Received: 11 February 2023 | Revised: 22 November 2023 | Accepted: 9 December 2023

DOI: 10.1112/blms.12991

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Nonautonomous double-phase equations with strong singularity and concave perturbation

Nikolaos S. Papageorgiou[1,2] | **Vicenţiu D. Rădulescu**[2,3,4,5,6] | **Shuai Yuan**[2,7]

[1]Department of Mathematics, National Technical University, Zografou Campus, Athens, Greece

[2]Department of Mathematics, University of Craiova, Craiova, Romania

[3]Faculty of Applied Mathematics, AGH University of Kraków, Kraków, Poland

[4]Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania

[5]Brno University of Technology, Faculty of Electrical Engineering and Communication, Brno, Czech Republic

[6]Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, China

[7]School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, P.R. China

Correspondence
Shuai Yuan, School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, P.R. China.
Email: ys950526@163.com

Funding information
Romanian Ministry of Research, Innovation and Digitization, Grant/Award Number: 22; China Scholarship Council, Grant/Award Number: 202106370097

Abstract

We consider a nonlinear Dirichlet problem driven by a nonautonomous double-phase differential operator and with a reaction consisting of a "strongly" singular term plus a concave perturbation. Using the Nehari method, we show the existence of a bounded strictly positive solution.

MSC 2020
35J20 (primary), 35J75, 35R01, 46E30, 58J32 (secondary)

1 | INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper, we study the following singular double-phase Dirichlet problem:

$$\begin{cases} -\Delta_p^\alpha u(z) - \Delta_q u(z) = \vartheta(z) u(z)^{-\eta} + \beta(z) u(z)^{\tau-1} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ 1 < \tau < q < p < N,\ 1 < \eta,\ \ u > 0. \end{cases} \tag{1}$$

For $\alpha \in L^\infty(\Omega) \setminus \{0\}$ with $\alpha(z) \geqslant 0$ for a.a. $z \in \Omega$ and for $r \in (1,\infty)$, we denote by Δ_r^α the weighted r-Laplace differential operator defined by

$$\Delta_r^\alpha u = \operatorname{div}\big(\alpha(z)|Du|^{r-2}Du\big).$$

Note that when $\alpha(z) \equiv 1$ for all $z \in \Omega$, then we have the standard r-Laplace differential operator denoted by Δ_r.

The interest in the study of this type of problem is twofold. On the one hand, there are physical motivations, since the double-phase operator has been applied to describe steady-state solutions of reaction–diffusion problems in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^a u(z) + \Delta_q u + g(x,u).$$

In this framework, the function u generally stands for a concentration, and the term $\Delta_p^a u(z) + \Delta_q u$ corresponds to the diffusion with coefficient $a(z)|Du|^{p-2} + |Du|^{q-2}$, whereas $g(x,u)$ represents the reaction term related to source and loss processes; see Cherfils–Il'yasov [5] and Singer [21]. On the other hand, such operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point, where the modulating coefficient $a(z)$ dictates the geometry of a composite made by two different materials.

In problem (1), the equation is driven by the sum of two such operators with different exponents. So, the differential operator (left-hand side) of problem (1) is not homogeneous. In the operator Δ_p^α, we do not assume that the weight function $\alpha(\cdot)$ is bounded away from zero, that is, we do not have that $\operatorname{ess\,inf}_\Omega \alpha > 0$. So, the integrand

$$\mu(z,x) = \alpha(z)x^p + x^q \text{ for all } z \in \Omega \text{ and for all } x \geqslant 0$$

associated with the energy functional of the differential operator, exhibits unbalanced growth with respect to the x-variable, namely, we have

$$x^q \leqslant \mu(z,x) \leqslant c_0[1 + x^p] \ \text{ for } \ a.a.\ z \in \Omega, \text{ all } x \geqslant 0, \text{ some } c_0 > 0.$$

Such integral functionals arise in the context of problems of mathematical physics (elasticity theory, fluid dynamics) and of the calculus of variations and were first investigated by Zhikov [24] and Marcellini [10, 11]. Recently, the interest for such functionals was revived with emphasis on the regularity properties of minimizers. We refer to the works of Baroni–Colombo–Mingione [1] and Marcellini [12, 13] and the references therein. We also mention the survey papers of Mingione–Rădulescu [14], Papageorgiou [15], and Rădulescu [20].

The double-phase problem (1) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born–Infeld equation [2] that appears in electromagnetism:

$$-\operatorname{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{1/2}}\right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!2^{n-1}}x^{n-1} + \cdots \ \text{for}\,|x| < 1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first-order approximation, we obtain problem (1) for $p = 4$ and $q = 2$. Furthermore, the nth-order approximation problem is driven by the multiphase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator:

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{3/4}}\,\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-3/4} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots.$$

This shows that the fourth-order relativistic operator can be approximated by the following autonomous double-phase operator:

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

In the reaction (right-hand side) of (1), we have the combined effects of two terms of different nature. One term is the singular function $u \mapsto \vartheta(z)u^{-\eta}$ and the other term is the $(q-1)$-sublinear perturbation $u \mapsto \beta(z)u^{\tau-1}(1<\tau<q)$. The singular term has two special features, which distinguish our work here from earlier ones on the subject. The first special feature is that the exponent $\eta > 1$. This means that the problem has a "strong" singularity. Such problems are more difficult to deal with compared to the so-called "weakly" singular problems in which $0<\eta<1$ and lead to regular solutions. In the context of purely singular equations (i.e., there is no perturbation term) driven by the Laplacian, Lazer–McKenna [9] proved that, if $\vartheta \in C^{\alpha}(\bar{\Omega}), 0<\alpha<1$, $\vartheta(z)>0$ for all $z \in \bar{\Omega}$ and $1<\eta$, then the problem $-\Delta u(z) = \vartheta(z)u(z)^{-\eta}$ in Ω, $u|_{\partial\Omega} = 0$ has a unique solution that is not in $C_0^1(\bar{\Omega})$ and it belongs to $H_0^1(\Omega)$ if and only if $\eta < 3$. So, for strongly singular problems, even in the semilinear case, we do not have regularity of the solutions and this leads to substantial difficulties in the analysis of the problem. The second special feature of the singular term is that the coefficient function $\vartheta(\cdot)$ need not be bounded. We have $\vartheta \in L^1(\Omega)$ and $\vartheta(z) > 0$ for a.a. $z \in \Omega$. In the perturbation term $u \mapsto \beta(z)u^{\tau-1}$, we have $\beta \in L^\infty(\Omega)$, $\beta(z) \geqslant 0$ for a.a. $z \in \Omega$ and $1<\tau<q$. So, the perturbation is strictly $(q-1)$-sublinear as $x \to +\infty$ (concave perturbation). Such strongly singular elliptic boundary value problems were studied by Sun [22] (semilinear problems driven by the Laplace differential operator) and by Sun–Tan [23] (semilinear Kirchhoff-type equations). Strongly singular double-phase equations with a $(p-1)$-superlinear perturbation were studied recently by Papageorgiou–Rădulescu–Zhang [17]. No works exist with

a $(p-1)$-linear perturbation. The recent work of Papageorgiou–Pudełko–Rădulescu [16] can be helpful in this direction.

Using an approach based on the Nehari method (see Brown–Wu [3] and Brown–Zhang [4]), we show that problem (1) admits a bounded positive solution $\hat{u}(z) > 0$ for a.a. $z \in \Omega$.

2 | MATHEMATICAL BACKGROUND AND HYPOTHESES

The unbalanced growth of the integrand $\mu(z,\cdot)$ implies that the standard Sobolev spaces are not enough to study (1) and we need to consider generalized Orlicz–Sobolev spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto–Hästö [7].

Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identify two such functions that differ on a Lebesgue-null set. Let $\alpha \in C^{0,1}(\bar{\Omega}) \setminus \{0\}$ with $\alpha(z) \geqslant 0$ for a.a. $z \in \Omega$, $1 < q < p < N$ and $\frac{p}{q} < 1 + \frac{1}{N}$. The last inequality is standard in Dirichlet, unbalanced double-phase problems and says that the exponents p, q cannot be far apart and $p < q^* = \frac{Nq}{N-q}$ that leads to useful embeddings of the relevant spaces. With $\mu(z,x) = \alpha(z)x^p + x^q$ for all $(z,x) \in \Omega \times \mathbb{R}_+$, the generalized Orlicz space $L^\mu(\Omega)$ is defined by

$$L^\mu(\Omega) = \left\{u \in L^0(\Omega) : \rho_\mu(u) < \infty\right\},$$

with $\rho_\mu(\cdot)$ being the modular function

$$\rho_\mu(u) = \int_\Omega \mu(z,|u|)\mathrm{d}z = \int_\Omega [\alpha(z)|u|^p + |u|^q]\mathrm{d}z.$$

The functional is continuous, convex, and so, it is also weakly lower semicontinuous. We equip $L^\mu(\Omega)$ with the so-called Luxemburg norm

$$\|u\|_\mu = \inf\left\{\lambda > 0 : \rho_\mu\left(\frac{u}{\lambda}\right) \leqslant 1\right\}.$$

Then, $L^\mu(\Omega)$ becomes a separable and uniformly convex Banach space (in particular, then $L^\mu(\Omega)$ is reflexive by the Milman–Pettis theorem, see Papageorgiou–Winkert [19], p. 225).

Next, we can define the generalized Orlicz–Sobolev space $W^{1,\mu}(\Omega)$ by setting

$$W^{1,\mu}(\Omega) = \{u \in L^\mu(\Omega) : |Du| \in L^\mu(\Omega)\}$$

where Du denotes the weak gradient of u. We equip this space with the following norm:

$$\|u\|_{1,\mu} = \|u\|_\mu + \|Du\|_\mu,$$

with $\|Du\|_\mu = \||Du|\|_\mu$.

Also, we set $W_0^{1,\mu}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mu}}$.

For this space the Poincare inequality holds (see [7, p. 138]), and so on, $W_0^{1,\mu}(\Omega)$, we can consider the following equivalent norm:

$$\|u\| = \|Du\|_\mu \text{ for all } u \in W_0^{1,\mu}(\Omega).$$

Both spaces $W^{1,\mu}(\Omega)$ and $W_0^{1,\mu}(\Omega)$ are separable and uniformly convex (thus reflexive). Moreover, the following embedding results hold.

Proposition 1.

(a) *The embeddings* $L^\mu(\Omega) \hookrightarrow L^s(\Omega)$, $W_0^{1,\mu}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ *are continuous for all* $s \in [1, q]$.
(b) *The embedding* $W_0^{1,\mu}(\Omega) \hookrightarrow L^s(\Omega)$ *is continuous if* $s \in [1, q^*]$ *and compact if* $s \in [1, q^*)$.
(c) *The embedding* $L^p(\Omega) \hookrightarrow L^\mu(\Omega)$ *is continuous.*

There is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_\mu(\cdot)$. Let $u \in W_0^{1,\mu}(\Omega)$. Then:

Proposition 2.

(a) $\|u\| = t \Leftrightarrow \rho_\mu(\frac{Du}{t}) = 1$;
(b) $\|u\| < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_\mu(Du) < 1$ $(resp. = 1, > 1)$;
(c) $\|u\| < 1 \Rightarrow \|u\|^p \leqslant \rho_\mu(Du) \leqslant \|u\|^q$;
(d) $\|u\| > 1 \Rightarrow \|u\|^q \leqslant \rho_\mu(Du) \leqslant \|u\|^p$;
(e) $\|u\| \to 0$ $(resp. \to \infty) \Leftrightarrow \rho_\mu(Du) \to 0$ $(resp. \to \infty)$.

We will also use another modular function $\rho_\alpha(\cdot)$ defined by

$$\rho_\alpha(Du) = \int_\Omega \alpha(z)|Du|^p \mathrm{d}z.$$

This too is continuous convex on $W_0^{1,\mu}(\Omega)$, hence weakly lower semicontinuous.
Let $A_p^\alpha, A_q : W_0^{1,\mu}(\Omega) \to (W_0^{1,\mu}(\Omega))^*$ be the nonlinear operators defined by

$$\langle A_p^\alpha(u), h\rangle = \int_\Omega \alpha(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z,$$

$$\langle A_q(u), h\rangle = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \text{ for all } u, h \in W_0^{1,\mu}(\Omega).$$

These are bounded, continuous, and strictly monotone (thus maximal monotone too) operators. We set $V(u) = A_p^\alpha(u) + A_q(u)$ for all $u \in W_0^{1,\mu}(\Omega)$. Evidently, $V(\cdot)$ is bounded, continuous, strictly monotone (thus maximal monotone too). If $\hat{\rho}(u) = \frac{1}{p}\rho_\alpha(Du) + \frac{1}{q}\|Du\|_q^q$ for all $u \in W_0^{1,\mu}(\Omega)$, then $\langle \hat{\rho}'(u), h\rangle = \langle V(u), h\rangle$ for all $u, h \in W_0^{1,\mu}(\Omega)$.
Our hypotheses on the data of problem (1) are the following:

H:
(i) $\alpha \in C^{0,1}(\bar{\Omega}) \setminus \{0\}, \alpha(z) \geqslant 0$ for all $z \in \bar{\Omega}$, $1 < \tau < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$;

(ii) $\vartheta \in L^1(\Omega), \vartheta(z) > 0$ for a.a. $z \in \Omega$ and there exists $\bar{u} \in W_0^{1,\mu}(\Omega)$ such that $\int_\Omega \vartheta(z)|\bar{u}|^{1-\eta}\mathrm{d}z < \infty$;

(iii) $\beta \in L^\infty(\Omega) \setminus \{0\}, \beta(z) \geqslant 0$ for a.a. $z \in \Omega$.

3 | POSITIVE SOLUTIONS

As we already mentioned in the Introduction, our approach is based on the Nehari method. So, we introduce the following two sets:

$$\mathcal{N} = \left\{ u \in W_0^{1,\mu}(\Omega) : \rho_\mu(Du) = \int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z + \int_\Omega \beta(z)|u|^\tau \mathrm{d}z \right\},$$

$$\mathcal{M} = \left\{ u \in W_0^{1,\mu}(\Omega) : \rho_\mu(Du) \geqslant \int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z + \int_\Omega \beta(z)|u|^\tau \mathrm{d}z \right\}.$$

Note that $\mathcal{N}$ is the Nehari manifold for the problem and $\mathcal{N} \subseteq \mathcal{M}$. Moreover, let $\varphi : W_0^{1,\mu}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi(u) = \frac{1}{p}\rho_\alpha(Du) + \frac{1}{q}\|Du\|_q^q + \frac{1}{\eta - 1}\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z - \frac{1}{\tau}\int_\Omega \beta(z)|u|^\tau \mathrm{d}z,$$
$$\text{for all } u \in W_0^{1,\mu}(\Omega).$$

Proposition 3. *If hypotheses H hold and $u \in W_0^{1,\mu}(\Omega)$ satisfies*

$$\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z < \infty,$$

then there exists a unique $t_0 > 0$ such that

$$t_0 u \in \mathcal{N},\, tu \in \mathcal{M} \text{ for all } t \geqslant t_0,$$

$$\varphi(t_0 u) \leqslant \varphi(tu) \text{ for all } t \geqslant 0.$$

Proof. We consider the fibering function $k_u(\cdot)$ corresponding to u defined by

$$k_u(t) = \frac{t^p}{p}\rho_\alpha(Du) + \frac{t^q}{q}\|Du\|_q^q + \frac{t^{1-\eta}}{\eta - 1}\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z - \frac{t^\tau}{\tau}\int_\Omega \beta(z)|u|^\tau \mathrm{d}z$$
$$\text{for all } t > 0.$$

Since $1 < \tau < q < p$ and $1 < \eta$, we see that

$$k_u(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

So, we can find $t_0 > 0$ such that

$$k_u(t_0) = \min_{t>0} k_u(t). \tag{2}$$

Clearly, $k_u \in C^1(0, \infty)$, and so, we have

$$\begin{aligned} & k_u'(t_0) = 0, \\ \Rightarrow\ & t_0^{p-1}\rho_\alpha(Du) + t_0^{q-1}\|Du\|_q^q - t_0^{-\eta}\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z - t_0^{\tau-1}\int_\Omega \beta(z)|u|^\tau \mathrm{d}z = 0 \\ \Rightarrow\ & t_0 u \in \mathcal{N}. \end{aligned}$$

Note that if $k_u'(t) = 0$, then

$$t^{p-\tau}\rho_\alpha(Du) + t^{q-\tau}\|Du\|_q^q - \frac{1}{t^{\eta+\tau-1}}\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z = \int_\Omega \beta(z)|u|^\tau \mathrm{d}z.$$

In this equation, the left-hand side is strictly increasing as a function of $t > 0$, whereas the right-hand side is independent of $t > 0$ (constant). So, it follows that $t_0 > 0$ must be unique. Moreover, we have $tu \in \mathcal{M}$ for all $t \geqslant t_0$. Finally, from (2), we infer that

$$\varphi(t_0 u) \leqslant \varphi(tu)$$

for all $t \geqslant 0$. □

Next, we show that the set $\mathcal{M}$ is bounded away from the origin (hence so does the Nehari manifold since $\mathcal{N} \subseteq \mathcal{M}$).

Proposition 4. *If hypotheses H hold, then there exists $\rho > 0$ such that $\|u\| \geqslant \rho$ for all $u \in \mathcal{M}$.*

Proof. We argue by contradiction. So, suppose that we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that

$$u_n \to 0 \text{ in } W_0^{1,\mu}(\Omega). \tag{3}$$

Since $\eta > 1$, we have $L^1(\Omega) \hookrightarrow L^{1/\eta}(\Omega)$ continuously (see Hewitt–Stromberg [8, p. 196]). Therefore, $\vartheta(\cdot)^{1/\eta} \in L^\eta(\Omega)$ and for the conjugate exponent of $\frac{1}{\eta} \in (0,1)$, we have

$$\left(\frac{1}{\eta}\right)' = \frac{\frac{1}{\eta}}{\frac{1}{\eta} - 1} = \frac{1}{1-\eta} < 0. \tag{4}$$

Applying Hölder's inequality for Lebesgue spaces with exponent in (0,1) (see Hewitt–Stromberg [8, p. 191]), we have

$$\left[\int_\Omega \vartheta(z)^{1/\eta}\mathrm{d}z\right]^\eta \left[\int_\Omega |u_n|\mathrm{d}z\right]^{1-\eta} \leqslant \int_\Omega \vartheta(z)|u_n|^{1-\eta}\mathrm{d}z \tag{5}$$

for all $n \in \mathbb{N}$ (see (4)).

Since $u_n \in \mathcal{M}$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_\Omega \vartheta(z)|u_n|^{1-\eta}\mathrm{d}z \leqslant \rho_\mu(Du_n) - \int_\Omega \beta(z)|u_n|^\tau \mathrm{d}z \text{ for all } n \in \mathbb{N}, \\
\Rightarrow &\int_\Omega \vartheta(z)|u_n|^{1-\eta}\mathrm{d}z \to 0 \text{ as } n \to \infty \text{ (see (3))}, \\
\Rightarrow &\left(\int_\Omega |u_n|\mathrm{d}z\right)^{1-\eta} \to 0 \text{ as } n \to \infty \text{ (see (5))}. \qquad (6)
\end{aligned}$$

Since $\eta > 1$, from (6), we infer that

$$\int_\Omega |u_n|\mathrm{d}z \to +\infty \text{ as } n \to \infty. \qquad (7)$$

Moreover, by virtue of Proposition 1, we know that $W_0^{1,\mu}(\Omega) \hookrightarrow L^1(\Omega)$ continuously, and from (3), we have

$$\int_\Omega |u_n|\mathrm{d}z \to 0 \text{ as } n \to \infty. \qquad (8)$$

Comparing (7) and (8), we have a contradiction. This proves that there exists $\rho > 0$ such that $\|u\| \geqslant \rho$ for all $u \in \mathcal{M}$. □

Proposition 5. *If hypotheses H hold, then $\varphi(\cdot)$ is coercive.*

Proof. Let $u \in W_0^{1,\mu}(\Omega)$ and without any loss of generality, we may assume that $\|u\| > 1$. We have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\rho_\alpha(Du) + \frac{1}{q}\|Du\|_q^q + \frac{1}{\eta - 1}\int_\Omega \vartheta(z)|u|^{1-\eta}\mathrm{d}z - \frac{1}{\tau}\int_\Omega \beta(z)|u|^\tau \mathrm{d}z \\
&\geqslant \frac{1}{p}\rho_\mu(Du) - \frac{1}{\tau}\int_\Omega \beta(z)|u|^\tau \mathrm{d}z \\
&\geqslant \frac{1}{p}\|u\|^p - c_1\|u\|^\tau \text{ for some } c_1 > 0. \quad \text{(see Propositions 1 and 2).} \qquad (9)
\end{aligned}$$

Since $1 < \tau < q < p$, from (9), it follows that $\varphi(\cdot)$ is coercive. □

We consider the following minimization problem:

$$\hat{m} = \inf\left[\varphi(u) : u \in \mathcal{M}\right].$$

Proposition 6. *If hypotheses H hold, then there exists $\hat{u} \in \mathcal{N} \cap L^\infty(\Omega)$ such that $\varphi(u) = \hat{m}$ and $\hat{u}(z) > 0$ for a.a. $z \in \Omega$.*

Proof. By the Ekeland variational principle (see Papageorgiou–Winkert [19, p. 564]), we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that

$$\varphi(u_n) \downarrow \hat{m} \text{ and } \varphi(u_n) \leqslant \varphi(y) + \frac{1}{n}\|y - u_n\| \text{ for all } y \in \mathcal{M}, \text{ all } n \in \mathbb{N}. \tag{10}$$

On account of Proposition 5, we have that $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mu}(\Omega)$ is bounded. So, using Proposition 1, we may assume that

$$u_n \xrightarrow{w} \hat{u} \text{ in } W_0^{1,\mu}(\Omega),\ u_n \to \hat{u} \text{ in } L^\tau(\Omega). \tag{11}$$

We will show that

$$u_n \to \hat{u} \text{ in } W_0^{1,\mu}(\Omega). \tag{12}$$

If (12) is not true, then at least one of the following strict inequalities holds:

$$\rho_\alpha(D\hat{u}) < \liminf_{n\to\infty} \rho_\alpha(Du_n),\ \|D\hat{u}\|_q^q < \liminf_{n\to\infty} \|Du_n\|_q^q. \tag{13}$$

Since $u_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, we have

$$\int_\Omega \vartheta(z)|u_n|^{1-\eta}\mathrm{d}z \leqslant \rho_\mu(Du_n) - \int_\Omega \beta(z)|u_n|^\tau \mathrm{d}z \leqslant c_2$$
$$\text{for some } c_2 > 0 \text{ and for all } n \in \mathbb{N} \text{ (see (11))}.$$

Then from (11) and Fatou's lemma, we obtain

$$\int_\Omega \vartheta(z)|\hat{u}|^{1-\eta}\mathrm{d}z \leqslant c_2,$$
$$\Rightarrow |\hat{u}(z)| > 0 \text{ for a.a. } z \in \Omega \text{ (recall } 1 < \eta).$$

Note that

$$\begin{aligned}
\hat{m} &= \lim_{n\to\infty} \varphi(u_n) \text{ (see (10))} \\
&= \liminf_{n\to\infty} \left[\frac{1}{p}\rho_\alpha(Du_n) + \frac{1}{q}\|Du_n\|_q^q + \frac{1}{\eta-1}\int_\Omega \vartheta(z)|u_n|^{1-\eta}\mathrm{d}z - \frac{1}{\tau}\int_\Omega \beta(z)|u_n|^\tau \mathrm{d}z\right] \\
&> \frac{1}{p}\rho_\alpha(D\hat{u}) + \frac{1}{q}\|D\hat{u}\|_q^q + \frac{1}{\eta-1}\int_\Omega \vartheta(z)|\hat{u}|^{1-\eta}\mathrm{d}z - \frac{1}{\tau}\int_\Omega \beta(z)|\hat{u}|^\tau \mathrm{d}z \\
&\qquad \text{(using Fatou's lemma and (11), (13))} \\
&= \varphi(\hat{u}) \\
&\geqslant \varphi(t_0\hat{u}) \text{ where } t_0 > 0 \text{ is such that } t_0\hat{u} \in \mathcal{N} \text{ (see Proposition 3)} \\
&\geqslant \inf_{\mathcal{N}} \varphi \geqslant \inf_{\mathcal{M}} \varphi = \hat{m} \text{ (recall } \mathcal{N} \subseteq \mathcal{M}),
\end{aligned} \tag{14}$$

which is a contradiction. Therefore, (12) is true and it implies that

$$\rho_\mu(Du_n) \to \rho_\mu(D\hat{u}), \int_\Omega \beta(z)|u_n|^\tau \mathrm{d}z \to \int_\Omega \beta(z)|\hat{u}|^\tau \mathrm{d}z. \tag{15}$$

Next, let us prove that $\hat{u} \in \mathcal{N}$. We distinguish two cases.

Case 1: $\{u_n\}_{n\geqslant n_0} \subseteq \mathcal{M} \setminus \mathcal{N}$ with $n_0 \in \mathbb{N}$.

Note that $\varphi(u) = \varphi(|u|)$ for all $u \in W_0^{1,\mu}(\Omega)$, and so, we may assume that $u_n \geqslant 0$ for all $n \geqslant n_0$. Since $u_n \in \mathcal{M} \setminus \mathcal{N}$, for all $h \in W_0^{1,\mu}(\Omega)$ with $h \geqslant 0$ and all $t > 0$, we have

$$\begin{aligned} &\int_\Omega \vartheta(z)[u_n + th]^{1-\eta}\mathrm{d}z \\ &\leqslant \int_\Omega \vartheta(z)u_n^{1-\eta}\mathrm{d}z < \rho_\mu(Du_n) - \int_\Omega \beta(z)u_n^\tau \mathrm{d}z \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Hence, for $t \in (0, 1)$ small, we have

$$\begin{aligned} &\int_\Omega \vartheta(z)[u_n + th]^{1-\eta}\mathrm{d}z \leqslant \rho_\mu(D(u_n + th)) - \int_\Omega \beta(z)[u_n + th]^\tau \mathrm{d}z, \\ &\Rightarrow u_n + th \in \mathcal{M} \text{ for } t \in (0, 1) \text{ small.} \end{aligned}$$

So, if in (10), we choose $y = u_n + th \in \mathcal{M}$, we obtain

$$\begin{aligned} &\frac{1}{1-\eta}\int_\Omega \vartheta(z)\left[(u_n + th)^{1-\eta} - u_n^{1-\eta}\right]\mathrm{d}z \\ &\leqslant \hat{\rho}(u_n + th) - \hat{\rho}(u_n) - \frac{1}{\tau}\int_\Omega \beta(z)\left[(u_n + th)^\tau - u_n^\tau\right]\mathrm{d}z + \frac{t}{n}\|h\| \\ &\qquad \text{for all } n \in \mathbb{N}, \text{ for } t \in (0, 1) \text{ small.} \end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0^+$, by Fatou's lemma, we have

$$\begin{aligned} &\int_\Omega \vartheta(z)u_n^{-\eta}h\mathrm{d}z \\ &= \int_\Omega \vartheta(z)\lim_{t\to 0^+}\left(\frac{(u_n + th)^{1-\eta} - u_n^{1-\eta}}{t(1-\eta)}\right)\mathrm{d}z \\ &\leqslant \underset{t\to 0^+}{l}\int_\Omega \vartheta(z)\left(\frac{(u_n + th)^{1-\eta} - u_n^{1-\eta}}{t(1-\eta)}\right)\mathrm{d}z \\ &\leqslant \langle V(u_n), h\rangle - \int_\Omega \beta(z)u_n^{\tau-1}h\mathrm{d}z \\ &\quad \text{for all } n \in \mathbb{N}\text{and for all } h \in W_0^{1,\mu}(\Omega) \text{ with } h \geq 0. \end{aligned}$$

We pass to the limit as $n \to \infty$ and use (12) and Fatou's lemma. We obtain

$$\begin{aligned} &\int_\Omega \vartheta(z)\hat{u}^{-\eta}h\mathrm{d}z \leqslant \langle V(\hat{u}), h\rangle - \int_\Omega \beta(z)\hat{u}^{\tau-1}h\mathrm{d}z \\ &\qquad \text{for all } h \in W_0^{1,\mu}(\Omega) \text{ with } h \geqslant 0. \end{aligned}$$

Choosing $h = \hat{u} \in W_0^{1,\mu}(\Omega)$, $\hat{u} \geqslant 0$, we obtain

$$\int_\Omega \vartheta(z)\hat{u}^{1-\eta}\mathrm{d}z \leqslant \rho_\mu(D\hat{u}) - \int_\Omega \beta(z)\hat{u}^\tau \mathrm{d}z,$$

$$\Rightarrow \hat{u} \in \mathcal{M}.$$

From (14), we have

$$\hat{m} = \varphi(\hat{u}) = \varphi(t_0\hat{u})$$

$$\Rightarrow t_0 = 1 \text{ and so } \hat{u} \in \mathcal{N}.$$

Case 2: At least for a subsequence, we have $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N}$.
For $t > 0$ and $h \in W_0^{1,\mu}(\Omega)$, $h \geqslant 0$, we have

$$\begin{aligned} &\int_\Omega \vartheta(z)(u_n + th)^{1-\eta}\mathrm{d}z \\ &\leqslant \int_\Omega \vartheta(z)u_n^{1-\eta}\mathrm{d}z \text{ (recall } \eta > 1) \\ &= \rho_\mu(Du_n) - \int_\Omega \beta(z)u_n^\tau \mathrm{d}z \text{ (since } u_n \in \mathcal{N}) \\ &\leqslant c_3 \text{ for some } c_3 > 0 \text{ and for all } n \in \mathbb{N}. \end{aligned}$$

Proposition 3 implies that there exists a unique $\hat{\mu}_n(t) > 0$ such that

$$\hat{\mu}_n(t)[u_n + th] \in \mathcal{N}. \tag{16}$$

From the definition of the Nehari manifold and using the Lebesgue dominated convergence theorem, we have that

$$\hat{\mu}_n(\cdot) \text{ is continuous.} \tag{17}$$

Moreover, since by hypothesis $u_n \in \mathcal{N}$ for all $n \in \mathbb{N}$, we see that

$$\hat{\mu}_n(0) = 1 \text{ for all } n \in \mathbb{N}. \tag{18}$$

Since $\hat{\mu}_n(t)[u_n + th] \in \mathcal{N}$ (see (16)), from the definition of the Nehari manifold, we have

$$\begin{aligned} 0 = {} &\hat{\mu}_n(t)^p\rho_\alpha(D(u_n + th)) + \hat{\mu}_n(t)^q\|D(u_n + th)\|_q^q \\ &- \hat{\mu}_n(t)^{1-\eta}\int_\Omega \vartheta(z)[u_n + th]^{1-\eta}\mathrm{d}z - \hat{\mu}_n(t)^\tau \int_\Omega \beta(z)(u_n + th)^\tau \mathrm{d}z \end{aligned} \tag{19}$$

In addition, since $u_n \in \mathcal{N}$ for all $n \in \mathbb{N}$, we have

$$0 = \rho_\mu(Du_n) - \int_\Omega \vartheta(z) u_n^{1-\eta} \mathrm{d}z - \int_\Omega \beta(z) u_n^\tau \mathrm{d}z. \tag{20}$$

From (19) and (20), and since $\hat{\mu}_n(0) = 1$ for all $n \in \mathbb{N}$ (see (18)) and $\hat{\mu}_n(\cdot)$ is continuous for all $n \in \mathbb{N}$ (see (17)), we have that

$$\begin{aligned} 0 \geqslant & \Big[\, p(1+o(1))^{p-1} \rho_\alpha(D(u_n+th)) + q(1+o(1))^{q-1} \|D(u_n+th)\|_q^q \\ & - (1-\eta)(1+o(1))^{-\eta} \int_\Omega \vartheta(z)(u_n+th)^{1-\eta} \mathrm{d}z \\ & - \tau(1+o(1))^{\tau-1} \int_\Omega \beta(z)(u_n+th)^\tau \mathrm{d}z \Big] (\hat{\mu}_n(t) - 1) \\ & + \hat{\rho}(u_n+th) - \hat{\rho}(u_n) \\ & - \int_\Omega \vartheta(z)\Big[(u_n+th)^{1-\eta} - u_n^{1-\eta}\Big] \mathrm{d}z - \int_\Omega \beta(z)\big[(u_n+th)^\tau - u_n^\tau\big] \mathrm{d}z \\ & \qquad \text{with } o(1) \to 0 \text{ as } t \to 0^+. \end{aligned} \tag{21}$$

In what follows we set

$$\gamma_n^* = \lim_{t \to 0^+} \tfrac{\hat{\mu}_n(t) - 1}{t} \in \mathbb{R} \cup \{\pm\infty\},$$

if this limit exists and if it does not exist, we simply take a sequence $t_m \to 0^+$ and define

$$\gamma_n^* = \lim_{m \to \infty} \frac{\hat{\mu}_n(t_m) - 1}{t_m} \in \mathbb{R} \cup \{\pm\infty\}.$$

So, if we divide (21) by $t > 0$ and let $t \to 0^+$, then we obtain

$$\begin{aligned} 0 \geqslant & \left[p\rho_\alpha(Du_n) + q\|Du_n\|_q^q + (\eta - 1) \int_\Omega \vartheta(z) u_n^{1-\eta} \mathrm{d}z - \tau \int_\Omega \beta(z) u_n^\tau \mathrm{d}z \right] \gamma_n^* \\ & + \langle V(u_n), h \rangle - \tau \int_\Omega \beta(z) u_n^{\tau-1} h \mathrm{d}z \text{ (since } 1 < \eta). \end{aligned} \tag{22}$$

Note that

$$\begin{aligned} & p\rho_\alpha(Du_n) + q\|Du_n\|_q^q \\ & \geqslant q\big[\rho_\alpha(Du_n) + \|Du_n\|_q^q\big] \\ & = q\rho_\mu(Du_n) \\ & = q\left[\int_\Omega \vartheta(z) u_n^{1-\eta} \mathrm{d}z + \int_\Omega \beta(z) u_n^\tau \mathrm{d}z\right] \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{23}$$

Also since $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mu}(\Omega)$ is bounded, we have

$$\langle V(u_n), h\rangle - \tau \int_\Omega \beta(z)u_n^\tau \mathrm{d}z \geqslant -c_4\|h\| \text{ for some } c_4 > 0 \text{ and for all } n \in \mathbb{N}. \tag{24}$$

Returning to (22) and using (23) and (24), we have

$$0 \geqslant (q-\tau)\left[\int_\Omega \vartheta(z)u_n^{1-\eta}\mathrm{d}z + \int_\Omega \beta(z)u_n^\tau \mathrm{d}z\right]\gamma_n^* - c_4\|h\|. \tag{25}$$

From Proposition 4, we know that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{N} \subseteq \mathcal{M}$ is bounded away from zero. So, from (25), we deduce that

$$\gamma_n^* \leqslant c_5 \text{ for some } c_5 > 0 \text{ and for all } n \in \mathbb{N}. \tag{26}$$

From (10) with $y = \hat{\mu}_n(t)[u_n + th] \in \mathcal{N}$ (see (16)), we have

$$\begin{aligned}
&\varphi(u_n) - \varphi(\hat{\mu}_n(t)(u_n + th)) \\
\leqslant\ &\frac{1}{n}\|u_n - \hat{\mu}_n(t)(u_n + th)\| \\
\leqslant\ &\frac{1}{n}\big|\hat{\mu}_n(t) - 1\big|\|u_n\| + \frac{t}{n}\hat{\mu}_n(t)\|h\|.
\end{aligned} \tag{27}$$

Using (27) and the fact that $u_n \in \mathcal{N}$ for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\frac{t}{n}\hat{\mu}_n(t)\|h\| \geqslant \Bigg[&-\left(1 + \frac{p}{\eta-1}\right)(1+o(1))^{p-1}\rho_\alpha(D(u_n + th)) \\
&-\left(1 + \frac{q}{\eta-1}\right)(1+o(1))^{q-1}\|D(u_n + th)\|_q^q \\
&+\left(1 + \frac{\tau}{\eta-1}\right)(1+o(1))^{\tau-1}\int_\Omega \beta(z)(u_n + th)^\tau \mathrm{d}z \\
&-\frac{\|u_n\|}{n}\operatorname{sign}(\hat{\mu}_n(t) - 1)\Bigg](\hat{\mu}_n(t) - 1) \\
&-\left[\frac{1}{p} + \frac{1}{\eta-1}\right](\rho_\alpha(D(u_n + th)) - \rho_\alpha(Du_n)) \\
&-\left[\frac{1}{q} + \frac{1}{\eta-1}\right]\left(\|D(u_n + th)\|_q^q - \|Du_n\|_q^q\right) \\
&+\left[\frac{1}{\tau} + \frac{1}{\eta-1}\right]\int_\Omega \beta(z)\big[(u_n + th)^\tau - u_n^\tau\big]\mathrm{d}z.
\end{aligned} \tag{28}$$

Note that as $t \to 0^+$, we have

$$\begin{aligned}
&\Bigg[-\left(1+\frac{p}{\eta-1}\right)(1+o(1))^{p-1}\rho_\alpha(D(u_n+th)) \\
&\quad -\left(1+\frac{q}{\eta-1}\right)(1+o(1))^{q-1}\|D(u_n+th)\|_q^q \\
&\quad +\left(1+\frac{\tau}{\eta-1}\right)(1+o(1))^{\tau-1}\int_\Omega \beta(z)(u_n+th)^\tau \mathrm{d}z \\
&\quad -\frac{\|u_n\|}{n}\,\mathrm{sign}(\hat{\mu}_n(t)-1)\Bigg] \\
&\to -\left(1+\frac{p}{\eta-1}\right)\rho_\alpha(Du_n)-\left(1+\frac{q}{\eta-1}\right)\|Du_n\|_q^q+\left(1+\frac{\tau}{\eta-1}\right)\int_\Omega \beta(z)u_n^\tau \mathrm{d}z.
\end{aligned}$$

But $u_n \in \mathcal{N}$ and $\tau < q < p$. Hence,

$$\begin{aligned}
&-\left(1+\frac{p}{\eta-1}\right)\rho_\alpha(Du_n)-\left(1+\frac{q}{\eta-1}\right)\|Du_n\|_q^q+\left(1+\frac{\tau}{\eta-1}\right)\int_\Omega \beta(z)u_n^\tau \mathrm{d}z \\
&\leqslant -\frac{\tau}{\eta-1}\left[\rho_\alpha(Du_n)+\|Du_n\|_q^q-\int_\Omega \beta(z)u_n^\tau \mathrm{d}z\right]-\int_\Omega \vartheta(z)u_n^{1-\eta}\mathrm{d}z \\
&\leqslant -\frac{\tau}{\eta-1}\left[\rho_\alpha(Du_n)+\|Du_n\|_q^q-\int_\Omega \beta(z)u_n^\tau \mathrm{d}z\right].
\end{aligned} \tag{29}$$

We divide (28) by $t > 0$ and let $t \to 0^+$. We obtain

$$\begin{aligned}
0 \geqslant &\Bigg[-\left(1+\frac{p}{\eta-1}\right)\rho_\alpha(Du_n)-\left(1+\frac{q}{\eta-1}\right)\|Du_n\|_q^q \\
&+\left(1+\frac{\tau}{\eta-1}\right)\int_\Omega \beta(z)u_n^\tau \mathrm{d}z\Bigg]\gamma_n^* \\
&-\left(\frac{1}{p}+\frac{1}{\eta-1}\right)\langle A_p^\alpha(u_n),h\rangle-\left(\frac{1}{q}+\frac{1}{\eta-1}\right)\langle A_q(u_n),h\rangle \\
&+\left(\frac{1}{\tau}+\frac{1}{\eta-1}\right)\int_\Omega \beta(z)u_n^{\tau-1}h\mathrm{d}z.
\end{aligned} \tag{30}$$

Using (29) and the boundedness of $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mu}(\Omega)$, from (30), we infer that

$$\gamma_n^* \geqslant -c_6 \text{ for some } c_6 > 0 \text{ and for all } n \in \mathbb{N}. \tag{31}$$

Then, (26) and (31) imply that

$$|\gamma_n^*| \leqslant c_7 \text{ for some } c_7 > 0 \text{ and for all } n \in \mathbb{N}. \tag{32}$$

From (27) as before, we have that

$$
\begin{aligned}
&\frac{1}{n}\left|\hat{\mu}_n(t)-1\right|\|u_n\| + \frac{t}{n}\hat{\mu}_n(t)\|h\| \\
&\geqslant \varphi(u_n) - \varphi(\hat{\mu}_n(t)(u_n+th)) \\
&\geqslant \Big[-(1+o(1))^{p-1}\rho_\alpha(D(u_n+th)) - (1+o(1))^{q-1}\|D(u_n+th)\|_q^q \\
&\quad + (1+o(1))^{-\eta}\int_\Omega \vartheta(z)(u_n+th)^{1-\eta}\mathrm{d}z \\
&\quad + (1+o(1))^{\tau-1}\int_\Omega \beta(z)(u_n+th)^{\tau}\mathrm{d}z\Big](\hat{\mu}_n(t)-1) \\
&\quad - \frac{1}{p}[\rho_\alpha(D(u_n+th)) - \rho_\alpha(Du_n)] - \frac{1}{q}\left[\|D(u_n+th)\|_q^q - \|Du_n\|_q^q\right] \\
&\quad + \frac{1}{1-\eta}\int_\Omega \vartheta(z)\left[(u_n+th)^{1-\eta} - u_n^{1-\eta}\right]\mathrm{d}z \\
&\quad + \frac{1}{\tau}\int_\Omega \beta(z)\left[(u_n+th)^{\tau} - u_n^{\tau}\right]\mathrm{d}z.
\end{aligned}
$$

We divide by $t > 0$ and then let $t \to 0^+$. We obtain

$$
\begin{aligned}
&\frac{1}{n}|\gamma_n^*| \\
&\geqslant \left[-\rho_\alpha(Du_n) - \|Du_n\|_q^q + \int_\Omega \vartheta(z)u_n^{1-\eta}\mathrm{d}z + \int_\Omega \beta(z)u_n^{\tau}\mathrm{d}z\right]\gamma_n^* \\
&\quad - \langle V(u_n), h\rangle + \int_\Omega \vartheta(z)u_n^{-\eta}h\mathrm{d}z + \int_\Omega \beta(z)u_n^{\tau-1}h\mathrm{d}z \\
&= - \langle V(u_n), h\rangle + \int_\Omega \vartheta(z)u_n^{-\eta}h\mathrm{d}z + \int_\Omega \beta(z)u_n^{\tau-1}h\mathrm{d}z \\
&\qquad\qquad \text{since } u_n \in \mathcal{N} \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Now, we let $n \to \infty$ and use (12) and (32). We have

$$
\langle V(\hat{u}), h\rangle \geqslant \int_\Omega \vartheta(z)\hat{u}^{-\eta}h\mathrm{d}z + \int_\Omega \beta(z)\hat{u}^{\tau-1}h\mathrm{d}z \qquad \text{for all } h \in W_0^{1,\mu}(\Omega) \text{ with } h \geqslant 0. \tag{33}
$$

If we choose $h = \hat{u} \in W_0^{1,\mu}(\Omega)$, we obtain

$$
\begin{aligned}
&\rho_\mu(D\hat{u}) \geqslant \int_\Omega \vartheta(z)\hat{u}^{1-\eta}\mathrm{d}z + \int_\Omega \beta(z)\hat{u}^{\tau}\mathrm{d}z \\
&\Rightarrow \hat{u} \in \mathcal{M}.
\end{aligned}
$$

Reasoning as in Case 1 and using (14), we conclude that $\hat{u} \in \mathcal{N}$.

As in Gasiński–Winkert [6, Theorem 3.1], using a Moser iteration argument, we show that $\hat{u} \in L^\infty(\Omega)$. Finally, Proposition 2.4 of Papageorgiou-Vetro–Vetro [18] implies that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$. □

Next, we show that this minimizer $\hat{u} \in \mathcal{N}$ is, in fact, a solution of (1). Hence, $\mathcal{M}$ is, in fact, a natural constraint for $\varphi(\cdot)$.

Proposition 7. *If hypotheses H hold and $\hat{u} \in \mathcal{N} \cap L^\infty(\Omega)$ is the minimizer given by Proposition 6, then $\hat{u}$ is a solution of (1).*

Proof. Let $\hat{h} \in W_0^{1,\mu}(\Omega)$ and $\varepsilon > 0$. We introduce the following subsets of the domain Ω:

$$\Omega_+^\varepsilon = \{z \in \Omega : (\hat{u} + \varepsilon\hat{h})(z) \geqslant 0\},$$

$$\Omega_-^\varepsilon = \{z \in \Omega : (\hat{u} + \varepsilon\hat{h})(z) < 0\}.$$

Using the test function $(\hat{u} + \varepsilon\hat{h})^+ \in W_0^{1,\mu}(\Omega)$ in (33), we have

$$\begin{aligned}
0 &\leqslant \langle V(\hat{u}), (\hat{u} + \varepsilon\hat{h})^+\rangle - \int_\Omega \vartheta(z)\hat{u}^{-\eta}(\hat{u} + \varepsilon\hat{h})^+ \mathrm{d}z - \int_\Omega \beta(z)\hat{u}^{\tau-1}(\hat{u} + \varepsilon\hat{h})^+ \mathrm{d}z \\
&= \int_{\Omega_+^\varepsilon} \alpha(z)|D\hat{u}|^{p-2}(D\hat{u}, D(\hat{u} + \varepsilon\hat{h}))_{\mathbb{R}^N} \mathrm{d}z + \int_{\Omega_+^\varepsilon} |D\hat{u}|^{q-2}(D\hat{u}, D(\hat{u} + \varepsilon\hat{h}))_{\mathbb{R}^N} \mathrm{d}z \\
&\quad - \int_{\Omega_+^\varepsilon} \vartheta(z)\hat{u}^{-\eta}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z - \int_{\Omega_+^\varepsilon} \beta(z)\hat{u}^{\tau-1}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z \\
&= \langle V(\hat{u}), (\hat{u} + \varepsilon\hat{h})\rangle - \int_{\Omega_-^\varepsilon} \alpha(z)|D\hat{u}|^{p-2}(D\hat{u}, D(\hat{u} + \varepsilon\hat{h}))_{\mathbb{R}^N} \mathrm{d}z \\
&\quad - \int_{\Omega_-^\varepsilon} |D\hat{u}|^{q-2}(D\hat{u}, D(\hat{u} + \varepsilon\hat{h}))_{\mathbb{R}^N} \mathrm{d}z \\
&\quad - \int_\Omega \vartheta(z)\hat{u}^{-\eta}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z + \int_{\Omega_-^\varepsilon} \vartheta(z)\hat{u}^{-\eta}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z \\
&\quad - \int_\Omega \beta(z)\hat{u}^{\tau-1}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z + \int_{\Omega_-^\varepsilon} \beta(z)\hat{u}^{\tau-1}(\hat{u} + \varepsilon\hat{h})\mathrm{d}z \\
&\leqslant \varepsilon\left[\langle V(\hat{u}), \hat{h}\rangle - \int_\Omega \vartheta(z)\hat{u}^{-\eta}\hat{h}\mathrm{d}z - \int_\Omega \beta(z)\hat{u}^{\tau-1}h\mathrm{d}z\right] \\
&\quad - \varepsilon\left[\int_{\Omega_-^\varepsilon} \alpha(z)|D\hat{u}|^{p-2}(D\hat{u}, D\hat{h})_{\mathbb{R}^N} \mathrm{d}z + \int_{\Omega_-^\varepsilon} |D\hat{u}|^{q-2}(D\hat{u}, D\hat{h})_{\mathbb{R}^N} \mathrm{d}z\right] \\
&\quad + \varepsilon\left[\int_{\Omega_-^\varepsilon} \vartheta(z)\hat{u}^{-\eta}\hat{h}\mathrm{d}z + \int_{\Omega_-^\varepsilon} \beta(z)\hat{u}^{\tau-1}\hat{h}\mathrm{d}z\right].
\end{aligned} \tag{34}$$

Since $\hat{u} > 0$, we see that

$$\left|\Omega_-^\varepsilon\right|_{\mathrm{N}} \to \text{as } \varepsilon \to 0^+$$

(here we denote by $|\cdot|_N$ the Lebesgue measure on $\mathbb{R}^N$).

So, if divide (34) by $\varepsilon > 0$ and then let $\varepsilon \to 0^+$, we obtain

$$0 \leqslant \langle V(\hat{u}), \hat{h}\rangle - \int_\Omega \vartheta(z)\hat{u}^{-\eta}\hat{h}\mathrm{d}z - \int_\Omega \beta(z)\hat{u}^{\tau-1}\hat{h}\mathrm{d}z.$$

Since $\hat{h} \in W_0^{1,\mu}(\Omega)$ is arbitrary, we infer that

$$\langle V(\hat{u}), h\rangle = \int_\Omega \vartheta(z)\hat{u}^{-\eta}h\mathrm{d}z + \int_\Omega \beta(z)\hat{u}^{\tau-1}h\mathrm{d}z$$
$$\text{for all } h \in W_0^{1,\mu}(\Omega).$$

Thus, $\hat{u}$ is a positive solution of (1). □

Summarizing we can state the following existence theorem for problem (1).

Theorem 8. *If hypotheses H hold, then problem (1) has a positive solution* $\hat{u} \in W_0^{1,\mu}(\Omega) \cap L^\infty(\Omega)$ *and* $\hat{u}(z) > 0$ *for a.a.* $z \in \Omega$.

ACKNOWLEDGMENTS

The first two authors are supported by the grant "Nonlinear Differential Systems in Applied Sciences" of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8 (Grant No. 22). Shuai Yuan would like to thank the China Scholarship Council (No. 202106370097) for the financial support and the Embassy of the People's Republic of China in Romania.

DECLARATION OF COMPETING INTEREST

The authors declare that they do not know any competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Vicenţiu D. Rădulescu https://orcid.org/0000-0003-4615-5537

REFERENCES

1. P. Baroni, M. Colombo, and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018), no. 62, 48.
2. D. Bonheure, P. d'Avenia, and A. Pomponio, *On the electrostatic Born-Infeld equation with extended charges*, Comm. Math. Phys. **346** (2016), 877–906.
3. K. J. Brown and T. Wu, *A fibering map approach to a semilinear elliptic boundary value problem*, Electron. J. Differential Equations **69** (2007), 1–9.
4. K. J. Brown and Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations **193** (2003), 481–499.
5. L. Cherfils and Y. Il'yasov, *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Commun. Pure Appl. Anal. **4** (2005), no. 1, 9–22.
6. L. Gasiński and P. Winkert, *Constant sign solutions for double phase problems with superlinear nonlinearity*, Nonlinear Anal. **195** (2020), 111739, 9 pp.
7. P. Harjulehto and P. Hästö, *Orlicz spaces and generalized Orlicz spaces*, Springer, Cham, 2019.
8. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer, New York-Heidelberg, 1975.
9. A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), 721–730.
10. P. Marcellini, *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Rational Mech. Anal. **105** (1989), 267–284.
11. P. Marcellini, *Regularity and existence of solutions of elliptic equations with p,q-growth conditions*, J. Differential Equations **90** (1991), no. 1, 1–30.
12. P. Marcellini, *Growth conditions and regularity for weak solutions to nonlinear elliptic PDEs*, J. Math. Anal. Appl. **501** (2021), 124408, 32 pp.
13. P. Marcellini, *Local Lipschitz continuity for p,q-PDEs with explicit u-dependence*, Nonlinear Anal. **226** (2023), Paper No. 113066, 26 pp.
14. G. Mingione and V. D. Rădulescu, *Recent developments in problems with nonstandard growth and nonuniform ellipticity*, J. Math. Anal. Appl. **501** (2021), 125197, 41 pp.
15. N. S. Papageorgiou, *Double phase problems: a survey of some recent results*, Opuscula Math. **42** (2022), no. 2, 257–278.
16. N. S. Papageorgiou, A. Pudełko, and V. D. Rădulescu, *Non-autonomous (p,q)-equations with unbalanced growth*, Math. Ann. **385** (2023), 1707–1745.
17. N. S. Papageorgiou, V. D. Rădulescu, and Y. Zhang, *Strongly singular double phase problems*, Mediterr. J. Math. **19** (2022), 82, 21.
18. N. S. Papageorgiou, C. Vetro, and F. Vetro, *Multiple solutions for parametric double phase Dirichlet problems*, Commun. Contemp. Math. **23** (2021), no. 4, 2050006, 18 pp.
19. N. S. Papageorgiou and P. Winkert, *Applied nonlinear functional analysis*, De Gruyter, Berlin, 2018.
20. V. D. Rădulescu, *Isotropic and anisotropic double-phase problems: old and new*, Opuscula Math. **39** (2019), no. 2, 259–279.
21. T. Singer, *Existence of weak solutions of parabolic systems with p,q-growth*, Manuscripta Math. **151** (2016), 87–112.
22. Y. Sun, *Compatibility phenomena in singular problems*, Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), 1321–1330.
23. Y. Sun and Y. Tan, *Kirchhoff type equations with strong singularities*, Commun. Pure Appl. Anal. **18** (2019), no. 1, 181–193.
24. V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675–710.