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Groundstates for magnetic Choquard equations with critical exponential growth

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ABSTRACT

This paper is dedicated to show the existence of ground state solution for a magnetic Choquard equation with critical exponential growth. By introducing a Moser type function involving magnetic potential and applying analytical techniques, we surmount the obstacles brought from the magnetic potential which makes it a complex-valued problem and the critical exponential growth nonlinearity which makes it difficult to show the non-vanishing of Cerami sequence. Our methods can be applied to related magnetic elliptic equations.

1. Introduction

In this paper, we are concerned with the following magnetic Choquard equation with critical exponential growth:

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\mu} * F(|u|)\right)\frac{f(|u|)}{|u|}u, \qquad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $0 < \mu < 2$, i is the imaginary unit, $V : \mathbb{R}^2 \to \mathbb{R}$ is an electric potential, $A \in L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ is a magnetic potential and $F(t) = \int_0^t f(s)ds$. The magnetic field $B := \text{curl } A = (B_{jk})$, $1 \le j,k \le 2$, where $B_{jk} := \partial_j A_k - \partial_k A_j$. Functions V, A and f satisfy the following basic assumptions respectively:

(VA) $V \in \mathcal{C}(\mathbb{R}^2 \to \mathbb{R})$ with $\inf_{x\in\mathbb{R}^2} V(x) > 0$, $A \in L^\infty(B(0,\rho), \mathbb{R}^2)$ where $\rho > 0$ is a constant, and $B(x)$, $V(x)$ are 1-periodic in x_1, x_2;

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(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{f(t)}{e^{\alpha t^2}} = 0 \ (+\infty), \qquad \text{for all } \alpha > \alpha_0 \ (\alpha < \alpha_0);$$

(F2) $f(t) = o(|t|^{(2-\mu)/2})$ as $|t| \to 0$.

Due to the relevance in semiconductor theory, condensed matter physics and plasma physics, the research on magnetic elliptic equations aroused great interests to scholars recently, see [1–7] and the references therein. For the magnetic Choquard equation (1.1), most of them are concerned with the case that the nonlinearity is of polynomial growth. By using Nehari manifold method and Ljusternik–Schnirelmann theory, Bueno–Mamani–Pereira [3] showed the existence of ground state solutions and multiple solutions for Eq. (1.1). When $f(u) = |u|^{p-1}(2 < p < \infty)$, Ji–Rădulescu [4] proved that the above equation has at least $2^k - 1$ multi-bump solutions if the zero set of V has several isolated connected components $\Omega_1, \ldots, \Omega_k$ such that the interior of Ω_j is non-empty and $\partial\Omega_j$ is smooth. Nevertheless, compared with the case that the nonlinearity is of polynomial growth in [3,4], dealing with Eq. (1.1) with exponential growth is much more difficult since it causes great obstacles in proving that the Cerami sequence is non-vanishing and showing that the weak limit of Cerami sequence is a solution of the original equation. As far as we know, there is no related results in this case. This is a basic research motivation of the present paper.

On the other hand, for the real-valued Choquard equation with critical exponential growth

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u),\ x \in \mathbb{R}^2, \tag{1.2}$$

by applying the critical point theorem established by Bartolo–Benci–Fortunato [8] and developing a direct approach, Qin–Tang [9] proved the existence of nontrivial solution for the above equation. Whereas, due to the existence of magnetic potential A, Eq. (1.1) cannot be changed into a real-valued problem, so the methods dealing with Eq. (1.2) cannot be applied to the complex-valued problem (1.1). Consequently, both the critical exponential growth nonlinearity and magnetic potential enforce the implementation of new tricks and techniques.

Define $\tilde{f}(t) \begin{cases} = f(t)/t, & \text{if } t \neq 0, \\ = 0, & \text{if } t = 0. \end{cases}$ Under the assumptions (F1) and (F2), we can write problem (1.1) in the form

$$-(\nabla + iA(x))^2 u + V(x)u = \left(\frac{1}{|x|^\mu} * F(|u|)\right) \tilde{f}(|u|)u, \qquad \text{in } \mathbb{R}^2, \tag{1.3}$$

and the corresponding energy functional can be defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla_A u|^2 + V(x)|u|^2]dx - \frac{1}{2}\int_{\mathbb{R}^2} \left(\frac{1}{|x|^\mu} * F(|u|)\right) F(|u|)dx,$$

where $\nabla_A u := \nabla u + iA(x)u$. Before stating our result, we introduce several mild assumptions on f.

(F3) there exists $\beta > 2$ such that $2tf(t) \geq \beta F(t) > 0$, $\forall\, t \in \mathbb{R}\backslash\{0\}$;
(F4) $f(t)$ is nondecreasing in $(0, \infty)$;
(F5) there exist $K_0 > 0$ and $t_0 > 0$ such that $F(t) \leq K_0|f(t)|$, $\forall\, |t| \geq t_0$;
(F6) $\underline{\lim}_{t\to\infty} \frac{f(t)}{e^{\alpha_0 t^2}} = \gamma > (4-\mu)\sqrt{(2-\mu)(3-\mu)/4\pi\rho^{4-\mu}}$, where ρ satisfies $(4-\mu)(V_\rho + 4\vartheta^2)\rho^2 < 4$ with $V_\rho = \max_{B(0,\rho)} V(x)$ and $\vartheta := \operatorname{esssup}_{B(0,\rho)} |A(x)|$.

Our main result is stated as follows.

Theorem 1.1. *Assume that* (VA), (F1)-(F6) *are satisfied. Then Eq.* (1.1) *has a ground state solution.*

Remark 1.2. By establishing an energy estimate inequality involving convolution terms and using non-Nehari manifold method developed in [10], we weaken the strict monotonicity condition

(WN) $f(t)/t$ is strictly increasing if $t>0$ and strictly decreasing in $t<0$,

which is used in [3,4] to (F4). Furthermore, we introduce a Moser type function involving the magnetic potential, with which we show that the minimax level associated with problem (1.1) is less than the threshold by subtle estimates, and then we prove successfully that the Cerami sequence does not vanish. Our results complement and generalize the previous ones in the literature and our methods can be applied to related magnetic elliptic equations with critical exponential growth.

Throughout this paper, we denote the open ball centered at x with radius R by $B(x,R)$. The norm of $L^s(\mathbb{R}^2,\mathbb{C})(1\le s<\infty)$ is denoted by $\|\cdot\|_s$ and $C_i(i=1,2,\dots,n)$ are different positive numbers in different places. For $x\in\mathbb{C}$, we use $Re(x)$ and $\bar{x}$ to denote the real part and the complex conjecture of x, respectively.

2. Variational settings and preliminaries

Consider the space $H_A:=\{u\in L^2(\mathbb{R}^2,\mathbb{C}):\nabla_A u\in L^2(\mathbb{R}^2,\mathbb{C})\}$ endowed with the scalar product $\langle u,v\rangle=Re\int_{\mathbb{R}^2}[\nabla_A u\overline{\nabla_A v}+V(x)u\bar{v}]dx$, and then $\|u\|=\langle u,u\rangle^{1/2}$. By the diamagnetic inequality (see [11]) $|\nabla|u|(x)|\le|\nabla u+iAu|$, a.e. on $\mathbb{R}^2$, for any $u\in H_A$, we have $|u|\in H^1(\mathbb{R}^2,\mathbb{R})$, and then the embedding $H_A\hookrightarrow L^s(\mathbb{R}^2,\mathbb{C})$ is continuous for $2\le s<\infty$ and locally compact for $1\le s<\infty$.

The Trudinger–Moser inequality is a crucial tool to deal with the exponential growth nonlinearity which is established firstly by Cao [12] and read as follows.

Lemma 2.1. *If $u\in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2\le 1$, $\|u\|_2\le M$ and $\alpha<4\pi$, then there exists a constant $C(M,\alpha)>0$ such that*

$$\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)dx\le C(M,\alpha).$$

By virtue of (F1) and (F2), for any $\varepsilon>0$, $\alpha>\alpha_0$ and fixed $q>1$, there exists $C_\varepsilon>0$ such that

$$|f(t)|\le\varepsilon|t|^{(2-\mu)/2}+C_\varepsilon|t|^{q-1}\left(e^{\alpha t^2-1}\right)\qquad\text{and}\qquad|F(t)|\le\varepsilon|t|^{(4-\mu)/2}+C_\varepsilon|t|^{q}\left(e^{\alpha t^2-1}\right),\tag{2.1}$$

which, together with Lemma 2.1 and the Hardy–Littlewood–Sobolev inequality (see [11]), implies $I\in\mathcal{C}^1(H_A,\mathbb{R})$ and for any $u,v\in H_A$

$$\langle I'(u),v\rangle=Re\left[\int_{\mathbb{R}^2}[\nabla_A u\overline{\nabla_A v}+V(x)u\bar{v}]dx-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|u(y)|)\tilde{f}(|u(x)|)u(x)\bar{v}(x)}{|x-y|^\mu}dxdy\right].$$

From now on, we always assume that (VA) and (F1)-(F6) are satisfied.

Lemma 2.2. *For any $u\in H_A$ and $t\ge 0$, there holds*

$$I(u)\ge I(tu)+\frac{1-t^2}{2}\langle I'(u),u\rangle.\tag{2.2}$$

Proof. Apparently, (2.2) holds when $u=0$. Next we assume $u\ne 0$. Note that

$$\begin{aligned}I(u)-I(tu)-\frac{1-t^2}{2}\langle I'(u),u\rangle=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|^\mu}&\left[\frac{1-t^2}{2}F(|u(y)|)f(|u(x)|)|u(x)|\right.\\&\left.+\frac{1}{2}F(|tu(y)|)F(|tu(x)|)-\frac{1}{2}F(|u(y)|)F(|u(x)|)\right]dxdy.\end{aligned}$$

Define a function $h(t):\mathbb{R}\to\mathbb{R}$ as follows.

$$h(t)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(1-t^2)F(|tu(y)|)f(|u(x)|)|u(x)|+F(|tu(y)|)F(|tu(x)|)-F(|u(y)|)F(|u(x)|)}{2|x-y|^{\mu}}dxdy.$$

By (F4), $F(t)/t$ is non-decreasing in $(0,\infty)$ and $f(t)t\geq F(t)\geq 0$, $\forall\ t\in\mathbb{R}$, which yields

$$\begin{aligned}h'(t)=&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{t|u(x)||u(y)|}{|x-y|^{\mu}}\left\{\frac{F(|tu(y)|)}{|tu(y)|}\left[f(|tu(x)|)-f(|u(x)|)\right]\right.\\&\left.+f(|u(x)|)\left[\frac{F(|tu(y)|)}{|tu(y)|}-\frac{F(|u(y)|)}{|u(y)|}\right]\right\}dxdy\begin{cases}\geq 0, t\geq 1,\\ <0, 0<t<1.\end{cases}\end{aligned}$$

This implies $h(t)\geq h(1)=0$. Therefore, we have that (2.2) holds. □

Lemma 2.3. *There exist a constant $c_*\in(0,m]$, and a sequence $\{u_n\}\subset H_A$ such that*

$$I(u_n)\to c_*,\quad \|I'(u_n)\|(1+\|u_n\|)\to 0, \tag{2.3}$$

where $m:=\inf_{\mathcal{N}} I$ and $\mathcal{N}:=\{u\in H_A\backslash\{0\}:\ \langle I'(u),u\rangle=0\}$.

Proof. It follows from (2.1) and Sobolev imbedding theorem that for any $u\in H_A$ with $\|u\|<\sqrt{(2-\mu)\pi/\alpha}$, there holds

$$\begin{aligned}\int_{\mathbb{R}^2}F(|u|)^{4/4-\mu}dx\leq&\int_{\mathbb{R}^2}\left[|u|^{(4-\mu)/2}+C_1|u|^q\left(e^{\alpha|u|^2}-1\right)\right]^{4/(4-\mu)}dx\\ \leq& C_2\left\{\|u\|_2^2+\|u\|_{2q}^{4q/(4-\mu)}\left[\int_{\mathbb{R}^2}\left(e^{4\alpha(2-\mu)^{-1}\|u\|^2(|u|/\|u\|)^2}-1\right)dx\right]^{\frac{2-\mu}{4-\mu}}\right\},\end{aligned}$$

which, together with the Hardy–Littlewood–Sobolev inequality and Lemma (2.1), implies

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|u(x)|)F(|u(y)|)}{|x-y|^{\mu}}dxdy\leq C_3\left[\int_{\mathbb{R}^2}|F(|u|)|^{4/(4-\mu)}dx\right]^{(4-\mu)/2}\leq C_4(\|u\|_2^{4-\mu}+\|u\|_{2q}^{2q}). \tag{2.4}$$

Therefore, by (2.4), one has

$$I(u)\geq\frac{1}{2}\|u\|^2-C_5\|u\|^{4-\mu}-C_6\|u\|^{2q},\ \forall\ \|u\|<\sqrt{(2-\mu)\pi/\alpha},$$

which yields that there exists small $\rho_0>0$ such that $I(u)>0$ on $S_{\rho_0}:=\{u\in H_A:\|u\|=\rho_0\}$. By virtue of (F1), we can derive that there exists $v\in H_A$ with $\|v\|>\rho_0$ such that $I(v)<0$. Then Mountain Pass Theorem implies there exists $\{u_n\}\subset H_A$ such that $I(u_n)\to c>0$ and $\|I'(u_n)\|(1+\|u_n\|)\to 0$ as $n\to\infty$. The rest part is similar to [10], so we omit here. □

Through a standard argument, we can derive the following lemma.

Lemma 2.4. *For any $u\in H_A\backslash\{0\}$, there exists $t_u>0$ such that $t_u u\in\mathcal{N}$.*

3. Proof of Theorem 1.1

Inspired by [2,13], we define a Moser type function involving the magnetic potential $w_n(x)=e^{i\phi(x)}u_n(x)$, where $\phi(x)=A(0)\cdot x=\sum_{j=1}^{2}A_j(0)x_j$ and $u_n(x)=\begin{cases}\frac{\sqrt{\log n}}{\sqrt{2\pi}}, & 0\leq|x|\leq\rho/n,\\ \frac{\log(\rho/|x|)}{\sqrt{2\pi\log n}}, & \rho/n\leq|x|\leq\rho,\\ 0, & |x|\geq\rho.\end{cases}$ with $\rho<2[(4-\mu)(V_\rho+$

$4\vartheta^2)]^{-1/2}$. Note that $|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2Re(z_1\overline{z_2})$ for any $z_1,z_2\in\mathbb{C}$ and $\nabla e^{i\phi(x)} = ie^{i\phi(x)}\nabla\phi(x) = ie^{i\phi(x)}A(0)$, then we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left|\nabla_A(e^{i\phi(x)}u_n)\right|^2 dx = \int_{\mathbb{R}^2}\left|\nabla(e^{i\phi(x)}u_n)+iA(x)e^{i\phi(x)}u_n\right|^2 dx\\
&= \int_{\mathbb{R}^2}\left|\nabla(e^{i\phi(x)}u_n)\right|^2 dx + \int_{\mathbb{R}^2}\left|A(x)e^{i\phi(x)}u_n\right|^2 dx - 2Re\int_{\mathbb{R}^2}\nabla(e^{i\phi(x)}u_n)\cdot iA(x)\overline{e^{i\phi(x)}u_n}dx.\\
&= \int_{\mathbb{R}^2}(|A(0)|^2u_n^2+|\nabla u_n|^2)dx + \int_{\mathbb{R}^2}|A(x)|^2|u_n|^2dx + 2\int_{\mathbb{R}^2}A(x)\cdot A(0)u_n^2dx\\
&\le \int_{B(0,\rho)}|\nabla u_n|^2dx + 4\vartheta^2\int_{B(0,\rho)}u_n^2dx<\infty,
\end{aligned}$$

from which we can deduce that $w_n\in H_A$. Moreover, via an elementary computation, one has

$$\int_{B(0,\rho)}|w_n|^2dx = \rho^2\left(\frac{1}{4\log n}-\frac{1}{4n^2\log n}-\frac{1}{2n^2}\right) := \rho^2\tau_n>0, \text{for } n\ge 2.$$

Hence we have

$$\begin{aligned}
\|w_n\|^2 &= \int_{\mathbb{R}^2}\left[\left|\nabla_A\left(e^{i\phi(x)}u_n\right)\right|^2 + V(x)\left|e^{i\phi(x)}u_n\right|^2\right]dx\\
&\le \int_{B(0,\rho)}|\nabla u_n|^2dx + 4\vartheta^2\int_{B(0,\rho)}u_n^2dx + \int_{B(0,\rho)}V(x)u_n^2dx \le 1+(V_\rho+4\vartheta^2)\rho^2\tau_n.
\end{aligned}$$

Lemma 3.1. *There exists $k\in\mathbb{N}$ such that* $\max_{t\ge0} I(tw_k) < \frac{(4-\mu)\pi}{2\alpha_0}$.

Proof. By virtue of (F6), we can obtain that there exist $\varepsilon>0$ and $t_\varepsilon>0$ such that

$$tF(x,t)\ge\frac{\gamma-\varepsilon}{2\alpha_0}e^{\alpha_0t^2},\ \forall\ x\ \in\mathbb{R}^2,\ |t|\ge t_\varepsilon,\ \text{and}\ \frac{(\gamma-\varepsilon)^2}{1+\varepsilon}>\frac{(2-\mu)(3-\mu)(4-\mu)^2}{4\pi\rho^{4-\mu}}. \tag{3.1}$$

In the following part, we discuss three cases, where the inequalities hold for $n\in\mathbb{N}$ large enough.

Case (i): $t\in\left[0,\sqrt{(4-\mu)\pi/2\alpha_0}\right]$. It follows from (F3) that

$$\begin{aligned}
I(tw_n) &= \frac{t^2}{2}\|w_n\|^2-\frac12\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|tw_n(x)|^2)F(|tw_n(y)|^2)}{|x-y|^\mu}dxdy\\
&\le\frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t^2\le\frac{(4-\mu)\pi}{4\alpha_0}+\frac{(4-\mu)(V_\rho+4\vartheta^2)\pi\rho^2}{16\alpha_0\log n}<\frac{(4-\mu)\pi}{2\alpha_0}.
\end{aligned}$$

Case (ii): $t\in\left[\sqrt{(4-\mu)\pi/2\alpha_0},\sqrt{(4-\mu)(1+\varepsilon)\pi/\alpha_0}\right]$. Since $|tw_n|\ge t_\varepsilon$ for $x\in B(0,\rho/n)$, by virtue of (3.1) and [14, (2.21)], one has

$$\begin{aligned}
I(tw_n) &= \frac{t^2}{2}\|w_n\|^2-\frac12\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|tw_n(x)|)F(|tw_n(y)|)}{|x-y|^\mu}dxdy\\
&\le\frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t^2-\frac{\pi(\gamma-\varepsilon)^2}{2\alpha_0^2t^2\log n}e^{\alpha_0\pi^{-1}t^2\log n}\int_{B(0,\rho/n)}dx\int_{B(0,\rho/n)}\frac{1}{|x-y|^\mu}dy\\
&\le\frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t^2-\frac{2\pi^2\rho^{4-\mu}(\gamma-\varepsilon)^2}{\alpha_0(1+\varepsilon)(2-\mu)(3-\mu)(4-\mu)^2n^{4-\mu}\log n}e^{\alpha_0\pi^{-1}t^2\log n}.
\end{aligned}$$

Let $t_n \in [0,\infty)$ such that $\varphi'(t_n)=0$, which means

$$t_n^2 = \frac{(4-\mu)\pi}{\alpha_0} \times \left[1 + \frac{\log[1+(V_\rho+4\vartheta^2)\rho^2\tau_n] + \log[(1+\varepsilon)(2-\mu)(3-\mu)(4-\mu)^2] - \log[4\pi\rho^{4-\mu}(\gamma-\varepsilon)^2]}{(4-\mu)\log n}\right].$$

Hence $\varphi_n(t)$ achieves its maximum at t_n, that is for each $n \in \mathbb{N}$,

$$\varphi_n(t) \le \varphi_n(t_n) = \frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t_n^2 - \frac{\pi[1+(V_\rho+4\vartheta^2)\rho^2\tau_n]}{2\alpha_0\log n},\ \forall\, t>0. \tag{3.2}$$

Using (3.1), we have

$$\begin{aligned}
&\frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t_n^2 \\
&\le \frac{(4-\mu)\pi}{2\alpha_0}\left[1+(V_\rho+4\vartheta^2)\rho^2\tau_n + \frac{\log\frac{(1+\varepsilon)(2-\mu)(3-\mu)(4-\mu)^2}{4\pi\rho^{4-\mu}(\gamma-\varepsilon)^2}}{(4-\mu)\log n}\right] + O\left(\frac{1}{\log^2 n}\right) \\
&\le \frac{(4-\mu)\pi}{2\alpha_0}\left(1+\frac{(V_\rho+4\vartheta^2)\rho^2}{4\log n}\right) + O\left(\frac{1}{\log^2 n}\right).
\end{aligned} \tag{3.3}$$

Combining (3.2) with (3.3), we derive

$$\begin{aligned}
I(tw_n) \le \varphi_n(t_n) &\le \frac{(4-\mu)\pi}{2\alpha_0}\left(1+\frac{(V_\rho+4\vartheta^2)\rho^2}{4\log n}\right) - \frac{\pi[1+(V_\rho+4\vartheta^2)\rho^2\tau_n]}{2\alpha_0\log n} + O\left(\frac{1}{\log^2 n}\right) \\
&\le \frac{(4-\mu)\pi}{2\alpha_0}\left[1-\frac{4-(4-\mu)(V_\rho+4\vartheta^2)\rho^2}{4(4-\mu)\log n}\right] + O\left(\frac{1}{\log^2 n}\right) < \frac{(4-\mu)\pi}{2\alpha_0}.
\end{aligned}$$

Case (iii): $t \in [\sqrt{(4-\mu)(1+\varepsilon)\pi/\alpha_0}, \infty]$. Similarly, we have $|tw_n| \ge t_\varepsilon$ for $x \in B(0,\rho/n)$ and

$$\begin{aligned}
I(tw_n) &\le \frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t^2 - \frac{2\pi^3\rho^{4-\mu}(\gamma-\varepsilon)^2}{\alpha_0^2(2-\mu)(3-\mu)(4-\mu)\log n t^2} e^{(4-\mu)\log n\left[\frac{\alpha_0}{(4-\mu)\pi}t^2-1\right]} \\
&\le \frac{\pi(4-\mu)(1+\varepsilon)[1+(V_\rho+4\vartheta^2)\rho^2\tau_n]}{2\alpha_0} - \frac{2\pi^2\rho^{4-\mu}(\gamma-\varepsilon)^2 n^{(4-\mu)\varepsilon}}{\alpha_0(1+\varepsilon)(2-\mu)(3-\mu)(4-\mu)^2\log n} < \frac{(4-\mu)\pi}{2\alpha_0},
\end{aligned}$$

where at the second inequality, we use the fact that the function

$$\psi_n(t) := \frac{1+(V_\rho+4\vartheta^2)\rho^2\tau_n}{2}t^2 - \frac{2\pi^3\rho^{4-\mu}(\gamma-\varepsilon)^2}{\alpha_0^2(2-\mu)(3-\mu)(4-\mu)\log n t^2} e^{(4-\mu)\log n\left[\frac{\alpha_0}{(4-\mu)\pi}t^2-1\right]}$$

is decreasing on $[\sqrt{(4-\mu)(1+\varepsilon)\pi/\alpha_0}, \infty]$ since its stagnation point of $\psi_n(t)$ tends to $\sqrt{(4-\mu)\pi/\alpha_0}$ as $n \to \infty$ and the last inequality can be deduced due to $\frac{n^{(4-\mu)\varepsilon}}{\log n} \to \infty$ as $n \to \infty$. The proof is now complete. $\square$

Proof of Theorem 1.1. In view of Lemma 2.3 and (F3), there exists a sequence $\{u_n\} \in H_A$ satisfying (2.3) and $\|u_n\| \le C_7$. If $\delta := \overline{\lim}_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B(y,1)}|u_n|^2dx = 0$, then Lions' concentration compactness lemma implies $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2,\infty)$. Through a similar argument in [9], we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|u_n(x)|)F(|u_n(y)|)}{|x-y|^\mu}dxdy = o(1) \quad\text{and}\quad \int_{\mathbb{R}^2}\int_{|u_n|\le 1}\frac{F(|u_n(y)|)f(|u_n(x)|)|u_n(x)|}{|x-y|^\mu}dxdy = o(1),$$

which, together with Lemmas 2.4 and 3.1, shows that there exists $\tilde{\varepsilon} > 0$ such that $\|u_n\|^2 = 2c_* + o(1) := \frac{(4-\mu)\pi}{2\alpha_0}(1-3\tilde{\varepsilon}) + o(1)$. Choose $q \in (1,2)$ such that $(1+\tilde{\varepsilon})(1-3\tilde{\varepsilon})q < 1-\tilde{\varepsilon}$. Then it follows from Lemma 2.1, (F1), Cauchy–Schwarz type inequality (see [15]) and Hardy–Littlewood–Sobolev inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{|u_n|\geq 1}\frac{F(|u_n(y)|)f(|u_n(x)|)|u_n(x)|}{|x-y|^{\mu}}dxdy\\
&\leq\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|u_n(x)|)F(|u_n(y)|)}{|x-y|^{\mu}}dxdy\right]^{\frac{1}{2}}\left[\int_{\mathbb{R}^2}[|x|^{\mu}*(f(|u_n|)|u_n|\chi_{|u_n|\geq 1})]f(|u_n|)|u_n|\chi_{|u_n|\geq 1}dx\right]^{\frac{1}{2}}\\
&\leq C_8\left[\int_{|u_n|\geq 1}|f(|u_n|)^{\frac{4q}{4-\mu}}dx\right]^{\frac{4-\mu}{4q}}\left[\int_{|u_n|\geq 1}|u_n|^{\frac{4q}{(q-1)(4-\mu)}}dx\right]^{\frac{(4-\mu)(q-1)}{4q}}\leq C_9\|u_n\|_{\frac{4q}{(4-\mu)(q-1)}}=o(1),
\end{aligned}$$

which, together with (2.3), implies

$$c_*+o(1)=I(u_n)-\frac{1}{2}\langle I'(u_n),u_n\rangle=\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|u_n(y)|)[f(|u_n(x)|)|u_n(x)|-F(|u_n(x)|)]}{|x-y|^{\mu}}dxdy=o(1).$$

This contradiction shows that $\delta>0$. Hence there exists $\{y_n\}\in\mathbb{R}^2$ such that $\int_{B(y_n,1)}|u_n|^2>\frac{\delta}{2}$. By virtue of (VA), we define a translation $\mathcal{T}:H_A\times\mathbb{Z}^2\to H_A$ by setting $(\mathcal{T}_z u)(x)=u(x+z)e^{i\varphi_z(x)}$, where $\varphi_z\in H^1_{loc}(\mathbb{R}^2)$ satisfying $A(x+z)-A(x)=\nabla\varphi_z(x)$. Then $\mathcal{T}$ is well defined and isometry (see [16]). Define $v_n=\mathcal{T}_{[y_n]}u_n$, where $[x]$ denotes the largest integer not exceeding x. By (VA) and (2.3), we have $\int_{B(0,1+\sqrt{2})}|v_n|^2dx>\frac{\delta}{2}$, $I(v_n)\to c_*$ and $\|I'(v_n)\|(1+\|v_n\|)\to 0$. The boundedness of $\{v_n\}$ can be yielded by (F3) and there exists $v\in H_A$ such that $v_n\rightharpoonup v$ in H_A and $v_n\to v$ in $L^s_{loc}(\mathbb{R}^2)$ for $s\in[2,\infty)$. By using a same argument in [9], we have $I'(v)=0$ with $v\neq 0$ which shows $I(v)\geq m$. From (F4), we can get

$$\begin{aligned}
m\geq c_*&=\lim_{n\to\infty}\left[I(v_n)-\frac{1}{2}\langle I'(v_n),v_n\rangle\right]\\
&=\lim_{n\to\infty}\frac{1}{2}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|v_n(y)|)[f(|v_n(x)|)|v_n(x)|-F(|v_n(x)|)]}{|x-y|^{\mu}}dxdy\right]\\
&\geq\frac{1}{2}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{F(|v(y)|)[f(|v(x)|)|v(x)|-F(|v(x)|)]}{|x-y|^{\mu}}dxdy\right]=I(v)-\frac{1}{2}\langle I'(v),v\rangle=I(v).
\end{aligned}$$

Therefore, $I(v)=m=\inf_{\mathcal{N}}I$. The proof is now complete. □

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