Research Article

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Asymmetric Robin Problems with Indefinite Potential and Concave Terms

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Abstract: We consider a parametric semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential. In the reaction, we have the competing effects of a concave term appearing with a negative sign and of an asymmetric asymptotically linear term which is resonant in the negative direction. Using variational methods together with truncation and perturbation techniques and Morse theory (critical groups), we prove two multiplicity theorems producing four and five, respectively, nontrivial smooth solutions when the parameter $\lambda > 0$ is small.

Keywords: Indefinite and Unbounded Potential, Concave Term, Asymmetric Reaction, Critical Groups, Multiple Solutions, Harnack Inequality

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric Robin problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) - \lambda|u(z)|^{q-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u(z) = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

In this problem, the potential function $\xi \in L^s(\Omega)$ $(s > N)$ is indefinite (that is, sign changing). In the reaction (right-hand side), the function $f(z, x)$ is Carathéodory (that is, for all $x \in \mathbb{R}$ the function $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous) and $f(z, \cdot)$ has linear growth near $\pm\infty$. However, the asymptotic behavior of $f(z, \cdot)$ as $x \to \pm\infty$ is asymmetric. More precisely, we assume that the quotient $\frac{f(z,x)}{x}$ as $x \to +\infty$ stays above the principal eigenvalue $\hat{\lambda}_1$ of the differential operator $u \mapsto -\Delta u + \xi(z)u$ with Robin boundary condition, while as $x \to -\infty$ the quotient $\frac{f(z,x)}{x}$ stays below $\hat{\lambda}_1$ with possible interaction (resonance) with respect to $\hat{\lambda}_1$ from the left. So, $f(z, \cdot)$ is a crossing (jumping) nonlinearity. In the term $-\lambda|u|^{q-2}u$, we suppose that $\lambda > 0$ is a parameter and $1 < q < 2$. Hence this term is a concave nonlinearity. Therefore, in the reaction we have the competing effects of resonant and concave terms. However, note that in our problem the concave nonlinearity enters with a negative sign. Such problems were

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considered by Perera [12], de Paiva and Massa [3] and de Paiva and Presoto [4] for Dirichlet problems with zero potential (that is, $\xi \equiv 0$). Of the aforementioned works, only de Paiva and Presoto [4] have an asymmetric reaction of special form, which is superlinear in the positive direction and linear and nonresonant in the negative direction. Recently, problems with asymmetric reaction have been studied by D'Agui, Marano and Papageorgiou [2] (Robin problems), Papageorgiou and Rădulescu [8, 11] (Neumann and Robin problems) and Recova and Rumbos [14] (Dirichlet problems).

We prove two multiplicity results in which we show that for all small $\lambda > 0$ the problem has four and five nontrivial smooth solutions, respectively. Our approach uses variational tools based on the critical point theory, together with suitable truncation, perturbation and comparison techniques and Morse theory (critical groups).

2 Mathematical Background and Hypotheses

Let X be a Banach space. We denote by X^* the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short) if the following property holds:

- Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This compactness-type condition on $\varphi(\cdot)$ is crucial in deriving the minimax theory of the critical values of φ. One of the main results in that theory is the so-called "mountain pass theorem", which we recall below.

Theorem 2.1. *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = r\} = m_r$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)) \quad \text{with} \quad \Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}.$$

Then $c \geqslant m_r$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi(u) = c$ and $\varphi'(u) = 0$).

Recall that a Banach space X has the "Kadec–Klee property" if the following holds:

$$u_n \xrightarrow{w} u \text{ in } X,\ \|u_n\| \to \|u\| \quad \Longrightarrow \quad u_n \to u \text{ in } X.$$

It is an easy consequence of the parallelogram law that every Hilbert space has the Kadec–Klee property (see [5]).

In the study of problem (P_λ), we will use the following three spaces:

$$H^1(\Omega),\ C^1(\overline{\Omega}),\ L^r(\partial\Omega) \quad (1 \leqslant r \leqslant \infty).$$

The Sobolev space $H^1(\Omega)$ is a Hilbert space with inner product given by

$$(u, h) = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\, dz + \int_\Omega uh\, dz \quad \text{for all } u, h \in H^1(\Omega).$$

We denote by $\|\cdot\|$ the corresponding norm on $H^1(\Omega)$. So, we have

$$\|u\| = \big[\|u\|_2^2 + \|Du\|_2^2\big]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior. Note that

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\} \subseteq \operatorname{int} C_+.$$

In fact, D_+ is the interior of C_+ when the latter is furnished with the relative $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the "boundary" Lebesgue spaces $L^r(\partial\Omega)$ (for $1 \leqslant r \leqslant \infty$). From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ known as the "trace map" such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns "boundary values" to every Sobolev function. The trace map is compact into $L^p(\partial\Omega)$ for all $1 \leqslant p < \frac{2(N-1)}{N-2}$ if $N \geqslant 3$, and into $L^p(\partial\Omega)$ for all $1 \leqslant p \leqslant \infty$ if $N = 1, 2$. Also, we have

$$\operatorname{im} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker \gamma_0 = H_0^1(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Next, we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda} u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This problem was studied by D'Agui, Marano and Papageorgiou [2]. We impose the following conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Remark. The potential function ξ is both unbounded and sign-changing.

Remark. If $\beta \equiv 0$, then we recover the Neumann problem.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2\,dz + \int_{\partial\Omega} \beta(z)u^2\,d\sigma \quad \text{for all } u \in H^1(\Omega).$$

Problem (2.1) admits a smallest eigenvalue $\hat{\lambda}_1 \in \mathbb{R}$ given by

$$\hat{\lambda}_1 = \inf\left\{ \frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega),\ u \neq 0 \right\}. \tag{2.2}$$

Moreover, there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geqslant c_0\|u\|^2 \quad \text{for some } c_0 > 0 \text{ and for all } u \in H^1(\Omega). \tag{2.3}$$

Using (2.3) and the special theorem for compact self-adjoint operators on Hilbert spaces, we produce the full spectrum of (2.2). This consists of a sequence $\{\hat{\lambda}_k\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$. Let $E(\hat{\lambda}_k)$ denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. By the regularity theory of Wang [15], we have

$$E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega}) \quad \text{for all } k \in \mathbb{N}.$$

Each eigenspace has the "Unique Continuation Property" (UCP for short). This means that if $u \in E(\hat{\lambda}_k)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$.

Let

$$\overline{H}_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k) \quad \text{and} \quad \hat{H}_m = \overline{H}_m^\perp = \overline{\bigoplus_{k\geqslant m+1} E(\hat{\lambda}_k)}.$$

We have

$$H^1(\Omega) = \overline{H}_m \oplus \hat{H}_m.$$

Moreover, for every $m \geqslant 2$, we have variational characterizations for the eigenvalues $\hat{\lambda}_m$ analogue to that for $\hat{\lambda}_1$ (see (2.2)):

$$\hat{\lambda}_m = \inf\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in \hat{H}_{m-1},\ u \neq 0\right\} = \sup\left\{\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_m,\ u \neq 0\right\}, \quad m \geqslant 2. \tag{2.4}$$

In (2.2) the infimum is realized on $E(\hat{\lambda}_1)$, while in (2.4) both the infimum and the supremum are realized on $E(\hat{\lambda}_m)$. We know that $\dim E(\hat{\lambda}_1) = 1$ (that is, the first eigenvalue $\hat{\lambda}_1$ is simple). Hence the elements of $E(\hat{\lambda}_1)$ have constant sign. We denote by $\hat{u}_1 \in C_+ \setminus \{0\}$ the positive L^2-normalized eigenfunction (that is, $\|\hat{u}_1\|_2 = 1$) corresponding to $\hat{\lambda}_1$. By the strong maximum principle, we have $\hat{u}_1(z) > 0$ for all $z \in \Omega$ and if $\xi^+ \in L^\infty(\Omega)$ (that is, the potential function is bounded above), by the Hopf boundary point theorem we have $\hat{u}_1 \in D_+$ (see [13, p. 120]).

Using (2.2), (2.4) and the above properties, we get the following useful inequalities.

Proposition 2.2. (i) *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \hat{\lambda}_m$ for almost all $z \in \Omega$, $\vartheta \not\equiv \hat{\lambda}_m$, $m \in \mathbb{N}$, then there exists $c_1 > 0$ such that*

$$c_1\|u\|^2 \leqslant \gamma(u) - \int_\Omega \vartheta(z)u^2\,dz \quad \text{for all } u \in \hat{H}_{m-1}.$$

(ii) *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geqslant \hat{\lambda}_m$ for almost all $z \in \Omega$, $\vartheta \not\equiv \hat{\lambda}_m$, $m \in \mathbb{N}$, then there exists $c_2 > 0$ such that*

$$\gamma(u) - \int_\Omega \vartheta(z)u^2\,dz \leqslant -c_2\|u\|^2 \quad \text{for all } u \in \overline{H}_m.$$

Note that if $\xi \equiv 0$ and $\beta \equiv 0$, then $\hat{\lambda}_1 = 0$, while if $\xi \geqslant 0$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$, then $\hat{\lambda}_1 > 0$. Also, the elements of $E(\hat{\lambda}_k)$ for $k \geqslant 2$ are nodal (that is, sign-changing).

In addition to the eigenvalue problem (2.1), we can consider its weighted version. So, let $m \in L^\infty(\Omega)$, $m(z) \geqslant 0$ for almost all $z \in \Omega$, $m \not\equiv 0$, and consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) + \xi(z)u(z) = \tilde{\lambda} m(z)u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

This eigenvalue problem exhibits the same properties as (2.1). So, the spectrum consists of a sequence $\{\tilde{\lambda}_k(m)\}_{k\in\mathbb{N}}$ of distinct eigenvalues such that $\tilde{\lambda}_k(m) \to +\infty$ as $k \to +\infty$. As for (2.1), the first eigenvalue $\tilde{\lambda}_1(m)$ is simple and the elements of $E(\tilde{\lambda}_1(m)) \subseteq C^1(\overline{\Omega})$ have fixed sign, while the elements of $E(\tilde{\lambda}_k(m)) \subseteq C^1(\overline{\Omega})$ (for all $k \geqslant 2$) are nodal. We have variational characterizations for all the eigenvalues as in (2.2) and (2.4) except that now the Rayleigh quotient is

$$\frac{\gamma(u)}{\int_\Omega m(z)u^2\,dz}.$$

Moreover, the eigenspaces have the UCP property. These properties yield the following monotonicity property for the map $m \mapsto \tilde{\lambda}_k(m)$, $k \in \mathbb{N}$.

Proposition 2.3. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leqslant m_1(z) \leqslant m_2(z)$ for almost all $z \in \Omega$, $m_1 \not\equiv 0$, $m_2 \not\equiv m_1$, then*

$$\tilde{\lambda}_k(m_2) < \tilde{\lambda}_k(m_1) \quad \text{for all } k \in \mathbb{N}.$$

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z,x) \leqslant a_0(z)[1 + |x|^{r-1}]| \quad \text{for almost all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and

$$1 < r \leqslant 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 3, \\ +\infty & \text{if } N = 1, 2 \end{cases}$$

(the critical Sobolev exponent). Let $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0 : H^1(\Omega) \to \mathbb{R}$

defined by

$$\varphi_0(u) = \frac{1}{2}\gamma(u) - \int_\Omega F_0(z, u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

As in [10, Proposition 8], using the regularity theory of Wang [15], we obtain the following result.

Proposition 2.4. *Assume that $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $\varphi_0(\cdot)$, that is, there exists $\rho_1 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \leqslant \rho_1.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$, and u_0 is also a local $H^1(\Omega)$-minimizer of φ_0, that is, there exists $\rho_2 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}),\ \|h\| \leqslant \rho_2.$$

Next, we recall some definitions and facts from Morse theory (critical groups). So, let X be a Banach space, let $\varphi \in C^1(X, \mathbb{R})$ and let $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leqslant c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Given a topological pair (Y_1, Y_2) such that $Y_2 \subseteq Y_1 \subseteq X$, for every $k \in \mathbb{N}_0$ we denote by $H_k(Y_1, Y_2)$ the k-th-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Suppose that $u \in K_\varphi^c$ is isolated. The critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood U. If u is a local minimizer of φ, then

$$C_k(\varphi, u) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Here, $\delta_{k,m}$ denotes the Kronecker symbol defined by

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

Next, let us fix our notation. If $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. For $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we denote by $N_g(\cdot)$ the Nemitsky (superposition) map defined by

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, $A \in \mathscr{L}(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in H^1(\Omega).$$

The hypotheses on the nonlinearity $f(z, x)$ are the following:

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) For every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leqslant a_\rho(z) \quad \text{for almost all } z \in \Omega \text{ and for all } |x| \leqslant \rho.$$

(ii) There exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)$ and $m \in \mathbb{N}$, $m \geqslant 2$, such that

$$\hat{\lambda}_1 \leqslant \eta(z) \leqslant \hat{\eta}(z) \leqslant \hat{\lambda}_m \qquad \text{for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_1,\ \hat{\eta} \not\equiv \hat{\lambda}_m,$$
$$\hat{\eta}(z) \leqslant \liminf_{x\to+\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to+\infty} \frac{f(z,x)}{x} \leqslant \hat{\eta}(z) \quad \text{uniformly for almost all } z \in \Omega,$$

and there exists $\tilde{\eta} > 0$ such that

$$-\tilde{\eta} \leqslant \liminf_{x\to-\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to-\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega.$$

(iii) If $F(z,x) = \int_0^x f(z,s)\,ds$, then

$$f(z,x)x - 2F(z,x) \to +\infty \qquad \text{uniformly for almost all } z \in \Omega \text{ as } x \to -\infty,$$
$$f(z,x)x - 2F(z,x) \geqslant 0 \qquad \text{for almost all } z \in \Omega \text{ and for all } x \geqslant M_0 > 0,$$
$$F(z,x) \leqslant \frac{\hat{\lambda}_m}{2} x^2 \quad \text{for almost all } z \in \Omega \text{ and for all } x \in \mathbb{R}.$$

(iv) There exist functions $\vartheta, \hat{\vartheta} \in L^\infty(\Omega)$ and $l \in \mathbb{N}$, $l \geqslant m$, such that

$$\hat{\lambda}_l \leqslant \vartheta(z) \leqslant \hat{\vartheta}(z) \leqslant \hat{\lambda}_{l+1} \qquad \text{for almost all } z \in \Omega,\ \vartheta \not\equiv \hat{\lambda}_l,\ \hat{\vartheta} \neq \hat{\lambda}_{l+1},$$
$$\vartheta(z) \leqslant \liminf_{x\to 0} \frac{f(z,x)}{x} \leqslant \limsup_{x\to 0} \frac{f(z,x)}{x} \leqslant \hat{\vartheta}(z) \quad \text{uniformly for almost all } z \in \Omega.$$

Remark. Hypothesis $H(f)$ (ii) implies that $f(z,\cdot)$ has asymmetric behavior as $x \to \pm\infty$ (jumping nonlinearity). Moreover, as $x \to -\infty$ we can have resonance with respect to the principal eigenvalue $\hat{\lambda}_1$. Hypothesis $H(f)$ (iii) implies that this resonance is from the left of $\hat{\lambda}_1$ in the sense that

$$\hat{\lambda}_1 x^2 - 2F(z,x) \to +\infty \quad \text{uniformly for almost all } z \in \Omega \text{ as } x \to -\infty.$$

Note that hypotheses $H(f)$ (i), (ii) and (iv) imply that

$$|f(z,x)| \leqslant c_3|x| \quad \text{for almost all } z \in \Omega \text{ for all } x \in \mathbb{R} \text{ and for some } c_3 > 0. \tag{2.6}$$

For every $\lambda > 0$, let $\varphi_\lambda : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{2}\gamma(u) + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(H^1(\Omega), \mathbb{R})$.

Let $\mu > 0$ be as in (2.3). We introduce the following truncations-perturbations of the reaction in problem (P_λ):

$$\begin{cases} k_\lambda^+(z,x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ f(z,x) - \lambda x^{q-1} + \mu x & \text{if } x > 0, \end{cases} \\ k_\lambda^-(z,x) = \begin{cases} f(z,x) - \lambda |x|^{q-2}x + \mu x & \text{if } x < 0, \\ 0 & \text{if } x \geqslant 0. \end{cases} \end{cases} \tag{2.7}$$

Both are Carathéodory functions. We set

$$K_\lambda^\pm(z,x) = \int_0^x k_\lambda^\pm(z,s)\,ds$$

and consider the C^1-functionals $\hat{\varphi}_\lambda^\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda^\pm(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega K_\lambda^\pm(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

3 Compactness Conditions for the Functionals

We consider the functionals $\hat{\varphi}_\lambda^\pm$ and φ_λ and we show that they satisfy the compactness-type condition.

Proposition 3.1. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then for every $\lambda > 0$ the functional $\hat{\lambda}_\lambda^+$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$\begin{aligned} |\hat{\varphi}_\lambda^+(u_n)| \leqslant M_1 \quad & \text{for some } M_1 > 0 \text{ and for all } n \in \mathbb{N}, \\ (1 + \|u_n\|)(\hat{\varphi}_\lambda^+)'(u_n) \to 0 \quad & \text{in } H^1(\Omega)^* \text{ as } n \to +\infty. \end{aligned} \tag{3.1}$$

From (3.1) we have

$$\left| \langle A(u_n), h\rangle + \int_\Omega [\xi(z) + \mu] u_n h \, dz + \int_{\partial\Omega} \beta(z) u_n h \, d\sigma - \int_\Omega k_\lambda^+(z, u_n) h \, dz \right| \leqslant \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \tag{3.2}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$. In (3.2) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\begin{aligned} & \gamma(u_n^-) + \mu\|u_n^-\|_2^2 \leqslant \epsilon_n && \text{for all } n \in \mathbb{N} \text{ (see (2.7))}, \\ \Longrightarrow \quad & c_0\|u_n^-\|^2 \leqslant \epsilon_n && \text{for all } n \in \mathbb{N} \text{ (see (2.3))}, \\ \Longrightarrow \quad & u_n^- \to 0 && \text{in } H^1(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we have

$$\left| \langle A(u_n^+), h\rangle + \int_\Omega \xi(z) u_n^+ h \, dz + \int_{\partial\Omega} \beta(z) u_n^+ h \, d\sigma - \int_\Omega [f(z, u_n^+) - \lambda (u_n^+)^{q-1}] h \, dz \right| \leqslant \epsilon_n' \|h\| \tag{3.4}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n' \to 0^+$ (see (2.7)).

We show that $\{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. Arguing by contradiction, suppose that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{3.5}$$

Let

$$y_n = \frac{u_n^+}{\|u_n^+\|}, \quad n \in \mathbb{N}.$$

Then $\|y_n\| = 1$ and $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega), \ y \geqslant 0. \tag{3.6}$$

Using (3.4), we obtain

$$\begin{aligned} & \left| \langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n h \, dz + \int_{\partial\Omega} \beta(z) y_n h \, d\sigma + \frac{\lambda}{\|u_n^+\|^{2-q}} \int_\Omega y_n^{q-1} h \, dz - \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|} h \, dz \right| \\ & \quad \leqslant \frac{\epsilon' \|h\|}{\|u_n^+\|} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

From (2.6) we see that

$$\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|} \right\}_{n\geqslant 1} \subseteq L^2(\Omega) \quad \text{is bounded.} \tag{3.8}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)$ (ii), we have (see [1, Proof of Proposition 16])

$$\frac{N_f(u_n^+)}{\|u_n^+\|} \xrightarrow{w} \nu(z) y \quad \text{in } L^2(\Omega), \ \eta(z) \leqslant \nu(z) \leqslant \hat{\eta}(z) \text{ for almost all } z \in \Omega. \tag{3.9}$$

If in (3.7) we choose $h = y_n - y \in H^1(\Omega)$, passing to the limit as $n \to \infty$ and using (3.5), (3.6), (3.8) and the fact that $q < 2$, we obtain

$$\begin{aligned} & \lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ \Longrightarrow \quad & \|Dy_n\|_2 \to \|Dy\|_2, \\ \Longrightarrow \quad & y_n \to y \quad \text{in } H^1(\Omega) \text{ (by the Kadec–Klee property), and hence } \|y\| = 1. \end{aligned} \tag{3.10}$$

In (3.7) we pass to the limit as $n \to \infty$ and use (3.9). We obtain

$$\langle A(y), h\rangle + \int_\Omega \xi(z)yh\,dz + \int_{\partial\Omega} \beta(z)yh d\sigma = \int_\Omega \nu(z)yh\,dz \quad \text{for all } h \in H^1(\Omega),$$

which implies

$$\begin{aligned} -\Delta y(z) + \xi(z)y(z) = \nu(z)y(z) \quad & \text{for almost all } z \in \Omega, \\ \frac{\partial y}{\partial n} + \beta(z)y = 0 \qquad & \text{on } \partial\Omega \text{ (see [9])}. \end{aligned} \tag{3.11}$$

From (3.9) and Proposition 2.3 we have

$$\tilde{\lambda}_1(\nu) < \tilde{\lambda}_1(\hat{\lambda}_1) = 1. \tag{3.12}$$

Then (3.11), (3.12) and the fact that $\|y\| = 1$ (see (3.10)) imply that $y(\cdot)$ must be nodal. But this contradicts (3.6). Therefore,

$$\begin{aligned} & \{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded}, \\ \Longrightarrow \quad & \{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded (see (3.3))}. \end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad u_n \to u \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.13}$$

In (3.2) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.13) and (2.6). Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ \Longrightarrow \quad & u_n \to u \quad \text{in } H^1(\Omega) \text{ (again by the Kadec–Klee property)}, \\ \Longrightarrow \quad & \hat{\varphi}_\lambda^+ \text{ satisfies the C-condition.} \end{aligned}$$

The proof is now complete. □

Proposition 3.2. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then for every $\lambda > 0$ the functional $\hat{\varphi}_\lambda^-$ is coercive.*

Proof. According to hypothesis $H(f)$ (iii), given any $\rho > 0$, we can find $M_2 = M_2(\rho) > 0$ such that

$$\rho \leqslant f(z,x)x - 2F(z,x) \quad \text{for almost all } z \in \Omega \text{ and for all } x \leqslant -M_2. \tag{3.14}$$

We have

$$\begin{aligned} \frac{d}{dx}\Big(\frac{F(z,x)}{x^2}\Big) &= \frac{f(z,x)x^2 - 2xF(z,x)}{x^4} \\ &= \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} \\ &\leqslant \frac{\rho}{|x|^2 x} \quad \text{for almost all } z \in \Omega \text{ and all } x \leqslant -M_2 \text{ (see (3.14))}, \end{aligned}$$

which implies

$$\frac{F(z,v)}{v^2} - \frac{F(z,y)}{y^2} \geqslant \frac{\rho}{2}\Big(\frac{1}{y^2} - \frac{1}{v^2}\Big) \quad \text{for almost all } z \in \Omega \text{ and for all } v \leqslant y \leqslant -M_2. \tag{3.15}$$

From hypothesis $H(f)$ (ii) we have

$$-\tilde{\eta} \leqslant \liminf_{x\to-\infty} \frac{2F(z,x)}{x^2} \leqslant \limsup_{x\to-\infty} \frac{2F(z,x)}{x^2} \leqslant \hat{\lambda}_1 \quad \text{uniformly for almost all } z \in \Omega. \tag{3.16}$$

If in (3.15) we let $v \to -\infty$ and use (3.16), then

$$\begin{aligned} &\hat{\lambda}_1 y^2 - 2F(z,y) \geqslant \rho && \text{for almost all } z \in \Omega \text{ and for all } y \leqslant -M_2, \\ \Longrightarrow\quad &\hat{\lambda}_1 y^2 - 2F(z,y) \to +\infty && \text{uniformly for almost all } z \in \Omega \text{ as } y \to -\infty. \end{aligned} \tag{3.17}$$

Suppose to the contrary that $\hat{\lambda}_\lambda^-$ is not coercive. This means that we can find $\{u_n\}_{n\geqslant1} \subseteq H^1(\Omega)$ such that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty \qquad \text{and} \qquad \hat{\varphi}_\lambda^-(u_n) \leqslant M_3 \quad \text{for some } M_3 > 0 \text{ and for all } n \in \mathbb{N}. \tag{3.18}$$

Let

$$v_n = \frac{u_n}{\|u_n\|}, \quad n \in \mathbb{N}.$$

Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$, and so we may assume that

$$v_n \xrightarrow{w} v \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad v_n \to v \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.19}$$

From (3.18) we have

$$\begin{aligned} &\frac{1}{2}\gamma(u_n) + \frac{\mu}{2}\|u_n\|_2^2 - \int_\Omega K_\lambda^-(z,u_n)\,dz \leqslant M_3 && \text{for all } n \in \mathbb{N}, \\ \Longrightarrow\quad &\frac{1}{2}\gamma(v_n) + \frac{\mu}{2}\|v_n\|_2^2 - \int_\Omega \frac{K_\lambda^-(z,u_n)}{\|u_n\|^2}\,dz \leqslant \frac{M_3}{\|u_n\|^2} && \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.20}$$

From (2.6) we obtain

$$\begin{aligned} &|F(z,x)| \leqslant \frac{c_3}{2}x^2 && \text{for almost all } z \in \Omega \text{ and for all } x \in \mathbb{R}, \\ \Longrightarrow\quad &\Big\{\frac{K_\lambda^-(\cdot,u_n(\cdot))}{\|u_n\|^2}\Big\}_{n\geqslant1} \subseteq L^1(\Omega) && \text{is uniformly integrable (see (2.7) and (3.19)).} \end{aligned}$$

Hence, by the Dunford–Pettis theorem and hypothesis $H(f)$ (ii) we have

$$\frac{K_\lambda^-(\cdot,u_n(\cdot))}{\|u_n\|^2} \xrightarrow{w} \frac{1}{2}[\tilde{e}(z)+\mu](v^-)^2 \quad \text{in } L^1(\Omega) \text{ as } n \to \infty \tag{3.21}$$

with $-\tilde{\eta} \leqslant \tilde{e}(z) \leqslant \hat{\lambda}_1$ for almost all $z \in \Omega$ (see [1]).

We return to (3.20) and pass to the limit as $n \to \infty$ in (3.18), (3.19) and (3.21). Since $\gamma(\cdot)$ is sequentially weakly lower semicontinuous on $H^1(\Omega)$, we obtain (see (2.3))

$$\begin{aligned} &\frac{1}{2}\gamma(v) + \frac{\mu}{2}\|v\|_2^2 \leqslant \frac{1}{2}\int_\Omega [\tilde{e}(z)+\mu](v^-)^2\,dz \\ \Longrightarrow\quad &\gamma(v^-) \leqslant \int_\Omega \tilde{e}(z)(v^-)^2\,dz. \end{aligned} \tag{3.22}$$

First, we assume that $\tilde{e} \not\equiv \hat{\lambda}_1$ (see (3.21)). Then by (3.22) and Proposition 2.2 we have $c_1\|v^-\|^2 \leqslant 0$, which implies

$$v \geqslant 0. \tag{3.23}$$

Then on account of (3.19) and (3.23) we have

$$v_n^- \xrightarrow{w} 0 \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad v_n^- \to 0 \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.24}$$

In (3.20) we pass to the limit as $n \to \infty$ and use (3.24), (3.22) and the sequential weak lower semicontinuity of $\gamma(\cdot)$. We obtain

$$\begin{aligned} & \gamma(v^+) + \mu\|v^+\|_2^2 \leqslant 0, \\ \Longrightarrow\quad & c_0\|v^+\|^2 \leqslant 0 \qquad \text{(see (2.3))}, \\ \Longrightarrow\quad & v = 0 \qquad \text{(see (3.23))}. \end{aligned}$$

From (3.20) we obtain $\|Dv_n\|_2 \to 0$, which implies $v_n \to 0$ in $H^1(\Omega)$ (see (3.19)), which contradicts the fact that $\|v_n\| = 1$ for all $n \in \mathbb{N}$.

Next, we assume that $\tilde{e}(z) = \hat{\lambda}_1$ for almost all $z \in \Omega$. From (3.22) and (2.2) we have $\gamma(v^-) = \hat{\lambda}_1\|v^-\|_2^2$, which implies

$$v^- = \tau\hat{u}_1 \quad \text{for some } \tau \geqslant 0. \tag{3.25}$$

If $\tau = 0$, then $v \geqslant 0$ and, arguing as above (see the part of the proof after (3.23)), we obtain $v = 0$, contradicting the fact that $\|v_n\| = 1$ for all $n \in \mathbb{N}$. If $\tau > 0$, then from (3.25) we have

$$v(z) < 0 \quad \text{for all } z \in \Omega.$$

This means that

$$\begin{aligned} & u_n^-(z) \to -\infty && \text{for almost all } z \in \Omega \text{ as } n \to \infty, \\ \Longrightarrow\quad & \hat{\lambda}_1 u_n^-(z)^2 - 2F(z, u_n^-(z)) \to +\infty && \text{for almost all } z \in \Omega \text{ as } n \to \infty \text{ (see (3.17))}, \\ \Longrightarrow\quad & \int_\Omega [\hat{\lambda}_1(u_n^-)^2 - 2F(z, u_n^-)]\, dz \to +\infty && \text{as } n \to \infty \text{ (by Fatou's lemma, see (3.17))}, \\ \Longrightarrow\quad & \gamma(u_n^-) - 2\int_\Omega F(z, -u_n^-)\, dz \to +\infty && \text{as } n \to \infty \text{ (see (2.2))}, \\ \Longrightarrow\quad & 2\hat{\varphi}_\lambda^-(u_n^-) \to +\infty && \text{as } n \to \infty \text{ (see (2.7))}. \end{aligned}$$

But this contradicts (3.18). We conclude that $\hat{\varphi}_\lambda^-$ is coercive. ☐

This proposition leads to the following corollary (see [6, Proposition 2.2]).

Corollary 3.3. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then for every $\lambda > 0$ the functional $\hat{\varphi}_\lambda^-$ satisfies the C-condition.*

Next, we turn our attention to the energy functional φ_λ, $\lambda > 0$.

Proposition 3.4. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then for every $\lambda > 0$ the functional φ_λ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$|\varphi_\lambda(u_n)| \leqslant M_4 \quad \text{for some } M_4 > 0 \text{ and for all } n \in \mathbb{N}, \tag{3.26}$$

$$(1 + \|u_n\|)\varphi_\lambda'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{3.27}$$

From (3.27) we have

$$\begin{aligned} & \left| \langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h\, dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma + \lambda \int_\Omega |u_n|^{q-2}u_n h d\sigma - \int_\Omega f(z, u_n)h\, dz \right| \\ & \quad \leqslant \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H^1(\Omega), \text{ with } \epsilon_n \to 0^+. \end{aligned} \tag{3.28}$$

In (3.28) we choose $h = u_n \in H^1(\Omega)$. Then

$$-\gamma(u_n) - \lambda\|u_n\|_q^q + \int_\Omega f(z, u_n)u_n dz \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.29}$$

On the other hand, from (3.26) we have

$$\gamma(u_n) + \frac{2\lambda}{q}\|u_n\|_q^q - \int_\Omega 2F(z, u_n)\,dz \leqslant 2M_4 \quad \text{for all } n \in \mathbb{N}. \tag{3.30}$$

We add (3.29) and (3.30). Recalling that $q < 2$, we obtain

$$\int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]\,dz \leqslant M_5 \quad \text{for all } n \in \mathbb{N}.$$

Using hypothesis $H(f)$ (iii), we see that

$$\int_\Omega [f(z, -u_n^-)(-u_n^-) - 2F(z, -u_n^-)]\,dz \leqslant M_5 \quad \text{for all } n \in \mathbb{N}. \tag{3.31}$$

We use (3.31) to show that $\{u_n^-\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. Arguing by contradiction, we may assume that

$$\|u_n^-\| \to \infty \quad \text{as } n \to \infty. \tag{3.32}$$

Let

$$y_n = \frac{u_n^-}{\|u_n^-\|}, \quad n \in \mathbb{N}.$$

Then $\|y_n\| = 1$ and $y_n \geqslant 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } H^1(\Omega) \qquad \text{and} \qquad y_n \to y \quad \text{in } L^2(\Omega) \text{ and in } L^2(\partial\Omega),\ y \geqslant 0. \tag{3.33}$$

In (3.28) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\gamma(u_n^-) + \lambda\|u_n^-\|_q^q - \int_\Omega f(z, -u_n^-)(-u_n^-)\,dz \leqslant \epsilon_n \qquad \text{for all } n \in \mathbb{N},$$

$$\implies \quad \gamma(y_n) + \frac{\lambda}{\|u_n^-\|^{2-q}}\|y_n\|_q^q - \int_\Omega \frac{N_f(-u_n^-)}{\|u_n^-\|} y_n dz \leqslant \frac{\epsilon_n}{\|u_n^-\|^2} \quad \text{for all } n \in \mathbb{N}. \tag{3.34}$$

From (2.6) we see that

$$\left\{\frac{N_f(-u_n^-)}{\|u_n^-\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)$ (ii), we have

$$\frac{N_f(-u_n^-)}{\|u_n^-\|} \xrightarrow{w} \tilde{e}(z)y \quad \text{in } L^2(\Omega) \text{ as } n \to \infty \tag{3.35}$$

with $-\tilde{\eta} \leqslant \tilde{e}(z) \leqslant \hat{\lambda}_1$ for almost all $z \in \Omega$.

Returning to (3.34), passing to the limit as $n \to \infty$ and using (3.32) (recall that $q < 2$), (3.33), (3.35) and the sequential weak lower semicontinuity of $\gamma(\cdot)$, we obtain

$$\gamma(y) \leqslant \int_\Omega \tilde{e}(z)y^2\,dz. \tag{3.36}$$

First, we assume that $\tilde{e} \not\equiv \hat{\lambda}_1$ (see (3.35)). Then from (3.36) and Proposition 2.2 we get $c_1\|y\|^2 \leqslant 0$, which implies $y = 0$. From this and (3.34) we infer that $\|Dy_n\|_2 \to 0$, which implies $y_n \to 0$ in $H^1(\Omega)$, which contradicts the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

We now assume that $\tilde{e}(z) = \hat{\lambda}_1$ for almost all $z \in \Omega$. Then from (3.36) and (2.2) we have

$$y = \tau\hat{u}_1 \quad \text{with } \tau \geqslant 0.$$

If $\tau = 0$, then $y = 0$ and, as above, we have

$$y_n \to 0 \quad \text{in } H^1(\Omega),$$

a contradiction since $\|y_n\| = 1$ for all $n \in \mathbb{N}$. If $\tau > 0$, then $y(z) > 0$ for all $z \in \Omega$, and so

$$\begin{aligned}
& u_n^-(z) \to +\infty && \text{for almost all } z \in \Omega, \\
\Longrightarrow\quad & f(z, -u_n^-(z))(-u_n^-)(z) - 2F(z, -u_n^-(z)) \to +\infty && \text{for almost all } z \in \Omega \text{ (see hypothesis } H(f)\text{ (iii))}, \\
\Longrightarrow\quad & \int_\Omega [f(z, -u_n^-)(-u_n^-) - 2F(z, -u_n^-)]\, dz \to +\infty && \text{(by Fatou's lemma).}
\end{aligned}$$

This contradicts (3.31). Therefore,

$$\{u_n^-\}_{n \geqslant 1} \subseteq H^1(\Omega) \quad \text{is bounded.} \tag{3.37}$$

Next, we show that $\{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. From (3.28) and (3.37) we have

$$\left| \langle A(u_n^+), h\rangle + \int_\Omega \xi(z) u_n^+ h\, dz + \int_{\partial\Omega} \beta(z) u_n^+ h d\sigma + \lambda \int_\Omega (u_n^+)^{q-1} h\, dz - \int_\Omega f(z, u_n^+) h\, dz \right| \leqslant M_6$$

for some $M_6 > 0$ and all $n \in \mathbb{N}$. Using this bound and a contradiction argument as in the proof of Proposition 3.1, we show that

$$\begin{aligned}
& \{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded}, \\
\Longrightarrow\quad & \{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega) \text{ is bounded (see (3.37))}.
\end{aligned}$$

From this, as before (see the proof of Proposition 3.1), via the Kadec–Klee property, we conclude that φ_λ satisfies the C-condition. □

4 Multiplicity Theorems

In this section, using variational methods, truncation and perturbation techniques and Morse theory, we prove two multiplicity theorems for problem (P_λ) when $\lambda > 0$ is small. In the first result, we produce four nontrivial smooth solutions, while in the second theorem, under stronger conditions on $f(z, \cdot)$, we establish the existence of five nontrivial smooth solutions.

We start with a result which allows us to satisfy the mountain pass geometry (see Theorem 2.1) and also to distinguish the solutions we produce from the trivial one.

Proposition 4.1. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then $u = 0$ is a local minimizer of φ_λ and of $\hat{\varphi}_\lambda^\pm$ for every $\lambda > 0$.*

Proof. We give the proof for the functional φ_λ. The proofs for the $\hat{\varphi}_\lambda^\pm$ are similar.

Recall that

$$|F(z, x)| \leqslant \frac{c_3}{2}|x|^2 \quad \text{for almost all } z \in \Omega \text{ and for all } x \in \mathbb{R} \text{ (see (2.6)).} \tag{4.1}$$

Then for $u \in C^1(\overline{\Omega}) \setminus \{0\}$ we have

$$\begin{aligned}
\varphi_\lambda(u) &\geqslant \frac{\lambda}{q}\|u\|_q^q - \left[\frac{c_8}{2} + \|\xi\|_\infty\right]\|u\|_2^2 && \text{(see (4.1) and hypotheses } H(\xi), H(\beta)). \\
&\geqslant \frac{\lambda}{q}\|u\|_q^q - c_4\|u\|_\infty^{2-q}\|u\|_q^q && \left(\text{with } c_4 = \left[\frac{c_1}{2} + \|\xi\|_\infty\right] > 0\right) \\
&= \left[\frac{\lambda}{q} - c_4\|u\|_\infty^{2-q}\right]\|u\|_q^q.
\end{aligned}$$

So, if

$$\|u\|_\infty \leqslant \|u\|_{C^1(\overline{\Omega})} < \left(\frac{\lambda}{qc_4}\right)^{\frac{1}{2-q}},$$

then $\varphi_\lambda(u) > 0 = \varphi_\lambda(0)$. Hence,

$$\begin{aligned} & u = 0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda(\cdot), \\ \implies\ & u = 0 \text{ is a local } H^1(\Omega)\text{-minimizer of } \varphi_\lambda(\cdot) \text{ (see Proposition 2.4).} \end{aligned}$$

The proofs for the functionals $\hat{\varphi}_\lambda^\pm$ are similar. □

With the next proposition we guarantee that for small $\lambda > 0$ the functional $\hat{\varphi}_\lambda^+(\cdot)$ satisfies the mountain pass geometry (see Theorem 2.1).

Proposition 4.2. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then we can find $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ there is $t_0 = t_0(\lambda) > 0$ for which we have $\hat{\varphi}_\lambda^+(t_0\hat{u}_1) < 0$.*

Proof. Let $r > 2$. From hypothesis $H(f)$ (iv) and (4.1), we see that given $\epsilon > 0$ we can find $c_5 = c_5(\epsilon, r) > 0$ such that

$$F(z, x) \geqslant \frac{1}{2}[\vartheta(z) - \epsilon]x^2 - c_5 x^r \quad \text{for almost all } z \in \Omega \text{ and for all } x \geqslant 0. \tag{4.2}$$

Then for all $t > 0$ we have

$$\begin{aligned} \hat{\varphi}_\lambda^+(t\hat{u}_1) &= \frac{t^2}{2}\gamma(\hat{u}_1) + \frac{\lambda t^q}{q}\|\hat{u}_1\|_q^q - \int_\Omega F(z, t\hat{u}_1)\,dz && \text{(see (2.7))} \\ &\leqslant \frac{t^2}{2}\Big[\gamma(\hat{u}_1) - \int_\Omega \vartheta(z)\hat{u}_1^2\,dz + \epsilon\Big] + \frac{\lambda t^q}{q}\|\hat{u}_1\|_q^q + c_5 t^r\|\hat{u}_1\|_r^r && \text{(see (4.2) and recall that } \|\hat{u}_1\|_2 = 1) \\ &= \frac{t^2}{2}\Big[\int_\Omega (\hat{\lambda}_1 - \vartheta(z))\hat{u}_1^2\,dz + \epsilon\Big] + \frac{\lambda t^q}{q}\|\hat{u}_1\|_q^q + c_5 t^r\|\hat{u}_1\|_r^r. \end{aligned} \tag{4.3}$$

Note that

$$k_* = \int_\Omega (\vartheta(z) - \hat{\lambda}_1)\hat{u}_1^2\,dz > 0 \quad \text{(see hypothesis } H(f)\text{ (iv)).}$$

Choosing $\epsilon \in (0, k_*)$, we see from (4.3) that

$$\hat{\varphi}_\lambda^+(t\hat{u}_1) \leqslant -c_6 t^2 + \lambda c_7 t^q + c_8 t^r = [-c_6 + \lambda c_7 t^{q-2} + c_8 t^{r-2}]t^2 \quad \text{for some } c_6, c_7, c_8 > 0. \tag{4.4}$$

Consider the function

$$\mathfrak{J}_\lambda(t) = \lambda c_7 t^{q-2} + c_8 t^{r-2} \quad \text{for all } t > 0.$$

Evidently, $\mathfrak{J}_\lambda \in C^1(0, +\infty)$, and since $1 < q < 2 < r$, we see that

$$\mathfrak{J}_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$\begin{aligned} & \mathfrak{J}_\lambda(t_0) = \min\{\mathfrak{J}(t) : 0 < t < +\infty\}, \\ \implies\ & \mathfrak{J}_\lambda'(t_0) = 0, \\ \implies\ & \lambda c_7(2 - q)t_0^{q-3} = c_8(r - 2)t_0^{r-3}, \\ \implies\ & t_0 = t_0(\lambda) = \Big[\frac{\lambda c_7(2 - q)}{c_8(r - 2)}\Big]^{\frac{1}{r-q}}. \end{aligned}$$

Then

$$\mathfrak{J}_\lambda(t_0) = \lambda c_7 \frac{[c_8(r - 2)]^{\frac{2-q}{r-q}}}{[\lambda c_2(2 - q)]^{\frac{2-q}{r-q}}} + c_8 \frac{[\lambda c_2(2 - q)]^{\frac{r-2}{2-q}}}{[c_8(r - 2)]^{\frac{r-2}{2-q}}}.$$

Since $\frac{2-q}{r-q} < 1$, we see that

$$\mathfrak{J}_\lambda(t_0) \to 0^+ \quad \text{as } \lambda \to 0^+.$$

So, we can find $\lambda^* > 0$ such that

$$\mathcal{J}_\lambda(t_0) < c_6 \quad \text{for all } \lambda \in (0, \lambda^*).$$

Then it follows from (4.4) that

$$\hat{\varphi}_\lambda^+(t_0 \hat{u}_1) < 0 \quad \text{for all } \lambda \in (0, \lambda^*).$$

This completes the proof of Proposition 4.2. □

Remark. In fact, a careful reading of the above proof reveals that

$$\hat{\varphi}_\lambda^-(-t_0 \hat{u}_1) < 0 \quad \text{for all } \lambda \in (0, \lambda^*). \tag{4.5}$$

Proposition 4.3. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold and $\lambda \in (0, \lambda^*)$, then there exists $u_0 \in C^1(\overline{\Omega})$ with $u_0(z) < 0$ for all $z \in \Omega$ and*

$$\hat{\varphi}_\lambda^-(u_0) = \inf\{\hat{\varphi}_\lambda^-(u) : u \in H^1(\Omega)\} < 0.$$

Proof. From Proposition 3.2 we know that $\hat{\varphi}_\lambda^-$ is coercive. Also, the Sobolev embedding theorem and the compactness of the trace map imply that $\hat{\varphi}_\lambda^-$ is sequentially weakly lower semicontinuous. Hence, by the Weierstrass–Tonelli theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$\hat{\varphi}_\lambda^-(u_0) = \inf\{\hat{\varphi}_\lambda^-(u) : u \in W^{1,p}(\Omega)\}. \tag{4.6}$$

From (4.5) we see that $\hat{\varphi}_\lambda^-(u_0) < 0 = \hat{\lambda}_\lambda^-(0)$, which implies $u_0 \neq 0$.

From (4.6) we have $(\hat{\varphi}_\lambda^-)'(u_0) = 0$, which implies

$$\langle A(u_0), h\rangle + \int_\Omega [\xi(z) + \mu] u_0 h \, dz + \int_{\partial\Omega} \beta(z) u_0 h \, d\sigma = \int_\Omega k_\lambda^-(z, u_0) h \, dz \quad \text{for all } h \in H^1(\Omega). \tag{4.7}$$

In (4.7) we choose $h = u_0^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
& \gamma(u_0^+) + \mu \|u_0^+\|_2^2 = 0 && \text{(see (2.7))},\\
\Longrightarrow\quad & c_0 \|u_0^+\|^2 \leqslant 0 && \text{(see (2.3))},\\
\Longrightarrow\quad & u_0 \leqslant 0, \quad u_0 \neq 0.
\end{aligned}$$

From (4.7) and (2.7) it follows that

$$\langle A(u_0), h\rangle + \int_\Omega \xi(z) u_0 h \, dz + \int_{\partial\Omega} \beta(z) u_0 h \, d\sigma = \int_\Omega [f(z, u_0) - \lambda |u_0|^{q-2} u_0] h \, dz \quad \text{for all } h \in H^1(\Omega),$$

which implies

$$\begin{aligned}
& -\Delta u_0(z) + \xi(z) u_0(z) = f(z, u_0(z)) - \lambda |u_0(z)|^{q-2} u_0(z) && \text{for almost all } z \in \Omega,\\
& \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 && \text{on } \partial\Omega \text{ (see [9])}.
\end{aligned} \tag{4.8}$$

Let

$$\tau_\lambda(z, x) = f(z, x) - \lambda |x|^{q-2} x \quad \text{and} \quad \hat{k}_\lambda(z) = \frac{\tau_\lambda(z, u_0(z))}{1 + |u_0(z)|} \qquad \text{for } \lambda > 0.$$

Hypotheses $H(f)$ (i) and (ii) imply that

$$\begin{aligned}
& |\tau_\lambda(z, x)| \leqslant c_9[1 + |x|] && \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R}, \text{ with } c_9 = c_9(\lambda) > 0,\\
\Longrightarrow\quad & |\hat{k}_\lambda(z)| = \frac{|\tau_\lambda(z, u_0(z))|}{1 + |u_0(z)|} \leqslant c_9 && \text{for almost all } z \in \Omega,\\
\Longrightarrow\quad & \hat{k}_\lambda \in L^\infty(\Omega).
\end{aligned}$$

From (4.8) we have

$$\begin{aligned}
& -\Delta u_0(z) = [\xi(z) - \hat{k}_\lambda(z)] u_0(z) + \hat{k}_\lambda(z) && \text{for almost all } z \in \Omega,\\
& \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 && \text{on } \partial\Omega
\end{aligned}$$

(recall that $u_0 \leqslant 0$). Since $(\xi - \hat{k}_\lambda)(\cdot) \in L^s(\Omega)$ (for $s > N$), we deduce by [15, Lemma 5.1] that

$$u_0 \in L^\infty(\Omega).$$

Then the Calderon–Zygmund estimates (see [15, Lemma 5.2]) imply that

$$u_0 \in (-C_+) \setminus \{0\}.$$

Moreover, the Harnack inequality (see [13, p. 163, Theorem 7.2.1]) implies that

$$u_0(z) < 0 \quad \text{for all } z \in \Omega.$$

This completes the proof. □

Remark. The negative sign of the concave term does not allow us to conclude that $u_0 \in -D_+$ when $\xi^+ \in L^\infty(\Omega)$ (by Hopf's boundary point theorem, see [13, p. 120]).

Now we can state and prove our first multiplicity theorem.

Theorem 4.4. *Assume that hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold. Then there exists $\hat{\lambda} > 0$ such that for all $\lambda \in (0, \hat{\lambda})$ problem (P_λ) has at least four nontrivial solutions*

$$\begin{aligned} &u_0, \hat{u} \in (-C_+) \setminus \{0\}, && u_0(z), \hat{u}(z) < 0 && \textit{for all } z \in \Omega,\\ &v_0 \in C_+ \setminus \{0\}, && v_0(z) > 0 && \textit{for all } z \in \Omega,\\ &y_0 \in C^1(\overline{\Omega}) \setminus \{0\}. \end{aligned}$$

Proof. From Proposition 4.3 and its proof (see (4.8)) we already have one solution

$$u_0 \in (-C_+) \setminus \{0\}, \qquad u_0(z) < 0 \quad \text{for all } z \in \Omega, \ \text{ when } \lambda \in (0, \lambda^*).$$

This solution is a global minimizer of the functional $\hat{\varphi}_\lambda^-$.

Claim. The solution u_0 is a local minimizer of the energy functional φ_λ.

We first show that u_0 is a local $C^1(\overline{\Omega})$-minimizer of φ_λ. Arguing by contradiction, suppose that we could find a sequence $\{u_n\}_{n \geqslant 1} \subseteq C^1(\overline{\Omega})$ such that

$$u_n \to u_0 \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty \text{ and } \varphi_\lambda(u_n) < \varphi_\lambda(u_0) \text{ for all } n \in \mathbb{N}. \tag{4.9}$$

Then for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &> \varphi_\lambda(u_n) - \varphi_\lambda(u_0)\\ &= \varphi_\lambda(u_n) - \hat{\varphi}_\lambda^-(u_0) \quad (\text{since } \varphi_\lambda|_{(-C_+)} = \hat{\varphi}_\lambda^-|_{(-C_+)}, \ \text{see (2.7)})\\ &\geqslant \varphi_\lambda(u_n) - \hat{\varphi}_\lambda^-(u_n) \quad (\text{recall that } u_0 \text{ is a global minimizer of } \hat{\varphi}_\lambda^-)\\ &= \frac{1}{2}\gamma(u_n) + \frac{\lambda}{q}\|u_n\|_q^q - \int_\Omega F(z, u_n)\,dz - \frac{1}{2}\gamma(u_n) - \frac{\mu}{2}\|u_n^+\|_2^2 - \frac{\lambda}{q}\|u_n^-\|_q^q + \int_\Omega F(z, -u_n^-)\,dz \quad (\text{see (2.7)})\\ &= \frac{\lambda}{q}\|u_n^+\|_q^q - \frac{\mu}{2}\|u_n^+\|_2^2 - \int_\Omega F(z, u_n^+)\,dz\\ &\geqslant \frac{\lambda}{q}\|u_n^+\|_q^q - \Big(\frac{\mu + c_3}{2}\Big)\|u_n^+\|_2^2 \quad (\text{see (4.1)})\\ &\geqslant \frac{\lambda}{q}\|u_n^+\|_q^q - c_{10}\|u_n^+\|_\infty^{2-q}\|u_n^+\|q_q\\ &= \Big[\frac{\lambda}{q} - c_{10}\|u_n^+\|_\infty^{2-q}\Big]\|u_n^+\|_q^q, \end{aligned} \tag{4.10}$$

where

$$c_{10} = \frac{\mu + c_3}{2} > 0.$$

From (4.9) we have

$$u_n^+ \to 0 \quad \text{in } C^1(\overline{\Omega}) \text{ (recall that } u_0|_\Omega < 0).$$

Therefore, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & \frac{\lambda}{q} > c_{10}\|u_n^+\|_\infty^{2-q} && \text{for all } n \geqslant n_0, \\ \Longrightarrow \quad & 0 > \varphi_\lambda(u_n) - \varphi(u_0) > 0 && \text{for all } n \geqslant n_0 \text{ (see (4.10))}, \end{aligned}$$

a contradiction. Hence we have that

$$\begin{aligned} & u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda, \\ \Longrightarrow \quad & u_0 \text{ is a local } H^1(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Proposition 2.4)}. \end{aligned}$$

This proves the claim.

Using (2.7) and the regularity theory of Wang [15], we can see that

$$K_{\hat{\lambda}_\lambda^-} \subseteq (-C_+) \quad \text{and} \quad K_{\hat{\varphi}_\lambda^+} \subseteq C_+ \qquad \text{for all } \lambda > 0. \tag{4.11}$$

On account of (4.11), we see that we may assume that both critical sets $K_{\hat{\varphi}_\lambda^-}$ and $K_{\hat{\varphi}_\lambda^+}$ are finite or, otherwise, we already have an infinity of nontrivial smooth solutions of constant sign and so we are done.

From Proposition 4.1 we know that $u = 0$ is a local minimizer of $\hat{\varphi}_\lambda^-$ for all $\lambda > 0$. Since $K_{\hat{\varphi}_\lambda^-}$ is finite, we can find $\rho \in (0, \|u_0\|)$ small such that (see [1, Proof of Proposition 29])

$$\hat{\varphi}_\lambda^-(u_0) < 0 = \hat{\varphi}_\lambda^-(0) < \inf\{\hat{\varphi}_\lambda^-(u) : \|u\| = \rho\} = \hat{m}_\rho^-. \tag{4.12}$$

From Corollary 3.3 we know that

$$\hat{\varphi}_\lambda^- \text{ satisfies the C-condition.} \tag{4.13}$$

Then (4.12) and (4.13) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_{\hat{\varphi}_\lambda^-} \subseteq (-C_+) \quad \text{(see (4.11))} \qquad \text{and} \qquad \hat{\varphi}_\lambda^-(u_0) < 0 = \hat{\varphi}_\lambda^-(0) < \hat{m}_\rho^- \leqslant \hat{\varphi}_\lambda^-(\hat{u}).$$

It follows that

$$\hat{u} \in (-C_+) \setminus \{0, u_0\} \quad \text{is a solution of } (P_\lambda) \text{ (see (2.7))}.$$

As before, Harnack's inequality implies that

$$\hat{u}(z) < 0 \quad \text{for all } z \in \Omega.$$

Now we use once more Proposition 4.1 to find $\rho_0 \in (0, t_0)$ small enough such that

$$0 = \hat{\varphi}_\lambda^+(0) < \inf\{\hat{\varphi}_\lambda^+(u) : \|u\| = \rho_0\} = \hat{m}_{\rho_0}^+, \quad \lambda > 0. \tag{4.14}$$

Proposition 4.2 implies that we can find $\lambda^* > 0$ such that

$$\hat{\varphi}_\lambda^+(t_0\hat{u}_1) < 0 \quad \text{for all } \lambda \in (0, \lambda^*) \text{ with } t_0 = t_0(\lambda) > 0. \tag{4.15}$$

Moreover, Proposition 2.5 implies that

$$\hat{\varphi}_\lambda^+ \text{ satisfies the C-condition for all } \lambda > 0. \tag{4.16}$$

Then, on account of (4.14)–(4.16), we can apply Theorem 2.1 (the mountain pass theorem) and produce $v_0 \in H^1(\Omega)$ such that

$$\begin{aligned} & v_0 \in K_{\hat{\varphi}_\lambda^+} \subseteq C_+ \quad \text{(see (4.11))} \qquad \text{and} \qquad 0 = \hat{\varphi}_\lambda^+(0) < \hat{m}_\rho^+ \leqslant \hat{\varphi}_\lambda^+(v_0), \\ \Longrightarrow \quad & v_0 \in C_+ \setminus \{0\} \quad \text{is a solution of } (P_\lambda),\ \lambda \in (0, \lambda^*) \text{ (see (2.7))}. \end{aligned}$$

Once again, Harnack's inequality guarantees that

$$v_0(z) > 0 \quad \text{for all } z \in \Omega.$$

Let $l \in \mathbb{N}$ be as in hypothesis $H(f)$ (iv) and set

$$\overline{H}_l = \bigoplus_{k=1}^{l} E(\hat{\lambda}_k), \quad \hat{H}_l = \overline{H}_l^{\perp} = \overline{\bigoplus_{k \geqslant l+1} E(\hat{\lambda}_k)}.$$

We have

$$H^1(\Omega) = \overline{H}_l \oplus \hat{H}_l \quad \text{and} \quad \dim \overline{H}_l < +\infty.$$

Consider $u \in \overline{H}_l$. We have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{2}\gamma(u) + \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(z,u)\,dz \\
&\leqslant \frac{1}{2}\Big[\gamma(u) - \int_\Omega \vartheta(z)u^2\,dz + \epsilon\|u\|^2\Big] + c_{11}[\lambda\|u\|^q + \|u\|^r] \qquad (4.17) \\
&\leqslant \frac{1}{2}[-c_2 + \epsilon]\|u\|^2 + c_{11}[\lambda\|u\|^q + \|u\|^r], \qquad (4.18)
\end{aligned}$$

where (4.17) holds for some $c_{11} > 0$ and follows from (4.2) and the fact that all norms on $\overline{H}_l$ are equivalent, and (4.18) follows from Proposition 2.2. Choosing $\epsilon \in (0, c_2)$, we have

$$\varphi_\lambda(u) \leqslant [-c_{12} + \lambda c_{11}\|u\|^{q-2} + c_{11}\|u\|^{r-2}]\|u\|^2 \quad \text{for some } c_{12} > 0.$$

Reasoning as in the proof of Proposition 4.3, we can find $\hat{\lambda} \in (0, \lambda^*]$ such that for all $\lambda \in (0, \hat{\lambda}]$ there exists $\rho_\lambda > 0$ for which we have

$$\varphi_\lambda(u) < 0 \quad \text{for all } u \in \overline{H}_l,\ \|u\| = \rho_\lambda. \qquad (4.19)$$

For $u \in \hat{H}_l$ we have

$$\begin{aligned}
\varphi_\lambda(u) &\geqslant \frac{1}{2}\gamma(u) + \frac{\lambda}{q}\|u\|_q^q - \frac{\hat{\lambda}_m}{2}\|u\|_2^2 \quad \text{(see hypothesis } H(f) \text{(iii))} \\
&\geqslant \frac{1}{2}[\gamma(u) - \hat{\lambda}_l\|u\|_2^2] + \frac{\lambda}{q}\|u\|_q^q \quad (\text{since } l \geqslant m) \\
&\geqslant 0. \qquad (4.20)
\end{aligned}$$

Finally, consider the half-space

$$H_+ = \{t\hat{u}_1 + \tilde{u} : t \geqslant 0,\ \tilde{u} \in \hat{H}_l\}.$$

Exploiting the orthogonality of $\hat{H}_l$ and $\overline{H}_l$, for every $u \in H_+$ we have

$$\begin{aligned}
\varphi_\lambda(u) &\geqslant \frac{1}{2}[t^2\gamma(\hat{u}_1) + \gamma(\tilde{u})] - \frac{\hat{\lambda}_m}{2}[t^2\|\hat{u}_1\|_2^2 + \|\tilde{u}\|_2^2] \quad \text{(see hypothesis } H(f) \text{(iii))} \\
&\geqslant 0 \quad (\text{since } \tilde{u} \in \hat{H}_l,\ l \geqslant m). \qquad (4.21)
\end{aligned}$$

Then (4.19)–(4.21) permit the use of [12, Theorem 3.1]. So, we can find $y_0 \in H^1(\Omega)$ such that

$$\begin{aligned}
&y_0 \in K_{\varphi_\lambda} \subseteq C^1(\overline{\Omega}) && \text{(by the regularity theory of Wang [15])}, \\
&\varphi_\lambda(y_0) < 0 = \varphi_\lambda(0) \quad \text{and} \quad C_{d_l-1}(\varphi_\lambda, y_0) \neq 0 && (d_l = \dim \overline{H}_l). \qquad (4.22)
\end{aligned}$$

From (4.22) it is clear that $y_0 \neq 0$. Recall that

$$0 < \varphi_\lambda(\hat{u}), \varphi_\lambda(v_0) \quad (\text{since } \varphi_\lambda = \hat{\varphi}_\lambda^-|_{(-C_+)} = \hat{\varphi}_\lambda^+|_{C_+}).$$

Therefore, it follows from (4.22) that

$$y_0 \notin \{\hat{u}, v_0, 0\}.$$

Also, by the claim we have that u_0 is a local minimizer of φ_λ. Hence,

$$C_k(\varphi_\lambda, u_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \qquad (4.23)$$

Note that $d_l \geqslant 2$ (since $l \geqslant m \geqslant 2$). Therefore,

$$d_{l-1} \geqslant 1,$$

and so from (4.22) and (4.23) we infer that

$$y_0 \neq u_0.$$

So, we conclude that $y_0 \in C^1(\overline{\Omega}) \setminus \{0\}$ is a fourth nontrivial solution of (P_λ) (for all $\lambda \in (0, \hat{\lambda})$) distinct from u_0, $\hat{u}$ and v_0. □

If we strengthen the hypotheses on $f(z, \cdot)$, we can improve the above multiplicity theorem and produce a fifth nontrivial smooth solution.

The new conditions on the nonlinearity $f(z, x)$ are the following:

$H(f)'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$, hypotheses $H(f)'$ (i), (ii) and (iii) are the same as the corresponding hypotheses $H(f)$ (i), (ii) and (iii), and, furthermore,

(iv) there exist $l \in \mathbb{N}$, $l \geqslant m$ such that

$$\begin{aligned} & f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} && \text{uniformly for almost all } z \in \Omega, \\ & f'_x(z,0) \in [\hat{\lambda}_l, \hat{\lambda}_{l+1}] && \text{for almost all } z \in \Omega, \\ & f'_x(\cdot,0) \not\equiv \hat{\lambda}_l, \quad f'_x(\cdot,0) \not\equiv \hat{\lambda}_{l+1}. \end{aligned}$$

Theorem 4.5. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)'$ hold, then there exists $\hat{\lambda} > 0$ such that for all $\lambda \in (0, \hat{\lambda})$ problem (P_λ) has at least five nontrivial solutions*

$$\begin{aligned} & u_0, \hat{u} \in (-C_+), && u_0(z) < 0 \quad \textit{for all } z \in \Omega, \\ & v_0 \in C_+, && v_0(z) > 0 \quad \textit{for all } z \in \Omega, \\ & y_0, \hat{y} \in C^1(\overline{\Omega}) \setminus \{0\}. \end{aligned}$$

Proof. Now we have $\varphi_\lambda \in C^2(H^1(\Omega) \setminus \{0\}, \mathbb{R})$. Similarly, $\hat{\varphi}_\lambda^\pm \in C^2(H^1(\Omega) \setminus \{0\}, \mathbb{R})$.

The solutions $u_0, \hat{u}, v_0, y_0$ are a consequence of Theorem 4.4. From Proposition 4.1 and (4.23) we have

$$C_k(\varphi_\lambda, u_0) = C_k(\varphi_\lambda, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0,\ \lambda \in (0, \hat{\lambda}). \tag{4.24}$$

Also, from the proof of Theorem 4.4 we know that $\hat{u}$ is a critical point of $\hat{\varphi}_\lambda^-$ of mountain pass type, and v_0 is a critical point of $\hat{\varphi}_\lambda^+$ of mountain pass type.

Invoking [7, Corollary 6.102], we have

$$C_k(\varphi_\lambda^-, \hat{u}) = C_k(\hat{\varphi}_\lambda^+, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.25}$$

The continuity in the C^1-norm of the critical groups (see [5, p. 836, Theorem 5.126]) implies that

$$C_k(\hat{\varphi}_\lambda^-, \hat{u}) = C_k(\varphi_\lambda, \hat{u}) \quad \text{for all } k \in \mathbb{N}_0, \tag{4.26}$$

$$C_k(\hat{\varphi}_\lambda^+, v_0) = C_k(\varphi_\lambda, v_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{4.27}$$

From (4.25)–(4.27) it follows that

$$C_k(\varphi_\lambda, \hat{u}) = C_k(\varphi_\lambda, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.28}$$

The fourth nontrivial solution $y_0 \in C^1(\overline{\Omega})$ was produced by using [12, Theorem 3.1]. According to that theorem, we can also find another function $\hat{y} \in H^1(\Omega)$, $\hat{y} \neq y_0$, such that

$$\hat{y} \in K_{\varphi_\lambda} \subseteq C^1(\overline{\Omega}) \quad \text{and} \quad C_{d_l}(\varphi_\lambda, \hat{y}) \neq 0 \quad (d_l \geqslant 2). \tag{4.29}$$

From (4.24)–(4.29) we conclude that

$$\hat{y} \in C^1(\overline{\Omega}) \setminus \{u_0, \hat{u}, v_0, y_0, 0\}$$

is the fifth nontrivial solution of problem (P_λ) for all $\lambda \in (0, \hat{\lambda})$. □

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