PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before April 30, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11537. *Proposed by Lang Withers, Jr., MITRE, McClean, VA.* Let p be a prime and a be a positive integer. Let X be a random variable having a Poisson distribution with mean a, and let M be the pth moment of X. Prove that $M \equiv 2a \pmod{p}$.

11538. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Prove that a finite commutative ring in which every element can be written as a product of two (not necessarily distinct) elements has a multiplicative identity.

11539. *Proposed by William C. Jagy, MSRI, Berkeley, CA.* Let E be the set of all positive integers not divisible by 2 or 3 or by any prime q represented by the quadratic form $4u^2 + 2uv + 7v^2$. (Thus, the first few members of E are 1, 5, 11, 17, 23, and 25.) Show that $4x^2 + 2xy + 7y^2 + z^3$ is not in $\{2n^3, -2n^3, 32n^3, -32n^3\}$ for $n \in E$ and $x, y, z \in \mathbb{Z}$.

11540. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Let n be an integer greater than 1, other than 4. Let p and q be positive integers less than n and relatively prime to n. Let $a = \frac{\cos(2\pi p/n)}{\cos(2\pi q/n)}$. Show that if a^k is rational for some positive integer k, then a^k is either 1 or -1.

11541. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Let $|MA|$, $|MB|$, and $|MC|$ be the distances from M to A, B, and C. Show that

$$\frac{|MA|}{R_b + R_c} + \frac{|MB|}{R_a + R_c} + \frac{|MC|}{R_a + R_b} \le \frac{3}{2}.$$

11542. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania.* Show that for $x, y, z > 1$, and for positive

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α, β, γ,

$$(2x^2 + yz)\Gamma(x) + (2y^2 + zx)\Gamma(y) + (2z^2 + xy)\Gamma(z) \\ \geq (x + y + z)(x\Gamma(x) + y\Gamma(y) + z\Gamma(z)),$$

and

$$B(x, \alpha)^{x^2+2yz} B(y, \beta)^{y^2+2zx} B(z, \gamma)^{z^2+2xy} \\ \geq (B(x, \alpha)B(y, \beta)B(z, \gamma))^{xy+yz+zx}.$$

Here, $B(x, \alpha)$ is Euler's beta function, defined by $B(x, \alpha) = \int_0^1 t^{x-1}(1-t)^{\alpha-1} dt$.

11543. *Proposed by Richard Stong, Center for Communications Research, San Diego, CA.* Let x, y, z be positive numbers with $xyz = 1$. Show that $(x^5 + y^5 + z^5)^2 \geq 3(x^7 + y^7 + z^7)$.

SOLUTIONS

A Euclidean Path

11390 [2008, 855]. *Proposed by Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Let G be the undirected graph on the vertex set V of all pairs (a, b) of relatively prime integers, with edges linking (a, b) to $(a + kab, b)$ and $(a, b + kab)$ for all integers k.

(**a**) Show that for all (a, b) in V, there is a path joining (a, b) and $(1, 1)$.

(**b**)* Call an edge linking (a, b) to $(a + kab, b)$ or $(a, b + kab)$ *positive* if $k > 0$, and *negative* if $k < 0$. Let the *reversal number* of a path from $(1, 1)$ to (a, b) be one more that the number of sign changes along the path, and let the *reversal value* of (a, b) be the minimal reversal number over all paths from $(1, 1)$ to (a, b). Are there pairs of arbitrarily high reversal value?

Solution by M. D. Meyerson and M. E. Kidwell, U.S. Naval Academy.

(**a**) Suppose first that a and b are positive; we may assume that $a < b$. Let $c = b - a$. Note that b and c are relatively prime (if d divides both, then it also divides a); hence there are integers m and n such that $mb + nc = 1$. We may choose m positive and n negative, since increasing m by c and decreasing n by b does not change $mb + nc$. We can link (a, b) to (a, c) via two negative edges, since $(a, b - mab) = (a, b - a(1 - nc)) = (a, b - a + nac) = (a, c + nac)$. We can similarly link (b, a) to (c, a) via two negative edges. By the Euclidean algorithm, we can thus reach $(1, 1)$ via only negative edges.

If $ab = 0$ then there is a negative edge from one of $(-1, 1)$, $(1, -1)$, or $(1, 1)$ to (a, b).

If exactly one of $\{a, b\}$ is negative, then we can add $(-2)ab$ to the negative component of (a, b) to reach a pair with positive components via a negative edge, followed by linking as above to $(1, 1)$. If both a and b are negative, then to make at least one coordinate positive we must use a sufficiently large positive multiple of their product, after which we can reach $(1, 1)$ via only negative edges. This process misses four points, $(0, \pm 1)$ and $(\pm 1, 0)$, which can easily be linked to $(1, 1)$ via at most two edges.

(**b**)* By the process in part (a), we can reach $(1, 1)$ via only negative edges unless a and b are both negative, in which case we only need to use one positive edge to start after which we can reach $(1, 1)$ using only negative edges. Thus there is always a path from (a, b) to $(1, 1)$ with reversal number at most 2, so there are no pairs (a, b) of arbitrarily high reversal value.

Editorial comment. The sign of an edge is well defined; if the link can be viewed from both ends, then the corresponding choices for k are equal and thus have the same sign.

Both parts also solved by P. Corn, K. Schilling, B. Schmuland (Canada), R. Stong, A. Vorobyov, and the Texas State University Problem Solvers Group. Part (a) also solved by D. Klyve & C. Storm, M. A. Prasad (India), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

A Congruence for Vanishing Modular Sums

11391 [2008, 855]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let p be a positive prime and s a positive integer. Let n and k be integers such that $n \geq k \geq p^s - p^{s-1}$, and let $x_1, \dots, x_n$ be integers. For $1 \leq j \leq n$, let m_j be the number of expressions of the form $x_{i_1} + \cdots + x_{i_j}$ with $1 \leq i_1 < \cdots < i_j \leq n$ that evaluate to 0 modulo p, and let n_j denote the number of such expressions that do not. (Set $m_0 = 1$ and $n_0 = 0$.) Apart from the cases $(s, k) = (1, p-1)$ and $s = p = k = 2$, show that

$$\sum_{j=0}^{k}(-1)^j \binom{n-k+j}{j} m_{k-j} \equiv 0 \pmod{p^s},$$

and show that the same congruence holds with n_{k-j} in place of m_{k-j}.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We prove a much stronger statement. Let $X = \{x_1, \dots, x_n\}$, and let q_j be the number of j-element subsets of X whose sum is congruent to a modulo p. For $n \geq k \geq 1 + s(p-1)$, we prove that

$$\sum_{j=0}^{k}(-1)^j \binom{n-k+j}{j} q_{k-j} \equiv 0 \pmod{p^s}, \tag{1}$$

except in the excluded cases. The desired result for m_{k-j} is the case $a = 0$, and the result for n_{k-j} follows by summing the remaining residue classes.

We first show that it suffices to prove the case $n = k$, which reduces to

$$\sum_{j=0}^{k}(-1)^j q_{k-j} \equiv 0 \pmod{p^s} \tag{2}$$

for $k \geq 1 + s(p-1)$. Assume (2), then, and let $[n]$ denote $\{1, \dots, n\}$. For $S \subseteq [n]$, let $S^* = \{T \subseteq S\colon \sum_{i \in T} x_i \equiv a \pmod{p}\}$. For general n and k, (2) implies, modulo p^s,

$$\begin{aligned} 0 &\equiv \sum_{\substack{S\subseteq[n]\\|S|=k}} \left(\sum_{j=0}^{k}(-1)^j \sum_{\substack{T\in S^*\\|T|=k-j}} 1 \right) \equiv \sum_{j=0}^{k}(-1)^j \sum_{\substack{T\in[n]^*\\|T|=k-j}} \sum_{\substack{T\subseteq S\subseteq[n]\\|S|=k}} 1 \\ &\equiv \sum_{j=0}^{k}(-1)^j \sum_{\substack{T\in[n]^*\\|T|=k-j}} \binom{n-(k-j)}{j} \equiv \sum_{j=0}^{k}(-1)^j \binom{n-k+j}{j} q_{k-j}. \end{aligned}$$

This proves that (1) follows from (2). To prove (2), we work in the ring $\mathbb{Z}[t]/(t^p - 1)$, where $t^p = 1$. In this ring, let

$$f(t) = \prod_{x\in X}(1 - t^x) = (1-t)^k \prod_{x\in X}(1 + t + \cdots + t^{x-1}).$$

The terms in the expansion of f have the form $(-1)^{|Y|}\prod_{y\in Y}t^y$, where $Y\subseteq X$. For fixed a, $(-1)^j q_j$ is the contribution to the coefficient of t^a in the expansion of f due to Y of size j and sum congruent to a mod p, and $\sum_{j=0}^{k}(-1)^j q_j=[t^a]f(t)$. We now show that each coefficient of f is a multiple of p^s, from which (2) follows. To see that each coefficient is a multiple of p^s, we show that when $k>(p-1)s$, every coefficient of $(1-t)^k$ is a multiple of p^s.

First we construct a polynomial $h(t)$ such that $(1-t)^p=p\cdot(1-t)h(t)$. For $p=2$ we have $(1-t)^2=1-2t+t^2=2-2t=2(1-t)$. For odd p, we have

$$(1-t)^p=1+\sum_{k=1}^{p-1}\binom{p}{k}(-1)^k t^k-t^p=\sum_{k=1}^{(p-1)/2}\binom{p}{k}(-1)^k t^k(1-t^{p-2k})$$
$$=p\cdot(1-t)\sum_{k=1}^{(p-1)/2}\binom{p}{k}/p(-1)^k t^k(1+t+\cdots+t^{p-2k-1}).$$

Now induction on s and the previous result imply when $k>s(p-1)$ that $(1-t)^k=p^s\cdot(1-t)^{k-s(p-1)}h_s(t)$ for some polynomial $h_s(t)$.

Also solved by R. Chapman (U.K.), D. Grinberg, J. H. Lindsey II, and the proposer.

Runs Versus Isolated Heads in Coin Tossing

11394 [2008, 856]. *Proposed by K. S. Bhanu, Institute of Science, Nagpur, India, and M. N. Deshpande, Nagpur, India.* A fair coin is tossed n times, with $n\geq 2$. Let R be the resulting number of runs of the same face, and X the number of isolated heads. Show that the covariance of the random variables R and X is $n/8$.

Solution by Michael Andreoli, Miami Dade College, Miami, FL. Define binary n-tuples U and V by letting $U_k=1$ if and only if an isolated head occurs at toss k, and $V_k=1$ if and only if a run begins at toss k. Now $X=\sum_k U_k$ and $R=\sum_k V_k$. Because $E(U_k)=P(U_k=1)$, we have $E(U_1)=E(U_n)=1/4$ and $E(U_k)=1/8$ for $2\leq k\leq n-1$. Similarly, $E(V_1)=1$ and $E(V_k)=1/2$ for $2\leq k\leq n$. It follows that $E(X)=(n+2)/8$ and $E(R)=(n+1)/2$.

Because $E(U_iV_j)=P(U_i=1 \text{ and } V_j=1)$, we obtain

- $E(U_1V_1)=E(U_1V_2)=1/4$ and $E(U_1V_j)=1/8$ for $3\leq j\leq n$;
- $E(U_nV_1)=E(U_nV_n)=1/4$ and $E(U_nV_j)=1/8$ for $2\leq j\leq n-1$; and
- for $2\leq i\leq n-1$ and $1\leq j\leq n$,

$$E(U_iV_j)=\begin{cases}1/8 & \text{if } j\in\{1,i,i+1\};\\ 1/16 & \text{otherwise.}\end{cases}$$

Therefore,

$$E(XR)=\sum_{i=1}^{n}\sum_{j=1}^{n}E(U_iV_j)=\sum_{j=1}^{n}E(U_1V_j)+\sum_{i=2}^{n-1}\sum_{j=1}^{n}E(U_iV_j)+\sum_{j=1}^{n}E(U_nV_j)$$
$$=\frac{n+2}{8}+\frac{(n-2)(n+3)}{16}+\frac{n+2}{8}=\frac{n^2+5n+2}{16}.$$

It follows that

$$\mathrm{Cov}(XR)=E(XR)-E(X)E(R)=\frac{n^2+5n+2}{16}-\frac{n+2}{8}\cdot\frac{n+1}{2}=\frac{n}{8}.$$

Also solved by D. Beckwith, M. A. Carlton, N. Caro (Brazil), R. Chapman (U.K.), M. P. Cohen, C. Curtis, P. J. Fitzsimmons, N. Grivaux (France), C. C. Heckman, S. J. Herschkorn, G. Keselman, J. H. Lindsey II, K. McInturff, E. Orney & S. Van Gulck (Belgium), A. Plaza & J. J. Gonzalez (Spain), M. A. Prasad (India), R. Pratt & E. Lada, K. Schilling, B. Schmuland (Canada), A. Stadler (Switzerland), J. H. Steelman, R. Stong, R. Tauraso (Italy), Armstrong Problem Solvers, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

Finite Subgroups of Continuous Bijections of [0,1]

11395 [2008, 856]. *Proposed by M. Farrokhi D.G., University of Tsukuba, Tsukuba Ibakari, Japan.* Prove that if H is a finite subgroup of the group G of all continuous bijections of $[0, 1]$ to itself, then the order of H is 1 or 2.

Solution by Jeffrey Bergen, DePaul University, Chicago, IL. If $g \in G$, then g is continuous and injective. Hence g is monotonic, by the intermediate value theorem. Therefore, either (i) $g(0) = 0$ and $g(1) = 1$ or (ii) $g(0) = 1$ and $g(1) = 0$.

Set $g^2 = g \circ g$ and $g^{n+1} = g \circ g^n$ for $n > 1$. If $g(0) = 0$ and $g(a) > a$ for some $a \in [0, 1]$, then the sequence $a, g(a), g^2(a), \ldots$ is increasing. Similarly, if $g(0) = 0$ and $g(a) < a$, then $a, g(a), g^2(a), \ldots$ is decreasing. Therefore, if $g(0) = 0$ and $g(x) \neq x$ for some $x \in [0, 1]$, then g does not have finite order. We conclude that if $g \in H$ and $g(0) = 0$, then g is the identity map.

Next, if $f_1, f_2 \in H$ with $f_1(0) = f_2(0) = 1$, then $f_1 \circ f_2 \in H$ with $f_1 \circ f_2(0) = 0$. Our previous argument shows that $f_1 \circ f_2(x) = x$, and so both f_2 and f_1 are inverses of f_1. Since inverses are unique in a group, it follows that $f_1 = f_2$. As a result, H contains at most one element other than the identity map, and so H has order either 1 or 2, as claimed.

Also solved by M. Barr (Canada), M. Bataille (France), D. R. Bridges, P. Budney, B. S. Burdick, N. Caro & F. Valenzuela (Brazil), R. Chapman (U.K.), L. Comerford, P. Corn, P. P. Dályay (Hungary), D. Grinberg, J. P. Grivaux (France), K. Hanes, E. A. Herman, S. P. Herschkorn, E. J. Ionascu, J. Konienczny, O. Kouba (Syria), J. Kujawa & K. Shankar, J. H. Lindsey II, O. P. Lossers (Netherlands), A. Magidin, R. Martin (Germany), S. Metcalfe, V. Pambuccian, J. W. Pfeffer, E. Pité (France), J. Schaer (Canada), B. Schmuland (Canada), N. C. Singer, V. Stakhovsky, J. H. Steelman, R. Stong, T. Tam, M. Tetiva (Romania) J. Vinuesa (Venezuela), G. Wene, M. Wildon (UK), N. Wodarz, Armstrong Problem Solvers, BSI Problems Group (Germany), Szeged Problem Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U.K.), McDaniel College Problems Group, Microsoft Research Problems Group, Missouri State University Problem Solving Group, Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposer.

A Riemann (Zeta) Sum

11400 [2008, 948]. *Proposed by Paul Bracken, University of Texas–Pan American, Edinburg, TX.* Let ζ be the Riemann zeta function. Evaluate $\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)}$ in closed form.

Solution by Oliver Guepel, Brühl, NRW, Germany. The sum is $\log(2\pi) - \frac{1}{2}$. Since summation of absolutely convergent series can be interchanged, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)} &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k^{2n} n(n+1)} \\ &= 1 + \sum_{k=2}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{k^2}\right)^n - \sum_{k=2}^{\infty}\left(k^2 \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\frac{1}{k^2}\right)^{n+1}\right)\end{aligned}$$

$$= 1 - \sum_{k=2}^{\infty} \log\left(1 - \frac{1}{k^2}\right) + \sum_{k=2}^{\infty}\left(1 + k^2 \log\left(1 - \frac{1}{k^2}\right)\right)$$

$$= 1 + \lim_{n\to\infty} \sum_{k=2}^{n} \left[1 + (k^2 - 1)\left(\log(k+1) - 2\log k + \log(k-1)\right)\right].$$

With $f(n) = n^2 - 1$ and $g(n) = \log n$, this last line can be written as

$$1 + \lim_{n\to\infty} \sum_{k=2}^{n} (1 + f(k)(g(k+1) - 2g(k) + g(k-1))).$$

Now put $h(n) = f(n-1)g(n) - f(n)g(n-1)$. In general, $h(n+1) - h(n) = f(n)(g(n+1) - 2g(n) + g(n-1)) - g(n)(f(n+1) - 2f(n) + f(n-1))$. Here, the second difference of f is identically 2, so

$$f(n)(g(n+1) - 2g(n) + g(n-1)) = h(n+1) - h(n) + 2\log n.$$

Thus

$$1 + \sum_{k=2}^{n} (1 + f(k)(g(k+1) - 2g(k) + g(k-1)))$$

$$= n + \sum_{k=2}^{n} (h(k+1) - h(k) + 2\log k) = n + h(n+1) - h(2) + 2\log(n!)$$

$$= n + (n^2 - 1)\log(n+1) - (n^2 + 2n)\log n + 2\log(n!).$$

A straightforward application of Stirling's formula yields $\log 2\pi - \frac{1}{2}$ as the limit. It also follows now from

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} = \log 2\pi - 1$$

(this MONTHLY **94** (1987), p. 467) that we have the rational sum

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+1)} = \frac{1}{2}.$$

Also solved by K. F. Andersen (Canada), R. Bagby, M. Bataille (France), D. Beckwith, B. S. Burdick, R. Chapman (U.K.), H. Chen, P. Corn, G. Crandall, P. P. Dályay (Hungary), B. E. Davis, Y. Dumont (France), O. Furdui (Romania), M. L. Glasser, G. C. Greubel, J. Grivaux (France), N. Grossman, J. A. Grzesik, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), G. Keselman, O. Kouba (Syria), G. Lamb, O. P. Lossers (Netherlands), K. McInturff, M. Omarjee (France), P. Perfetti (Italy), E. Pité (France), Á. Plaza & S. Falcón (Spain), C. Pohoata (Romania), M. A. Prasad (India), P. R. Refolio (Spain), O. G. Ruehr, V. Rutherfoord, B. Schmuland (Canada), N. C. Singer, S. Singh, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), D. R. Teske, M. Tetiva (Romania), J. Vinuesa (Spain), M. Vowe (Switzerland), BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

A Characterization of the Identity Matrix

11401 [2008, 949]. *Proposed by Marius Cavachi, "Ovidius" University of Constanţa, Constanţa, Romania.* Let A be a nonsingular square matrix with integer entries. Suppose that for every positive integer k, there is a matrix X with integer entries such that $X^k = A$. Show that A must be the identity matrix.

Solution by Microsoft Research Problems Group, c/o Peter Montgomery, Redmond, WA. For $k \in \mathbb{N}$, let X_k be an integer matrix such that $X_k^k = A$. Let p be a prime that does not divide $\det A$. Viewing $X_k \bmod p$ as an element of the general linear group G over the field $\mathbb{F}_p$, Legendre's theorem implies that $X_k^{|G|} \equiv I \bmod p$ for all k. Setting $k = |G|$ yields $A = X_{|G|}^{|G|} \equiv I \bmod p$. That is, all entries of $A - I$ are divisible by p. Since there are infinitely many choices for p, we obtain $A = I$.

Also solved by P. Budney, N. Caro (Brazil), R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, J. Grivaux (France), E. A. Herman, J. Konieczny, K. Koo, T. Laffey & H. Šmigoc (Ireland), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Nakhash, S. Pierce, E. Pité (France), C. Pohoata (Romania), V. Rutherfoord, R. A. Simon (Chile), N. C. Singer, R. Stong, T. Tam, M. Tetiva (Romania), T. Thomas (U.K.), Z. Vörös (Hungary), J. Young, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Double Factorial Sum

11406 [2009, 82]. *Proposed by A. A. Dzhumadil'daeva, Almaty, Republics Physics and Mathematics School, Almaty, Kazakhstan.* Let $n!!$ denote the product of all positive integers not greater than n and congruent to n mod 2, and let $0!! = (-1)!! = 1$. Thus, $7!! = 105$ and $8!! = 384$. For positive integer n, find

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2(n-i)-1)!!$$

in closed form.

Solution I by Kenneth F. Andersen, University of Alberta, Edmonton, Alberta, Canada. The sum is $2^n n!$. To see this, let $f(x) = (1-2x)^{-1/2}$ and $g(x) = (1-2x)^{-1}$ for $|x| < 1/2$. Induction shows that the ith derivatives of f and g are given by

$$\begin{aligned} f^{(i)}(x) &= (2i-1)!!\,(1-2x)^{-1/2-i} \\ g^{(i)}(x) &= 2^i i!\,(1-2x)^{-1-i} \end{aligned} \tag{3}$$

for each nonnegative integer i. In particular, $f^{(i)}(0) = (2i-1)!!$, so

$$\sum_{i=0}^{n} \binom{n}{i} (2i-1)!!\,(2n-2i-1)!! = \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(0) f^{(n-i)}(0).$$

Since $g = f^2$, the Leibniz rule for the nth derivative of a product shows that the latter sum is $g^{(n)}(0)$. In view of (3), this equals $2^n n!$.

Solution II by Ulrich Abel, University of Applied Sciences Giessen-Friedberg, Friedberg, Germany. First note that

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2n-2i}{n-i} = [z^n]\left(\sum_{i=0}^{\infty}\binom{2i}{i} z^i\right)^2 = [z^n]((1-4z)^{-1/2})^2 = 4^n.$$

Using $(2k-1)!! = (2k)!/(2^k k!)$, the original sum becomes

$$\sum_{i=0}^{n} \binom{n}{i} \frac{(2i)!}{2^i i!}\,\frac{(2n-2i)!}{2^{n-i}(n-i)!} = n!\,2^{-n} \sum_{i=0}^{n} \binom{2i}{i}\binom{2n-2i}{n-i} = n!\,2^{-n} 4^n = n!\,2^n.$$

Also solved by 65 other readers.

Some Intermediate Value Variants

11429 [2009, 365]. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanta, Romania.* For a continuous real-valued function ϕ on $[0, 1]$, let $T\phi$ be the function mapping $[0, 1] \to \mathbb{R}$ given by $T\phi(t) = \phi(t) - \int_0^t \phi(u)\,du$, and similarly define S by $S\phi(t) = t\phi(t) - \int_0^t u\phi(u)\,du$. Show that if f and g are continuous real-valued functions on $[0, 1]$, then there exist numbers a, b, and c in $(0, 1)$ such that each of the following is true:

$$Tf(a) = Sf(a),$$

$$Tg(b)\int_{u=0}^{1} f(u)\,du = Tf(b)\int_{u=0}^{1} g(u)\,du,$$

$$Sg(c)\int_{u=0}^{1} f(u)\,du = Sf(c)\int_{u=0}^{1} g(u)\,du.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA.

Lemma. *If h is continuous on $[0, \alpha]$, and $h(\alpha) = 0$, then there exists $a \in (0, \alpha)$ such that $h(a) = \int_0^a h(u)\,du$.*

Proof. Let $H(t) = e^{-t}\int_0^t h(u)\,du$. Note that $H(0) = 0$, and H is continuously differentiable with $H'(t) = e^{-t}(h(t) - \int_0^t h(u)\,du)$. Thus it suffices to find an $a \in (0, \alpha)$ with $H'(a) = 0$. If no such a exists, then $H(t)$ is monotone, and hence $J(t) = H(t)^2$ is monotone increasing and in particular $J(\alpha) > 0$. This gives the contradiction $J'(\alpha) = 2H(\alpha)H'(\alpha) = -2e^{-2\alpha}\left(\int_0^\alpha h(u)\,du\right)^2 = -2J(\alpha)^2 < 0$. ■

Let $F = \int_0^1 f(t)\,dt$, $G = \int_0^1 g(t)\,dt$. Applying the lemma to $h(t) = (1 - t)f(t)$ with $\alpha = 1$ gives $a \in (0, 1)$ such that $(1 - a)f(a) = \int_0^a (1 - u)f(u)\,du$ or $Tf(a) = Sf(a)$. Applying the lemma to $h(t) = f(t)G - g(t)F$, and noting that $\int_0^1 h(t)\,dt = 0$ implies the existence of some $\alpha \in (0, 1)$ with $h(\alpha) = 0$, gives $b \in (0, 1)$ such that

$$f(b)G - g(b)F = \int_0^b f(u)\,du G - \int_0^b g(u)\,du F,$$

or $Tf(b)G = Tg(b)F$. Applying the lemma to $h(t) = tf(t)G - tg(t)F$, and noting that the α found in the previous case still works, gives $c \in (0, 1)$ such that

$$cf(c)G - cg(c)F = \int_0^c uf(u)\,du G - \int_0^c ug(u)\,du F$$

or $Sf(c)G = Sg(c)F$.

Also solved by K. F. Andersen (Canada), R. Bagby, R. Chapman (U.K.), W. J. Cowieson, P. P. Dályay (Hungary), E. A. Herman, B.-I. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, P. Perfetti (Italy), GCHQ Problem Solving Group, and the proposers.