Research Article

Nikolaos S. Papageorgiou and Vicenţiu D. Rădulescu*

Infinitely Many Nodal Solutions for Nonlinear Nonhomogeneous Robin Problems

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Abstract: We consider a nonlinear Robin problem driven by a nonhomogeneous differential operator which incorporates the p-Laplacian as special case. The reaction $f(z, x)$ is a Carathéodory function which need not satisfy a global growth condition and is only assumed to be odd near zero. Using variational tools, we show that the problem has a whole sequence of distinct nodal (that is, sign-changing) solutions.

Keywords: Nonhomogeneous Differential Operator, Nodal Solution, Nonlinear Regularity Theory, Nonlinear Maximum Principle

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the nonlinear, nonhomogeneous Robin problem

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In this problem, $a : \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone, continuous map which satisfies certain other regularity and growth conditions listed in the hypotheses $H(a)$ below. These conditions are general enough to incorporate in our framework many differential operators of interest, such as the p-Laplace operator, $1 < p < \infty$, and the sum of a p-Laplacian with a q-Laplacian, $1 < q < p < \infty$. The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the map $z \mapsto f(z, x)$ is measurable, while, for almost all $z \in \Omega$, the function $x \mapsto f(z, x)$ is continuous).

The interesting feature of our work here is that we do not impose any global growth condition on $f(z, \cdot)$. Instead, we assume a local symmetry condition, namely, we require that, for almost all $z \in \Omega$, the function $x \mapsto f(z, x)$ is odd in the bounded interval $[-\eta, \eta]$. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the generalized normal derivative corresponding to the differential operator $\operatorname{div} a(Du)$ and is defined by

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \quad \text{for all } u \in W^{1,p}(\Omega)$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This kind of normal derivative is dictated by the nonlinear Green's identity (see, for example, Gasiński and Papageorgiou [7, p. 210]) and can be also found in the work

Nikolaos S. Papageorgiou: Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece, e-mail: npapg@math.ntua.gr
***Corresponding author: Vicenţiu D. Rădulescu:** Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, 21589 Jeddah, Saudi Arabia; and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania, e-mail: vicentiu.radulescu@imar.ro

of Lieberman [13]. The boundary weight function $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ satisfies $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, then we have the Neumann problem.

Under these general hypotheses on the data of (1.1), we show that there exists a whole sequence $\{u_n\}_{n\geq 1} \subseteq C^1(\overline{\Omega})$ of distinct nodal (that is, sign-changing) solutions. Our approach uses variational tools together with suitable truncation-perturbation techniques. Recently, nodal solutions for nonlinear, nonhomogeneous Robin problems were produced by Papageorgiou and Rădulescu [20, 22]. However, in the aforementioned works, the authors establish the existence of only one nodal solution.

2 Mathematical Background and Hypotheses

Let X be a Banach space and let $\varphi \in C^1(X,\mathbb{R})$. We say that φ satisfies the Palais–Smale condition (PS-condition for short) if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

Our main variational tool will be a variant due to Heinz [10] of a classical result of Clark [4]. The next result is essentially due to Heinz [10] and can be found in Wang [28]. Further extensions with applications to semilinear elliptic Dirichlet problems and to Hamiltonian systems can be found in the works of Liu and Wang [15] and Kajikiya [12].

Theorem 2.1. *Let X be a Banach space and assume that $\varphi \in C^1(X,\mathbb{R})$ satisfies the PS-condition, it is even, bounded from below, $\varphi(0) = 0$ and, for every $n \in \mathbb{N}$, there exist an n-dimensional subspace Y_n of X and $\rho_n > 0$ such that*

$$\sup\{\varphi(u) : u \in Y_n \cap \partial B_{\rho_n}\} < 0,$$

where $\partial B_{\rho_n} = \{u \in X : \|u\| = \rho_n\}$. Then, there exists a sequence $\{u_n\}_{n\geq 1}$ of critical points of φ such that

$$\varphi(u_n) < 0 \quad \textit{for all } n \in \mathbb{N}$$

and

$$\varphi(u_n) \to 0 \quad \textit{as } n \to \infty.$$

Let $\vartheta \in C^1(0,+\infty)$ with $\vartheta(t) > 0$ for all $t > 0$ and assume that there exists $p > 1$ such that

$$0 < \hat{c} \leq \frac{\vartheta'(t)t}{\vartheta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2(1 + t^{p-1}) \tag{2.1}$$

for all $t > 0$ and for some $c_1, c_2 > 0$. Then, our hypotheses on the map $a(\cdot)$ involved in the definition of the differential operator are that

$H(a)$ $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,+\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,+\infty)$, $a_0(t)t \to 0$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \leq c_3 \frac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\vartheta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)s\,ds$ for $t > 0$, then there exists $q \in (1,p)$ such that

$$\limsup_{t\to 0^+} \frac{qG_0(t)}{t^q} \leq c^* \quad \text{and} \quad t \mapsto G_0(t^{1/q}) \text{ is convex.}$$

Remark 2.2. Hypotheses $H(a)$ (i)–(iii) come from the nonlinear regularity theory of Lieberman [13] and the nonlinear maximum principle of Pucci and Serrin [26]. Hypothesis $H(a)$ (iv) serves the needs of our problem, but it is a mild condition which is satisfied in all the main cases of interest, as the examples which follow illustrate.

From the above hypotheses it is clear that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then, $G(\cdot)$ is convex, $G(0) = 0$ and

$$\nabla G(0) = 0 \quad \text{and} \quad \nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

So, $G(\cdot)$ is the primitive of $a(\cdot)$. The convexity of $G(\cdot)$, since $G(0) = 0$, implies that

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$. It is a straightforward consequence of hypotheses $H(a)$ (i)–(iii) and of (2.1).

Lemma 2.3. *If hypotheses $H(a)$ (i)–(iii) hold, then*
(a) *$y \mapsto a(y)$ is continuous and strictly monotone, hence, maximal monotone too;*
(b) *$|a(y)| \leq c_4(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$ and for some $c_4 > 0$;*
(c) *$(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

The last lemma and (2.2) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.4. *If hypotheses $H(a)$ (i)–(iii) hold, then $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$ for all $y \in \mathbb{R}^N$ and for some $c_5 > 0$.*

The examples that follow illustrate that our conditions on the map $a(\cdot)$ cover many cases of interest.

Example 2.5. The following maps satisfy the hypotheses $H(a)$.
(i) The map $a(y) = |y|^{p-2}y$ with $1 < p < \infty$, which corresponds to the p-Laplacian differential operator defined by
$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$
(ii) The map $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$, which corresponds to the (p, q)-differential operator defined by
$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$
Such operators arise in problems of mathematical physics. We mention the works of Benci, D'Avenia, Fortunato and Pisani [1] (quantum physics) and Cherfils and Ilyasov [2] (plasma physics). Recently, existence and multiplicity results for such equations with Dirichlet boundary conditions were proved by Cingolani and Degiovanni [3], Gasiński and Papageorgiou [9], Mugnai and Papageorgiou [17], Papageorgiou and Rădulescu [19, 21, 23] and Sun, Zhang and Su [27].
(iii) The map $a(y) = (1 + |y|^2)^{(p-2)/2}y$ with $1 < p < \infty$, which corresponds to the generalized p-mean curvature differential operator defined by
$$\operatorname{div}((1 + |Du|^2)^{(p-2)/2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$
(iv) The map $a(y) = |y|^{p-2}y(1 + \frac{1}{1+|y|^p})$ with $1 < p < \infty$, which corresponds to the differential operator
$$\Delta_p u + \operatorname{div}\Big(\frac{|Du|^{p-2}Du}{1 + |Du|^p}\Big) \quad \text{for all } u \in W^{1,p}(\Omega),$$
which is used in problems of plasticity.

Finally, we impose the hypothesis that
$H(\beta)$ $\beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$
and our hypotheses on the reaction term $f(z, x)$ are that
$H(f)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot)$ is odd on $[-\eta, \eta]$ for some $\eta > 0$ with $f(z, \eta) \leq 0 \leq f(z, -\eta)$ and
(i) there exists $a_\eta \in L^\infty(\Omega)_+$ such that $|f(z, x)| \leq a_\eta(z)$ for almost all $z \in \Omega$ and all $|x| \leq \eta$;

(ii) if $q \in (1, p)$ is as in $H(a)$ (iv), then we have

$$\lim_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} = +\infty \quad \text{uniformly for almost all } z \in \Omega.$$

Remark 2.6. We stress that the above hypotheses do not impose any global growth condition on $f(z,\cdot)$. Instead, we assume that $f(z,\cdot)$ has a kind of oscillatory behavior near zero and that it is symmetric in that interval. Hypothesis $H(f)$ (ii) implies the presence of a "concave" term near zero. We mention the work of Liu and Wang [14] who produced infinitely many nodal solutions for a semilinear Schrödinger equation without assuming the existence of zeros. We should point out that the idea of using cut-off techniques to produce an infinity of solutions converging to zero goes back to the work of Wang [28] who modified the reaction term in the interval $[-\eta, \eta]$ and applied the result of Clark and Heinz to the modified functional (see Wang [28, Lemma 2.3]).

Using hypothesis $H(f)$ (ii), we see that, given any $\xi > 0$ and recalling that $q < p$, we can find $\delta = \delta(\xi) \in (0, \hat{\eta})$ with $\hat{\eta} = \min\{1, \eta\}$ such that

$$f(z,x)x \geq \xi|x|^q \geq \xi|x|^p \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \delta. \tag{2.3}$$

Then, given $r \in (p, +\infty)$, we can find $c_6 = c_6(r, \delta) > 0$ such that

$$f(z,x)x \geq \xi|x|^q - c_6|x|^r \quad \text{for almost all } z \in \Omega \text{ and all } x \in [-\eta, \eta]. \tag{2.4}$$

From (2.3) we have

$$F(z,x) \geq \frac{\xi}{q}|x|^q \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \delta. \tag{2.5}$$

In our analysis of (1.1), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior and, if $u \in C_+$ with $u(z) > 0$ for all $z \in \overline{\Omega}$, then $u \in \operatorname{int} C_+$. On $\partial\Omega$, we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define the "boundary" Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. We have

$$\operatorname{Im}\gamma_0 = W^{1/p',p}(\partial\Omega)\Big(\frac{1}{p} + \frac{1}{p'} = 1\Big) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

Moreover, the trace map γ_0 is compact into $L^q(\partial\Omega)$ for $q \in [1, \frac{(N-1)p}{N-p})$. Hereafter, for the sake of notational simplicity, we will drop the use of the trace map γ_0. The restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces. By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

For every $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u = u^+ - u^-, \quad |u| = u^+ + u^- \quad \text{and} \quad u^+, u^- \in W^{1,p}(\Omega).$$

Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).$$

The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ is continuous, monotone and of type $(S)_+$, that is,

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0 \text{ implies that } u_n \to u \text{ in } W^{1,p}(\Omega).$$

Here, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$ (see Gasiński and Papageorgiou [8]). Finally, for any $\varphi \in C^1(X, \mathbb{R})$, by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

3 Nodal Solutions

Using (2.4), we introduce the truncation

$$e(z,x)=\begin{cases} -\xi\eta^{q-1}+c_6\eta^{r-1} & \text{if } x<-\eta,\\ \xi|x|^{q-2}x-c_6|x|^{r-2}x & \text{if } -\eta\le x\le\eta,\\ \xi\eta^{q-1}-c_6\eta^{r-1} & \text{if } \eta<x \end{cases} \tag{3.1}$$

of the right-hand side of (2.4) and, for all $(z,x)\in\partial\Omega\times\mathbb{R}$, the truncation

$$b(z,x)=\begin{cases} -\beta(z)\eta^{p-1} & \text{if } x<-\eta,\\ \beta(z)|x|^{p-2}x & \text{if } -\eta\le x\le\eta,\\ \beta(z)\eta^{p-1} & \text{if } \eta<x \end{cases} \tag{3.2}$$

of the boundary term $\beta(z)|x|^{p-2}x$. Both are Carathéodory functions. We consider the auxiliary nonlinear, nonhomogeneous Robin problem

$$\begin{cases} -\operatorname{div} a(Du(z))=e(z,u(z)) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial n_a}+b(z,u)=0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proposition 3.1. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then* (3.3) *admits a unique positive solution $\bar u\in \operatorname{int} C_+$ and $\bar v=-\bar u\in -\operatorname{int} C_+$ is its unique negative solution.*

Proof. We introduce the Carathéodory function $\tau:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$\tau(z,x)=\begin{cases} -\eta^{p-1} & \text{if } x<-\eta,\\ |x|^{p-2}x & \text{if } -\eta\le x\le\eta,\\ \eta^{p-1} & \text{if } \eta<x. \end{cases} \tag{3.4}$$

Let

$$T(z,x)=\int_0^x \tau(z,s)\,ds,\quad E(z,x)=\int_0^x e(z,s)\,ds\quad\text{and}\quad B(z,x)=\int_0^x b(z,s)\,ds,$$

and consider the C^1-functional $\psi_+:W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_+(u)=\int_\Omega G(Du)\,dz+\frac{1}{p}\|u\|_p^p+\int_{\partial\Omega}B(z,u^+)\,d\sigma-\int_\Omega E(z,u^+)\,dz-\int_\Omega T(z,u^+)\,dz\quad\text{for all } u\in W^{1,p}(\Omega).$$

From Corollary 2.4 and (3.1), (3.2) and (3.4) it is clear that ψ_+ is coercive. Also, using the Sobolev embedding theorem and the trace theorem, we see that ψ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\bar u\in W^{1,p}(\Omega)$ such that

$$\psi_+(\bar u)=\inf\{\psi_+(u): u\in W^{1,p}(\Omega)\}. \tag{3.5}$$

Hypothesis $H(a)$ (iv) implies that we can find $c_1^*>c^*$ and $\delta_1\in(0,\hat\eta)$ such that

$$G_0(t)\le\frac{c_1^*}{q}t^q\quad\text{for all } t\in[0,\delta_1]. \tag{3.6}$$

Let $u\in\operatorname{int}C_+$ and choose $t\in(0,1)$ small such that

$$tu(z)\le\delta_1\quad\text{and}\quad t|Du(z)|\le\delta_1\quad\text{for all } z\in\overline{\Omega}. \tag{3.7}$$

Using (3.6), (3.7), (3.1), (3.2) and (3.4), we have (see hypothesis $H(\beta)$ and the trace theorem)

$$\begin{aligned}\psi_+(tu)&\le\frac{t^pc_1^*}{p}\|Du\|_p^p+\frac{t^p}{p}\int_{\partial\Omega}\beta(z)u^p\,d\sigma-\frac{t^q\xi}{q}\|u\|_q^q+\frac{t^rc_6}{r}\|u\|_r^r\\ &\le\Big(\frac{t^{p-q}c_1^*}{p}\|Du\|_p^p+\frac{t^{p-q}}{p}c_8\|u\|^p-\frac{\xi}{q}\|u\|_q^q+\frac{t^{r-q}}{r}c_8\|u\|_r^r\Big)t^q\end{aligned}$$

for some $c_8 > 0$. Since $1 < q < p < r$, choosing $t \in (0,1)$ even smaller if necessary, we have that $\psi_+(tu) < 0$ implies (see (3.5))

$$\psi_+(\bar{u}) < 0 = \psi_+(0)$$

and, hence, $\bar{u} \neq 0$. From (3.5) we have that $\psi_+'(\bar{u}) = 0$ implies

$$\langle A(\bar{u}), h\rangle + \int_\Omega |\bar{u}|^{p-2}\bar{u}h\,dz + \int_{\partial\Omega} b(z,u^+)h\,d\sigma = \int_\Omega (e(z,u^+) + \tau(z,u^+))h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.8}$$

In (3.8), first we choose $h = -\bar{u}^- \in W^{1,p}(\Omega)$ and then we have that (see Lemma 2.3 and (3.1), (3.2) and (3.4))

$$\frac{c_1}{p-1}\|D\bar{u}^-\|_p^p + \|u^-\|_p^p \le 0$$

implies $\bar{u} \ge 0$ and $\bar{u} \neq 0$. Also, in (3.8), we choose $h = (\bar{u} - \eta)^+ \in W^{1,p}(\Omega)$ and then we have (see (3.1), (3.2) and (3.4) for the equality and (2.4) for the first inequality)

$$\begin{aligned}
&\langle A(\bar{u}), (\bar{u}-\eta)^+\rangle + \int_\Omega \bar{u}^{p-1}(\bar{u}-\eta)^+\,dz + \int_{\partial\Omega} \beta(z)\eta^{p-1}(\bar{u}-\eta)^+\,d\sigma \\
&= \int_\Omega (\xi\eta^{q-1} - c_6\eta^{r-1} + \eta^{p-1})(\bar{u}-\eta)^+\,dz \\
&\le \int_\Omega (f(z,\eta) + \eta^{p-1})(\bar{u}-\eta)^+\,dz \\
&\le \langle A(\eta), (\bar{u}-\eta)^+\rangle + \int_\Omega \eta^{p-1}(\bar{u}-\eta)^+\,dz + \int_{\partial\Omega} \beta(z)\eta^{p-1}(\bar{u}-\eta)^+\,d\sigma
\end{aligned}$$

since $A(\eta) = 0$ and $f(z,\eta) \le 0$ for almost all $z \in \Omega$, which implies that

$$\langle A(\bar{u}) - A(\eta), (\bar{u}-\eta)^+\rangle + \int_\Omega (\bar{u}^{p-1} - \eta^{p-1})(\bar{u}-\eta)^+\,dz \le 0.$$

Therefore,

$$|\{\bar{u} > \eta\}|_N = 0,$$

that is,

$$\bar{u} \le \eta.$$

Thus, we have proved that

$$\bar{u} \in [0,\eta] = \{u \in W^{1,p}(\Omega) : 0 \le u(z) \le \eta \text{ for almost all } z \in \Omega\} \quad \text{and} \quad \bar{u} \neq 0. \tag{3.9}$$

Then, using (3.1), (3.2), (3.4) and (3.9), we see that (3.8) becomes

$$\langle A(\bar{u}), h\rangle + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}h\,d\sigma = \int_\Omega e(z,\bar{u})h\,dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

which gives (see Papageorgiou and Rădulescu [18])

$$\begin{cases} -\operatorname{div} a(D\bar{u}(z)) = e(z,\bar{u}(z)) & \text{for almost all } z \in \Omega, \\ \dfrac{\partial \bar{u}}{\partial n_a} + \beta(z)\bar{u}^{p-1} = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, $\bar{u}$ is a positive solution of (3.3). From Papageorgiou and Rădulescu [24] we have that $\bar{u} \in L^\infty(\Omega)$ and, then, the nonlinear regularity result of Lieberman [13, p. 320]) implies that $\bar{u} \in C_+ \setminus \{0\}$. Because of (3.9) we have

$$-\operatorname{div} a(D\bar{u}(z)) = \xi\bar{u}(z)^{q-1} - c_6\bar{u}(z)^{r-1} \quad \text{for almost all } z \in \Omega,$$

which gives

$$\operatorname{div} a(D\bar{u}(z)) \leq c_6 \eta^{r-p} \bar{u}(z)^{p-1} \quad \text{for almost all } z \in \Omega,$$

that is (see Pucci and Serrin [26, pp. 111, 120]),

$$\bar{u} \in \operatorname{int} C_+ .$$

Next, we show the uniqueness of this positive solution. To this end, we consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega G(Du^{1/q})\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^{p/q}\,d\sigma & \text{if } u \geq 0,\ u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of j). We set

$$u = ((1-t)u_1 + tu_2)^{1/q} \quad \text{for } t \in [0,1].$$

Using Díaz and Saá [5, Lemma 1], we have

$$|Du(z)| \leq \big[(1-t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q\big]^{1/q} \quad \text{for almost all } z \in \Omega$$

and because $G_0(\cdot)$ is increasing and from hypothesis $H(a)$ (iv), for almost all $z \in \Omega$, we have

$$G_0(|Du(z)|) \leq G_0\big(\big[(1-t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q\big]^{1/q}\big) \leq (1-t)G_0(|Du_1(z)^{1/q}|) + tG_0(|Du_2(z)^{1/q}|),$$

which gives

$$G(Du(z)) \leq (1-t)G(Du_1(z)^{1/q}) + tG(Du_2(z)^{1/q}) \quad \text{for almost all } z \in \Omega,$$

that is, $j(\cdot)$ is convex (recall that $q < p$ and see hypothesis $H(\beta)$) By Fatou's lemma, $j(\cdot)$ is lower semicontinuous.

Let $\bar{y} \in W^{1,p}(\Omega)$ be another positive solution of (3.3). As we did for $\bar{u}$ in the first part of the proof, we can show that

$$\bar{y} \in [0, \eta] \cap \operatorname{int} C_+.$$

For any $h \in C^1(\overline{\Omega})$ and for $|t| < 1$ small, we have

$$\bar{u}^q + th \in \operatorname{dom} j \quad \text{and} \quad \bar{y}^q + th \in \operatorname{dom} j.$$

Then, we see that the functional $j(\cdot)$ is Gâteaux differentiable at $\bar{u}^q$ and $\bar{y}^q$ in the direction h. Moreover, via the chain rule and the nonlinear Green's identity, we have

$$j'(\bar{u}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\bar{u})}{\bar{u}^{q-1}} h\,dz \quad \text{and} \quad j'(\bar{y}^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(D\bar{y})}{\bar{y}^{q-1}} h\,dz.$$

Choose $h = \bar{u}^q - \bar{y}^q$. Since $j(\cdot)$ is convex, $j'(\cdot)$ is monotone, and so we have (see (3.1))

$$0 \leq \int_\Omega \Big(\frac{-\operatorname{div} a(D\bar{u})}{\bar{u}^{q-1}} - \frac{-\operatorname{div} a(D\bar{y})}{\bar{y}^{q-1}}\Big)(\bar{u}^q - \bar{y}^q)\,dz = \int_\Omega c_6(\bar{y}^{r-q} - \bar{u}^{r-q})(\bar{u}^q - \bar{y}^q)\,dz,$$

which gives

$$\bar{u} = \bar{y}$$

and, then, $\bar{u} \in [0, \eta] \cap \operatorname{int} C_+$ is the unique positive solution of (3.3). Evidently, since $x \mapsto \xi|x|^{q-2}x - c_6|x|^{r-2}x$ is odd, we have that $\bar{v} = -\bar{u} \in [-\eta, 0] \cap (-\operatorname{int} C_+)$ is the unique negative solution of (3.3). □

We introduce the sets

$$S_+ = \{u \in W^{1,p}(\Omega) : u \text{ is a positive solution of (1.1) with } u \in [0,\eta]\},$$
$$S_- = \{v \in W^{1,p}(\Omega) : v \text{ is a negative solution of (1.1) with } v \in [-\eta,0]\}.$$

As before, the nonlinear maximum principle implies that

$$S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad S_- \subseteq -\operatorname{int} C_+.$$

Moreover, as in Filippakis and Papageorgiou [6], we have that

$$S_+ \text{ is downward directed,}$$

that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \le \min\{u_1, u_2\}$, and

$$S_- \text{ is upward directed,}$$

that is, if $v_1, v_2 \in S_-$, then we can find $v \in S_-$ such that $v \ge \max\{v_1, v_2\}$ (see also Motreanu, Motreanu and Papageorgiou [16, p. 421]).

Proposition 3.2. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then $\bar{u} \le u$ for all $u \in S_+$ and $v \le \bar{v}$ for all $v \in S_-$.*

Proof. Let $u \in S_+$. We consider the Carathéodory functions $k_+(z,x)$, $\hat{b}_+(z,x)$ and $\hat{\tau}_+(z,x)$ defined by

$$k_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ e(z,x) & \text{if } 0 \le x \le u(z), \\ e(z,u(z)) & \text{if } u(z) < x, \end{cases} \tag{3.10}$$

$$\hat{b}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \beta(z)x^{p-1} & \text{if } 0 \le x \le u(z), \\ \beta(z)u(z)^{p-1} & \text{if } u(z) < x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbb{R}, \tag{3.11}$$

$$\hat{\tau}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ x^{p-1} & \text{if } 0 \le x \le u(z), \\ u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{3.12}$$

We set

$$K_+(z,x) = \int_0^x k_+(z,s)\,ds, \quad \hat{B}_+(z,x) = \int_0^x \hat{b}_+(d,s)\,ds \quad \text{and} \quad \hat{T}_+(z,x) = \int_0^x \hat{\tau}_+(z,s)\,ds.$$

Consider the C^1-functional $\gamma_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_+(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\|u\|_p^p + \int_{\partial\Omega} \hat{B}_+(z,u)\,d\sigma - \int_\Omega K_+(z,u)\,dz - \int_\Omega \hat{T}_+(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From Corollary 2.4 and (3.10), (3.11) and (3.12) we see that γ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_0 \in W^{1,p}(\Omega)$ such that

$$\gamma_+(\bar{u}_0) = \inf\{\gamma_+(u) : u \in W^{1,p}(\Omega)\}. \tag{3.13}$$

As before (see the proof of Proposition 3.1), since $1 < q < p < r$, for $\tilde{u} \in \operatorname{int} C_+$ and $t \in (0,1)$ small, we have (see hypothesis $H(a)$ (iv))

$$\gamma_+(t\tilde{u}) < 0 = \gamma_+(0),$$

which implies (see (3.13))

$$\gamma_+(\bar{u}_0) < 0 = \gamma_+(0)$$

and, hence, $\bar{u}_0 \neq 0$. From (3.13) we have that $\gamma_+'(\bar{u}_0) = 0$ implies

$$\langle A(\bar{u}_0), h\rangle + \int_\Omega |\bar{u}_0|^{p-2}\bar{u}_0 h\,dz + \int_{\partial\Omega} \hat{b}_+(z,\bar{u}_0)h\,dz = \int_\Omega (k_+(z,\bar{u}_0) + \hat{\tau}_+(z,\bar{u}_0))h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.14}$$

In (3.14), we choose $h = -\bar{u}_0^-$. Using Lemma 2.3 and (3.10), (3.11) and (3.12), we obtain that

$$\frac{c_1}{p-1}\|D\bar{u}_0^-\|_p^p + \|\bar{u}_0^-\|_p^p \leq 0$$

implies $\bar{u}_0 \geq 0$ and $\bar{u}_0 \neq 0$. Also, in (3.14), we choose $h = (\bar{u}_0 - u)^+ \in W^{1,p}(\Omega)$. Then, we have (see (3.10), (3.11) and (3.12) for the first equality, see (3.1) and recall that $u \in [0,\eta]$ for the second one and see (2.4) for the first inequality)

$$\begin{aligned}
&\langle A(\bar{u}_0), (\bar{u}_0 - u)^+\rangle + \int_\Omega \bar{u}_0^{p-1}(\bar{u}_0 - u)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\bar{u}_0 - u)^+\,d\sigma \\
&\quad= \int_\Omega (e(z,u) + u^{p-1})(\bar{u}_0 - u)^+\,dz \\
&\quad= \int_\Omega (\xi u^{q-1} - c_6 u^{r-1} + u^{p-1})(\bar{u}_0 - u)^+\,dz \\
&\quad\leq \int_\Omega (f(z,u) + u^{p-1})(\bar{u}_0 - u)^+\,dz \\
&\quad= \langle A(u), (\bar{u}_0 - u)^+\rangle + \int_\Omega u^{p-1}(\bar{u}_0 - u)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}(u_0 - u)^+\,d\sigma
\end{aligned}$$

since $u \in S_+$, which implies that

$$\langle A(\bar{u}_0) - A(u), (\bar{u}_0 - u)^+\rangle + \int_\Omega (\bar{u}_0^{p-1} - u^{p-1})(\bar{u}_0 - u)^+\,dz \leq 0.$$

Therefore,

$$|\{\bar{u}_0 > u\}|_N = 0,$$

that is,

$$\bar{u}_0 \leq u.$$

Thus, we have proved that

$$\bar{u}_0 \in [0,u] = \{y \in W^{1,p}(\Omega) : 0 \leq y(z) \leq u(z) \text{ for almost all } z \in \Omega\} \quad \text{and} \quad \bar{u}_0 \neq 0. \tag{3.15}$$

Because of (3.10), (3.11), (3.12) and (3.15) we have that (3.14) becomes

$$\langle A(\bar{u}_0), h\rangle + \int_{\partial\Omega} \beta(z)\bar{u}_0^{p-1}h\,d\sigma = \int_\Omega e(z,\bar{u}_0)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

which implies that $\bar{u}_0$ is a positive solution of (3.3) (see Papageorgiou and Rădulescu [18]). Then, from Proposition 3.1 we have $\bar{u}_0 = \bar{u}$ and, as a result, $\bar{u} \leq u$ for all $u \in S_+$.

In a similar fashion, we show that $v \leq \bar{v}$ for all $v \in S_+$. □

Next, we produce extremal constant-sign solutions, that is, a smallest positive solution and a biggest negative solution.

Proposition 3.3. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then* (1.1) *admits a smallest positive solution*

$$u_* \in [0,\eta] \cap \operatorname{int} C_+$$

and a biggest negative solution

$$v_* \in [-\eta, 0] \cap (-\operatorname{int} C_+).$$

Proof. Evidently, we restrict ourselves to the sets S_+ and S_-. From Hu and Papageorgiou [11, Lemma 3.10] we know that we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geq 1} u_n.$$

For all $n \in \mathbb{N}$, we have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, dz = \int_\Omega f(z, u_n) h\, dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.16}$$

Clearly, $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.17}$$

In (3.16), we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, we pass to the limit as $n \to \infty$ and we use (3.17). Then, we have that

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0$$

implies

$$u_n \to u_* \quad \text{in } W^{1,p}(\Omega) \tag{3.18}$$

since $A(\cdot)$ is of type $(S)_+$. So, if in (3.16) we pass to the limit as $n \to \infty$ and use (3.18), then

$$\langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h\, d\sigma = \int_\Omega f(z, u_*) h\, dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.19}$$

Also (see Proposition 3.2),

$$\bar{u} \leq u_*. \tag{3.20}$$

From (3.19) and (3.20) we infer that

$$u_* \in S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_* = \inf S_+.$$

Similarly, we produce

$$v_* \in S_- \quad \text{and} \quad v_* = \sup S_-.$$ □

Using these two extremal constant-sign solutions, we introduce the Carathéodory functions

$$\mu(z, x) = \begin{cases} f(z, v_*(z)) + |v_*(z)|^{p-2} v_*(z) & \text{if } x < v_*(z), \\ f(z, x) + |x|^{p-2} x & \text{if } v_*(z) \leq x \leq u_*(z), \\ f(z, u_*(z)) + u_*(z)^{p-1} & \text{if } u_*(z) < x, \end{cases} \tag{3.21}$$

$$\bar{b}(z, x) = \begin{cases} \beta(z) |v_*(z)|^{p-2} v_*(z) & \text{if } x < v_*(z), \\ \beta(z) |x|^{p-2} x & \text{if } v_*(z) \leq x \leq u_*(z), \\ \beta(z) u_*(z)^{p-1} & \text{if } u_*(z) < x. \end{cases} \tag{3.22}$$

We set

$$M(z, x) = \int_0^x \mu(z, s)\, ds \quad \text{and} \quad \bar{B}(z, x) = \int_0^x \bar{b}(z, s)\, ds,$$

and we consider the C^1-functional $\bar{\varphi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\bar{\varphi}(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}\|u\|_p^p + \int_{\partial\Omega} \bar{B}(z, u)\, d\sigma - \int_\Omega M(z, u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 3.4. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then $\bar{\varphi}$ satisfies the PS-condition, it is even, bounded from below, $\bar{\varphi}(0) = 0$ and $K_{\bar{\varphi}} \subseteq [v_*, u_*]$.*

Proof. From (3.21) and (3.22) it is clear that $\bar{\varphi}$ is coercive. So, it is bounded from below and satisfies the PS-condition (see Papageorgiou and Winkert [25]). Hypotheses $H(f)$ imply that $\bar{\varphi}$ is even (recall that $u_* \in S_+$ and $v_* \in S_-$) and $\bar{\varphi}(0) = 0$. Finally, let $u \in K_{\bar{\varphi}}$. Then, $\bar{\varphi}'(u) = 0$ implies that

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uh\,dz + \int_{\partial\Omega} \bar{b}(z,u)h\,d\sigma = \int_\Omega \mu(z,u)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{3.23}$$

In (3.23), we first choose $h = (u - u_*)^+ \in W^{1,p}(\Omega)$. Then, we have (see (3.21) and (3.22) for the first equality and recall that $u_* \in S_+$ for the second one)

$$\begin{aligned}\langle A(u), (u-u_*)^+\rangle &+ \int_\Omega u^{p-1}(u-u_*)^+\,dz + \int_{\partial\Omega} \beta(z)u_*^{p-1}(u-u_*)^+\,d\sigma \\ &= \int_\Omega (f(z,u_*) + u_*^{p-1})(u-u_*)^+\,dz \\ &= \langle A(u_*), (u-u_*)^+\rangle + \int_\Omega u_*^{p-1}(u-u_*)^+\,dz + \int_{\partial\Omega} \beta(z)u_*^{p-1}(u-u_*)^+\,d\sigma,\end{aligned}$$

which implies that

$$\langle A(u) - A(u_*), (u-u_*)^+\rangle + \int_\Omega (u^{p-1} - u_*^{p-1})(u-u_*)^+\,dz = 0.$$

Therefore,

$$|\{u > u_*\}|_N = 0,$$

that is,

$$u \le u_*.$$

Similarly, if in (3.23) we choose $h = (v_* - u)^+ \in W^{1,p}(\Omega)$, then we obtain that $v_* \le u$ implies

$$K_{\bar{\varphi}} \subseteq [v_*, u_*]. \qquad \square$$

The extremality of $v_* \in -\operatorname{int} C_+$ and of $u_* \in \operatorname{int} C_+$ implies the following property.

Corollary 3.5. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then the elements of $K_{\bar{\varphi}} \setminus \{0, v_*, u_*\}$ are nodal solutions of* (1.1).

Now, we are ready to produce a whole sequence of distinct nodal solutions for (1.1).

Theorem 3.6. *Assume that hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold. Then,* (1.1) *has a whole sequence $\{u_n\}_{n\ge1} \subseteq C^1(\overline{\Omega})$ of distinct nodal solutions.*

Proof. Let $\bar{\eta} = \min\{\min_{\overline{\Omega}} u_*, -\max_{\overline{\Omega}} v_*\}$ (recall that $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$). Hypothesis $H(a)$ (iv) implies that we can find $\delta_0 \in (0, \bar{\eta}]$ such that

$$G(y) \le c_9|y|^q \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \le \delta_0 \text{ and for some } c_9 > 0. \tag{3.24}$$

Also, from (2.5) we have

$$F(z,x) \ge \frac{\xi}{q}|x|^q \quad \text{for almost all } z \in \Omega \text{ and for all } |x| \le \delta \text{ with } \xi > 0. \tag{3.25}$$

Let $n \in \mathbb{N}$ and let $Y_n \subseteq W^{1,p}(\Omega)$ be an n-dimensional subspace. Then, all norms are equivalent on Y_n. So, we can find $\rho_n > 0$ such that $u \in Y_n$ and $\|u\| \le \rho_n$ imply

$$|u(z)| \le \delta \quad \text{for almost all } z \in \Omega. \tag{3.26}$$

Using (3.24), (3.25) and (3.26) together with (3.21) and (3.22), for all $u \in Y_n$ with $\|u\| \le \rho_n$, we have

$$\bar{\varphi}(u) \le c_9 \|Du\|_q^q + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma - \frac{\xi}{q} \|u\|_q^q \le (c_{10} - \xi c_{11}) \|u\|^q \tag{3.27}$$

with $c_{10}, c_{11} > 0$ independent of $\xi > 0$ (use the trace theorem and recall that all norms are equivalent on Y_n). Recall that $\xi > 0$ is arbitrary (see (2.3)). So, we choose $\xi > \frac{c_{10}}{c_{11}}$ and we have that

$$\varphi(u) < 0 \quad \text{for all } u \in Y_n \text{ with } \|u\| = \rho_n.$$

Because of Proposition 3.4 we can apply Theorem 2.1 to find $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega) \setminus \{0\}$ such that

$$u_n \in K_{\bar{\varphi}} \setminus \{0\} \quad \text{for all } n \in \mathbb{N}$$

and

$$\bar{\varphi}(u_n) \to 0 \quad \text{as } n \to \infty.$$

Since $K_{\bar{\varphi}} \subseteq C^1(\overline{\Omega})$ (nonlinear regularity theory), we have

$$u_n \in C^1(\overline{\Omega}) \setminus \{0\} \quad \text{for all } n \in \mathbb{N}.$$

Finally, Corollary 3.5 implies that $\{u_n\}_{n\ge1} \subseteq C^1(\overline{\Omega})$ is a sequence of distinct nodal solutions for (1.1). □

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