HARTMAN-STAMPACCHIA RESULTS FOR STABLY PSEUDOMONOTONE OPERATORS AND NONLINEAR HEMIVARIATIONAL INEQUALITIES *

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ABSTRACT. We are concerned with two classes of nonstandard hemivariational inequalities. In the first case we establish a Hartman-Stampacchia type existence result in the framework of stably pseudomonotone operators. Next, we prove an existence result for a class of nonlinear perturbations of canonical hemivariational inequalities. Our analysis includes both the cases of compact sets and of closed convex sets in Banach spaces. Applications to noncoercive hemivariational and variational-hemivariational inequalities illustrate the abstract results of this paper.
2000 Mathematics Subject Classification: 35J85, 47H05, 49J53.
Key words: hemivariational inequality; Clarke's generalized gradient; stably pseudomonotone operator; fixed point theorem; set valued mappings.

1 Introduction

Inequality theory plays an important role in many fields, such as mechanics, engineering sciences, economics, optimal control, etc. Because of their wide applicability, inequality problems have become an important area of investigation in the past several decades, an important part of this research focusing on the existence of the solutions. Inequality problems can be divided into two main classes: that of variational inequalities and that of hemivariational inequalities. The study of variational inequality problems began in the early sixties with the work of G. Fichera [2], J. L. Lions and G. Stampacchia [5]. The most basic result is due to Hartman and Stampacchia [4], which states that if X is a finite dimensional Banach space, $K \subset X$ is compact and convex, and A is a continuous operator, then the

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variational inequality problem of finding $u \in K$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in K \tag{1}$$

has a solution. When K is not compact, or X is infinite dimensional, certain monotonicity properties are required to prove the existence of solution.

By replacing the subdifferential of a convex function by the generalized gradient (in the sense of F. H. Clarke) of a locally Lipschitz functional, hemivariational inequalities arise whenever the energetic functional associated to a concrete problem is nonconvex. This new type of inequalities appears as a generalization of the variational inequalities, but hemivariational inequalities are much more general, in the sense that they are not equivalent to minimum problems but, give rise to substationarity problems. The theory of hemivariational inequalities can be viewed as a new field of Nonsmooth Mechanics since the main ingredient used in the study of these inequalities is the notion of Clarke subdifferential of a locally Lipschitz functional. The mathematical theory hemivariational inequalities, as well as their applications in Mechanics, Engineering or Economics were introduced and developed by P. D. Panagiotopoulos [14]-[18] in the case of nonconvex energy functions. For a treatment of this theory and further comments we recommend the monographs by Z. Naniewicz and P. D. Panagiotopoulos [13], D. Motreanu and P. D. Panagiotopoulos [10] and by D. Motreanu and V. Rădulescu [12].

Throughout this paper V will denote a real Banach space and let $T : V \to L^p(\Omega; \mathbb{R}^k)$ be a linear and continuous operator, where $1 < p < \infty$, $k \geq 1$, and Ω is a bounded open set in $\mathbb{R}^N$. We shall denote $Tu = \hat{u}$ and by p' the conjugated exponent of p. Let K be a subset of V, $A : K \to V^*$ a nonlinear operator and $j = j(x, y) : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a function such that the mapping

$$j(\cdot, y) : \Omega \to \mathbb{R} \quad \text{is measurable, for every } y \in \mathbb{R}^k. \tag{2}$$

We assume that at least one of the following conditions hold true: either there exist $l \in L^{p'}(\Omega; \mathbb{R})$ such that

$$|j(x, y_1) - j(x, y_2)| \leq l(x)|y_1 - y_2|, \quad \forall\, x \in \Omega,\ \forall\, y_1, y_2 \in \mathbb{R}^k, \tag{3}$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz, } \forall\, x \in \Omega, \tag{4}$$

and there exist $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \quad \forall\, x \in \Omega, \forall\, z \in \partial j(x, y). \tag{5}$$

Recall that $j^{\,0}(x, y; h)$ denotes the Clarke's generalized directional derivative of the locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $h \in \mathbb{R}^k$, where $x \in \Omega$, while $\partial j(x, y)$ is the Clarke's generalized gradient of this mapping at $y \in \mathbb{R}^k$, that is (see e.g. [1], [12], [13])

$$j^{\,0}(x, y; h) = \limsup_{\substack{y' \to y \\ t \downarrow 0}} \frac{j(x, y' + th) - j(x, y')}{t} \ ,$$

$$\partial j(x,y) = \{z \in \mathbb{R}^k : \ z \cdot h \leq j^0(x,y;h), \ \text{for all } h \in \mathbb{R}^k\},$$

where the symbol "·" means the inner product on $\mathbb{R}^k$.

The euclidian norm in $\mathbb{R}^k$, $k \geq 1$, resp. the duality pairing between a Banach space and its dual will be denoted by $|\cdot|$, respectively $\langle \cdot, \cdot \rangle$.

Definition 1. *The operator $A : K \to V^*$ is said to be*

(i) hemicontinuous if A is continuous from line segments in K to the w^* topology of V^*;

(ii) monotone if $\langle Av - Au, v - u \rangle \geq 0$, for all $u, v \in K$;

(iii) pseudomonotone if $\langle Au, v - u \rangle \geq 0$ implies $\langle Av, v - u \rangle \geq 0$, for all $u, v \in K$;

(iv) stably pseudomonotone with respect to a set $U \subset V^*$ if, A and $A(\cdot) - z$ are pseudomonotone for every $z \in U$.

Remark 1. *Clearly a monotone map is pseudomonotone and stably pseudomonotone with respect to any set $U \subset V^*$. Stably pseudomonotone mappings are introduced and discussed in [6, 7, 8]. Moreover, Yiran He showed in [7] (see Example 2, p.460) that the class of of stably pseudomonotone mappings is strictly broader than the class of monotone mappings (we point out the fact that the operator in this example is also hemicontinuous).*

The following fixed point theorem will be used in the proof of the main results. We denote by 2^K the family all nonempty subsets of a set K.

Theorem 1. *(Tarafdar [20]) Let $K \neq \emptyset$ be a convex subset of a Hausdorff topological vector space E. Let $F : K \to 2^K$ be a set valued map such that*

(1) for each $u \in K$, $F(u)$ is a nonempty convex subset of K;

(2) for each $v \in K$, $F^{-1}(v) = \{ u \in K : \ v \in F(u)\}$ contains an open set O_v which may be empty;

(3) $\bigcup_{v \in K} O_v = K$;

(4) there exists a nonempty set V_0 contained in a compact convex subset V_1 of K such that $D = \bigcap_{v \in V_0} O_v^c$ is either empty or compact (where O_v^c is the complement of O_v in K).

Then there exist a point $u_0 \in K$ such that $u_0 \in F(u_0)$.

2 Hemivariational inequalities of Hartman-Stampacchia type involving stably pseudomonotone operators

This section is devoted to the study of the *hemivariational inequality problem of Hartman-Stampacchia type* (as it was called by Panagiotopoulos, Fundo and Rădulescu in [19] to show that this kind of inequalities are a generalization of those studied by Hartman and Stampacchia in [4]):

($\mathbf{P}$) Find $u \in K$ such that

$$\langle Au, v-u\rangle + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0, \quad \forall\, v \in K.$$

Throughout this section V will denote a real reflexive Banach space and $T : V \to L^p(\Omega; \mathbb{R}^k)$ will stand for a linear and compact operator.

Let us consider a Banach space Y such that there exists a linear and compact operator $L : V \to Y$ and let $J : Y \to \mathbb{R}$ be an arbitrary locally Lipschitz functional. For each $u \in V$ there exists (see e.g. [1], [12], [13]) $\mathbf{z}_u \in \partial J(Lu)$ such that

$$J^0(Lu; \zeta) = \langle \mathbf{z}_u, \zeta\rangle = \max\left\{ \langle w, \zeta\rangle : \ w \in \partial J(Lu)\right\}. \tag{6}$$

Denoting by $L^* : Y^* \to V^*$ the adjoint operator of L, we define the subset $U(J, L)$ of V^* as follows

$$U(J, L) = \{-\mathbf{Z}_u : \ u \in K \text{ and } \mathbf{Z}_u = L^* \mathbf{z}_u\}. \tag{7}$$

Lemma 1. *Let K be a nonempty, closed and convex subset of V. Assume that $A : K \to V^*$ is a hemicontinuous and stably pseudomonotone map with respect to the set $U(J, L)$ defined in (7). Further we assume that there exists a nonempty subset V_0 contained in a weakly compact subset V_1 of K such that the set*

$$D = \{\, u \in K : \ \langle Av, v-u\rangle + J^{\,0}(Lu; Lv - Lu) \geq 0, \quad \forall\, v \in V_0\}$$

is weakly compact or empty. Then the problem

($\mathbf{P}'$) *Find $u \in K$ such that*

$$\langle Au, v-u\rangle + J^{\,0}(Lu; Lv - Lu) \geq 0, \quad \forall\, v \in K,$$

admits at least one solution.

As we will see problem (P') closely links to the following problem:

($\mathbf{P}^\star$) Find $u \in K$ such that

$$\langle Av, v-u\rangle + J^{\,0}(Lu; Lv - Lu) \geq 0, \quad \forall\, v \in K.$$

We denote by S' and $S^\star$ the solution sets of problem ($\mathbf{P}'$) and problem ($\mathbf{P}^\star$), respectively. Now we establish the relationship between problems ($\mathbf{P}'$) and ($\mathbf{P}^\star$).

Proposition 1. *Let K be a nonempty closed and convex subset of V. If $A : K \to V^*$ is stably pseudomonotone with respect to the set $U(J, L)$ defined in (7), then $S' \subseteq S^\star$. In addition, if A is hemicontinuous, then $S' = S^\star$.*

Proof. Let $u_0 \in S'$. For any $v \in K$ we have

$$\begin{aligned} 0 &\leq \langle Au_0, v-u_0\rangle + J^0(Lu_0; Lv - Lu_0) \\ &= \langle Au_0, v-u_0\rangle + \langle \mathbf{z}_{u_0}, L(v-u_0)\rangle \\ &= \langle Au_0, v-u_0\rangle + \langle L^*\mathbf{z}_{u_0}, v-u_0\rangle \\ &= \langle Au_0, v-u_0\rangle + \langle \mathbf{Z}_{u_0}, v-u_0\rangle. \end{aligned}$$

Using the stably pseudomonotonicity of A we get that

$$\langle Av, v - u_0\rangle + \langle \mathbf{Z}_{u_0}, v - u_0\rangle \geq 0, \quad \forall v \in K.$$

The above inequality is equivalent to

$$\langle Av, v - u_0\rangle + J^0(Lu_0; Lv - Lu_0) \geq 0, \quad \forall v \in K,$$

which means that $u_0 \in S^\star$.

In addition, if A is hemicontinuous, we will show that $S' = S^\star$. Suppose $u_0 \in S^\star$. For any $v \in K$, let

$$v_t = tv + (1-t)u_0, \quad t \in (0,1).$$

Then $v_t \in K$ and

$$\langle Av_t, t(v - u_0)\rangle + J^0(Lu_0; t(Lv - Lu_0)) \geq 0.$$

Using the positive homogeneity of the map $J^0(\cdot\,;\cdot)$ and dividing by t we obtain that

$$\langle Av_t, v - u_0\rangle + J^0(Lu_0; Lv - Lu_0) \geq 0.$$

Letting $t \to 0$, by the hemicontinuity of A, we get

$$\langle Au_0, v - u_0\rangle + J^0(Lu_0; Lv - Lu_0) \geq 0, \quad \forall v \in K,$$

which means $u_0 \in S'$. $\square$

Proof of Lemma 1. Arguing by contradiction, suppose that for every $u \in K$, there exists $v \in K$ such that

$$\langle Au, v - u\rangle + J^0(Lu; Lv - Lu) < 0.$$

This implies, by Proposition 1 that for every $u \in K$ there exists $v \in K$ such that

$$\langle Av, v - u\rangle + J^0(Lu; Lv - Lu) < 0. \tag{8}$$

We define the set valued map $F : K \to 2^K$ as follows

$$F(u) = \{\, v \in K : \langle Au, v - u\rangle + J^0(Lu; Lv - Lu) < 0\}$$

We will prove next that F satisfies the conditions of Theorem 1.

Let $u \in K$ be arbitrary but fixed. Obviously $F(u)$ is nonempty for each $u \in K$ since we have assumed that the problem $(\mathbf{P'})$ has no solutions. Let $v_1, v_2 \in F(u)$ and $t \in [0,1]$. Taking into account that L is linear and the application $Lv \longmapsto J^0(Lu; Lv)$ is convex we have

$$\langle Au, tv_1 + (1-t)v_2 - u\rangle + J^0(Lu; L(tv_1 + (1-t)v_2) - Lu) \leq$$
$$t\langle Au, v_1 - u\rangle + (1-t)\langle Au, v_2 - u\rangle + tJ^0(Lu; Lv_1 - Lu) + (1-t)J^0(Lu; Lv_2 - Lu) < 0,$$

which shows that $F(u)$ is a convex subset of K.

Let, now $v \in K$ be arbitrary but fixed.

$$\begin{aligned} F^{-1}(v) &= \{ u \in K : \; v \in F(u)\} \\ &= \{ u \in K : \; \langle Au, v-u\rangle + J^0(Lu; Lv - Lu) < 0\} \\ &\supseteq \{ u \in K : \; \langle Av, v-u\rangle + J^0(Lu; Lv - Lu) < 0\} := O_v. \end{aligned}$$

We shall prove that $[F^{-1}(v)]^c \subseteq O_v^c$ which implies $O_v \subseteq F^{-1}(v)$. Let $u \in [F^{-1}(v)]^c$. Then

$$\langle Au, v-u\rangle + J^0(Lu; Lv - Lu) \geq 0.$$

Using the fact that A is stably pseudomonotone with respect to the set $U(J, L)$ we get that

$$\langle Av, v-u\rangle + J^0(Lu; Lv - Lu) \geq 0,$$

which leads to the conclusion that $u \in O_v^c$.

We claim that O_v is weakly open. Indeed, if $O_v \ni u_n \rightharpoonup u$ then, since L is linear and compact $Lu_n \to Lu$ and by the upper semicontinuity of $J^0(\cdot,\cdot)$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \left[\langle Av, v-u_n\rangle + J^0(Lu_n; Lv - Lu_n)\right] &\leq \lim_{n\to\infty}\langle Av, v-u_n\rangle + \limsup_{n\to\infty} J^0(Lu_n; Lv - Lu_n) \\ &\leq \langle Av, v-u\rangle + J^0(Lu; Lv - Lu). \end{aligned}$$

The above relation implies that the application $u \longmapsto \langle Av, v-u\rangle + J^0(Lu; Lv - Lu)$ is weakly upper semicontinuous. Thus by the definition of the upper semicontinuity, O_v is an weakly open set.

Obviously $\bigcup_{v\in K} O_v \subseteq K$. Now, let $u \in K$. By (8) there exists $v \in K$ such that $\langle Av, v-u\rangle + J^0(Lu; Lv - Lu) < 0$ and thus $u \in O_v$. It follows that $K \subseteq \bigcup_{v\in K} O_v$.

By our hypothesis the set

$$\begin{aligned} D &= \{ u \in K : \; \langle Av, v-u\rangle + J^0(Lu; Lv - Lu) \geq 0, \quad \forall\, v \in V_0\} \\ &= \bigcap_{v\in V_0} \{ u \in K : \; \langle Av, v-u\rangle + J^0(Lu; Lv - Lu) \geq 0\} \\ &= \bigcap_{v\in V_0} O_v^c \end{aligned}$$

is weakly compact or empty.

Thus the mapping F satisfies the conditions of Theorem 1 in the weak topology, so there exists a point $u_0 \in K$ such that $u_0 \in F(u_0)$; that is $0 = \langle Au_0, u_0 - u_0\rangle + J^0(Lu_0; Lu_0 - Lu_0) < 0$ and this contradiction completes the proof of Lemma 1.

We will derive a result applicable to the inequality problem $(\mathbf{P})$ which constitutes the main result of this section.

Theorem 2. *Assume that the hypotheses of Lemma 1 are fulfilled for $Y = L^p(\Omega; \mathbb{R}^k)$, $L = T$ and $J : L^p(\Omega; \mathbb{R}^k) \to \mathbb{R}$*

$$J(u) = \int_\Omega j(x, u(x))\, dx.$$

Then the problem $(\mathbf{P})$ has at least one solution.

Proof. Conditions (2) and (3) or (2) and (4)-(5) on j ensure that J is locally Lipschitz on Y (see e.g. Clarke [1], p. 83 or Motreanu-Rădulescu [12] Theorem 1.3) and

$$\int_\Omega j^{\,0}(x,u(x);v(x))\,dx \ge J^{\,0}(u;v), \quad \forall\, u,v \in L^p(\Omega;\mathbb{R}^k).$$

It follows that

$$\int_\Omega j^{\,0}(x,\hat{u}(x);\hat{v}(x))\,dx \ge J^{\,0}(\hat{u};\hat{v}), \quad \forall\, u,v \in V. \tag{9}$$

It results that if $u \in K$ is a solution of the problem $(\mathbf{P'})$ then u solves inequality problem $(\mathbf{P})$, too. □

If in addition K is bounded it follows that K is a weakly compact set. Setting $V_0 = V_1 = K$, we notice that the set D in Theorem 2 is weakly compact as it is the intersection of weakly closed sets O_v^c. The following result follows.

Corollary 1. *Let $Y = L^p(\Omega;\mathbb{R}^k)$, $L = T$ and $J : L^p(\Omega;\mathbb{R}^k) \to \mathbb{R}$ be the function*

$$J(u) = \int_\Omega j(x,u(x))\,dx.$$

Assume that K is a nonempty, closed, bounded and convex subset of V and let $A : K \to V^*$ be a hemicontinuous and stably pseudomonotone map with respect to the set $U(J,L)$ defined in (7). Then the problem $(\mathbf{P})$ *has at least one solution.*

3 Existence results for nonlinear hemivariational inequalities and applications

This section is concerned with the study of the following *nonlinear hemivariational inequality problem*

Find $u \in K$ such that

$$\Lambda(u,v) + \int_\Omega j^{\,0}(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))\,dx \ge \langle f, v-u\rangle, \quad \forall\, v \in K, \tag{10}$$

where $\Lambda : K \times K \to \mathbb{R}$ is a given function and $f \in V^*$.

We point out the fact that the study of inequality problems involving nonlinear terms has captured special attention in the last few years. We just refer to the prototype problem of finding $u \in K$ such that

$$\Lambda(u,v) \ge \langle f, v-u\rangle, \quad \forall\, v \in K. \tag{11}$$

Nonlinear inequality problems of the type (11) model some equilibrium problems drawn from operations research, as well as some unilateral boundary value problems stemming from mathematical physics and where introduced by J. Gwinner [3] who investigated the existence theory and abstract stability analysis in the setting of reflexive Banach spaces.

The main object of this section is to establish existence results for the nonlinear hemivariational inequality (10) for general maps, without monotonicity assumptions. As a consequence to our theorems, we will derive some existence results for hemivariational inequalities that have been studied in [11], [13] and [19] as it will be seen at the end of this section.

3.1 Existence results

In this subsection we shall establish two existence results for the inequality problem (10). The first result is given by the following theorem.

Theorem 3. *Let K be a nonempty, closed and convex subset of V and $f \in V^*$. Let $\Lambda : K \times K \to \mathbb{R}$ a function vanishing on the diagonal, that is, $\Lambda(u,u) = 0$, for all $u \in K$, which satisfies the following assumptions:*

(1) Λ is convex with respect to the second variable;

(2) Λ is upper semicontinuous with respect to the first variable, that is, $\limsup_{n\to\infty} \Lambda(u_n, v) \leq \Lambda(u,v)$, *for all $v \in K$ whenever $u_n \to u$ in K;*

(3) There exists a compact convex subset V_1 of K such that for each $u \in K \setminus V_1$ there is some v in V_1 for which we have

$$\Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle.$$

If j satisfies the conditions (2) and (4)-(5) and $T : V \to L^p(\Omega; \mathbb{R}^k)$ is linear and continuous, then the nonlinear hemivariational inequality problem (10) has at least one solution in K.

Proof. For each $v \in K$ we define the set

$$N(v) := \left\{ u \in K :\ \Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle \right\}.$$

We point out the fact that the solution set of (10) is $S = \bigcap_{v\in K} N(v)$.

First we prove that for each $v \in K$ the set $N(v)$ is closed. Let $\{u_n\} \subset N(v)$ be a sequence which converges to u as $n \to \infty$. We show that $u \in N(v)$. Since j satisfies the conditions (2) and (4)-(5), by part (a) of Lemma 1 in [19] (pp. 44) the application

$$(u,v) \longmapsto \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx$$

is upper semicontinuous. Since T is linear and continuous, $\hat{u}_n \to \hat{u}$ and by the fact that $u_n \in N(v)$ for each n, we have

$$\begin{aligned}
\langle f, v-u\rangle &= \limsup_{n\to\infty} \langle f, v-u_n\rangle \leq \limsup_{n\to\infty} \left[\Lambda(u_n, v) + \int_\Omega j^{\,0}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))\, dx \right] \\
&\leq \limsup_{n\to\infty} \Lambda(u_n, v) + \limsup_{n\to\infty} \int_\Omega j^{\,0}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))\, dx \\
&\leq \Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx.
\end{aligned}$$

This is equivalent to $u \in N(v)$.

Arguing by contradiction, suppose that $S = \emptyset$. Then for each $u \in K$ there exists $v \in K$ such that

$$\Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle. \tag{12}$$

We define the set valued map $F : K \to 2^K$ by $u \longmapsto F(u)$ where

$$F(u) = \left\{ v \in K :\ \Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle \right\}.$$

Taking (12) into account we deduce that $F(u)$ is nonempty for each $u \in K$. Using the fact that Λ is convex with respect to the second variable, T is linear and the application $\hat{v} \longmapsto j^{\,0}(x, \hat{u}; \hat{v})$ is also convex, we obtain that $F(u)$ is a convex set.

Now, for each $v \in K$, the set

$$\begin{aligned}
F^{-1}(v) &= \{u \in K :\ v \in F(u)\} \\
&= \left\{ u \in K :\ \Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle \right\} \\
&= \left\{ u \in K :\ \Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle \right\}^c \\
&= [N(v)]^c = O_v
\end{aligned}$$

is open in K. We claim next that $\bigcup_{v\in K} O_v = K$. To prove that, let $u \in K$. As $F(u)$ is nonempty it follows that there exists $v \in F(u)$ which implies $u \in F^{-1}(v)$. Thus $K \subseteq \bigcup_{v\in K} O_v$, the converse inclusion being obvious.

Finally, from the last conditions of the theorem, for each $u \in K \setminus V_1$ there exists $v \in V_1$ such that

$$\Lambda(u,v) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle$$

which means that $u \notin N(v)$. This implies that the set $D = \bigcap_{v\in V_1} O_v^c = \bigcap_{v\in V_1} N(v) \subset V_1$ is empty or compact as a closed subset of the compact set V_1. Taking $V_0 = V_1$ we have proved that the set valued map F satisfies the conditions of Theorem 1, hence there exists $u_0 \in K$ such that $u_0 \in F(u_0)$, that is,

$$0 = \Lambda(u_0, u_0) + \int_\Omega j^{\,0}(x, \hat{u}_0(x); \hat{u}_0(x) - \hat{u}_0(x))\, dx < \langle f, u_0 - u_0\rangle = 0,$$

which is a contradiction. Hence the solution set S of problem (10) is nonempty. $\square$

In case K is compact, the last condition of Theorem 3 is automatically fulfilled, since we can set $V_1 = K$. Thus the following Corollary has been proved.

Corollary 2. *Let K be a compact convex subset of V and $f \in V^*$. Let $\Lambda : K \times K \to \mathbb{R}$ be a function which vanishes on the diagonal such that Λ is convex with respect to the second variable and upper semicontinuous with respect to the first variable. Then, if j satisfies (2) and (4)-(5) and $T : V \to L^p(\Omega; \mathbb{R}^k)$ is linear and continuous, the problem (10) has a solution in K.*

Remark 2. *If V is reflexive and the operator $T : V \to L^p(\Omega; \mathbb{R}^k)$ is linear and compact the conclusion of Theorem 3 still holds if Λ is weakly upper semicontinuous with respect to the first variable instead of being upper semicontinuous, because in these conditions, by part (b) of Lemma 1 in [19] the application*

$$(u, v) \longmapsto \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx$$

is weakly upper semicontinuous. The proof is identical to that of Theorem 3, but the conditions of Theorem 1 are satisfied in the weak topology.

Taking into account the above remark, we can state a variant of Corollary 2 for reflexive Banach spaces as follows:

Corollary 3. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space V and $f \in V^*$. Let $\Lambda : K \times K \to \mathbb{R}$ be a function which vanishes on the diagonal such that Λ is convex with respect to the second variable and weakly upper semicontinuous with respect to the first variable. Then, if j satisfies (2) and (4)-(5) and $T : V \to L^p(\Omega; \mathbb{R}^k)$ is linear and compact, the inequality problem (10) has a solution in K.*

In order to establish another result concerning the existence of solutions for the nonlinear hemivariational problem (10) we need the following result which is due to Mosco (see [9]):

Mosco's Theorem. Let K be a nonempty convex and compact subset of a topological vector space V. Let $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function such that $D(\Phi) \cap K \neq \emptyset$. Let $g, h : V \times V \to \mathbb{R}$ be two functions such that

(i) $g(x, y) \leq h(x, y)$, for every $x, y \in V$;

(ii) the mapping $h(\cdot, y)$ is concave, for any $y \in V$;

(iii) the mapping $g(x, \cdot)$ is lower semicontinuous, for every $x \in V$.

Let λ be an arbitrary real number. Then the following alternative holds: either

-there exists $y_0 \in D(\Phi) \cap K$ such that $g(x, y_0) + \Phi(y_0) - \Phi(x) \leq \lambda$, for any $x \in V$,

or

-there exists $x_0 \in V$ such that $h(x_0, x_0) > \lambda$.

We notice that a particular case of interest for the above result is if $\lambda = 0$ and $h(x, x) \leq 0$, for every $x \in V$.

Lemma 2. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space V and $f \in V^*$ be arbitrary but fixed. Consider a Banach space Y such that there exists a linear and compact mapping $L : V \to Y$ and let $J : Y \to \mathbb{R}$ be an arbitrary locally Lipschitz function. Suppose in addition that $\Lambda : V \times V \to \mathbb{R}$ is a function which satisfies the following conditions:*

(1) $\Lambda(u, u) = 0, \quad \forall\, u \in V$;

(2) $\Lambda(u,v)+\Lambda(v,u)\geq 0, \quad \forall\, u,v\in V$*;*

(3) Λ *is weakly upper semicontinuous and concave with respect to the first variable.*

Then it exists $u\in K$ *such that for all* $v\in K$ *we have*

$$\Lambda(u,v)+J^{\,0}(Lu;Lv-Lu)\geq\langle f,v-u\rangle.$$

Proof. Set

$$g(v,u)=-\Lambda(u,v)-\langle f,u-v\rangle-J^{\,0}(Lu;Lv-Lu)$$

and

$$h(v,u)=\Lambda(v,u)-\langle f,u-v\rangle-J^{\,0}(Lu;Lv-Lu).$$

By condition (2) we have

$$g(v,u)-h(v,u)=-[\Lambda(u,v)+\Lambda(v,u)]\leq 0, \quad \forall\, u,v\in V.$$

The mapping $u\longmapsto g(v,u)$ is weakly lower semicontinuous for each $v\in V$, while the mapping $v\longmapsto h(v,u)$ is concave for each $u\in V$. We shall apply Mosco's Theorem with $\lambda=0$ and $\Phi=I_K$, where I_K denotes the indicator function of the set K, that is $I_K(u)=0$ if $u\in K$ and $I_K(u)=\infty$ otherwise. Clearly I_K is proper, convex and lower semicontinuous since K is nonempty, convex and closed. We obtain that exists $u\in K$ satisfying

$$g(v,u)+I_K(u)-I_K(v)\leq 0, \quad \forall v\in V;$$

A simple computation yields that there exists $u\in K$ such that

$$\Lambda(u,v)+J^{\,0}(Lu;Lv-Lu)\geq\langle f,v-u\rangle, \quad \forall v\in K.$$

□

The second existence result concerning the nonlinear hemivariational inequality problem can now be stated as follows:

Theorem 4. *Assume that the hypotheses of Lemma 2 are fulfilled for* $Y=L^p(\Omega;\mathbb{R}^k)$*,* $L=T$ *and* $J:L^p(\Omega;\mathbb{R}^k)\to\mathbb{R}$

$$J(u)=\int_\Omega j(x,u(x))\,dx.$$

Then the inequality problem (10) has at least one solution.

3.2 Applications

3.2.1 Noncoercive hemivariational and variational-hemivariational inequalities

Let us consider the noncoercive forms of the coercive and semicoercive hemivariational inequalities investigated in [13] (pp. 65-77, 80-85). For this, let assume that V is a real Hilbert space with the property that

$$V \subset \left[L^2(\Omega;\mathbb{R}^k)\right]^N \subset V^*$$

and the injections are continuous and dense. Moreover, let $T : V \to L^2(\Omega;\mathbb{R}^k)$, $Tu = \hat{u}$, $\hat{u}(x) \in \mathbb{R}^k$ be linear and continuos. It is also assumed that $f \in V^*$ and $a : V \times V \to \mathbb{R}$ is a bilinear symmetric and continuous form. We consider the problem:

($\mathbf{P_1}$) Find $u \in K$ such that

$$a(u, v-u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle, \quad \forall\, v \in K.$$

We take $\Lambda : V \times V \to \mathbb{R}$ defined by $\Lambda(u,v) = a(u, v-u)$.

Case 1. If K is compact and convex it suffices to apply Corollary 2 to conclude that the problem ($\mathbf{P_1}$) has at least one solution. We point out that this result is more general from the point of view of the absence of the coercivity and semicoercivity assumption, but less general from the point of view of the compactness of K.

Case 2. If K is the entire space V (as in [13]) we can replace the coercivity and semicoercivity assumptions by the following assumption:
There exists a compact convex subset V_1 of V such that for each $u \in V \setminus V_1$ there is some $v \in V_1$ for which we have

$$a(u, v-u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u\rangle.$$

It suffices in this case to apply Theorem 3 to conclude that the problem ($\mathbf{P_1}$) admits at least one solution.

Let us consider now a functional $\Phi : V \to (-\infty, +\infty]$, which is convex, lower semicontinuous and proper and formulate the following variational-hemivariational problem:

($\mathbf{P_2}$) Find $u \in K$ such that

$$a(u, v-u) + \Phi(v) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle, \quad \forall\, v \in K.$$

Here V, a and j satisfy the same conditions as in the case of the hemivariational problem ($\mathbf{P_1}$).

Defining $\Lambda : V \times V \to \mathbb{R}$ by $\Lambda(u,v) = a(u, v-u) + \Phi(v) - \Phi(u)$ the conclusions from Case 1 and Case 2 still hold with the specification that the condition which replaces coercivity and semicoercivity assumptions, when we take $K = V$, becomes:

There exists a compact convex subset V_1 of V such that for each $u \in V \setminus V_1$ there is some $v \in V_1$ for which we have

$$a(u, v-u) + \Phi(v) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < \langle f, v-u \rangle.$$

3.2.2 Hartman-Stampacchia type hemivariational and variational-hemivariational inequalities

Let us consider we are in the framework of [19] where it is studied a hemivariational inequality of Hartman-Stampacchia type. Let V be a real Banach space and let $T : V \to L^p(\Omega; \mathbb{R}^k)$ be a linear continuous operator, where $1 \leq p < \infty$, $k \geq 1$. K is a subset of V, and $A : K \to V^*$ an operator while j satisfies conditions (2) and (4)-(5). We consider the problem:

($\mathbf{P_3}$) Find $u \in K$ such that

$$\langle Au, v-u \rangle + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0, \quad \forall\, v \in K.$$

If A is w^*-demicontinuous, that is, for any sequence $\{u_n\} \subset K$ converging to u, the sequence $\{Au_n\}$ converges to Au for the w^* topology of V^*, then accordingly to Remark 3 in [19]

$$\lim_{n\to\infty} \langle Au_n, u_n \rangle = \langle Au, u \rangle, \qquad \text{whenever } u_n \to u.$$

We define $\Lambda(u, v) = \langle Au, v-u \rangle$ and assume that A is w^*-demicontinuous (or equivalently the application $u \longmapsto \langle Au, v \rangle$ is weakly upper semicontinuos), then Λ satisfies conditions (1)-(2) of Theorem 3.

Case 1. Let K be a compact convex subset of the infinite dimensional Banach space V. Applying Corollary 2 we obtain that the problem ($\mathbf{P_3}$) has a solution.

Case 2. Weaking the hypotheses on K by assuming that K is a nonempty, closed and convex subset of the Banach space V we need an extra condition to "balance" the lack of compactness. Theorem 3 states that a sufficient condition such that ($\mathbf{P_3}$) has a solution is:
There exists a compact convex subset V_1 of K such that for each $u \in K$ there is some $v \in K$ for which we have

$$\langle Au, v-u \rangle + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < 0.$$

In [11] it is studied the following variational-hemivariational inequality problem:

($\mathbf{P_4}$) Find $u \in K$ such that

$$\langle Au - f, v-u \rangle + \Phi(v) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0, \quad \forall\, v \in K,$$

where $f \in V^*$, K is a subset of the real reflexive Banach space V, $A : V \to V^*$ is a nonlinear operator and $\Phi : V \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuos function. We also consider that $T : L^p(\Omega; \mathbb{R}^k) \to V$ is a linear and compact operator and $Tu = \hat{u}$.

If A is monotone and hemicontinuous, and K is nonempty, bounded, closed and convex we shall rediscover the result of Theorem 2 in [11] by means of Theorem 4. We take $\Lambda : V \times V \to \mathbb{R}$ to be $\Lambda(u,v) = \langle Av, v-u\rangle + \Phi(v) - \Phi(u)$. It is easy to observe that $\Lambda(v,u) = -\langle Au, v-u\rangle + \Phi(u) - \Phi(v)$. Then by the monotonicity of A we have

$$\Lambda(u,v) + \Lambda(v,u) = \langle Av - Au, v-u\rangle \geq 0.$$

Clearly the application Λ defined as above is concave and weakly upper semicontinuous with respect to the first variable. Thus by Theorem 4 there exists $u \in K$ such that

$$\langle Aw, w-u\rangle + \Phi(w) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{w}(x) - \hat{u}(x))\, dx \geq \langle f, w-u\rangle, \quad \forall\, w \in K. \tag{13}$$

We fix $v \in K$ and set $w = tv + (1-t)u \in K$, for $t \in (0,1)$. So, by (13),

$$t\langle A(tv+(1-t)u), v-u\rangle + \Phi(tv+(1-t)u) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); t(\hat{v}(x) - \hat{u}(x)))\, dx \geq t\langle f, v-u\rangle.$$

Using the convexity of Φ, the fact that $j^{\,0}(x,u;\cdot)$ is positive homogeneous and dividing by t we find

$$\langle A(tv+(1-t)u), v-u\rangle + \Phi(v) - \Phi(u) + \int_\Omega j^{\,0}(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq \langle f, v-u\rangle.$$

Now, letting $t \to 0$ and using the hemicontinuity of A we find that u is a solution of (P_4).

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