Proposed to The American Mathematical Monthly [1]

(a) *Find a sequence of distinct complex numbers $(z_n)_{n\geq 1}$ and a sequence of nonzero real numbers $(\alpha_n)_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty} \alpha_n \, |z-z_n|^{-1}$ either converges to a positive number or diverges to $+\infty$ for almost all complex numbers z, but not all α_n are positive.*

(b) *Let $(z_n)_{n\geq 1}$ be a sequence of distinct complex numbers. Assume that $\sum_{n=1}^{\infty} \alpha_n$ is an absolutely convergent series of real numbers such that $\sum_{n=1}^{\infty} \alpha_n \, |z-z_n|^{-1}$ converges to a nonnegative number, for almost all $z \in \mathbb{C}$. Prove that α_n are nonnegative for all $n \geq 1$.*

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Solution. **(a)** We prove that the series

$$-\frac{1}{|z|} + \frac{1}{\left|z+\frac{1}{2}\right|} + \frac{1}{\left|z-\frac{1}{2}\right|} + \cdots + \frac{1}{\left|z+\frac{1}{n}\right|} + \frac{1}{\left|z-\frac{1}{n}\right|} + \cdots \tag{1}$$

diverges to $+\infty$ for all $z \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{2}, \ldots, \pm\frac{1}{n}, \ldots\right\}$.

Indeed, we first observe that, for any fixed $z \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{2}, \ldots, \pm\frac{1}{n}, \ldots\right\}$, the above series has the same nature as the series $-1+1+1+1+\ldots$, which diverges. Next, we observe that

$$-\frac{1}{|z|} + \frac{1}{\left|z+\frac{1}{n}\right|} \geq 0 \quad \text{if and only if} \quad \operatorname{Re} z \leq -\frac{1}{2n}$$

and

$$-\frac{1}{|z|} + \frac{1}{\left|z-\frac{1}{n}\right|} \geq 0 \quad \text{if and only if} \quad \operatorname{Re} z \geq \frac{1}{2n}\,.$$

The above relations show that for any $z \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{2}, \ldots, \pm\frac{1}{n}, \ldots\right\}$ with $\operatorname{Re} z \neq 0$ there exists $N \in \mathbb{N}$ such that $-\frac{1}{|z|} + \sum_{k=1}^{N} \left(\frac{1}{\left|z+\frac{1}{k}\right|} + \frac{1}{\left|z-\frac{1}{k}\right|} \right) > 0$. It remains to prove that this is also true if $z = iy$, $y \in \mathbb{R} \setminus \{0\}$. For this purpose we observe that

$$\begin{aligned} &\frac{1}{\left|z+\frac{1}{2}\right|} + \frac{1}{\left|z-\frac{1}{2}\right|} + \cdots + \frac{1}{\left|z+\frac{1}{n}\right|} + \frac{1}{\left|z-\frac{1}{n}\right|} = \\ &\frac{2}{\sqrt{y^2+\frac{1}{4}}} + \cdots + \frac{2}{\sqrt{y^2+\frac{1}{n^2}}} \geq \frac{2n-2}{\sqrt{y^2+\frac{1}{4}}} \geq \frac{1}{|y|} = \frac{1}{|z|}\,, \end{aligned}$$

provided $2|y| \geq (4n^2-8n+3)^{-1}$. In conclusion, the series (1) diverges to $+\infty$ for all $z \in \mathbb{C} \setminus \left\{0, \pm\frac{1}{2}, \ldots, \pm\frac{1}{n}, \ldots\right\}$.

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Another example of series with the above properties is

$$-\frac{1}{|z|}+\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{1}{|z+\ln n|}+\frac{1}{|z-\ln n|}\right), \qquad z\in\mathbb{C}\setminus\{0,\pm\ln 2;\pm\ln 3,\ldots\}.$$

(b) It is sufficient to focus on an arbitrary term of the sequence, say α_1, and to show that $\alpha_1\geq 0$. We can assume, without loss of generality, that $z_1=0$. Fix arbitrarily $\varepsilon\in(0,1)$. Since $\sum_{n=1}^{\infty}|\alpha_n|<\infty$, there exists a positive integer N such that $\sum_{i=N+1}^{\infty}|\alpha_i|<\varepsilon$. Next, we choose $r>0$ small enough so that $|a_i|>r/\varepsilon$, for all $i\in\{2,\ldots,N\}$. Set

$$f(z)=\sum_{n=1}^{\infty}\frac{\alpha_n}{|z-z_n|}.$$

It follows that

$$\begin{aligned}
0\ &\leq \int_{B_r(0)}f(z)dz=\alpha_1\int_{B_r(0)}\frac{dz}{|z|}+\sum_{i=2}^{N}\alpha_i\int_{B_r(0)}\frac{dz}{|z-z_i|}+\sum_{i=N+1}^{\infty}\alpha_i\int_{B_r(0)}\frac{dz}{|z-z_i|}\\
&\leq \alpha_1\int_{B_r(0)}\frac{dz}{|z|}+\sum_{i=2}^{N}|\alpha_i|\int_{B_r(0)}\frac{dz}{|z-z_i|}+\sum_{i=N+1}^{\infty}|\alpha_i|\int_{B_r(0)}\frac{dz}{|z-z_i|}\\
&=2\pi r\alpha_1+\sum_{i=2}^{N}|\alpha_i|\int_{B_r(0)}\frac{dz}{|z-z_i|}+\sum_{i=N+1}^{\infty}|\alpha_i|\int_{B_r(0)}\frac{dz}{|z-z_i|}\\
&\leq 2\pi r\alpha_1+\sum_{i=2}^{N}|\alpha_i|\int_{B_r(0)}\frac{dz}{|z-z_i|}+\varepsilon\sup_{i\geq N+1}\int_{B_r(0)}\frac{dz}{|z-z_i|}.
\end{aligned}\tag{2}$$

For every $i\in\{2,\ldots N\}$ we have $|z-z_i|\geq|z_i|-|z|\geq\frac{r}{\varepsilon}-r=r(1-\varepsilon)/\varepsilon$, so

$$\int_{B_r(0)}\frac{dz}{|z-z_i|}\leq\frac{\varepsilon}{r(1-\varepsilon)}\int_{B_r(0)}dz=\frac{\varepsilon\pi r}{1-\varepsilon}.\tag{3}$$

If $i\geq N+1$ we distinguish two cases: either $|z_i|\geq 2r$ or $|z_i|<2r$. In the first situation we deduce that $|z-z_i|\geq r$, for any $z\in B_r(0)$. Thus

$$\int_{B_r(0)}\frac{dz}{|z-z_i|}\leq\frac{1}{r}\int_{B_r(0)}dz=\pi r.$$

If $|z_i|<2r$ then

$$\int_{B_r(0)}\frac{dz}{|z-z_i|}\leq\int_{B_{4r}(z_i)}\frac{dz}{|z-z_i|}=8\pi r.$$

The above two relations show that

$$\sup_{i\geq N+1}\int_{B_r(0)}\frac{dz}{|z-z_i|}\leq 8\pi r.\tag{4}$$

Using (2), (3) and (4) we obtain

$$0\leq 2\pi r\alpha_1+\sum_{i=2}^{N}|\alpha_i|\cdot\frac{\varepsilon\pi r}{1-\varepsilon}+8\varepsilon\pi r.$$

Dividing by r and letting $\varepsilon \to 0$ we deduce that $\alpha_1 \geq 0$. $\square$

Remark. For part **(a)** of this proposal, we have **not** been able to find an example of series $\sum_{n=1}^{\infty} \alpha_n \, |z - z_n|^{-1}$ which converges to a positive number for almost all complex numbers z, but not all α_n being positive. It might be possible that such a series does not exist and the unique situation which can occur is that, under our assumptions described in **(a)**, the series $\sum_{n=1}^{\infty} \alpha_n \, |z - z_n|^{-1}$ always diverges to $+\infty$. We let at your choice to decide if this assertion could be included as an open problem in this proposal.