x|^p \geq 0$ for a.a. $z \in \Omega$, all $|x| \leq \rho$. Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ as just mentioned. We have

$$\begin{aligned} &-\Delta_p u_0(z) - \Delta u_0(z) + \hat{\xi}_\rho u_0(z)^{p-1} = f(z, u_0(z)) + \hat{\xi}_\rho u_0(z)^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega \\ &\Rightarrow \Delta_p u_0(z) + \Delta u_0(z) \leq \xi_\rho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.35}$$

Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be the C^1-map defined by

$$a(y) = \|y\|^{p-2} y + y \quad (\text{recall } p > 2).$$

We have $\operatorname{div} a(Du) = \Delta_p u + \Delta u$ for all $u \in W_0^{1,p}(\Omega)$ and

$$\begin{aligned} \nabla a(y) &= \|y\|^{p-2}\left(I + (p-2)\frac{y \otimes y}{\|y\|^2}\right) + I \quad \text{for all } y \in \mathbb{R}^N \\ \Rightarrow (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} &\geq \|\xi\|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N. \end{aligned}$$

Then the tangency principle of [26, p. 35] implies that

$$u_0(z) > 0 \quad \text{for all } z \in \Omega.$$

So, from (3.35) and the boundary point theorem of [26, p. 120], we conclude that $u_0 \in \operatorname{int} C_+$.

Note that $\varphi|_{C_+} = \varphi_+|_{C_+}$. So, $u_0 \in \operatorname{int} C_+$ is a local $C_0^1(\bar{\Omega})$-minimizer of φ. Invoking Proposition 2.2, we conclude that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ.

□

To produce a second nontrivial solution, we need to restrict the behavior of $f(z,\cdot)$ near zero. More precisely, the new hypotheses on the reaction $f(z,x)$ are the following.

<u>H_2</u>: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_2(i), (ii), (iii), (v) are the same as the corresponding hypotheses H_1(i), (ii), (iii), (v) and

(iv) there exist an integer $m \geq 2$ and functions $\beta, \hat{\beta} \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_m(2) \leq \beta(z) \leq \hat{\beta}(z) \leq \hat{\lambda}_{m+1}(2) \text{ a.e. in } \Omega,$$

$$\hat{\lambda}_m(2) \neq \beta, \quad \hat{\lambda}_{m+1}(2) \neq \hat{\beta} \quad \text{and}$$

$$\beta(z) \leq \liminf_{x \to 0} \frac{f(z,x)}{x} \leq \limsup_{x \to 0} \frac{f(z,x)}{x} \leq \hat{\beta}(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Theorem 3.4. *Assume that hypotheses H_2 hold. Then problem* (1.1) *admits at least two nontrivial solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad \hat{u} \in C_0^1(\bar{\Omega}), \quad u_0 \neq \hat{u}.$$

Proof. From Proposition 3.3 we already have one nontrivial solution $u_0 \in \operatorname{int} C_+$, which is a local minimizer of φ. Hence as in [1] (see the proof of Proposition 29), we can find $\rho \in (0, 1)$ small such that

$$\varphi(u_0) < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho. \tag{3.36}$$

Hypothesis H_2(iii) implies that

$$\varphi(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to -\infty. \tag{3.37}$$

Recall that φ satisfies the C-condition (see Proposition 3.1). This fact, together with (3.36) and (3.37), permits the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_\varphi \quad \text{and} \quad \varphi(u_0) < m_\rho \leq \varphi(\hat{u}). \tag{3.38}$$

From (3.38) it is clear that $\hat{u} \neq u_0$ and it is a solution of problem (1.1). We need to show that $\hat{u} \neq 0$.

Since $\hat{u}$ is a critical point of φ of mountain pass type, we have

$$C_1(\varphi, \hat{u}) \neq 0. \tag{3.39}$$

Claim 1. $C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z}$ *for all* $k \geq 0$, *with* $d_m = \dim \bigoplus_{i=1}^m E(\hat{\lambda}_i(2))$.

Let $\mu \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and consider the C^2-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\mu}{2}\|Du\|_2^2 \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently ψ is coercive (recall that $p > 2$) and so it satisfies the C-condition.

We consider the homotopy $h : [0, 1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h(t, u) = (1 - t)\varphi(u) + t\psi(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{gathered} t_n \to t \text{ in } [0,1], \quad u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \\ h'_u(t_n, u_n) = 0 \quad \text{for all } n \geq 1. \end{gathered} \tag{3.40}$$

We have

$$A_p(u_n) + A(u_n) = (1 - t_n)N_f(u_n) + t_n\mu u_n \quad \text{for all } n \geq 1. \tag{3.41}$$

Let $y_n = \frac{u_n}{\|u_n\|} n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^2(\Omega). \tag{3.42}$$

From (3.41) we have

$$\|u_n\|^{p-2}A_p(y_n) + A(y_n) = (1-t_n)\frac{N_f(u_n)}{\|u_n\|} + t_n\mu y_n \quad \text{for all } n \geq 1. \tag{3.43}$$

Evidently $\{\frac{N_f(u_n)}{\|u_n\|}\}_{n\geq 1} \subseteq L^2(\Omega)$ is bounded (see H_2(i), (iv)) and by virtue of hypothesis H_2(iv) and (3.40), we have (at least for a subsequence)

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} gy \text{ in } L^2(\Omega) \quad \text{with } \beta(z) \leq g(z) \leq \hat{\beta}(z) \text{ a.e. in } \Omega. \tag{3.44}$$

Since A_p is bounded (see Proposition 2.3) and $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, if in (3.43) we pass to the limit as $n \to \infty$ and use (3.42) and (3.44), we obtain

$$\begin{aligned} &A(y) = [(1-t)g + t\mu]y \\ &\Rightarrow -\Delta y(z) = [(1-t)g(z) + t\mu]y(z) \quad \text{a.e. in } \Omega, y|_{\partial\Omega} = 0. \end{aligned} \tag{3.45}$$

Note that

$$\hat{\lambda}_m(2) \leq (1-t)g(z) + t\mu = g_t(z) \leq \hat{\lambda}_{m+1}(2) \text{ a.e. in } \Omega,$$
$$\hat{\lambda}_m(2) \neq g_t, \quad \hat{\lambda}_{m+1}(2) \neq g_t.$$

By virtue of the unique continuation property, we have

$$\hat{\lambda}_m(2, g_t) < \hat{\lambda}_m(2, \hat{\lambda}_m(2)) = 1 \quad \text{and} \quad 1 = \hat{\lambda}_{m+1}(2, \hat{\lambda}_{m+1}(2)) < \hat{\lambda}_{m+1}(2, g_t).$$

Then from (3.45) it follows that $y = 0$.

From (3.43), we have

$$\begin{aligned} &-\|u_n\|^{p-2}\Delta_p y_n(z) - \Delta y_n(z) \\ &\quad = (1-t_n)\frac{f(z, u_n(z))}{\|u_n\|} + t_n\mu y_n(z) \quad \text{a.e. in } \Omega, y_n|_{\partial\Omega} = 0. \end{aligned}$$

From [15], we know that we can find $M_7 > 0$ such that

$$\|y_n\|_\infty \leq M_7.$$

Then the regularity result of [17] implies that we can find $\gamma \in (0,1)$ and $M_8 > 0$ such that

$$y_n \in C_0^{1,\gamma}(\bar{\Omega}) \text{ and } \|y_n\|_{C_0^{1,\gamma}(\bar{\Omega})} \leq M_8 \quad \text{for all } n \geq 1.$$

Exploiting the compact embedding of $C_0^{1,\gamma}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, we have

$$\begin{aligned} &y_n \to 0 \quad \text{in } C_0^1(\bar{\Omega}) \quad \text{(see (3.42) and recall } y = 0) \\ &\Rightarrow y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega) \end{aligned}$$

which contradicts the fact that $\|y_n\| = 1$ for all $n \geq 1$. This implies that (3.40) cannot happen. Then the homotopy invariance property of critical groups implies that

$$C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \geq 0.$$

But since $\mu \in (\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2))$, by Theorem 1 of [10] we deduce that

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0.$$

This proves the claim.

From (3.39) and the claim, we conclude that $\hat{u} \neq 0$. Then $\hat{u}$ is the second nontrivial solution of (1.1) and by the nonlinear regularity theory (see [15, 17]), we have $\hat{u} \in C_0^1(\bar{\Omega})$. ☐

4. Three Solutions Theorem

In this section, we produce a third nontrivial solution for problem (1.1) (three solutions theorem). To do this we need to improve the regularity of $f(z, \cdot)$ and also avoid complete resonance at $+\infty$. So, the new hypotheses on the reaction $f(z, x)$ are the following:

$\underline{H_3}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega, f(z, 0) = 0, f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z, x)| \leq a(z)(1 + |x|^{p-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) there exists a function $\vartheta \in L^\infty(\Omega)_+$ such that $\vartheta(z) \leq \hat{\lambda}_1(p)$ a.e. in $\Omega, \vartheta \neq \hat{\lambda}_1(p)$ and

$$\limsup_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \leq \vartheta(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exists a function $\eta \in L^\infty(\Omega)_+$ such that

$$\hat{\lambda}_1(p) \leq \eta(z) \text{ for a.a. } z \in \Omega, \quad \hat{\lambda}_1(p) \neq \eta \quad \text{and}$$

$$\eta(z) \leq \liminf_{x \to -\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \limsup_{x \to -\infty} \frac{f(z, x)}{|x|^{p-2}x} \leq \hat{\lambda}_2(p) \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exists integer $m \geq 2$ such that

$$f'_x(z, 0) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ a.e. in } \Omega,$$

$$f'_x(\cdot, 0) \neq \hat{\lambda}_m(2), \quad f'_x(\cdot, 0) \neq \hat{\lambda}_{m+1}(2);$$

(v) $f(z, x)x - pF(z, x) \geq 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

Remark 4.1. Now at $+\infty$ we allow only nonuniform nonresonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ (see hypothesis H_3(ii)). The reason for this is the computation of the critical groups of φ at infinity based on Proposition 2.6 (see Proposition 4.2 below).

Proposition 4.2. *Assume that hypotheses H_3 hold. Then $C_k(\varphi, \infty) = 0$ for all $k \geq 0$.*

Proof. Let $\beta \in L^{\infty}(\Omega)_+, \beta \neq 0$ and $\mu \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$. We consider the following one-parameter family of C^1-functionals defined by

$$\begin{aligned} h_t(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - t\int_\Omega F(z,u(z))dz \\ &\quad - \frac{1-t}{p}\mu\|u^-\|_p^p + (1-t)\int_\Omega \beta(z)u(z)dz \quad \text{for all } (t,u) \in [0,1]\times W_0^{1,p}(\Omega). \end{aligned}$$

We have

$$\begin{aligned} &h_0(u) = \psi(u) = \frac{1}{p}\|Du\|_p^p - \frac{\mu}{p}\|u^-\|_p^p + \int_\Omega \beta(z)u(z)dz \quad \text{for all } u \in W_0^{1,p}(\Omega), \\ &h_1(u) = \varphi(u) \quad \text{for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

Since $\mu \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$, we can easily check that ψ satisfies the C-condition. Also, from Proposition 3.1 we know that φ satisfies the C-condition.

Claim 2. *There exist $\gamma_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that*

$$h_t(u) \leq \gamma_0 \Rightarrow (1+\|u\|)\|(h_t)'(u)\|_* \geq \delta_0(\|u\|^2 + \|u\|^p) \quad \textit{for all } t \in [0,1).$$

We proceed by contradiction. So, suppose that the claim is not true. Since the function $(t,u) \mapsto h_t(u)$ maps bounded sets to bounded sets, we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &t_n \to t, \quad \|u_n\| \to \infty, \quad h_{t_n}(u_n) \to -\infty \quad \text{and} \\ &(1+\|u_n\|)\|(h_{t_n})'(u_n)\|_* \leq \frac{1}{n}(\|u_n\|^2 + \|u_n\|^p) \quad \text{for all } n \geq 1. \end{aligned} \tag{4.1}$$

Let $y_n = \frac{u_n}{\|u_n\|} n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega). \tag{4.2}$$

From (4.1) we have

$$\begin{aligned} &\Big|\langle A_p(u_n), v\rangle + \langle A(u_n), v\rangle - t_n\int_\Omega f(z,u_n)vdz + (1-t_n)\mu\int_\Omega (u_n^-)^{p-1}vdz \\ &\qquad + (1-t_n)\int_\Omega \beta(z)vdz\Big| \leq \frac{1}{n}\frac{\|v\|}{1+\|u_n\|}(\|u_n\|^2 + \|u_n\|^p) \\ &\Rightarrow \Big|\langle A_p(y_n), v\rangle + \frac{1}{\|u_n\|^{p-2}}\langle A(y_n), v\rangle - t_n\int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}}vdz \\ &\qquad + (1-t_n)\mu\int_\Omega (y_n^-)^{p-1}vdz + \frac{1-t_n}{\|u_n\|^{p-1}}\int_\Omega \beta(z)vdz\Big| \leq \frac{\|v\|}{n} \quad \text{for all } n \geq 1. \end{aligned} \tag{4.3}$$

In (4.3) we choose $v = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.2). Since $p > 2$, we obtain

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0$$
$$\Rightarrow y_n \to y \quad \text{in } W_0^{1,p}(\Omega) \text{ and so } \|y\| = 1. \tag{4.4}$$

Hypotheses H_3(i), (iv) imply that

$$\begin{aligned} |f(z,x)| &\le c_1(|x| + |x|^{p-1}) \\ &\le c_2(1 + |x|^{p-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \\ &\qquad \text{and some } c_2 > 0 \text{ (recall } p > 2) \end{aligned}$$
$$\Rightarrow \left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\ge1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

Hence we may assume that

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^{p'}(\Omega). \tag{4.5}$$

Hypotheses H_3(ii), (iii) imply that

$$g(z) = \tilde{\vartheta}(z)y^+(z)^{p-1} - \tilde{\xi}(z)y^-(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{4.6}$$

So, if we return to (4.3), pass to the limit as $n \to \infty$ and use (4.4), (4.5), (4.6), we obtain

$$\begin{aligned} \langle A_p(y), v\rangle &= t\int_\Omega \tilde{\vartheta}(z)(y^+)^{p-1} v dz \\ &\quad - \int_\Omega [t\tilde{\xi}(z) + (1-t)\mu](y^-)^{p-1} v dz \quad \text{for all } v \in W_0^{1,p}(\Omega) \end{aligned}$$
$$\Rightarrow A_p(y) = t\tilde{\vartheta}(y^+)^{p-1} - \tilde{\xi}_t(y^-)^{p-1} \quad \text{where } \tilde{\xi}_t = t\tilde{\xi} + (1-t)\mu, \tag{4.7}$$
$$\Rightarrow -\Delta_p y(z) = t\tilde{\vartheta}(z)y^+(z)^{p-1} - \tilde{\xi}_t(z)y^-(z)^{p-1} \quad \text{a.e. in } \Omega, y|_{\partial\Omega} = 0. \tag{4.8}$$

On (4.7) first we act with $y^+ \in W_0^{1,p}(\Omega)$. Then

$$\|Dy^+\|_p^p = t\int_\Omega \tilde{\vartheta}(z)(y^+)^p dz \le \int_\Omega \vartheta(z)(y^+)^p dz$$
$$\Rightarrow c_0\|Dy^+\|_p^p \le 0 \quad \text{(see Lemma 2.5)}, \quad \text{hence } y^+ = 0.$$

From (4.8) and since $\tilde{\xi}_t(z) \in [\eta(z), \hat{\lambda}_2(p)]$ a.e. in Ω, it follows that y^- must be nodal, a contradiction. This proves the claim.

Invoking Proposition 2.6, we infer that

$$C_k(\varphi, \infty) = C_k(\psi, \infty) \quad \text{for all } k \ge 0. \tag{4.9}$$

Now let $u \in K_\psi$. Then

$$A_p(u) = -\mu(u^-)^{p-1} - \beta. \tag{4.10}$$

On (4.10) we act with $u^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\|Du^+\|_p^p = -\int_\Omega \beta(z)u^+ dz \le 0$$
$$\Rightarrow u^+ = 0, \quad \text{hence } u \le 0$$

Equation (4.10) becomes

$$A_p(u) = \mu|u|^{p-2}u - \beta$$
$$\Rightarrow -\Delta_p u(z) = \mu|u(z)|^{p-2}u(z) - \beta(z) \text{ a.e. in } \Omega, \quad u|_{\partial\Omega} = 0.$$

The nonlinear regularity theory implies that $u \in (-C_+)\backslash\{0\}$. Moreover, as before from the tangency principle and the boundary point theorem of [26, pp. 35 and 120], we have

$$u \in -\operatorname{int} C_+.$$

Let $v \in \operatorname{int} C_+$ and consider the function

$$R(v,-u)(z) = \|Dv(z)\|^p - \|D(-u)(z)\|^p \left(D(-u)(z), D\left(\frac{v^p}{(-u)^{p-1}}\right)(z)\right)_{\mathbb{R}^N}.$$

From the nonlinear Picone's identity of [3], we have

$$\begin{aligned}
0 &\le \int_\Omega R(v-u)dz \\
&= \|Dv\|_p^p - \int_\Omega -\Delta_p(-u)\frac{v^p}{(-u)^{p-1}}dz \quad \text{(by Green's theorem)} \\
&= \|Dv\|_p^p - \int_\Omega \mu(-u)^{p-1}\frac{v^p}{(-u)^{p-1}}dz - \int_\Omega \beta\frac{v^p}{(-u)^{p-1}}dz \quad \text{(see (4.11))} \\
&\le \|Dv\|_p^p - \mu\|v\|_p^p.
\end{aligned} \tag{4.11}$$

Choose $v = \hat{u}_1(p) \in \operatorname{int} C_+$, to reach a contradiction (recall that $\mu \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$). Hence $K_\psi = \emptyset$ and so we have

$$C_k(\psi,\infty) = 0 \quad \text{for all } k \ge 0$$
$$\Rightarrow C_k(\varphi,\infty) = 0 \quad \text{for all } k \ge 0 \quad \text{(see (4.9))}.$$

This completes the proof. □

Now we are ready for the three solutions theorem.

Theorem 4.3. *Assume that hypotheses H_3 hold. Then problem* (1.1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad \hat{u}, \tilde{u} \in C_0^1(\bar{\Omega})\backslash\{0\}.$$

Proof. From Theorem 3.4, we already have two nontrivial solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad \hat{u} \in C_0^1(\bar{\Omega}) \setminus \{0\}.$$

From Proposition 3.3 we know that u_0 is a local minimizer of the energy functional φ. Therefore

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.12}$$

From the proof of Theorem 3.4, we know that $\hat{u}$ is a critical point φ of mountain pass type. Then from [22, 24], we have

$$C_k(\varphi, \hat{u}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.13}$$

From the proof of the Theorem 3.4 (see the claim), we have

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.14}$$

Finally, Proposition 4.2 implies that

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \geq 0. \tag{4.15}$$

Suppose that $K_\varphi = \{0, u_0, \hat{u}\}$. Then from (4.12)–(4.15) and the Morse relation (see (2.4)) with $t = -1$, we have

$$(-1)^{d_m} + (-1)^0 + (-1)^1 = 0$$
$$\Rightarrow (-1)^{d_m} = 0, \quad \text{a contradiction.}$$

So, we can find $\tilde{u} \in K_\varphi$, $\tilde{u} \notin \{0, u_0, \hat{u}\}$. Then $\tilde{u}$ is the third nontrivial solution of (1.1) and $\tilde{u} \in C_0^1(\bar{\Omega})$ (nonlinear regularity). □

Remark 4.4. It is an interesting open problem if this three solutions theorem remains valid when we allow complete resonance at $+\infty$.

Acknowledgments

V. Rădulescu acknowledges the support through Grant CNCS PCE-47/2011.

References

[1] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, *Mem. Amer. Math. Soc.* **196** (2008), no. 915, vi+70 pp.

[2] S. Aizicovici, N. S. Papageorgiou and V. Staicu, On *p*-superlinear equations with a nonhomogeneous differential operator, (*NoDEA*) *Nonlinear Differential Equations Appl.* **20** (2013) 151–175.

[3] W. Allegretto and Y. X. Huang, A Picone's identity for the *p*-Laplacian and applications, *Nonlinear Anal.* **32** (1998) 819–830.

[4] V. Benci, P. D'Avenia, D. Fortunato and L. Pisani, Solitons in several space dimensions: Dirrick's problem and infinitely many solutions, *Arch. Ration. Mech. Anal.* **154** (2000) 297–324.

[5] J. Chabrowski and J. Yang, On the Neumann problem with combined nonlinearities, *Ann. Polon. Math.* **85** (2005) 239–250.

[6] X. Chang, Multiplicity of solutions for semilinear elliptic problems with combined nonlinearities, *Commun. Contemp. Math.* **13** (2011) 389–405.

[7] L. Cherfils and Y. Ilyasov, On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian, *Commun. Pure Appl. Anal.* **4** (2005) 9–22.

[8] P. G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications* (SIAM, Philadelphia, PA, 2013).

[9] S. Cingolani and M. Degiovanni, Nontrivial solutions for p-Laplacian equations with right-hand side having p-linear growth, *Commun. Partial Differential Equations* **30** (2005) 1191–1203.

[10] S. Cingolani and G. Vannella, Critical groups computations on a class of Sobolev Banach spaces via Morse index, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003) 271–292.

[11] F. O. de Paiva and E. Massa, Multiple solutions for some elliptic equations with a nonlinearity concave near the origin, *Nonlinear Anal.* **66** (2007) 2940–2946.

[12] F. O. de Paiva and A. E. Presoto, Semilinear elliptic problems with asymmetric nonlinearities, *J. Math. Anal. Appl.* **409** (2014) 254–262.

[13] L. Gasinski and N. S. Papageorgiou, *Nonlinear Analysis* (Chapman & Hall/CRC, Boca Raton, 2006).

[14] A. Granas and J. Dugundji, *Fixed Point Theory* (Springer, New York, 2003).

[15] O. Ladyzhenskaya and N. Uraltseva, *Linear and Quasilinear Elliptic Equations* (Academic Press, New York, 1968).

[16] Z. Liang and J. Su, Multiple solutions for semilinear elliptic boundary value problems with double resonance, *J. Math. Anal. Appl.* **354** (2009) 147–158.

[17] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988) 1203–1219.

[18] D. Motreanu, V. Motreanu and N. S. Papageorgiou, Multiple solutions for Dirichlet problems which are superlinear at $+\infty$ and (sub-)linear at $-\infty$, *Commun. Appl. Nonlinear Anal.* **13** (2009) 341–358.

[19] D. Motreanu, V. Motreanu and N. S. Papageorgiou, On p-Laplacian equations with concave terms and asymmetric perturbations, *Proc. Roy. Soc. Edinburgh, Sect. A* **141** (2011) 171–192.

[20] N. S. Papageorgiou and S. Kyritsi, *Handbook of Applied Analysis* (Springer, New York, 2009).

[21] N. S. Papageorgiou and V. Rădulescu, Semilinear Neumann problems with indefinite and unbounded potential and crossing nonlinearity, in *Recent Trends in Nonlinear Partial Differential Equations II: Stationary Problems*, Contemporary Mathematics, Vol. 595 (American Mathematical Society, 2013), pp. 293–315.

[22] N. S. Papageorgiou and V. Rădulescu, Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance, *Appl. Math. Optim.* (2013); DOI: 10.1007/s00245-013-9227-z.

[23] N. S. Papageorgiou and V. Rădulescu, Solutions with sign information for nonlinear nonhomogeneous elliptic equations, to appear in *Topol. Methods. Nonlinear Anal.*

[24] N. S. Papageorgiou and G. Smyrlis, On nonlinear nonhomogeneous resonant Dirichlet equations, *Pacific J. Math.* **264** (2013) 421–453.

[25] K. Perera, Multiplicity results for some elliptic problems with concave nonlinearities, *J. Differential Equations* **140** (1997) 133–141.

[26] P. Pucci and J. Serrin, *The Maximum Principle* (Birkhäuser, Basel, 2007).

[27] V. Rădulescu, *Qualitative Analysis of Nonlinear Partial Differential Equations* (Hindawi Publishing Corporation, New York, 2006).
[28] M. Sun, Multiplicity of solutions for a class of quasilinear elliptic equations at resonance, *J. Math. Anal. Appl.* **386** (2012) 661–668.