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Infinitely many nodal solutions for semilinear Robin problems with an indefinite linear part

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ABSTRACT

We consider a semilinear Robin problem driven by the Laplacian plus an indefinite potential and with a Carathéodory reaction $f(z,x)$ with no growth restriction on the x-variable. We only assume that $f(z,\cdot)$ is odd and superlinear near zero. Using a variant of the symmetric mountain pass theorem, we show that the problem has a whole sequence of distinct smooth nodal solutions converging to the trivial one.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem:

$$\left\{\begin{array}{l} -\Delta u(z) + \xi(z)u(z) = f(z,u(z)) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{in } \partial\Omega\,. \end{array}\right\} \tag{1}$$

In this problem, $\xi \in L^s(\Omega)$ $(s > N)$ is an indefinite (that is, sign-changing) potential function and the reaction term $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous). No global growth condition is imposed on $f(z,\cdot)$, which can

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be arbitrary near $\pm\infty$. The only conditions on $f(z,\cdot)$ concern its behavior near zero and we require that it is superlinear there. In the boundary condition, $\frac{\partial u}{\partial n}$ is the usual normal derivative defined by extension of the linear map

$$u \mapsto \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(\cdot)$ belongs to $W^{1,\infty}(\partial\Omega)$ and we assume that $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

We are looking for nodal (that is, sign-changing) solutions of problem (1). Using an abstract multiplicity result of Heinz [1], Wang [2] and Kajikiya [3], we show that problem (1) admits a whole sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ of distinct nodal solutions such that

$$u_n \in C^1(\overline{\Omega}) \quad \text{for all } n \in \mathbb{N} \text{ and } u_n \to 0 \text{ in } C^1(\overline{\Omega}).$$

Recently multiplicity results for semilinear elliptic problems with indefinite linear part were proved by Castro, Cossio and Vélez [4], Papageorgiou and Papalini [5], Qin, Tang and Tang [6], Wu and An [7], Zhang and Liu [8], Zhang, Tang and Zhang [9] (Dirichlet problems), Papageorgiou and Rădulescu [10,11] (Neumann problems) and Papageorgiou and Rădulescu [12] (Robin problems). None of the aforementioned works produces a whole sequence of nodal solutions and all impose a subcritical growth condition on the reaction term $f(z,\cdot)$. We mention also the very recent work of Papageorgiou and Rădulescu [13], which deals with nonlinear nonhomogeneous Robin problems with no potential term (that is, $\xi \equiv 0$) and a reaction term $f(z,x)$ of arbitrary growth in $x \in \mathbb{R}$. The authors of [13] produce a sequence of nodal solutions but under more restrictive conditions on $f(z,\cdot)$.

2. Mathematical background

Let X be a Banach space. By X^* we denote its topological dual and by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Palais–Smale condition" (the "PS-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

As we already mentioned, our main tool is the following abstract multiplicity theorem of Heinz [1], Wang [2] and Kajikiya [3]. The result is a variant of the classical symmetric mountain pass theorem (see, for example, Gasinski and Papageorgiou [14, p. 688]).

Theorem 1. *Assume that X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ satisfies the PS-condition, it is even, bounded below, $\varphi(0) = 0$ and for every $n \in \mathbb{N}$, there exist an n-dimensional subspace $V_n \subseteq X$ and $\rho_n > 0$ such that*

$$\sup[\varphi(u) : u \in V_n,\ \|u\| = \rho_n] < 0 \quad \textit{for all } n \in \mathbb{N}.$$

Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that

$$\varphi'(u_n) = 0 \quad \textit{for all } n \in \mathbb{N} \textit{ and } u_n \to 0 \textit{ in } X.$$

The analysis of problem (1) involves the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the Lebesgue "boundary" spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$.

The Sobolev space $H^1(\Omega)$ is a Hilbert space with inner product

$$(u,v)_{H^1(\Omega)} = \int_\Omega uv dz + \int_\Omega (Du, Dv)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in H^1(\Omega)$$

and corresponding norm

$$\|u\| = [\|u\|_2^2 + \|Du\|_2^2]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \Omega\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \leqslant p \leqslant \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ known as "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

We know that

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = H_0^1(\Omega).$$

Moreover, the trace map γ_0 is compact into $L^p(\partial\Omega)$ for all $p \in \left[1, \frac{2N-2}{N-2}\right)$ if $N \geqslant 3$ and into $L^p(\partial\Omega)$ for all $p \geqslant 1$ if $N = 1, 2$. In the sequel for notational economy, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Consider the following linear eigenvalue problem

$$\begin{Bmatrix} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda}u(z) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \end{Bmatrix} \tag{2}$$

We assume that

- $\xi \in L^{\frac{N}{2}}(\Omega)$ if $N \geqslant 3, \xi \in L^p(\Omega)$ with $p \in (1, +\infty)$ if $N = 2$ and $\xi \in L^1(\Omega)$ if $N = 1$;
- $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

From Papageorgiou and Rădulescu [10,12], we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|^2 \geqslant c_0 \|u\|_2^2 \quad \text{for all } u \in H^1(\Omega), \text{ some } c_0 > 0. \tag{3}$$

Using (3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we produce the spectrum of (2) which consists of a sequence $\{\hat{\lambda}_k\}_{k\geqslant 1}$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$. By

$E(\hat{\lambda}_k)$ ($k \in \mathbb{N}$) we denote the corresponding eigenspace and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{\underset{k\geqslant 1}{\oplus} E(\hat{\lambda}_k)}.$$

Concerning the first eigenvalue $\hat{\lambda}_1$, we have

- $\hat{\lambda}_1$, is simple (that is, $\dim E(\hat{\lambda}_1) = 1$);
- $\hat{\lambda}_1 = \inf \left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0 \right]$. (4)

The infimum in (4) is realized on $E(\hat{\lambda}_1)$. From the above properties, it is clear that the elements of $E(\hat{\lambda}_1)$ do not change sign. Let $\hat{u}_1$ denote the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1$. If $\xi \in L^s(\Omega)$ with $s > N$, then the regularity theory of Wang [15] implies that $\hat{u}_1 \in C_+ \setminus \{0\}$. In fact, if $\xi^+ \in L^\infty(\Omega)$ then the strong maximum principle implies that $\hat{u}_1 \in D_+$. We mention that $\hat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (that is, sign-changing) eigenfunctions.

We conclude this section, by introducing some notation which we will use in sequel.

By $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ we denote the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H^1(\Omega).$$

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in H^1(\Omega)$ we define

$$u^\pm(\cdot) = v(\cdot)^\pm.$$

We know that

$$u^\pm \in H^1(\Omega), \qquad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

3. Nodal solutions

The hypotheses on the data of problem (1), are the following:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark 1. When $\beta = 0$, we recover the Neumann problem.

$H(f)$: $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot)$ is odd on $[-c, c]$ with $c > 0$ and

(i) for every $M > 0$, there exists $a_M \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leqslant a_M(z) \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant M;$$

(ii) $\lim_{x\to 0} \frac{f(z,x)}{x} = +\infty$ uniformly for almost all $z \in \Omega$.

Remark 2. We stress that no global growth condition is imposed on $f(z,\cdot)$. Also note that $f(z,\cdot)$ is superlinear near zero (presence of a concave term near zero). Note that the function $f(x) = x(1 - \ln|x|)$ for $|x| \leqslant c$ satisfies hypotheses $H(f)$, but does not fit in the framework of Papageorgiou and Rădulescu [16].

Hypothesis $H(f)$ (ii) implies that given $\eta > |\hat{\lambda}_1|$, we can find $c_0 \in (0, c]$ such that

$$f(z,x)x \geqslant \eta x^2 \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant c_0. \tag{5}$$

Recall that $\hat{u}_1 \in D_+$. Hence we can find $t > 0$ such that $t\hat{u}_1 \leqslant c_0$. Let $\tau > 0$ be the maximum such positive real. Let $\hat{\mu} > \max\{\|\xi^+\|_\infty, \mu\}$ (see hypothesis $H(\xi)$ and (3)). We introduce the Carathéodory function $\hat{f} : \Omega\times\mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(z,x) = \begin{cases} f(z, -\tau\hat{u}_1(z)) - \hat{\mu}(\tau\hat{u}_1(z)) & \text{if } x < \tau\hat{u}_1(z) \\ f(z,x) + \hat{\mu}x & \text{if } |x| \leqslant \tau\hat{u}_1(z) \\ f(z, \tau\hat{u}_1(z)) + \hat{\mu}(\tau\hat{u}_1(z)) & \text{if } \tau\hat{u}_1(z) < x. \end{cases} \tag{6}$$

We set $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)ds$ and consider the C^1-functional $\hat{\varphi} : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \frac{1}{2}\gamma(u) + \frac{\hat{\mu}}{2}\|u\|_2^2 - \int_\Omega \hat{F}(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

From hypothesis $H(f)$, the choice of $\hat{\mu} > 0$ and (3), (6) we infer that $\hat{\varphi}$ has the following properties.

Proposition 2. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\hat{\varphi}$ is even, $\hat{\varphi}(0) = 0$ and $\hat{\varphi}(\cdot)$ is coercive.*

From Proposition 2 and Papageorgiou and Winkert [17, Proposition 2.13], we infer that:

Corollary 3. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $\hat{\varphi}$ is bounded from below and satisfies the PS-condition.*

Let $\hat{\varphi}_\pm : H^1(\Omega) \to \mathbb{R}$ be the positive and negative truncations of $\hat{\varphi}$, that is

$$\hat{\varphi}_\pm(u) = \frac{1}{2}\gamma(u) + \frac{\hat{\mu}}{2}\|u\|_2^2 - \int_\Omega \hat{F}(z, \pm u^\pm)dz \quad \text{for all } u \in H^1(\Omega).$$

We know that $\hat{\varphi}_\pm \in C^1(H^1(\Omega), \mathbb{R})$. We introduce the critical sets of the functionals $\hat{\varphi}$, $\hat{\varphi}_\pm$, that is, the sets

$$K_{\hat{\varphi}} = \{u \in H^1(\Omega) : \hat{\varphi}'(u) = 0\},$$
$$K_{\hat{\varphi}_\pm} = \{u \in H^1(\Omega) : \hat{\varphi}_\pm'(u) = 0\}.$$

From the regularity theory of Wang [15], we have

$$K_{\hat{\varphi}} \subseteq C^1(\overline{\Omega}). \tag{7}$$

Similarly, using the regularity theory of Wang [15] and the strong maximum principle, we have

$$K_{\hat{\varphi}_+} \subseteq D_+ \cup \{0\} \quad \text{and} \quad K_{\hat{\varphi}_-} \subseteq (-D_+) \cup \{0\}. \tag{8}$$

Proposition 4. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists $M > 0$ such that*

$$-M \leqslant u(z) \leqslant M \quad \textit{for all } z \in \overline{\Omega}, \textit{ all } u \in K_{\hat{\varphi}}.$$

Proof. From (6) we see that we can find $M > 0$ such that

$$|\hat{f}(z,x)| \leqslant (\xi(z) + \hat{\mu})M \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R} \tag{9}$$

(recall that $\hat{\mu} > \max\{\|\xi^+\|_\infty, \mu\}$).

Let $u \in K_{\hat{\varphi}}$. Then we have

$$\begin{aligned}\langle A(u), h\rangle + \int_\Omega (\xi(z)+\hat{\mu})uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma &= \int_\Omega \hat{f}(z,u)hdz \\ &\leqslant \int_\Omega |\hat{f}(z,u)||h|dz \\ &\leqslant \int_\Omega (\xi(z)+\hat{\mu})M|h|dz \quad \text{for all } h \in H^1(\Omega) \text{ (see (9))}.\end{aligned}$$

Choose $h = (u-M)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}&\langle A(u), (u-M)^+\rangle + \int_\Omega (\xi(z)+\hat{\mu})u(u-M)^+dz + \int_{\partial\Omega} \beta(z)u(u-M)^+d\sigma \\ &\leqslant \int_\Omega (\xi(z)+\hat{\mu})M(u-M)^+dz + \int_{\partial\Omega}\beta(z)M(u-M)^+d\sigma \text{ (see hypothesis } H(\beta)), \\ &\Rightarrow \langle A(u)-A(M), (u-M)^+\rangle + \int_\Omega (\xi(z)+\hat{\mu})(u-M)(u-M)^+dz + \int_{\partial\Omega}\beta(z)(u-M)(u-M)^+d\sigma \leqslant 0, \\ &\Rightarrow \|D(u-M)^+\|_2^2 + \int_\Omega (\xi(z)+\hat{\mu})((u-M)^+)^2dz + \int_{\partial\Omega}\beta(z)((u-M)^+)^2d\sigma \leqslant 0, \\ &\Rightarrow u \leqslant M.\end{aligned}$$

Similarly, choosing $h = (-M-u)^+ \in H^1(\Omega)$, we obtain

$$\begin{aligned}&-M \leqslant u, \\ &\Rightarrow u \in [-M, M] \cap C^1(\overline{\Omega}) \text{ (see (7))}. \quad \square\end{aligned}$$

Proposition 5. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then*

(a) $\tau\hat{\mu}_1 \leqslant u$ *for all* $u \in K_{\hat{\varphi}_+} \setminus \{0\}$;
(b) $v \leqslant -\tau\hat{u}_1$ *for all* $v \in K_{\hat{\varphi}_-} \setminus \{0\}$.

Proof.

(a) Let $u \in K_{\hat{\varphi}_+} \setminus \{0\}$ and consider the set

$$\mathcal{S}_u = \{t > 0 : t\hat{u}_1 \leqslant u\}.$$

Since $u \in D_+$ (see (8)), we infer that

$$\mathcal{S}_u \neq \varnothing.$$

Let $t^* = \sup \mathcal{S}_u$ and suppose that $t^* < \tau$. Set

$$\Omega_+^1 = \{0 < u \leqslant \tau\hat{u}_1\} \quad \text{and} \quad \Omega_+^2 = \{u > \tau\hat{u}_1\}.$$

We have

$$\begin{aligned}\hat{f}(z,u) &\geqslant (\eta+\hat{\mu})u > (|\hat{\lambda}_1|+\hat{\mu})u \geqslant (|\hat{\lambda}_1|+\hat{\mu})(t^*\hat{u}_1) \\ &\qquad \text{for almost all } z \in \Omega_+^1 \text{ (see (6) and recall } t^* < \tau) \qquad (10)\\ \hat{f}(z,u) &= f(z,\tau\hat{u}_1) + \hat{\mu}(\tau u_1) \text{ (see (6))} \\ &\geqslant (\eta+\hat{\mu})(\tau\hat{\mu}_1) \text{ (see (5) and recall the definition of } \tau) \\ &> (|\hat{\lambda}_1|+\hat{\mu})(t^*\hat{u}_1) \quad \text{for almost all } z \in \Omega_+^2 \text{ (recall } t^* < \tau). \qquad (11)\end{aligned}$$

Then

$$\langle A(u), h\rangle + \int_\Omega (\xi(z)+\hat{\mu})uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma = \int_\Omega \hat{f}(z,u)hdz \quad \text{for all } h \in H^1(\Omega)$$

$$\begin{aligned}
\Rightarrow -\Delta u(z) + (\xi(z)+\hat{\mu})u(z) &= \hat{f}(z,u(z)) \\
&> (|\hat{\lambda}_1| + \hat{\mu})(t^*\hat{u}_1(z)) \text{ (see (10), (11))} \\
&\geqslant -\Delta(t^*\hat{u}_1(z)) + (\xi(z)+\hat{\mu})(t^*\hat{u}_1(z)) \quad \text{for almost all } z \in \Omega \\
&\quad \text{(see (10), (11))} \\
\Rightarrow \Delta(u - t^*\hat{u}_1)(z) &\leqslant (\|\xi^+\|_\infty + \hat{\mu})(u - t^*\hat{u}_1)(z) \quad \text{for almost all } z \in \Omega \\
&\quad \text{(see hypothesis } H(\xi) \text{ and recall that } t^* < \tau) \\
\Rightarrow u - t^*\hat{u}_1 \in D_+ &\quad \text{(by the strong maximum principle).}
\end{aligned}$$

This contradicts the maximality of t^*. Therefore

$$\begin{aligned}
&\tau \leqslant t^*, \\
&\Rightarrow \tau\hat{u}_1 \leqslant u \quad \text{for all } u \in K_{\hat{\varphi}_+}.
\end{aligned}$$

(b) Similarly we show that

$$v \leqslant -\tau\hat{u}_1 \quad \text{for all } v \in K_{\hat{\varphi}_-}. \quad \square$$

With the next proposition we satisfy the geometry of Theorem 1.

Proposition 6. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, and V_n is an n-dimensional subspace of $H^1(\Omega)$, then we can find $\rho_n \in (0,1)$ small such that*

$$\sup[\hat{\varphi}(u) : u \in V_n,\ \|u\| = \rho_n] < 0.$$

Proof. All norms on V_n are equivalent. So, we can find $\rho_n \in (0,1)$ small such that

$$u \in V_n,\ \|u\| \leqslant \rho_n \Rightarrow |u(z)| \leqslant m_* \quad \text{for almost all } z \in \Omega, \tag{12}$$

with $m_* = \min_{\overline{\Omega}} \tau\hat{u}_1$ (recall that $\hat{u}_1 \in D_+$). So, for $u \in V_n$, with $\|u\| \leqslant \rho_n$, we have

$$\begin{aligned}
\hat{\varphi}(u) &\leqslant \hat{c}\|u\|^2 - \eta\,\|u\|_2^2 \quad \text{for some } \hat{c} > 0 \\
&\quad \text{(see hypothesis } H(\xi), H(\beta), \text{ (5) and (12))} \\
&\leqslant \hat{c} - \eta\hat{c}_1\|u\|^2 \quad \text{for some } \hat{c}_1 > 0 \\
&\quad \text{(since all norms on } V_n \text{ are equivalent).}
\end{aligned}$$

Then choosing $\eta > |\hat{\lambda}_1|$ even bigger if necessary (so that $\eta > \frac{\hat{c}_1}{\hat{c}}$), we see that

$$\sup[\hat{\varphi}(u) : u \in V_n, \|u\| = \rho_n] < 0. \quad \square$$

Now we are ready for the main result of this paper, which shows that problem (1) admits a whole sequence of distinct nodal smooth solutions which converge to the trivial solution.

Theorem 7. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then problem (1) has a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ of nodal solutions such that*

$$u_n \in C^1(\overline{\Omega}) \quad \textit{for all } n \in \mathbb{N} \textit{ and } u_n \to 0 \textit{ in } C^1(\overline{\Omega}).$$

Proof. Proposition 2, Corollary 3 and Proposition 6 permit the use of Theorem 1. So, we can find

$$\{u_n\}_{n\geqslant 1} \subseteq K_{\hat{\varphi}} \quad \text{such that } u_n \to 0 \text{ in } H^1(\Omega). \tag{13}$$

Recall that $K_{\hat{\varphi}} \subseteq C^1(\overline{\Omega})$ (see (7)). So, we have $u_n \in C^1(\overline{\Omega})$.

Proposition 4 and the regularity theory of Wang [15] imply that there exist $\alpha \in (0,1)$ and $\hat{c}_2 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant \hat{c}_2 \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Then from (13), (14) and the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we have that

$$u_n \to 0 \quad \text{in } C^1(\overline{\Omega}).$$

So, we will have

$$u_n \in [-\tau\hat{u}_1, \tau\hat{u}_1] \setminus \{\pm\tau\hat{u}_1\} \quad \text{for all } n \geqslant n_0. \tag{15}$$

From Proposition 5 and (6), (15), it follows that

$$\{u_n\}_{n\geqslant n_0} \subseteq C^1(\overline{\Omega}) \text{ are nodal solutions of (1)}$$

and we have

$$u_n \to 0 \quad \text{in } C^1(\overline{\Omega}). \quad \square$$

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