Research Article

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$(p, 2)$-equations asymmetric at both zero and infinity

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Abstract: We consider a $(p, 2)$-equation, that is, a nonlinear nonhomogeneous elliptic equation driven by the sum of a p-Laplacian and a Laplacian with $p > 2$. The reaction term is $(p - 1)$-linear, but exhibits asymmetric behavior at $\pm\infty$ and at $0^{\pm}$. Using variational tools, together with truncation and comparison techniques and Morse theory, we prove two multiplicity theorems, one of them providing sign information for all the solutions (positive, negative, nodal).

Keywords: p-Laplacian, asymmetric reaction, resonance, Fučik spectrum, constant sign solutions, nodal solution, critical groups, Morse relation

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous Dirichlet problem:

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0,\ 2 < p. \tag{1.1}$$

Here, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u(z) = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

If $p = 2$, then $\Delta_2 = \Delta$ the Laplacian.

In problem (1.1), the reaction term $f(z, x)$ is a Carathéodory function such that $f(z, 0) = 0$. We assume that $f(z, \cdot)$ exhibits $(p - 1)$-linear growth near $\pm\infty$. However, the growth of $f(z, \cdot)$ is asymmetric near $\pm\infty$. More precisely, the quotient

$$\frac{f(z, x)}{|x|^{p-2} x}$$

crosses at least the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ as we move from $-\infty$ to $+\infty$ (crossing or jumping nonlinearity). In the negative direction we allow resonance with respect to $\hat{\lambda}_1(p) > 0$, while in the positive direction resonance can occur with respect to any nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We have a similar asymmetric behavior when $x \to 0^{\pm}$. This time the quotient $\frac{f(z,x)}{x}$ crosses $\hat{\lambda}_1(2) > 0$. Under this double asymmetric setting, we prove a multiplicity theorem producing three nontrivial smooth solutions and

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provide sign information for all of them. A second multiplicity theorem is also proved without sign information for the third solution.

Equations involving the sum of a Laplacian and a p-Laplacian arise in problems of mathematical physics; see Cherfils and Ilyasov [9] (plasma physics) and Benci, D'Avenia, Fortunato and Pisani [6] (quantum physics). Recently, there have been existence and multiplicity results for different classes of such equations. We mention the works of Aizicovici, Papageorgiou and Staicu [3], Cingolani and Degiovanni [10], Gasinski and Papageorgiou [13, 15], Papageorgiou and Rădulescu [22, 23], Papageorgiou, Rădulescu and Repovš [25], Sun [30], Sun, Zhang and Su [31] and Yang and Bai [32]. In the aforementioned works, only Papageorgiou and Rădulescu [23] deal with an asymmetric p-sublinear reaction term. They consider a reaction term $f(z, x)$ such that the quotient

$$\frac{f(z,x)}{|x|^{p-2}x}$$

crosses only the first eigenvalue $\hat{\lambda}_1(p)$ as we move from $-\infty$ to $+\infty$, and resonance is allowed at $-\infty$. At zero, the behavior of the quotient $\frac{f(z,x)}{x}$ is symmetric. Finally, in [23] the multiplicity result does not produce nodal solutions. Concerning asymmetric sublinear problems, we should also mention the semilinear works of D'Agui, Marano and Papageorgiou [11] (Robin problems with an indefinite and unbounded potential) and Recova and Rumbos [28] (Dirichlet problems with zero potential).

Our approach is variational, based on the critical point theory combined with suitable truncation and comparison techniques and Morse theory (critical groups).

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short) if the following holds: *Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and*

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

This is a compactness-type condition on the functional. It leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. One of the main results in this theory is the so-called "mountain pass theorem" of Ambrosetti and Rabinowitz [5], stated here in a slightly more general form (see Gasinski and Papageorgiou [12]).

Theorem 2.1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r > 0$,*

$$\max\{\varphi(u_o), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = r] = m_r$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)) \quad \textit{with } \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

Then $c \geqslant m_r$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi(u) = c$, $\varphi'(u) = 0$).

In the study of (1.1), we will use the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. By Poincaré's inequality, the norm of $W_0^{1,p}(\Omega)$ can be defined by

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Sobolev space $H_0^1(\Omega)$ is a Hilbert space and, as above, the Poincaré inequality implies that we can choose as inner product

$$(u, h) = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in H_0^1(\Omega).$$

The corresponding norm is

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_2 \quad \text{for all } u \in H_0^1(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \Big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\Big\}.$$

Here, by $\frac{\partial u}{\partial n}$ we denote the normal derivative of u, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Recall that $C_0^1(\overline{\Omega})$ is dense in both $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$.

We consider a function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ which is Carathéodory function, that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f_0(z, x)$ is measurable and for almost all $z \in \Omega$ the function $x \mapsto f_0(z, x)$ is continuous. We assume that

$$|f_0(z, x)| \leqslant a_0(z)[1 + |x|^{r-1}] \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and

$$1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p \end{cases}$$

(the critical Sobolev exponent for p). We set

$$F_0(z, x) = \int_0^x f_0(z, s)\, ds$$

and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z, u)\, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Aizicovici, Papageorgiou and Staicu [2, Proposition 2]. See also Papageorgiou and Rădulescu [21, 24] for corresponding results for the Neumann and Robin problems. The result is essentially a byproduct of the regularity theory of Lieberman [18, Theorem 1].

Proposition 2.2. *Assume that $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}) \textit{ with } \|h\|_{C_0^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega) \textit{ with } \|h\| \leqslant \rho_1.$$

For any $r \in (1, +\infty)$, let

$$A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*\Big(\frac{1}{r} + \frac{1}{r'} = 1\Big)$$

be the map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

From Motreanu, Motreanu and Papageorgiou [19, Proposition 2.72, p. 40] we have the following property.

Proposition 2.3. *The map A_r is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (hence, maximal monotone, too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,r}(\Omega) \qquad \text{and} \qquad \limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \leqslant 0$$

imply

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Note that if $p = 2$, then $A_2 = A \in \mathscr{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

We will use the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$ and the Fučik spectrum of $(-\Delta, H_0^1(\Omega))$. So, let us recall some basic facts about them.

We start with the following nonlinear eigenvalue problem:

$$-\Delta_r u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0,\ 1 < r < \infty. \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_r, W_0^{1,r}(\Omega))$ if problem (2.1) admits a nontrivial solution $\hat{u} \in W_0^{1,r}(\Omega)$, known as an "eigenfunction" corresponding to $\hat{\lambda}$. There is a smallest eigenvalue $\hat{\lambda}_1(r) > 0$ such that the following conditions hold:

- $\hat{\lambda}_1(r)$ is isolated in the spectrum $\hat{\sigma}(r)$ of $(-\Delta_r, W_0^{1,r}(\Omega))$, that is, there exists $\epsilon > 0$ such that
$$(\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \epsilon) \cap \hat{\sigma}(r) = \emptyset.$$
- $\hat{\lambda}_1(r)$ is simple, that is, if $\hat{u}, \tilde{u} \in W_0^{1,r}(\Omega)$ are eigenfunctions corresponding to $\hat{\lambda}_1(r)$, then there exists $\xi \in \mathbb{R} \setminus \{0\}$ such that $\hat{u} = \xi\tilde{u}$.
- The equation
$$\hat{\lambda}_1(r) = \inf\Big[\frac{\|Du\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega),\ u \neq 0\Big] \tag{2.2}$$
holds.

In (2.2), the infimum is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(r)$. The aforementioned properties imply that the elements of this eigenspace have fixed sign. Moreover, the nonlinear regularity theory (see, for example, Gasinski and Papageorgiou [12, pp. 737–738]) implies that all the eigenfunctions of $(-\Delta_r, W_0^{1,r}(\Omega))$ belong in $C_0^1(\overline{\Omega})$. By $\hat{u}_1(r)$ we denote the positive L^r-normalized (that is, $\|\hat{u}_1(r)\|_r = 1$) eigenfunction corresponding to $\hat{\lambda}_1(r) > 0$. The nonlinear strong maximum principle (see, for example, Gasinski and Papageorgiou [12, p. 738]) implies that $\hat{u}_1(r) \in \operatorname{int} C_+$. An eigenfunction $\hat{u} \in C_0^1(\overline{\Omega})$ corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(r)$ is necessarily nodal (sign-changing). It is easily seen that the set $\hat{\sigma}(r)$ is closed. Since $\hat{\lambda}_1(r) > 0$ is isolated, the second eigenvalue $\hat{\lambda}_2(r) > 0$ is well-defined by

$$\hat{\lambda}_2(r) = \min[\hat{\lambda} \in \hat{\sigma}(r) : \hat{\lambda} \neq \hat{\lambda}_1(r)].$$

To produce additional eigenvalues, we can use the Ljusternik–Schnirelmann minimax scheme. In this way, we obtain a whole nondecreasing sequence of eigenvalues $\{\hat{\lambda}_k(r)\}_{k\geqslant 1}$ of $(-\Delta_r, W_0^{1,r}(\Omega))$ such that $\hat{\lambda}_k(r) \to +\infty$ as $k \to \infty$. These eigenvalues are known as "variational eigenvalues", and $\hat{\lambda}_1(r)$ and $\hat{\lambda}_2(r)$ are as described above. We do not know if the variational eigenvalues exhaust the spectrum of $(-\Delta_r, W_0^{1,r}(\Omega))$. This is the case if $r = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equations). In the linear case ($r = 2$), the eigenspaces $E(\hat{\lambda}_k(2))$, $k \in \mathbb{N}$, are finite-dimensional subspaces of $C_0^1(\overline{\Omega})$ and we have the following orthogonal direct sum decomposition:

$$H_0^1(\Omega) = \overline{\bigoplus_{k\geqslant 1} E(\hat{\lambda}_k(2))}.$$

When $r \neq 2$ (nonlinear eigenvalue problem), the eigenspaces are only cones and there is no decomposition of the space $W_0^{1,r}(\Omega)$ in terms of them. This makes the study of problems driven by $-\Delta_r$ and resonant at higher parts of the spectrum difficult to deal with.

We will also encounter a weighted version of the eigenvalue problem (2.1). So, let $m \in L^\infty(\Omega)$, $m(z) \geqslant 0$ for almost all $z \in \Omega$, $m \not\equiv 0$. We consider the following nonlinear eigenvalue problem:

$$-\Delta_r u(z) = \tilde{\lambda} m(z)|u(z)|^{r-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.3}$$

Again $\tilde{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega), m)$ if problem (2.3) admits a nontrivial solution. We have a smallest eigenvalue $\tilde{\lambda}_1(r, m) > 0$ which is isolated, simple and satisfies

$$\tilde{\lambda}_1(r, m) = \inf\Big[\frac{\|Du\|_r^r}{\int_\Omega m|u|^r\,dz} : u \in W_0^{1,r}(\Omega),\ u \neq 0\Big]. \tag{2.4}$$

As before, the infimum is realized on the corresponding one-dimensional eigenspace, the elements of which do not change sign. This fact and (2.4) lead to the following monotonicity property of $m \to \tilde{\lambda}_1(r, m)$.

Proposition 2.4. *Suppose $m, m' \in L^\infty(\Omega) \setminus \{0\}$, $0 \leqslant m(z) \leqslant m'(z)$ for almost all $z \in \Omega$, and $m \not\equiv m'$. Then $\tilde{\lambda}_1(r, m') < \tilde{\lambda}_1(r, m)$.*

Remark 1. For the linear eigenvalue problem (that is, $r = 2$), the spectrum consists of a sequence

$$\{\tilde{\lambda}_k(2, m) = \tilde{\lambda}_k(m)\}_{k\in\mathbb{N}}$$

of distinct eigenvalues such that

$$\tilde{\lambda}_k(m) \to +\infty \quad \text{as } k \to \infty.$$

The eigenspaces $E(\tilde{\lambda}_k(2, m))$ have the unique continuation property, that is, if $u \in E(\tilde{\lambda}_k(2, m))$ and $u(\cdot)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. This property leads to the following strict monotonicity property of $\tilde{\lambda}_k(2, \cdot)$:

$$m, m' \in L^\infty(\Omega) \setminus \{0\}, \qquad 0 \leqslant m(z) \leqslant m'(z) \quad \text{for almost all } z \in \Omega, \qquad m \not\equiv m'$$

imply

$$\tilde{\lambda}_k(2, m') < \tilde{\lambda}_k(2, m) \quad \text{for all } k \in \mathbb{N}.$$

Another related result is the following lemma, which is a consequence of the properties of $\hat{\lambda}_1(p)$ (see Motreanu, Motreanu and Papageorgiou [19, Lemma 11.3, p. 305]).

Lemma 2.5. *If $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leqslant \hat{\lambda}_1(p)$ for almost all $z \in \Omega$, $\vartheta \not\equiv \hat{\lambda}_1(p)$, then there exists $\hat{c} > 0$ such that*

$$\hat{c}\|u\|^p \leqslant \|Du\|_p^p - \int_\Omega \vartheta(z)|u|^p\,dz \quad \textit{for all } u \in W_0^{1,p}(\Omega).$$

Since our problem is also asymmetric at zero, in our analysis we will use the Fučik spectrum of $(-\Delta, H_0^1(\Omega))$. So, we consider the following linear eigenvalue problem:

$$-\Delta u(z) = \alpha u^+(z) - \beta u^-(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0, \tag{2.5}$$

where $u^\pm(\cdot) = \max\{\pm u(\cdot), 0\}$ (the positive and negative parts of u). By Σ_2 we denote the set of points $(\alpha, \beta) \in \mathbb{R}^2$ for which problem (2.5) admits a nontrivial solution. The set Σ_2 is called the “Fučik spectrum” of $(-\Delta, H_0^1(\Omega))$. Let $\{\hat{\lambda}_k(2)\}_{k\in\mathbb{N}}$ be the sequence of distinct eigenvalues of $(-\Delta, H_0^1(\Omega))$. While the spectrum of $(-\Delta, H_0^1(\Omega))$ is a sequence of points, the frame of the Fučik spectrum Σ_2 consists of a family of curves. In particular, the lines $(\{\hat{\lambda}_1(2)\} \times \mathbb{R}) \cup (\mathbb{R} \times \{\hat{\lambda}_1(2)\})$ can be considered as the first curve of Σ_2. In fact, this curve is isolated in Σ_2. For every $\ell \in \mathbb{N}$, $\ell \geqslant 2$, there are two decreasing curves $C_{\ell,1}$, $C_{\ell,2}$ (which may coincide) which pass through the point $(\hat{\lambda}_\ell(2), \hat{\lambda}_\ell(2))$ such that all points in the square $Q_\ell = (\hat{\lambda}_{\ell-1}(2), \hat{\lambda}_{\ell+1}(2))^2$ which are either in the region $I_{\ell,1}$ below both curves or in the region $I_{\ell,2}$ above the curves, do not belong to Σ_2 (these are the regions of type I). The status of the points between the two curves (when they do not coincide) is unknown in general. However, when $\hat{\lambda}_\ell(2)$ is a simple eigenvalue, points between the two curves are not in Σ_2. We mention that $\Sigma_2 \subseteq \mathbb{R}^2$ is closed with respect to the diagonal (that is, $(\alpha, \beta) \in \Sigma_2$ if and only if $(\beta, \alpha) \in \Sigma_2$). Also, $(\lambda, \lambda) \in \Sigma_2$ if and only if $\lambda = \hat{\lambda}_n(2)$ for some $n \in \mathbb{N}$. As we have already mentioned the lines $\{\hat{\lambda}_1(2)\} \times \mathbb{R}$ and $\mathbb{R} \times \{\hat{\lambda}_1(2)\}$ are contained in Σ_2. In the scalar case (that is, $N = 1$), we have a complete description of the Fučik spectrum. For more information about Σ_2, we refer to Schechter [29].

Next, we recall some basic definitions and facts from Morse theory (critical groups). So, as before, X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$
\begin{aligned}
K_\varphi &= \{u \in X : \varphi'(u) = 0\},\\
K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\},\\
\varphi^c &= \{u \in X : \varphi(u) \leqslant c\}.
\end{aligned}
$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Suppose that $u \in K_\varphi^c$ is isolated. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the isolating neighborhood U. Suppose that φ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, suppose $c' < c < \inf \varphi(K_\varphi)$. From Motreanu, Motreanu and Papageorgiou [19, Corollary 6.35, p. 115], we have that

$$\varphi^{c'} \text{ is a strong deformation retract of } \varphi^c,$$

which implies

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see [19, Corollary 6.15 (a), p. 145]). Therefore, indeed $C_k(\varphi, \infty)$ is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Now suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and that K_φ is finite. We define

$$
\begin{aligned}
M(t, u) &= \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, u) t^k \quad &&\text{for all } t \in \mathbb{R} \text{ and all } u \in K_\varphi,\\
P(t, \infty) &= \sum_{k \in \mathbb{N}_0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad &&\text{for all } t \in \mathbb{R}.
\end{aligned}
$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

where

$$Q(t) = \sum_{k \in \mathbb{N}_0} \beta_k t^k$$

is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

We conclude this section by fixing our notation and introducing the hypotheses on the reaction term $f(z, x)$. Recall that if $u \in W_0^{1,p}(\Omega)$, we define

$$u^\pm(z) = \max\{\pm u(z), 0\}.$$

We know that $u^\pm \in W_0^{1,p}(\Omega)$, and we have $u = u^+ - u^-$ and $|u| = u^+ + u^-$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, and given $f(z, x)$ a measurable function (for example, a Carathéodory function), we denote by $N_f(\cdot)$ the Nemitsky (superposition) map corresponding to $f(\cdot, \cdot)$ and defined by

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The hypotheses on $f(z, x)$ are the following.

Hypotheses H(f). $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and the following conditions hold:

(i) For every $r > 0$, there exists $a_r \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_r(z) \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leqslant r.$$

(ii) There exist $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant \hat{\lambda}_1(p)$ for almost all $z \in \Omega$, $\eta \neq \hat{\lambda}_1(p)$, and $\hat{\eta}, \hat{\vartheta} > 0$ such that

$$-\hat{\vartheta} \leqslant \liminf_{x\to-\infty} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \limsup_{x\to-\infty} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \hat{\lambda}_1(p),$$
$$\eta(z) \leqslant \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leqslant \hat{\eta}$$

uniformly for almost all $z \in \Omega$.

(iii) If $F(z,x) = \int_0^x f(z,s)\,ds$, then $f(z,x)x - pF(z,x) \to +\infty$ as $x \to -\infty$ uniformly for almost all $z \in \Omega$ and there exists $M_0 > 0$ such that

$$f(z,x)x - pF(z,x) \geqslant 0 \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant M_0.$$

(iv) There exist $0 < \alpha < \hat{\lambda}_1(2) < \beta < \hat{\lambda}_2(2)$ such that

$$\lim_{x\to0^+} \frac{f(z,x)}{x} = \alpha, \qquad \lim_{x\to0^-} \frac{f(z,x)}{x} = \beta$$

uniformly for almost all $z \in \Omega$, and for every $\rho > 0$ there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the mapping $x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 2. Hypothesis H(f) (ii) implies that $f(z,\cdot)$ is a crossing nonlinearity. In fact, we can cross any finite number of variational eigenvalues, starting with $\hat{\lambda}_1(p) > 0$. Note that in the negative direction we can have resonance with respect to $\hat{\lambda}_1(p) > 0$, while in the positive direction resonance is possible with respect to any nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. As we will see in the proof of Proposition 3.3, Hypothesis H(f) (iii) guarantees that at $-\infty$ the resonance with respect to $\hat{\lambda}_1(p) > 0$ is from the left of the principal eigenvalue in the sense that

$$\hat{\lambda}_1(p)|x|^p - pF(z,x) \to +\infty \quad \text{as } x \to -\infty, \text{ uniformly for almost all } z \in \Omega.$$

This makes the negative truncation of the energy functional of (1.1) coercive. So, we can use the direct method of the calculus of variations. Hypothesis H(f) (iv) implies that at zero, too, we have an asymmetric behavior of the quotient $\frac{f(z,x)}{x}$.

3 Solutions of constant sign

In this section, using variational tools, we show that problem (1.1) admits two nontrivial smooth solutions of constant sign (one positive and the other one negative).

So, let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\varphi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Also, we consider the positive and negative truncations of $f(z,\cdot)$, that is, the Carathéodory function

$$f_\pm(z,x) = f(z, \pm x^\pm).$$

We set $F_\pm(z,x) = \int_0^x f_\pm(z,s)\,ds$ and consider the C^1-functionals $\varphi_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_\pm(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.1. *If Hypotheses* H(f) *hold, then φ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \leqslant W_0^{1,p}(\Omega)$ such that

$$|\varphi(u_n)| \leqslant M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \in \mathbb{N}, \tag{3.1}$$

$$(1+\|u_n\|)\varphi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z,u_n)h\,dz\right| \leqslant \frac{\epsilon_n\|h\|}{1+\|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+. \tag{3.3}$$

In (3.3), we chose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$-\|Du_n\|_p^p - \|Du_n\|_2^2 + \int_\Omega f(z,u_n)u_n\,dz \leqslant \xi_n \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Also, from (3.1) we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \int_\Omega pF(z,u_n)\,dz \leqslant pM_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

We add (3.4) and (3.5). Recalling that $p > 2$, we obtain

$$\int_\Omega [f(z,u_n)u_n - pF(z,u_n)]\,dz \leqslant M_2 \quad \text{for some } M_2 > 0 \text{ and all } n \in \mathbb{N},$$

which implies

$$\int_\Omega [f(z,-u_n^-)(-u_n^-) - pF(z,-u_n^-)]\,dz \leqslant M_3 \quad \text{for some } M_3 > 0 \text{ and all } n \in \mathbb{N} \tag{3.6}$$

(see Hypotheses H(f) (i) and (iii)).

Claim 1. $\{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ *is bounded.*

We argue by contradiction. So, suppose that Claim 1 is not true. By passing to a subsequence if necessary, we may assume that

$$\|u_n^-\| \to \infty \text{ as } n \to \infty. \tag{3.7}$$

Let $y_n = \frac{u_n^-}{\|u_n^-\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ and $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega), \qquad y \geqslant 0. \tag{3.8}$$

In (3.3), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 - \int_\Omega f(z,-u_n^-)(-u_n^-)\,dz \leqslant \xi_n \quad \text{for all } n \in \mathbb{N},$$

which implies

$$\|Dy_n\|_p^p + \frac{1}{\|u_n^-\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} y_n\,dz \leqslant \frac{\xi_n}{\|u_n^-\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Hypotheses H(f) (i), (ii) and (iii) imply that

$$|f(z,x)| \leqslant c_1[1+|x|^{p-1}] \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and for some } c_1 > 0. \tag{3.10}$$

From (3.10) it follows that

$$\left\{\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded } \Big(\frac{1}{p} + \frac{1}{p'} = 1\Big). \tag{3.11}$$

On account of (3.11) and Hypothesis H(f) (ii), at least for a subsequence we have

$$\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}} \xrightarrow{w} \vartheta(z)y^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ with } -\hat{\vartheta} \leqslant \vartheta(z) \leqslant \hat{\lambda}_1(p) \text{ for almost all } z \in \Omega \tag{3.12}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 16]). We pass to the limit as $n \to \infty$ in (3.9). Using (3.7), (3.8), (3.12) and the fact that $2 < p$, we obtain

$$\|Dy\|_p^p \leqslant \int_\Omega \vartheta(z)y^p \, dz. \tag{3.13}$$

If $\vartheta \not\equiv \hat{\lambda}_1(p)$, then from (3.7) and Lemma 2.5 we have $\hat{c}\|y\|^p \leqslant 0$, which implies

$$y = 0. \tag{3.14}$$

Then from (3.9), using as before (3.7), (3.8) and (3.12) (the last two relations with $y = 0$, see (3.14)) and the fact that $p > 2$, we infer that $\|Dy_n\|_p \to 0$, which implies $y_n \to 0$ in $W_0^{1,p}(\Omega)$, which contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Next we assume that $\vartheta(z) = \hat{\lambda}_1(p)$ for almost all $z \in \Omega$ (resonant case). Then from (3.13) and (2.2) we have

$$\|Dy\|_p^p = \hat{\lambda}_1(p)\|y\|_p^p,$$

which implies

$$y = \tilde{\vartheta}\hat{u}_1(p) \quad \text{with } \tilde{\vartheta} \geqslant 0$$

(recall that $y \geqslant 0$, see (3.8)).

If $\tilde{\vartheta} = 0$, then $y = 0$ and as above we have

$$y_n \to 0 \quad \text{in } W_0^{1,p}(\Omega),$$

again contradicting the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

If $\tilde{\vartheta} > 0$, then $y(z) > 0$ for all $z \in \Omega$, and so

$$u_n^-(z) \to +\infty \quad \text{for all } z \in \Omega,$$

which implies

$$f_n(z, -u_n^-(z))(-u_n^-(z)) - pF(z, -u_n^-(z)) \to 0 \quad \text{for almost all } z \in \Omega \text{ as } n \to \infty$$

(see Hypothesis H(f) (iii)), which implies

$$\int_\Omega [f(z, -u_n^-)(-u_n^-) - pF(z, -u_n^-)] \, dz \to +\infty \quad \text{(by Fatou's lemma).} \tag{3.15}$$

We compare (3.15) and (3.6) and have a contradiction. This proves Claim 1.

Claim 2. *$\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded.*

Again we argue indirectly. So, suppose that Claim 2 is not true. Then at least for a subsequence we have

$$\|u_n^+\| \to +\infty \quad \text{as } n \to \infty. \tag{3.16}$$

Let $v_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|v_n\| = 1$ and $v_n \geqslant 0$ for all $n \in \mathbb{N}$, and so we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad v_n \to v \quad \text{in } L^p(\Omega), \qquad v \geqslant 0. \tag{3.17}$$

From (3.3) and Claim 1 we have

$$\left| \langle A_p(u_n^+), h\rangle + \langle A(u_n^+), h\rangle - \int_\Omega f(z, u_n^+)h \, dz \right| \leqslant M_4\|h\|$$

for some $M_4 > 0$, all $n \in \mathbb{N}$ and all $h \in W_0^{1,p}(\Omega)$ (see Hypothesis H(f) (i)), which implies

$$\left|\langle A_p(v_n), h\rangle + \frac{1}{\|u_n^+\|^{p-2}}, \langle A(v_n), h\rangle - \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} h\, dz\right| \leqslant \frac{M_4\|h\|}{\|u_n^+\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \tag{3.18}$$

Using the growth condition from (3.10), we see that

$$\left\{\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{3.19}$$

In (3.18), we choose $h = v_n - v \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.13), (3.16), (3.17) and the fact that $p > 2$. Then

$$\lim_{n\to\infty} \langle A_p(v_n), v_n - v\rangle = 0,$$

which implies

$$v_n \to v \text{ in } W_0^{1,p}(\Omega) \tag{3.20}$$

(see Proposition 2.3).

From (3.19) and Hypothesis H(f) (ii) we see that at least for a subsequence we have

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \tilde{\eta}(z)v^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ with } \eta(z) \leqslant \tilde{\eta}(z) \leqslant \hat{\eta} \text{ for almost all } z \in \Omega \text{ (see [1]).} \tag{3.21}$$

So, if in (3.18) we pass to the limit as $n \to \infty$ and use (3.16), (3.20), (3.21) and the fact that $p > 2$, then

$$\langle A_p(v), h\rangle = \int_\Omega \tilde{\eta}(z)v^{p-1} h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

which implies

$$-\Delta_p v(z) = \tilde{\eta}(z)v(z)^{p-1} \quad \text{for almost all } z \in \Omega,\ v|_{\partial\Omega} = 0. \tag{3.22}$$

From Proposition 2.4 we have

$$\tilde{\lambda}_1(p, \tilde{\eta}) < \tilde{\lambda}_1(p, \hat{\lambda}_1) = 1. \tag{3.23}$$

From (3.22) and (3.23) and since $\|v\| = 1$ (see (3.20)), it follows that $v(\cdot)$ must be nodal, contradicting (3.17). This proves Claim 2.

From Claims 1 and 2 we deduce that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{3.24}$$

In (3.3), we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.24) and the fact that $\{N_f(u_n)\}_{n\geqslant 1} \subseteq L^{p'}(\Omega)$ is bounded (see (3.10)). Then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0,$$

which implies

$$\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \leqslant 0$$

(since $A(\cdot)$ is monotone), which then implies

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.3).

Therefore, the energy functional φ satisfies the C-condition. □

Next, we show that φ_+ satisfies the C-condition, too.

Proposition 3.2. *If Hypotheses* H(f) *hold, then* φ_+ *satisfies the* C*-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} |\varphi_+(u_n)| &\leqslant M_5 \quad \text{for some } M_5 > 0 \text{ and for all } n \in \mathbb{N}, \\ (1 + \|u_n\|)\varphi'_+(u_n) &\to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{3.25}$$

From (3.25) we have

$$\Big|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f_+(z, u_n)h\,dz\Big| \leqslant \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+. \tag{3.26}$$

In (3.26), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p + \|Du_n^-\|_2^2 \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N},$$

which implies

$$u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.27}$$

From (3.26) and (3.27) it follows that

$$\Big|\langle A_p(u_n^+), h\rangle + \langle A(u_n^+), h\rangle - \int_\Omega f(z, u_n^+)h\,dz\Big| \leqslant \epsilon'_n\|h\| \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon'_n \to 0.$$

Suppose that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. So, we may assume that $\|u_n^+\| \to \infty$. We set $v_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$, and have $\|v_n\| = 1$ and $v_n \geqslant 0$ for all $n \in \mathbb{N}$. Hence we can say (at least for a subsequence) that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad v_n \to v \quad \text{in } L^p(\Omega).$$

Then, reasoning as in the proof of Proposition 3.1 (see the part of the proof after (3.17)), we show that

$$\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.} \tag{3.28}$$

From (3.27) and (3.28) it follows that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.29}$$

In (3.26), we choose $h = u_n - u \in W_0^{1,p}(\Omega)$. Passing to the limit as $n \to \infty$, using (3.29) and following the argument in the last part of the proof of Proposition 3.1 (see the part of the proof after (3.24)), we obtain $u_n \to u$ in $W_0^{1,p}(\Omega)$. We conclude that φ_+ satisfies the C-condition. □

For the functional φ_-, we have the following result.

Proposition 3.3. *If Hypotheses* H(f) *hold, then* φ_- *is coercive.*

Proof. Hypothesis H(f) (iii) implies that given any $\xi > 0$, we can find $M_6 = M_6(\xi) > 0$ such that

$$f(z, x)x - pF(z, x) \geqslant \xi \quad \text{for almost all } z \in \Omega \text{ and all } x \leqslant -M_6. \tag{3.30}$$

We have

$$\begin{aligned} \frac{d}{dx}\Big[\frac{F(z, x)}{|x|^p}\Big] &= \frac{f(z, x)|x|^p - p|x|^{p-2}xF(z, x)}{|x|^{2p}} \\ &= \frac{f(z, x)x - pF(z, x)}{|x|^p x} \\ &\leqslant \frac{\xi}{|x|^p x} \end{aligned}$$

for almost all $z \in \Omega$ and all $x \leqslant M_6$ (see (3.30)), which implies

$$\frac{F(z,y)}{|y|^p} - \frac{F(z,w)}{|w|^p} \geqslant \frac{\xi}{p}\left[\frac{1}{|w|^p} - \frac{1}{|y|^p}\right] \quad \text{for almost all } z \in \Omega \text{ and all } y \leqslant w \leqslant -M_6. \tag{3.31}$$

Hypothesis H(f) (iii) implies that

$$-\hat{\vartheta} \leqslant \liminf_{x\to-\infty} \frac{pF(z,x)}{|x|^p} \leqslant \limsup_{x\to-\infty} \frac{pF(z,x)}{|x|^p} \leqslant \hat{\lambda}_1(p) \quad \text{uniformly for almost all } z \in \Omega. \tag{3.32}$$

If in (3.31) we pass to the limit as $y \to -\infty$ and use (3.32), then

$$\hat{\lambda}_1(p)|w|^p - pF(z,w) \geqslant \xi \quad \text{for almost all } z \in \Omega \text{ and all } w \leqslant -M_6.$$

But $\xi > 0$ is arbitrary. So, we infer that

$$\hat{\lambda}_1(p)|w|^p - pF(z,w) \to +\infty \quad \text{as } w \to -\infty, \text{ uniformly for almost all } z \in \Omega. \tag{3.33}$$

We will use (3.33) to show that φ_- is coercive. We argue by contradiction. So, suppose that φ_- is not coercive. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ and $M_7 > 0$ such that

$$\|u_n\| \to \infty \qquad \text{and} \qquad \varphi_-(u_n) \leqslant M_7 \quad \text{for all } n \in \mathbb{N}. \tag{3.34}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega). \tag{3.35}$$

From (3.34) we have

$$\|Dy_n\|_p^p + \frac{p}{2\|u_n\|^{p-2}}\|Dy_n\|_2^2 - \int_\Omega \frac{pF_-(z,u_n)}{\|u_n\|^p}\,dz \leqslant \frac{M_7}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{3.36}$$

From (3.10) we have

$$|F(z,x)| \leqslant c_3[1 + |x|^p] \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_3 > 0,$$

which implies that

$$\left\{\frac{F_-(\cdot, u_n(\cdot))}{\|u_n\|^p}\right\}_{n\geqslant 1} \subseteq L^1(\Omega) \quad \text{is uniformly integrable.}$$

By the Dunford–Pettis theorem and (3.32), we have (at least for a subsequence) that

$$\frac{F_-(\cdot, u_n(\cdot))}{\|u_n\|^p} \xrightarrow{w} \frac{1}{p}\vartheta(z)(y^-)^p \quad \text{in } L^1(\Omega) \text{ with } -\hat{\vartheta} \leqslant \vartheta(z) \leqslant \hat{\lambda}_1(p) \text{ for almost all } z \in \Omega. \tag{3.37}$$

We return to (3.36), pass to the limit as $n \to \infty$ and use (3.34), (3.35) and (3.37) together with the fact that $p > 2$. Then

$$\|Dy\|_p^p \leqslant \int_\Omega \vartheta(z)(y^-)^p\,dz, \tag{3.38}$$

which implies

$$\|Dy^-\|_p^p \leqslant \int_\Omega \vartheta(z)(y^-)^p\,dz. \tag{3.39}$$

If $\vartheta \not\equiv \hat{\lambda}_1(p)$, then from (3.39) and Lemma 2.5 we have $\hat{c}\|y^-\|^p \leqslant 0$, which implies $y \geqslant 0$. Then (3.38) implies that $y = 0$. So, from (3.36) it follows that

$$y_n \to 0 \text{ in } W_0^{1,p}(\Omega),$$

which contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Next, assume that $\vartheta(z) = \hat{\lambda}_1(p)$ for almost all $z \in \Omega$. From (3.39) and (2.2) we have

$$\|Dy_-\|_p^p = \hat{\lambda}_1(p)\|y^-\|_p^p,$$

which implies

$$y^- = \tilde{\vartheta}\hat{u}_1(p) \quad \text{with } \tilde{\vartheta} \geqslant 0.$$

If $\tilde{\vartheta} = 0$, then $y \geqslant 0$ and, as above, we reach a contradiction.
If $\tilde{\vartheta} > 0$, then $y(z) < 0$ for all $z \in \Omega$, and so

$$u_n(z) \to -\infty \quad \text{for all } z \in \Omega \text{ as } n \to \infty,$$

which implies

$$u_n^-(z) \to +\infty \quad \text{for all } z \in \Omega \text{ as } n \to \infty,$$

thus

$$\hat{\lambda}_1(p)u_n^-(z)^p - pF(z, -u_n^-(z)) \to +\infty \quad \text{for almost all } z \in \Omega$$

(see (3.33)), which implies

$$\int_\Omega [\hat{\lambda}_1(p)(u_n^-)^p - pF(z, -u_n^-)]\, dz \to +\infty$$

by Fatou's lemma.
Since $\hat{\lambda}_1(p)\|u_n^-\|_p^p \leqslant \|Du_n^-\|_p^p$ for all $n \in \mathbb{N}$ (see (2.2)), it follows that

$$p\varphi_-(u_n) \to +\infty \quad \text{as } n \to \infty,$$

which contradicts (3.34). Therefore φ_- is coercive. □

Remark 3. From (3.33) we see that the resonance with respect to $\hat{\lambda}_1(p) > 0$ at $-\infty$ is from the left of the principal eigenvalue.

From the above proposition we infer the following fact about the functional φ_- (see [19]).

Corollary 3.4. *If Hypotheses* H(f) *hold, then φ_- satisfies the C-condition.*

Next, we determine the nature of the critical point $u = 0$ for φ_+.

Proposition 3.5. *If Hypotheses* H(f) *hold, then $u = 0$ is a local minimizer for φ_+.*

Proof. Hypothesis H(f) (iv) implies that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z, x) \leqslant \frac{1}{2}(\alpha + \epsilon)x^2 \quad \text{for almost all } z \in \Omega \text{ and all } 0 \leqslant x \leqslant \delta. \tag{3.40}$$

Let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$. Then

$$\begin{aligned}
\varphi_+(u) &\geqslant \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\alpha + \epsilon}{2}\|u^+\|_2^2 && \text{(see (3.40))}\\
&\geqslant \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du^-\|_2^2 + \frac{1}{2}[\|Du^+\|_2^2 - \alpha\|u^+\|_2^2] - \frac{\epsilon}{2\hat{\lambda}_1(2)}\|Du^+\|_2^2 && \text{(see (2.2))}\\
&\geqslant \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du^-\|_2^2 + \frac{1}{2}\Big[c_4 - \frac{\epsilon}{\hat{\lambda}_1(2)}\Big]\|Du^+\|_2^2 && \text{for some } c_4 > 0
\end{aligned}$$

(recall that $\alpha < \hat{\lambda}_1(2)$).
Choosing $\epsilon \in (0, \hat{\lambda}_1(2)c_4)$, we have

$$\varphi_+(u) \geqslant \frac{1}{p}\|Du\|_p^p \quad \text{for all } u \in C_0^1(\overline{\Omega}), \text{ with } \|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta,$$

which implies that

$$u = 0 \quad \text{is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_+,$$

which then implies

$$u = 0 \quad \text{is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_+$$

(see Proposition 2.2). The proof is complete. □

Now we are ready to produce constant sign smooth solutions for problem (1.1).

Proposition 3.6. *If Hypotheses* H(f) *hold, then problem* (1.1) *admits two constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. We can easily check that $K_{\varphi_+} \subseteq C_+$. So, we may assume that K_{φ_+} is finite or otherwise we already have an infinity of positive solutions for problem (1.1). Then on account of Proposition 3.5 we can find small $\rho \in (0, 1)$ such that

$$0 = \varphi_+(0) < \inf[\varphi_+(u) : \|u\| = \rho] = m_+ \tag{3.41}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29]). Hypotheses H(f) (i) and (ii) imply that given $\epsilon > 0$, we can find $c_3 > 0$ such that

$$F(z, x) \geqslant \frac{1}{p}[\eta(z) - \epsilon]x^p - c_5 \quad \text{for almost all } z \in \Omega \text{ and all } x \geqslant 0. \tag{3.42}$$

Then for $t > 0$ we have

$$\begin{aligned}\varphi_+(t\hat{u}_1(p)) &\leqslant \frac{t^p}{p}\hat{\lambda}_1(p) + \frac{t^2}{2}\|D\hat{u}_1(p)\|_2^2 - \frac{t^p}{p}\int_\Omega \eta(z)\hat{u}_1(p)^p\,dz + \frac{t^p}{p}\epsilon + c_5|\Omega|_N \\ &= \frac{t^p}{p}\Big[\int_\Omega [\hat{\lambda}_1(p) - \eta(z)]\hat{u}_1(p)^p\,dz + \epsilon\Big] + \frac{t^2}{2}\|D\hat{u}_1(p)\|_2^2 + c_5|\Omega|_N\end{aligned}$$

(for the inequality, see (3.42) and recall that $\|\hat{u}_1(p)\|_p = 1$).

Since $\hat{u}_1(p) \in \operatorname{int} C_+$, we have

$$k_0 = \int_\Omega [\eta(z) - \hat{\lambda}_1(p)]\hat{u}_1(p)^p\,dz > 0.$$

Choosing $\epsilon \in (0, k_0)$, we have

$$\varphi_+(t\hat{u}_1(p)) \leqslant -\frac{t^p}{p}c_6 + \frac{t^2}{2}\|D\hat{u}_1(p)\|_2^2 \quad \text{for some } c_6 > 0.$$

Since $p > 2$, it follows that

$$\varphi_+(t\hat{u}_1(p)) \to -\infty \text{ as } t \to +\infty. \tag{3.43}$$

Also, from Proposition 3.2 we know that

$$\varphi_+ \text{ satisfies the } C\text{-condition.} \tag{3.44}$$

Then relations (3.41), (3.43) and (3.44) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that $u_0 \in K_{\varphi_+}$ and $\varphi_+(0) = 0 < m_+ \leqslant \varphi_+(u_0)$, which implies $u_0 \neq 0$.

Since $K_{\varphi_+} \subseteq C_+$, we have that $u_0 \in C_+ \setminus \{0\}$. With $\rho = \|u_0\|_\infty$, let $\hat{\xi}_p > 0$ be as postulated by Hypothesis H(f) (iv), that is, for almost all $z \in \Omega$ the function

$$x \mapsto f(z, x) + \xi_p x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Consider the map $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$a(y) = |y|^{p-2}y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Then $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

which implies

$$\nabla a(y) = |y|^{p-2} y\Big[\mathrm{id}_N + (p-2)\frac{y \otimes y}{|y|^2}\Big] + \mathrm{id}_N,$$

with id_N being the identity map on $\mathbb{R}^N$. For all $\xi \in \mathbb{R}^N$, we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 > 0 \quad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Also, for $0 \leqslant x \leqslant v \leqslant \rho$ we have

$$\begin{aligned} f(z,v) - f(z,x) &\geqslant -\xi_\rho(v^{p-1} - x^{p-1}) \\ &\geqslant -\hat{\xi}_\rho(v - x) \end{aligned}$$

for almost all $z \in \Omega$ and some $\hat{\xi}_\rho > 0$ (recall that $p > 2$). So, we can apply the tangency principle of Pucci and Serrin [27, Theorem 2.5.2, p. 35] and have

$$u_0(z) > 0 \quad \text{for all } z \in \Omega.$$

Then the boundary point theorem of Pucci and Serrin [27, Theorem 5.5.1, p. 120] implies that

$$u_0 \in \operatorname{int} C_+.$$

By the Sobolev embedding theorem, φ_- is sequentially weakly lower semicontinuous. Also, from Proposition 3.3 we know that φ_- is coercive. Hence by the Weierstrass–Tonelli theorem, we can find $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_-(v_0) = \inf[\varphi_-(u) : u \in W_0^{1,p}(\Omega)], \tag{3.45}$$

which implies

$$v_0 \in K_{\varphi_-} \subseteq -C_+.$$

Hypotheses H(f) (i), (ii) and (iv) imply that given $\epsilon > 0$, we can find $c_7 = c_7(\epsilon) > 0$ such that

$$F(z,x) \geqslant \frac{1}{2}(\beta - \epsilon)x^2 - c_7|x|^p \quad \text{for almost all } z \in \Omega \text{ and all } x \leqslant 0. \tag{3.46}$$

Then for $t > 0$ we have

$$\begin{aligned} \varphi_-(-t\hat{u}_1(2)) &\leqslant \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p + \frac{t^2}{2}\hat{\lambda}_1(2) - \frac{t^2}{2}(\beta - \epsilon) + c_7 t^p\|\hat{u}_1(2)\|_p^p \\ &= t^p\Big[\frac{1}{p}\|D\hat{u}_1(2)\|_p^p + \|\hat{u}_1(2)\|_p^p\Big] - \frac{t^2}{2}[\beta - \epsilon - \hat{\lambda}_1(2)] \end{aligned}$$

(see (3.46) and recall that $\|\hat{u}_1(2)\|_2 = 1$).

We choose $0 < \epsilon < \beta - \hat{\lambda}_1(2)$ (see Hypothesis H(f) (iv)). Then, since $2 < p$ for $t \in (0,1)$ small, we can see that

$$\varphi_-(-t\hat{u}_1(2)) < 0,$$

which implies

$$\varphi_-(v_0) < 0 = \varphi_-(0),$$

(see (3.45)), which then implies

$$v_0 \neq 0.$$

Moreover, as for u_0, using the nonlinear strong maximum principle, we have $v_0 \in -\operatorname{int} C_+$, and this is the second constant sign solution of (1.1). □

4 Nodal solutions – multiplicity theorems

In this section, using tools from Morse theory (critical groups), we show the existence of a nodal (sign changing) smooth solution and formulate our multiplicity theorems.

To produce a nodal sign, changing solution, we will need one more hypothesis which is the following one.

Hypothesis H_0. Problem (1.1) has a finite number of solutions of constant sign.

Remark 4. This condition is equivalent to saying that K_{φ_+} and K_{φ_-} are finite sets.

We start by computing the critical groups of φ at infinity.

Proposition 4.1. *If Hypotheses* $\mathrm{H}(f)$ *hold, then* $C_k(\varphi, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. Let $\lambda > \hat{\lambda}_1(p)$, $\lambda \notin \hat{\sigma}(p)$, and consider the C^1-functional $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \frac{1}{p}\|Du\|_p^p - \frac{\lambda}{p}\|u^+\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We consider the homotopy

$$h(t, u) = (1 - t)\varphi(u) + t\Psi(u) \quad \text{for all } t \in [0, 1] \text{ and all } u \in W_0^{1,p}(\Omega).$$

Claim. *There exist* $\gamma \in \mathbb{R}$ *and* $\tau > 0$ *such that*

$$h(t, u) \leqslant \gamma \textit{ implies } (1 + \|u\|)\|h_u'(t, u)\|_* \geqslant \tau \quad \textit{for all } t \in [0, 1].$$

We argue by contradiction. Since $h(\cdot\,,\cdot)$ maps bounded sets to bounded sets, if the claim is not true, then we can find two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, \quad \|u_n\| \to \infty, \quad h(t_n, u_n) \to -\infty, \quad (1 + \|u_n\|)h_u'(t_n, u_n) \to 0. \tag{4.1}$$

From the last convergence in (4.1) we have

$$\left|\langle A_p(u_n), h\rangle + (1 - t)\langle A(u_n), h\rangle - (1 - t_n)\int_\Omega f(z, u_n)h\,dz - \lambda t_n \int_\Omega (u_n^+)^{p-1} h\,dz\right| \leqslant \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \tag{4.2}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\epsilon_n \to 0^+$.

From the third convergence in (4.1) we see that we can find $n_0 \in \mathbb{N}$ such that

$$\|Du_n\|_p^p + \frac{(1 - t_n)p}{2}\|Du_n\|_2^2 - (1 - t_n)\int_\Omega pF(z, u_n)\,dz - \lambda t_n\|u_n^+\|_p^p \leqslant -1 \quad \text{for all } n \geqslant n_0. \tag{4.3}$$

In (4.2), we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then

$$-\|Du_n\|_p^p - (1 - t_n)\|Du_n\|_2^2 + (1 - t_n)\int_\Omega f(z, u_n)u_n\,dz + \lambda t_n\|u_n^+\|_p^p \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

Adding (4.3) and (4.4), we obtain

$$(1 - t_n)\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz \leqslant 0 \quad \text{for all } n \geqslant n_1 \geqslant n_0 \tag{4.5}$$

(recall that $p > 2$ and $\epsilon_n \to 0^+$ as $n \to +\infty$).

We claim that $t < 1$. If $t_n \to 1$, then let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$ for all $n \in \mathbb{N}$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega). \tag{4.6}$$

From (4.2) we have

$$\left|\langle A_p(y_n), h\rangle + \frac{1-t_n}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle - (1-t_n)\int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}} h\,dz - \lambda t_n \int_\Omega (y_n^+)^{p-1} h\,dz\right| \leqslant \frac{\epsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \tag{4.7}$$

for all $n \in \mathbb{N}$.

In (4.7) we choose $h = y_n - y$, pass to the limit as $n \to \infty$ and use (4.6), (3.10) and $t_n \to 1$, $p > 2$. Then

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0,$$

which implies

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega), \text{ and so } \|y\| = 1 \tag{4.8}$$

(see Proposition 2.3). So, if in (4.7) we pass to the limit as $n \to \infty$ and use (4.8), then

$$\langle A_p(y), h\rangle = \lambda \int_\Omega (y^+)^{p-1} h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega) \tag{4.9}$$

(recall that $t_n \to 1$). Choosing $h = -y^- \in W_0^{1,p}(\Omega)$, we have $\|Dy^-\|_p^p = 0$, which implies $y \geqslant 0$ and $y \neq 0$ (see (4.8)).

From (4.9) we have

$$-\Delta_p y(z) = \lambda y(z)^{p-1} \quad \text{for almost all } z \in \Omega,\ y|_{\partial\Omega} = 0.$$

Since $\lambda > \hat{\lambda}_1(p)$, $\lambda \notin \hat{\sigma}(p)$, from (4.9) we infer that

$$y = 0,$$

which contradicts (4.8). Therefore, $t < 1$ and we have

$$\int_\Omega [f(z, u_n)u_n - pF(z, u_n)]\,dz \leqslant 0 \quad \text{for all } n \geqslant n_1,$$

which implies

$$\int_\Omega [f(z, -u_n^-)(-u_n^-) - pF(z, -u_n^-)]\,dz \leqslant c_8 \tag{4.10}$$

for some $c_8 > 0$ and all $n \geqslant n_1$ (see Hypothesis H(f) (iii)).

Using (4.10) and reasoning as in Claims 1 and 2 in the proof of Proposition 3.1, we establish that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

This contradicts (4.1). Therefore, we have proven the claim.

Then by the claim and Chang [8, Theorem 5.1.21, p. 334] (see also Liang and Su [17, Proposition 3.2]) we have

$$C_k(h(0,\cdot),\infty) = C_k(h(1,\cdot),\infty) \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k(\varphi,\infty) = C_k(\Psi,\infty) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.11}$$

So, our task is now to compute $C_k(\Psi,\infty)$. To this end, first note that since $\lambda > \hat{\lambda}_1(p)$, we have

$$K_\Psi = \{0\}. \tag{4.12}$$

Consider the C^1-functional $\hat{h} : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{h}(t,u) = \Psi(u) - t\int_\Omega u(z)\,dz \quad \text{for all } t \in [0,1] \text{ and all } u \in W_0^{1,p}(\Omega).$$

Suppose that $u \in K_{\hat{h}(t,\cdot)}$, $t \in (0, 1]$. Then

$$\langle A_p(u), h\rangle = \lambda \int_\Omega (u^+)^{p-1} h\, dz + t \int_\Omega h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{4.13}$$

Choosing $h = -u^- \in W_0^{1,p}(\Omega)$, we obtain $\|Du^-\|_p^p \leqslant 0$, which implies $u \geqslant 0$ and $u \neq 0$.
From (4.13) we have

$$-\Delta_p u(z) = \lambda u(z)^{p-1} + t \quad \text{for almost all } z \in \Omega,\ u|_{\partial\Omega} = 0, \tag{4.14}$$

which implies $u \in \operatorname{int} C_+$ by the nonlinear strong maximum principle (see [12, p. 738]).

Let $v \in \operatorname{int} C_+$ and consider the function

$$R(v, u)(z) = |Dv(z)|^p - |Du(z)|^{p-2}\Big(Du(z), D\Big(\frac{v^p}{u^{p-1}}\Big)(z)\Big)_{\mathbb{R}_N}.$$

From the nonlinear Picone's identity of Allegretto and Huang [4] we have

$$\begin{aligned} 0 &\leqslant \int_\Omega R(v, u)\, dz \\ &= \|Dv\|_p^p - \int_\Omega (-\Delta_p u)\frac{v^p}{u^{p-1}}\, dz \\ &= \|Dv\|_p^p - \int_\Omega [\lambda u^{p-1} + t]\frac{v^p}{u^{p-1}}\, dz \\ &\leqslant \|Dv\|_p^p - \int_\Omega \lambda v^p\, dz, \end{aligned}$$

where the first equation uses the nonlinear Green's identity (see [12, p. 211]), the second equation follows from (4.14) and the last inequality holds since $u, v \in \operatorname{int} C_+$.

Choosing $v = \hat{u}_1(p) \in \operatorname{int} C_+$, we have

$$0 \leqslant \hat{\lambda}_1(p) - \lambda < 0$$

(recall that $\|\hat{u}_1(p)\|_p = 1$), a contradiction. Therefore,

$$K_{\hat{h}(t,\cdot)} = \emptyset \quad \text{for all } t \in (0, 1]. \tag{4.15}$$

From the homotopy invariance of singular homology, for $r > 0$ small we have

$$H_k(\hat{h}(0,\cdot)^0 \cap B_r, \hat{h}(0,\cdot)^0 \cap B_r \setminus \{0\}) = H_k(\hat{h}(1,\cdot)^0 \cap B_r, \hat{h}(1,\cdot)^0 \cap B_r \setminus \{0\}) \quad \text{for all } k \in \mathbb{N}. \tag{4.16}$$

Then by (4.15) and the noncritical interval theorem (see Chang [8, Theorem 5.1.6, p. 320] and Motreanu, Motreanu and Papageorgiou [19, Corollary 5.35, p. 115]), we have

$$H_k(\hat{h}(1,\cdot)^0 \cap B_r, \hat{h}(1,\cdot)^0 \cap B_r \setminus \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{4.17}$$

Also, from the definition of critical groups we have

$$H_k(\hat{h}(0,\cdot)^0 \cap B_r, \hat{h}(0,\cdot)^0 \cap B_r \setminus \{0\}) = H_k(\Psi^0 \cap B_r, \Psi^0 \cap B_r \setminus \{0\}) = C_k(\Psi, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.18}$$

From (4.16)–(4.18) we conclude that

$$C_k(\Psi, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k(\Psi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0$$

(see (4.12) and [19, Proposition 6.61 (c), p. 160]), which then implies

$$C_k(\varphi, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{see } (4.11)). \qquad \square$$

Next, we compute the critical groups at infinity for the functional $\varphi_\pm$.

Proposition 4.2. *If Hypotheses* H(f) *hold, then* $C_k(\varphi_+, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. Let $\Psi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ be as in the proof of Proposition 4.1 and consider the homotopy

$$h_+(t, u) = (1-t)\varphi_+(u) + t\Psi(u) \quad \text{for all } t \in [0,1] \text{ and all } u \in W_0^{1,p}(\Omega).$$

Claim. *There exist* $\gamma \in \mathbb{R}$ *and* $\tau > 0$ *such that for all* $t \in [0,1]$,

$$h_+(t, u) \leqslant \gamma \quad \textit{implies} \quad (1 + \|u\|)\|(h_+)'(t, u)\|_* \geqslant \tau.$$

As in the proof of Proposition 4.1, we argue by contradiction. So, we can find two sequences

$$\{t_n\}_{n\geqslant 1} \subseteq [0,1] \quad \text{and} \quad \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$$

such that

$$t_n \to t, \quad \|u_n\| \to \infty, \quad h_+(t_n, u_n) \to -\infty, \quad (1 + \|u_n\|)(h_+)'(t_n, u_n) \to 0. \tag{4.19}$$

From the last convergence in (4.19) we have

$$\left|\langle A_p(u_n), h\rangle + (1-t_n)\langle A(u_n), h\rangle - (1-t_n)\int_\Omega f_+(z, u_n)h\,dz - \lambda t_n \int_\Omega (u_n^+)^{p-1}h\,dz\right| \leqslant \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \tag{4.20}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\epsilon_n \to 0^+$.

In (4.20), we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p + (1-t)\|Du_n^-\|_2^2 \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N},$$

which implies

$$u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega). \tag{4.21}$$

From (4.20) and (4.21) we infer that

$$\left|\langle A_p(u_n^+), h\rangle + (1-t_n)\langle A(u_n^+), h\rangle - (1-t_n)\int_\Omega f(z, u_n^+)h\,dz - \lambda t_n \int_\Omega (u_n^+)^{p-1}h\,dz\right| \leqslant \epsilon_n'\|h\|$$

for all $h \in W_0^{1,p}(\Omega)$, with $\epsilon_n' \to 0^+$.

Suppose that $\|u_n^+\| \to +\infty$. We set $v_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then $\|v_n\| = 1$ and $v_n \geqslant 0$ for all $n \in \mathbb{N}$, and so we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad v_n \to v \quad \text{in } L^p(\Omega).$$

Reasoning as in the proof of Proposition 3.1 (see the proof of Claim 2), we reach a contradiction, and so we infer that

$$\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded},$$

which implies that

$$\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded}$$

(see (4.21)).

But this contradicts (4.19). Hence the claim holds and, as before (see the proof of Proposition 4.1), we have

$$C_k(h_+(0,\cdot), \infty) = C_k(h_+(1,\cdot), \infty) \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k(\varphi_+, \infty) = C_k(\Psi, \infty) \quad \text{for all } k \in \mathbb{N}_0,$$

which then implies

$$C_k(\varphi_+, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0$$

(see the end of the proof of Proposition 4.1).

The proof is complete. □

Proposition 4.3. *If Hypotheses* H(f) *hold, then* $C_k(\varphi_-, \infty) = \delta_{k,0}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$.

Proof. From Proposition 3.3 we know that φ_- is coercive. So, it is bounded below and satisfies the C-condition (see Corollary 3.4). Hence [19, Proposition 6.64 (a), p. 116] implies that

$$C_k(\varphi_-, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,$$

as desired. □

Next, we compute the critical groups of φ at $u = 0$.

Proposition 4.4. *If Hypotheses* H(f) *hold, then* $C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z}$ *for all* $k \in \mathbb{N}_0$.

Proof. Let $\alpha \in (0, \hat{\lambda}_1(2))$ and $\beta \in (\hat{\lambda}_1(2), \hat{\lambda}_2(2))$ be as postulated by Hypothesis H(f) (iv). We consider the C^1-functional $\hat{\Psi}_0 : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\Psi}_0(u) = \frac{1}{2}\|Du\|_2^2 - \frac{\alpha}{2}\|u^+\|_2^2 - \frac{\beta}{2}\|u^-\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Let $\Psi_0 = \hat{\Psi}_0|_{W_0^{1,p}(\Omega)}$ (recall that $2 < p$) and consider the homotopy

$$h(t, u) = (1 - t)\varphi(u) + t\Psi_0(u) \quad \text{for all } t \in [0, 1] \text{ and all } u \in W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \in [0, 1], \quad u_n \to 0 \text{ in } W_0^{1,p}(\Omega), \quad h_u'(t_n, u_n) = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{4.22}$$

From the equality in (4.22) we have

$$(1 - t_n)A_p(u_n) + A(u_n) = (1 - t_n)N_f(u_n) + t_n[\alpha u_n^+ - \beta u_n^-] \quad \text{in } W^{-1,p'}(\Omega) \text{ for all } n \in \mathbb{N}. \tag{4.23}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$, and so we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^p(\Omega). \tag{4.24}$$

From (4.23) we have

$$(1 - t_n)\|u_n\|^{p-2}A_p(y_n) + A(y_n) = (1 - t_n)\frac{N_f(u_n)}{\|u_n\|} + t_n[\alpha y_n^+ - \beta y_n^-] \quad \text{for all } n \in \mathbb{N}. \tag{4.25}$$

Note that Hypotheses H(f) (i), (ii) and (iv) imply that for some $c_9 > 0$,

$$|f(z, x)| \leqslant c_9[|x| + |x|^{p-1}] \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},$$

which implies that

$$\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded.}$$

This fact and Hypothesis H(f) (iv) imply that

$$\frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} \alpha y^+ - \beta y^- \quad \text{in } L^{p'}(\Omega) \text{ as } n \to \infty \tag{4.26}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 16]). From (4.25) we have

$$-(1 - t_n)\|u_n\|^{p-2}\Delta_p y_n(z) - \Delta y_n(z) = (1 - t_n)\frac{f(z, u_n(z))}{\|u_n\|} + t_n[\alpha y_n^+(z) - \beta y_n^-(z)]$$

for almost all $z \in \Omega$, $y_n|_{\partial\Omega} = 0$, $n \in \mathbb{N}$.

Then by (4.22), (4.24), (4.26) (recall that $p > 2$) and [16, Theorem 7.1, p. 286] we can find $M_8 > 0$ such that

$$\|y_n\|_\infty \leqslant M_8 \quad \text{for all } n \in \mathbb{N}.$$

Then invoking [18, Theorem 1], we can find $\vartheta \in (0, 1)$ and $M_9 > 0$ such that

$$y_n \in C_0^{1,\vartheta}(\overline{\Omega}) \quad \text{and} \quad \|y_n\|_{C_0^{1,\vartheta}(\overline{\Omega})} \leqslant M_9 \qquad \text{for all } n \in \mathbb{N}.$$

From (4.5) and the compact embedding of $C_0^{1,\vartheta}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, we have

$$y_n \to y \text{ in } C_0^1(\overline{\Omega}), \tag{4.27}$$

which implies $\|y\| = 1$, and so $y \neq 0$.

If in (4.25) we pass to the limit as $n \to \infty$ and use (4.22), (4.26), (4.27) and the fact that $p > 2$, we obtain

$$A(y) = \alpha y^+ - \beta y^- \quad \text{in } W^{-1,p'}(\Omega),$$

which implies

$$-\Delta y(z) = \alpha y^+(z) - \beta y^-(z) \quad \text{for almost all } z \in \Omega,\ y|_{\partial\Omega} = 0.$$

Since $0 < \alpha < \hat{\lambda}_1(2) < \beta < \hat{\lambda}_2(2)$ (see Hypothesis H(f) (iv)), we have $(\alpha, \beta) \notin \Sigma_2$, and so

$$y = 0,$$

a contradiction (recall that $\|y\| = 1$). Therefore (4.22) cannot occur, and then the homotopy invariance of critical groups (see Gasinski and Papageorgiou [14, Theorem 5.125, p. 836]) implies that

$$C_k(\varphi, 0) = C_k(\Psi_0, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.28}$$

Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$ from [20, Theorem 16] (see also Chang [7, p. 14]), we have

$$C_k(\Psi_0, 0) = C_k(\hat{\Psi}_0, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.29}$$

But [26, Theorem 1.1 (a)] implies that

$$C_k(\hat{\Psi}_0, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0$$

(see (4.28) and (4.29)). □

Also, we have the following property.

Proposition 4.5. *If Hypotheses* H(f) *hold, then* $C_k(\varphi_-, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Proof. In this case, we consider the C^1-functional $\hat{\Psi} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\Psi}_-(u) = \frac{1}{2}\|Du\|_2^2 - \beta\|u^-\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

We set $\Psi_- = \hat{\Psi}_-|_{W_0^{1,p}(\Omega)}$ (recall that $p > 2$) and consider the homotopy

$$h_-(t, u) = (1 - t)\varphi_-(u) + t\Psi_-(u) \quad \text{for all } t \in [0, 1] \text{ and all } u \in W_0^{1,p}(\Omega).$$

As in the proof of Proposition 4.4, via the homotopy invariance of critical groups we have

$$C_k(\varphi_-, 0) = C_k(\Psi_-, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

Recalling that the nonprincipal eigenfunctions of $(-\Delta, H_0^1(\Omega))$ are nodal and since $\beta > \hat{\lambda}_1(2)$, we infer that $K_{\Psi_-} = \{0\}$. Moreover, as in the last part of the proof of Proposition 4.1, using Picone's identity, we have

$$C_k(\Psi_-, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0,$$

which implies

$$C_k(\varphi_-, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0.$$

This completes the proof. □

Now we are ready for our first multiplicity theorem. Recall that at the beginning of this section we have introduced an extra Hypothesis H_0, which says that the constant sign solutions of (1.1) are finite. This is equivalent to saying that

$$K_{\varphi_-} = \{v_i\}_{i=1}^m \cup \{0\} \subseteq (-\operatorname{int} C_+) \cup \{0\},$$
$$K_{\varphi_+} = \{u_l\}_{l=1}^n \cup \{0\} \subseteq \operatorname{int} C_+ \cup \{0\}.$$

From Proposition 3.6 we know that

$$m, n \in \mathbb{N}.$$

Then we have the following multiplicity theorem.

Theorem 4.6. *If Hypotheses* $\mathrm{H}(f)$ *and* H_0 *hold, then problem* (1.1) *has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in C_0^1(\overline{\Omega}) \qquad \textit{nodal}.$$

Proof. From Proposition 3.6 we already have two nontrivial constant sign smooth solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

By Hypothesis H_0, we have

$$K_{\varphi_-} = \{v_i\}_{i=1}^m \cup \{0\} \subseteq (-\operatorname{int} C_+) \cup \{0\},$$
$$K_{\varphi_+} = \{u_l\}_{l=1}^n \cup \{0\} \subseteq \operatorname{int} C_+ \cup \{0\}.$$

We set

$$\chi_-(v_i) = \sum_{k\geqslant 0} (-1)^k \operatorname{rank} C_k(\varphi_-, v_i), \quad i = 1, \dots, m,$$
$$\chi_+(u_l) = \sum_{k\geqslant 0} (-1)^k \operatorname{rank} C_k(\varphi_+, u_l), \quad l = 1, \dots, n,$$
$$\chi(v_i) = \sum_{k\geqslant 0} (-1)^k \operatorname{rank} C_k(\varphi, v_i),$$
$$\chi(u_l) = \sum_{k\geqslant 0} (-1)^k \operatorname{rank} C_k(\varphi, u_l).$$

Since $v_i \in -\operatorname{int} C_+$ and $u_l \in \operatorname{int} C_+$, we have

$$\chi_-(v_i) = \chi(v_i) \quad \text{for all } i = 1, \dots, m \qquad \text{and} \qquad \chi_+(u_l) = \chi(u_l) \quad \text{for all } l = 1, \dots, n. \tag{4.30}$$

From Propositions 3.5, 4.4 and 4.5 we have

$$\chi_+(0) = 1, \quad \chi(0) = 1, \quad \chi_-(0) = 0. \tag{4.31}$$

Let $\{y_i\}_{j=1}^d \subseteq K_\varphi \subseteq C_0^1(\overline{\Omega})$ (nonlinear regularity theory) be the set of nodal solutions of (1.1) (if there are no nodal solutions, then $d = 0$). From the Morse relation (see (2.6)) we have

$$\chi_-(0) + \sum_{i=1}^m \chi_-(v_i) = 0 + \sum_{i=1}^m \chi(v_j) = 1 \quad \text{(see (4.30), (4.31) and Proposition 4.3)}, \tag{4.32}$$
$$\chi_+(0) + \sum_{l=1}^n \chi_+(u_l) = 1 + \sum_{l=1}^n \chi(u_l) = 0 \quad \text{(see (4.30), (4.31) and Proposition 4.2)}, \tag{4.33}$$
$$\chi(0) + \sum_{i=1}^m \chi(v_i) + \sum_{l=1}^n \chi(u_l) + \sum_{j=1}^d \chi(y_j) = 0 \quad \text{(see Proposition 4.1)},$$

where the last line implies

$$1 + 1 - 1 + \sum_{j=1}^{d} \chi(y_j) = 0$$

(see (4.31)–(4.33)), which then implies

$$\sum_{j=1}^{d} \chi(y_j) = -1,$$

and so $d \geqslant 1$.

This means that problem (1.1) admits at least one nodal solution $y_0 \in C_0^1(\overline{\Omega})$. □

We can drop Hypothesis H_0 at the expense of strengthening the regularity of $f(z,\cdot)$. Then we can still have a three nontrivial solutions multiplicity theorem, but without providing sign information about the third nontrivial smooth solution.

The new hypotheses on $f(z,x)$ are the following.

Hypothesis H(*f*)′. $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ and $f(z,\cdot) \in C^1(\mathbb{R} \setminus \{0\})$ for almost all $z \in \Omega$, and Hypotheses $\mathrm{H}(f)'$ (i)–(iv) are the same as the corresponding Hypotheses $\mathrm{H}(f)$ (i)–(iv).

Theorem 4.7. *If Hypotheses* $\mathrm{H}(f)'$ *hold, then problem* (1.1) *admits at least three nontrivial smooth solutions*

$$v_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in C_0^1(\overline{\Omega}).$$

Proof. Again, from Proposition 3.6 we already have two nontrivial constant sign smooth solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

From the proof of Proposition 3.6 we know that $u_0 \in K_{\varphi_+}$ is of mountain pass type. Since $u_0 \in \operatorname{int} C_+$, we have

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \quad \text{for all } k \in \mathbb{N} \qquad \text{and} \qquad C_1(\varphi_+, u_0) \neq 0$$

(see Motreanu, Motreanu and Papageorgiou [19, Corollary 6.81, p. 168]). Therefore,

$$C_1(\varphi, u_0) \neq 0. \tag{4.34}$$

But $\varphi \in C^2(W_0^{1,p}(\Omega) \setminus \{0\})$. Hence from (4.34) and Papageorgiou and Rădulescu [22] we have

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.35}$$

The negative solution $v_0 \in K_{\varphi_-}$ is a global minimizer of φ_-. Since

$$\varphi_-|_{-C_+} = \varphi|_{-C_+} \quad \text{and} \quad v_0 \in -\operatorname{int} C_+,$$

it follows that v_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Invoking Proposition 2.2, we infer that v_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ. Therefore,

$$C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.36}$$

From Propositions 4.1 and 4.4 we have

$$C_k(\varphi, \infty) = 0 \qquad \text{for all } k \in \mathbb{N}_0, \tag{4.37}$$

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.38}$$

Suppose that $K_\varphi = \{u_0, v_0, 0\}$. Then the Morse relation with $t = -1$ and (4.35)–(4.38) imply that

$$(-1)^1 + (-1)^0 + (-1)^0 = 0,$$

which implies $(-1)^1 = 0$, a contradiction.

So, there exists $y_0 \in K_\varphi$, $y_0 \notin \{u_0, v_0, 0\}$. Hence y_0 is the third nontrivial solution of problem (1.1) and the nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{\Omega})$. □

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