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The inviscid limit for the incompressible stationary magnetohydrodynamics equations in three dimensions

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This paper is concerned with the zero-viscosity limit of the three-dimensional (3D) incompressible stationary magnetohydrodynamics (MHD) equations in the 3D unbounded domain $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$. The main result of this paper establishes that the solution of 3D incompressible stationary MHD equations converges to the solution of the 3D incompressible stationary Euler equations as the viscosity coefficient goes to zero.

Keywords: MHD equations; Euler equations; zero viscosity limit.

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1. Introduction and Main Results

1.1. *Introduction*

We consider the three-dimensional (3D) incompressible stationary magnetohydrodynamics (MHD) equations:

$$\begin{aligned} -\nu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P &= (\mathbf{H}\cdot\nabla)\mathbf{H} + f^{\nu}, \\ -\nu\Delta\mathbf{H} + (\mathbf{v}\cdot\nabla)\mathbf{H} &= (\mathbf{H}\cdot\nabla)\mathbf{v}, \\ \nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{H} &= 0, \end{aligned} \tag{1.1}$$

where $x \in \Omega := \mathbb{R}^{+}\times\mathbb{R}^{2}$, $\mathbf{v} := (v_1, v_2, v_3)$ denotes the 3D velocity field of the fluid, P stands for the pressure in the fluid, $\mathbf{H} := (H_1, H_2, H_3)$ is the magnetic field, and $\nu > 0$ denotes the viscosity constant and the magnetic diffusion constant, respectively. The vector field f^{ν} is an external force and $f^{\nu}|_{x\in\partial\Omega} = 0$. The divergence free condition in second equations of problem (1.1) guarantees the incompressibility of the fluid. The pressure takes the form

$$P = -\Delta^{-1}\mathrm{div}(\mathbf{v}\cdot\nabla\mathbf{v} - (\mathbf{H}\cdot\nabla)\mathbf{H}). \tag{1.2}$$

We supplement the 3D incompressible stationary MHD equations (1.1) with the boundary condition

$$\mathbf{v}(x)|_{x\in\partial\Omega} = 0, \quad \mathbf{H}(x)|_{x\in\partial\Omega} = 0, \tag{1.3}$$

that is, in x_1 direction

$$\begin{aligned} v_i(x)|_{x_1=0} &= 0, \quad H_i(x)|_{x_1=0} = 0, \quad i = 1, 2, 3, \\ \lim_{x_1\to+\infty} v_i(x) &= 0, \quad \lim_{x_1\to+\infty} H_i(x) = 0, \end{aligned}$$

and the vanishing boundary condition in $\bar{x} := (x_2, x_3)$ direction

$$\lim_{|\bar{x}|\to+\infty} v_i(x) = 0, \quad \lim_{|\bar{x}|\to+\infty} H_i(x) = 0, \quad i = 1, 2, 3.$$

It is easy to check that the solutions of the 3D incompressible stationary MHD equations (1.1) admit the scaling invariant property. More precisely, if $(\mathbf{v}, \mathbf{H}, P)$ is an arbitrary solution of problem (1.1), then for any constant $\lambda > 0$, the functions

$$\mathbf{v}_{\lambda}(x) = \lambda\mathbf{v}(\lambda x), \quad \mathbf{H}_{\lambda}(x) = \lambda\mathbf{H}(\lambda x), \quad P_{\lambda}(x) = \lambda^{2}P(\lambda x),$$

are also solutions of the 3D incompressible stationary MHD equations (1.1).

The incompressible MHD equations describe the dynamics of electrically conducting fluids arising in plasmas or some other physical phenomena. These equations are a combination of the Navier–Stokes equations of fluid dynamics and the Maxwell equations of electromagnetism. An interesting problem in fluid mechanics is the study of the zero-viscosity limit in the presence of a boundary with certain boundary conditions, such as the non-slip boundary condition and the Dirichlet boundary condition. Toward this direction, for the unsteady equations and in the absence of the boundary, it has been proved that the Navier–Stokes equations converge to the Euler equations in various functional settings; see [1, 3, 20, 36, 41]. However, in the presence of the boundary, the inviscid limit problem will become very complicated due to the appearance of the boundary layer. Masmoudi and Rousset [29] introduced the conormal functional space to justify the limit from the incompressible Navier–Stokes equations to the incompressible Euler equations for the Navier slip boundary condition. We refer to [17, 18, 30] for more results on this boundary condition. For the non-slip boundary condition, Sammartino and Caflisch [37, 38] proved the inviscid limit of the incompressible Navier–Stokes equations for well-prepared data with analytic regularity in the half-space. Wang *et al.* [45] developed an energy method for the inviscid limit problem in the analytic setting to deal with the inviscid limit problem in general domains. Nguyen and Nguyen [33] gave a direct proof of the inviscid limit for general analytic data without having to construct Prandtl's boundary layer correctors. Very recently, Kukavica *et al.* [21] obtained the inviscid limit for the Navier–Stokes equations in a half space, and they only required that the initial datum is analytic only close to the boundary of the domain. Meanwhile, it has Sobolev regularity in the complement. Recently, Liu *et al.* [26] established the well-posedness theory for the MHD boundary layer. Next, they justified [27] the high Reynolds numbers limit for the MHD system with Prandtl boundary layer expansion with no-slip boundary condition. We refer to [6, 28, 34, 35, 43, 42, 44, 46] and references therein for more relevant results.

For the 2D stationary equations, Iyer [19] considered the validity of the Prandtl boundary layer theory for steady incompressible Navier–Stokes flows over a rotating disk. Guo and Nguyen [15] constructed general boundary layer expansions to the steady Navier–Stokes equations in a half plane, subject to a positive Dirichlet datum for the horizontal velocity. Gerard-Varet and Maekawa [13] gave the inviscid limit problem in Sobolev regularity (H^1-regularity) for a nontrivial class of steady 2D Navier–Stokes flows with no-slip boundary condition. Recently, Li *et al.* [22] showed the vanishing viscosity limit for homogeneous axisymmetric no-swirl solutions of stationary Navier–Stokes equations.

The question of finite time singularity/global regularity for 3D incompressible Navier–Stokes equations is the most important open problem in mathematical fluid mechanics [9]. Thus, this question is also a natural important problem for the 3D incompressible MHD equations. Sermange and Temam [40] established the local well-posedness of classical solutions for fully viscous MHD equations, in which the global well-posedness is also proved in two dimensions. Fefferman *et al.* [10] proved a local existence result for the MHD equations (1.1) taking arbitrary initial data in $\mathbb{H}^s(\mathbb{R}^d)$ with $s > d/2$ for $d = 2, 3$. Chemin *et al.* [5] obtained the local existence of solutions to the viscous, non-resistive MHD equations with initial data $(\mathbf{v}_0, \mathbf{H}_0)$ in the Besov space $B^{\frac{d}{2}}_{2,1}(\mathbb{R}^n) \times B^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^n)$ with $n = 2, 3$. Fefferman *et al.* [11] improved the initial data $(\mathbf{v}_0, \mathbf{H}_0) \in \mathbb{H}^{s-1+\varepsilon}(\mathbb{R}^n)\times\mathbb{H}^s(\mathbb{R}^n)$ for $s > d/2$ and $0 < \varepsilon < 1$. Next, Li *et al.* [23] improved these results in homogeneous Besov spaces. For global existence results, Lin *et al.* [24] established the global well-posedness of a 2D incompressible MHD system with smooth initial data close to some nontrivial steady state. We also refer to Lin and Zhang [25] who extended the global well-posedness of a 2D incompressible MHD system to the 3D case with small initial data. Abidi and Zhang [2] considered a more general initial data closed to the nontrivial equilibrium state $(x_3, 0)$. Cai and Lei [4] showed the global well-posedness for the incompressible MHD system with or without viscosity with the initial data near a constant vector (Alfvén waves) by means of the ghost weight technique. Their result depends on the inherent strong null structure of the system, which can ensure the decay in time of the energy. Deng and Zhang [8] constructed the smooth solutions near the trivial equilibrium state $(e_3, 0)$ by using the Nash-Moser iteration scheme, where e_3 is a constant. Yan [50] found a family of stable infinite energy blowup solutions for 3D incompressible MHD systems. Recently, Yan and Rădulescu [52] proved the existence of global small finite energy solution for the 3D incompressible viscous MHD systems. To authors' best knowledge, there are very few results on the 3D vanishing viscosity limit for the steady MHD equations in the general case.

The problem. An interesting problem to consider is the relationship between the solution of MHD equations and the solution of Euler equations, that is, the inviscid limit of MHD equations. When the viscosity goes to zero, a natural problem is to study the convergence of the steady solution of the 3D incompressible MHD equations (1.1) to the solution of the following steady Euler equations with an external force:

$$\begin{cases} \mathbf{v}\cdot\nabla\mathbf{v}+\nabla P = f^{\nu}, & x\in\Omega,\\ \nabla\cdot\mathbf{v}=0, & x\in\Omega,\\ \mathbf{v}|_{x\in\partial\Omega}=0. \end{cases} \tag{1.4}$$

More precisely, let $(\mathbf{v}^{\nu}, \mathbf{H}^{\nu}, P^{\nu})$ and $(\mathbf{v}^{e}, P^{e})$ be the solutions of the MHD equations (1.1) and the Euler equations (1.4), respectively. Is it true that

$$(\mathbf{v}^{\nu}, \mathbf{H}^{\nu}, P^{\nu}) \to (\mathbf{v}^{e}, \mathbf{0}, P^{e}) \quad \text{as } \nu\to 0, \text{ in some Sobolev space?}$$

Here, the external force f^{ν} is the same with the force given in (1.1).

One can check that there exist two functions P^e and f^{ν} such that the following function is an exact solution of problem (1.4):

$$\mathbf{v}(t,x) = (v_1(x), v_2(x), v_3(x)), \quad \forall\, x\in\mathbb{R}^+\times\mathbb{R}^2,$$

where

$$v_1(x) = x_1^q x_2^{2p+1} x_3^{2p+1} e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},$$

$$v_2(x) = 2^{-1}(1+p)^{-1}(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_3^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},$$

$$v_3(x) = -(1+p)^{-1}(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_2^{2p+1}e^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})}.$$

1.2. *Main results*

Let the smooth function $(\mathbf{v}^e(x), P^e(x))$ be a solution of the incompressible 3D steady Euler equations (1.4) (see [7, 12] for the existence of smooth solutions of steady 3D Euler equations). Consider the smooth vector function $\mathbf{v}^e(x) = (v_1^e(x), v_2^e(x), v_3^e(x))$ and assume that

$$\begin{aligned} \sum_{k=0}^{s} \|\partial_{x_i}^k v_j^e\|_{\mathbb{L}^\infty(\Omega)} &\lesssim c_0, \quad \forall\, i,j = 1,2,3, \\ \|v^e\|_{H^{s+2}(\Omega)} &\lesssim c_0, \\ \partial_{x_i}^l v^e|_{x\in\partial\Omega} &= 0, \quad 0 \le l \le s, \end{aligned} \tag{1.5}$$

for a fixed positive constant c_0.

We now state the main result of this paper.

Theorem 1.1. *Assume that conditions* (1.5) *hold. Then for any fixed constant* $s > 1$, *the steady incompressible MHD equations* (1.1) *with the boundary conditions* (1.3) *admit a Sobolev regular solution with finite energy* $(\boldsymbol{v}^\nu, \boldsymbol{H}^\nu, P^\nu) \in H^s(\Omega) \times H^s(\Omega) \times \mathbb{H}^s(\Omega)$ *such that*

$$\lim_{\nu\to 0} (\|\boldsymbol{v}^\nu - \boldsymbol{v}^e\|_{H^s(\Omega)} + \|\boldsymbol{H}^\nu\|_{H^s(\Omega)}) = 0,$$

and

$$\lim_{\nu\to 0} \|P^\nu - P^e\|_{H^s(\Omega)} = 0.$$

Remark 1.1. Let the parameter λ satisfy $1 < \max\{\nu^{-\frac{1}{2}}, c_0\} \le \lambda < \varepsilon^{-1}$. We will construct the small Sobolev regular solutions of (1.1) by means of the explicit representation formula as follows:

$$\begin{aligned} \mathbf{v}^{(\infty)}(x) &= \mathbf{v}^{(e)}(x) + \mathbf{u}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(\lambda x) = \mathbf{v}^{(e)}(x) + \mathcal{O}(\sqrt{\nu}), \\ \mathbf{H}^{(\infty)}(x) &= \mathbf{H}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{w}^{(m)}(\lambda x) = \mathcal{O}(\sqrt{\nu}), \end{aligned} \tag{1.6}$$

where the function $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$ satisfies the assumptions

$$\begin{cases} \nabla \cdot \mathbf{u}^{(0)}(t,x) = 0, \\ \|\mathbf{u}^{(0)}\|_{H^s(\Omega)} \lesssim \varepsilon < \nu^{\frac{1}{2}}, \quad \text{for } 0 < \nu \ll 1, \\ \mathbf{u}^{(0)}(x)|_{x\in\partial\Omega} = 0, \end{cases} \tag{1.7}$$

and

$$\begin{cases} \nabla \cdot \mathbf{H}^{(0)}(t,x) = 0, \\ \|\mathbf{H}^{(0)}\|_{H^s(\Omega)} \lesssim \varepsilon < \nu^{\frac{1}{2}}, \quad \text{for } 0 < \nu \ll 1, \\ \mathbf{H}^{(0)}(x)|_{x\in\partial\Omega} = 0, \end{cases} \tag{1.8}$$

and

$$\begin{aligned}&\sum_{k=0}^{s}\|\partial_{x_i}^k u_j^{(0)}(x)\|_{\mathbb{L}^\infty}\lesssim\varepsilon_0<\varepsilon,\quad\forall\, i,j=1,2,3,\\&\sum_{k=0}^{s}\|\partial_{x_i}^k H_j^{(0)}(x)\|_{\mathbb{L}^\infty}\lesssim\varepsilon_0<\varepsilon,\quad\forall\, i,j=1,2,3,\end{aligned}\tag{1.9}$$

where $(\mathbf{h}^{(m)}(\lambda x),\mathbf{w}^{(m)}(\lambda x))$ $(m=1,2,3,\ldots)$ is obtained by solving the linearized problem with the Dirichlet boundary condition in the Sobolev space $H^s(\Omega)\times H^s(\Omega)$ with $s>1$ as follows:

$$\begin{cases}\mathcal{L}_1[\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=E^{(m-1)}(x),\\ \mathcal{L}_2[\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})=\overline{E}^{(m-1)}(x),\\ \nabla\cdot\mathbf{h}^{(m)}=0,\quad \nabla\cdot\mathbf{w}^{(m)}=0\\ \mathbf{h}^{(m)}(\lambda x)|_{x\in\partial\Omega}=0,\quad \mathbf{w}^{(m)}(\lambda x)|_{x\in\partial\Omega}=0.\end{cases}\tag{1.10}$$

Here, the linear operators $\mathcal{L}_1[\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})$ and $\mathcal{L}_2[\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)}](\mathbf{h}^{(m)},\mathbf{w}^{(m)})$ are defined in (2.9), while $E^{(m-1)}(x)$ and $\overline{E}^{(m-1)}(x)$ denote the error term. The index s of the Sobolev regularity depends on the higher derivative estimate of the solution to the linearized equations. By (1.6), one can see that the solution depends on the initial approximation function $(\mathbf{u}^{(0)}(x),\mathbf{H}^{(0)})(x)$ strongly. Our proof is based on the Nash–Moser iteration scheme, see also [47, 48, 53, 49, 51, 54]. For the general Nash–Moser implicit function theorem, we refer to the seminal papers by Nash [32], Moser [31] and Hörmander [16].

Remark 1.2. In each approximation step, the approximation function should satisfy the higher order boundary conditions (2.33). Let us consider the linear elliptic equation with an external force:

$$-\Delta u=f(x),\tag{1.11}$$

with the non-slip boundary condition. We observe that there exists an external force $f(x)$ satisfying

$$\partial_{x_i}^l f(x)|_{\partial\Omega}=0,\quad 0\le l\le s,$$

such that (1.11) admits a solution $u(x)$ satisfying

$$\partial_{x_i}^l u(x)|_{\partial\Omega}=0,\quad 0\le l\le s.$$

Indeed, choosing the external force of the form

$$f(x)=x_1^p(p(p-1)x_1^{-2}-4+4(x_1^2+x_2^2))e^{-x_2^2-x_3^2},\quad\forall\, p>s>1,$$

which satisfies

$$\partial_{x_i}^l f(x)|_{\partial\Omega}=0,\quad 0\le l\le s,$$

then by direct computation, the linear elliptic equation admits an exact solution

$$u^*(x)=x_1^p e^{-x_2^2-x_3^2},$$

which satisfies the non-slip boundary condition. Moreover, it holds

$$\partial_{x_i}^l u^*(x)|_{x\in\partial\Omega} = 0, \quad \forall\, 0 \le l \le s.$$

1.3. *Sketch of the proof*

We set the solution of the incompressible steady MHD equations (1.1) by

$$\begin{aligned}
\mathbf{v}^\nu(x) &= \mathbf{v}^e(x) + \mathbf{u}(x),\\
\mathbf{H}^\nu(x) &= \mathbf{H}(x),\\
P^\nu(x) &= P^e(x) + \overline{P}(x),
\end{aligned}$$

It follows that

$$\begin{cases}
-\nu\Delta\mathbf{u} + \mathbf{v}^e\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v}^e + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\overline{P} - (\mathbf{H}\cdot\nabla)\mathbf{H} = g^\nu,\\
-\nu\Delta\mathbf{H} + (\mathbf{v}^e(x) + \mathbf{u}(x))\cdot\nabla\mathbf{H} - (\mathbf{H}\cdot\nabla)(\mathbf{v}^e(x) + \mathbf{u}(x)) = 0,\\
\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{u} = 0,\\
\mathbf{u}|_{x\in\partial\Omega} = 0, \quad \mathbf{H}|_{x\in\partial\Omega} = 0,
\end{cases} \tag{1.12}$$

where

$$g^\nu = \nu\Delta U^e,$$

which satisfies $\nabla\cdot g^\nu = 0$ by $\nabla\cdot\mathbf{v}^e = 0$.

The pressure takes the form

$$\overline{P}(x) = -\Delta^{-1}\nabla(\mathbf{v}^e\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v}^e + \mathbf{u}\cdot\nabla\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{H}). \tag{1.13}$$

In order to solve the nonlinear problem (1.12), the main difficulty is to find suitable dissipative terms, which can cause the time decay of the energy of solutions for the linearized equations. Assume that we have chosen a suitable initial approximation function $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$. Then we linearize the nonlinear equation around $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$ to get the linearized operator $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2)$ defined by

$$\begin{aligned}
&\mathcal{J}_1[\mathbf{u}^{(0)}, \mathbf{H}^{(0)}](\mathbf{h}^{(1)}, \mathbf{w}^{(1)})\\
&\quad := -\nu\Delta\mathbf{h}^{(1)} + \Pi_{N_1}[((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{h}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e)\\
&\qquad + \nabla(\mathcal{D}_{\mathbf{u}^{(0)}}\overline{P})\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}\overline{P})\mathbf{w}^{(1)}\\
&\qquad - (\mathbf{H}^{(0)}\cdot\nabla)\mathbf{w}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)\mathbf{H}^{(0)}],\\
&\mathcal{J}_2[\mathbf{u}^{(0)}, \mathbf{H}^{(0)}](\mathbf{h}^{(1)}, \mathbf{w}^{(1)})\\
&\quad := -\nu\Delta\mathbf{w}^{(1)} + \Pi_{N_1}[((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{w}^{(1)}\\
&\qquad + (\mathbf{h}^{(1)}\cdot\nabla)\mathbf{H}^{(0)} - (\mathbf{H}^{(0)}\cdot\nabla)\mathbf{h}^{(1)}\\
&\qquad - (\mathbf{w}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e)],
\end{aligned}$$

where $\mathcal{D}_{\mathbf{u}^{(0)}}$ denotes the Fréchet derivative at the function $\mathbf{u}^{(0)}$.

Furthermore, if we set $(\mathbf{h}^{(1)}(\lambda x), \mathbf{w}^{(1)}(\lambda x))$ with a big constant (for instance, we can choose $\lambda \geq \nu^{-\frac{1}{2}}$), then the linearized operator is changed into

$$\begin{aligned}
&\mathcal{J}_1[\mathbf{u}^{(0)}, \mathbf{H}^{(0)}](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\
&\quad := -\nu\lambda^2\Delta\mathbf{h}^{(1)} + \Pi_{N_1}[\lambda((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{h}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e) \\
&\qquad + \nabla(\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)} - \lambda(\mathbf{H}^{(0)}\cdot\nabla)\mathbf{w}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)\mathbf{H}^{(0)}], \\
&\mathcal{J}_2[\mathbf{u}^{(0)}, \mathbf{H}^{(0)}](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\
&\quad := -\nu\lambda^2\Delta\mathbf{w}^{(1)} + \Pi_{N_1}[\lambda((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{w}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)\mathbf{H}^{(0)} \\
&\qquad - \lambda(\mathbf{H}^{(0)}\cdot\nabla)\mathbf{h}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e)].
\end{aligned}$$

Here, we observe that the terms $-\nu\lambda^2\Delta\mathbf{h}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e)$ and $-\nu\lambda^2\Delta\mathbf{w}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)\mathbf{H}^{(0)}$ can be dissipative if we choose a suitable initial approximation function $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$ and a big positive constant λ. We denoted by Π_{N_1} the smoothing operator. Thus, the decay in time of the solution for the linearized equation

$$\begin{aligned}
\mathcal{J}_1[\mathbf{u}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) &= \Pi_{N_1} E^{(0)}, \\
\mathcal{J}_2[\mathbf{u}^0, \mathbf{H}^0](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) &= \Pi_{N_1} \overline{E}^{(0)}, \\
\nabla\cdot\mathbf{h}^{(1)} = 0, \quad \nabla\cdot\mathbf{w}^{(1)} &= 0,
\end{aligned}$$

is as desired.

Consequently, the Nash–Moser iteration scheme can be used to construct the solution of (1.12) as follows:

$$\mathbf{u}^{(\infty)}(x) = \mathbf{u}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(\lambda x) \in H^s(\Omega),$$

$$\mathbf{H}^{(\infty)}(x) = \mathbf{H}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{w}^{(m)}(\lambda x) \in H^s(\Omega),$$

where $(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x))$ is obtained by solving the linearized equations (1.10).

Notations. Throughout this paper, we set $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$ and we denote the usual norms of the Lebesgue space $\mathbb{L}^2(\Omega)$ and of Sobolev space $\mathbb{H}^s(\Omega)$ by $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$, respectively. The norm of the Sobolev space $H^s(\mathbb{R}^3) := (\mathbb{H}^s(\Omega))^3$ is denoted by $\|\cdot\|_{H^s}$. The symbol $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$. We denote by (a, b, c) the column vector in $\mathbb{R}^3$. The letter C with subscripts to denote dependencies stands for a positive constant that might change its value at each occurrence.

This paper is organized as follows. In Sec. 2, we show how to choose suitable initial approximation functions, which lead to the partial dissipative structure of the linearized equations. Next, we give the existence of Sobolev solution for the linearized equations of first approximation step. In Sec. 3, we establish the general approximation step for the construction of the Nash–Moser iteration scheme. In Sec. 4, we give the proof of Theorem 1.1.

2. The First Approximation Step

For $m = 1, 2, \ldots$, by setting $N_m = 2^m$, we introduce the family $\Pi_{N_m} : L^2 \to C^\infty$ of smoothing operators (see [39] for more details) such that

$$\begin{aligned} \|\Pi_{N_m} U\|_{H^{s_1}} &\lesssim N_m^{s_1 - s_2} \|U\|_{H^{s_2}}, \quad \forall\, s_1 \geq s_2 \geq 0, \\ \|\Pi_{N_m} U - U\|_{H^{s_1}} &\lesssim N_m^{s_1 - s_2} \|U\|_{H^{s_2}}, \quad \forall\, 0 \leq s_1 \leq s_2. \end{aligned} \tag{2.1}$$

We now consider the approximation problem of incompressible MHD equations (1.1) as follows:

$$\begin{aligned} &\mathcal{L}_1(\mathbf{u}, \mathbf{H}) := -\nu \Delta \mathbf{u} + \mathbf{v}^e \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{v}^e + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \overline{P} - (\mathbf{H} \cdot \nabla)\mathbf{H} - g^\nu, \\ &\mathcal{L}_2(\mathbf{u}, \mathbf{H}) := -\nu \Delta \mathbf{H} + (\mathbf{v}^e(x) + \mathbf{u}(x)) \cdot \nabla \mathbf{H} - (\mathbf{H} \cdot \nabla)(\mathbf{v}^e(x) + \mathbf{u}(x)), \end{aligned} \tag{2.2}$$

with the non-slip boundary condition

$$\mathbf{u}|_{x \in \partial\Omega} = 0, \quad \mathbf{H}|_{x \in \partial\Omega} = 0, \tag{2.3}$$

and the incompressible condition

$$\nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{H} = 0.$$

2.1. *The initial approximation function*

The general condition of initial approximation functions. We now give an abstract condition on the initial approximation function. Let $s \geq 1$ be a fixed finite constant and $0 < \varepsilon_0 < \varepsilon^2 \ll 1$. For any $x \in \Omega$, we choose the initial approximation functions

$$\begin{aligned} \mathbf{u}^{(0)}(x) &= (u_1^{(0)}(x), u_2^{(0)}(x), u_3^{(0)}(x)) \in H^s(\Omega), \\ \mathbf{H}^{(0)}(x) &= (H_1^{(0)}(x), H_2^{(0)}(x), H_3^{(0)}(x)) \in H^s(\Omega), \end{aligned}$$

where we require

$$\begin{cases} \nabla \cdot \mathbf{u}^{(0)}(x) = 0, \\ \|\mathbf{u}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ \partial_{x_i}^l \mathbf{u}^{(0)}(x)|_{x \in \partial\Omega} = 0, \quad 0 \leq l \leq s, \end{cases} \tag{2.4}$$

and

$$\begin{cases} \nabla \cdot \mathbf{H}^{(0)}(x) = 0, \\ \|\mathbf{H}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ \partial_{x_i}^l \mathbf{H}^{(0)}(x)|_{x \in \partial\Omega} = 0, \quad 0 \leq l \leq s. \end{cases} \tag{2.5}$$

Moreover, for any fixed $s \geq 1$ and $x \in \Omega$ and $i, j = 1, 2, 3$, we also need the conditions

$$\sum_{k=0}^{s} \|\partial_{x_i}^k \mathbf{u}_j^{(0)}(x)\|_{L^\infty} \lesssim \varepsilon_0, \tag{2.6}$$

and

$$\sum_{k=0}^{s} \|\partial_{x_i}^k \mathbf{H}_j^{(0)}(x)\|_{L^\infty} \lesssim \varepsilon_0, \tag{2.7}$$

and the initial error term

$$\begin{gathered} \partial_{x_i}^l E^{(0)}(x)|_{x\in\partial\Omega} = 0, \quad \partial_{x_i}^l \overline{E}^{(0)}(x)|_{x\in\partial\Omega} = 0, \quad 0 \leq l \leq s, \\ \|E^{(0)}\|_{H^s} \lesssim \varepsilon_0, \quad \|\overline{E}^{(0)}\|_{H^s} \lesssim \varepsilon_0, \end{gathered} \tag{2.8}$$

where $E^{(0)}$ and $\overline{E}^{(0)}$ denote the error term taking the form

$$\begin{aligned} E^{(0)} &:= \mathcal{L}_1(\mathbf{u}^{(0)}, \mathbf{H}^{(0)}) \\ &= \nu\Delta\mathbf{u}^{(0)} + \mathbf{v}^e \cdot \nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla\mathbf{v}^e + \mathbf{u}^{(0)} \cdot \nabla\mathbf{u}^{(0)} + \nabla\overline{P}^{(0)} \\ &\quad - (\mathbf{H}^{(0)} \cdot \nabla)\mathbf{H}^{(0)} - g^\nu, \\ \overline{E}^{(0)} &:= \mathcal{L}_2(\mathbf{u}^{(0)}, \mathbf{H}^{(0)}) \\ &= -\nu\Delta\mathbf{H}^{(0)} + (\mathbf{v}^e(x) + \mathbf{u}^{(0)}(x)) \cdot \nabla\mathbf{H}^{(0)} - (\mathbf{H}^{(0)} \cdot \nabla)(\mathbf{v}^e(x) + \mathbf{u}^{(0)}(x)), \end{aligned}$$

with the vector form

$$E^{(0)} = (E_1^{(0)}, E_2^{(0)}, E_3^{(0)}), \quad \overline{E}^{(0)} = (\overline{E}_1^{(0)}, \overline{E}_2^{(0)}, \overline{E}_3^{(0)}).$$

2.2. *The time decay of the first approximation step*

Let λ be a positive constant such that

$$1 < \max\{\nu^{-\frac{1}{2}}, c_0\} \leq \lambda < \varepsilon^{-1}.$$

We now construct the first approximation solution of problem (2.2). This solution is denoted by $(\mathbf{u}^{(1)}(\lambda x), \mathbf{H}^{(1)}(\lambda x))$. The first approximation step between the initial approximation function and the first approximation solution is denoted by

$$\mathbf{h}^{(1)}(\lambda x) := \mathbf{u}^{(1)}(\lambda x) - \mathbf{u}^{(0)}(x), \quad \mathbf{w}^{(1)}(\lambda x) := \mathbf{H}^{(1)}(\lambda x) - \mathbf{H}^{(0)}(x).$$

Next, we linearize the nonlinear system (2.2) around $(\mathbf{u}^{(0)}, \mathbf{H}^{(0)})^T$ and we obtain the linearized operators as follows:

$$\begin{aligned} &\mathcal{J}_1[\mathbf{u}^{(0)}, \mathbf{H}^{(0)}](\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \\ &\quad := -\nu\lambda^2\Delta\mathbf{h}^{(1)} + \Pi_{N_1}[\lambda((\mathbf{u}^{(0)} + \mathbf{v}^e) \cdot \nabla)\mathbf{h}^{(1)} + (\mathbf{h}^{(1)} \cdot \nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e) \\ &\qquad + \nabla(\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + \nabla(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)} - \lambda(\mathbf{H}^{(0)} \cdot \nabla)\mathbf{w}^{(1)} - (\mathbf{w}^{(1)} \cdot \nabla)\mathbf{H}^{(0)}], \end{aligned}$$

$$\begin{aligned}&\mathcal{J}_2[\mathbf{u}^{(0)},\mathbf{H}^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)})\\&\quad:= -\nu\lambda^2\Delta\mathbf{w}^{(1)} + \Pi_{N_1}[\lambda((\mathbf{u}^{(0)}+\mathbf{v}^e)\cdot\nabla)\mathbf{w}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)\mathbf{H}^{(0)}\\&\qquad - \lambda(\mathbf{H}^{(0)}\cdot\nabla)\mathbf{h}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)}+\mathbf{v}^e)],\end{aligned} \tag{2.9}$$

where $\mathcal{D}_{\mathbf{u}^{(0)}}$ denotes the Fréchet derivative on $\mathbf{u}^{(0)}$.

We now consider the linear system

$$\begin{aligned}&\mathcal{J}_1[\mathbf{u}^{(0)},\mathbf{H}^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) = \Pi_{N_1}E^{(0)},\\&\mathcal{J}_2[\mathbf{u}^{(0)},\mathbf{H}^{(0)}](\mathbf{h}^{(1)},\mathbf{w}^{(1)}) = \Pi_{N_1}\overline{E}^{(0)}, \quad \nabla\cdot\mathbf{h}^{(1)}=0, \quad \nabla\cdot\mathbf{w}^{(1)}=0,\end{aligned} \tag{2.10}$$

and the boundary condition

$$\mathbf{h}^{(1)}(\lambda x)|_{x\in\partial\Omega}=0, \quad \mathbf{w}^{(1)}(\lambda x)|_{x\in\partial\Omega}=0. \tag{2.11}$$

The solution of this problem gives the first approximation step of the problem (1.12).

Before we carry out some *a priori* estimates, we rewrite equations of (2.10) into a coupled system as follows:

$$\begin{aligned}&-\nu\lambda^2\Delta h_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^3(u_i^{(0)}+v_i^e)\partial_{x_i}h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^3 h_i^{(1)}(\partial_{x_i}u_j^{(0)}+\partial_{x_i}v_j^e)\\&\quad + \Pi_{N_1}\partial_{x_j}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}) - \lambda\Pi_{N_1}\sum_{i=1}^3 H_i^{(0)}\partial_{x_j}w_j^{(1)}\\&\quad - \Pi_{N_1}\sum_{i=1}^3 w_i^{(1)}\partial_{x_i}H_j^{(0)} = \Pi_{N_1}E_j^{(0)}, \quad \text{for } j=1,2,3,\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&-\nu\lambda^2\Delta w_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^3(u_i^{(0)}+v_i^e)\partial_{x_i}w_j^{(1)} + \Pi_{N_1}\sum_{i=1}^3 h_i^{(1)}\partial_{x_i}H_j^{(0)}\\&\quad - \lambda\Pi_{N_1}\sum_{i=1}^3 H_i^{(0)}\partial_{x_i}h_j^{(1)} - \Pi_{N_1}\sum_{i=1}^3 w_i^{(1)}(\partial_{x_i}u_j^{(0)}\\&\quad + \partial_{x_i}v_j^e) = \Pi_{N_1}\overline{E}_j^{(0)}, \quad \text{for } j=1,2,3,\end{aligned} \tag{2.13}$$

with the boundary condition (2.11), where

$$\begin{aligned}&\partial_{x_j}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})\\&\quad = -\partial_{x_j}\Delta^{-1}\sum_{i,j=1}^3(\partial_{x_j}h_i^{(1)}(\partial_{x_i}u_j^{(0)}+\partial_{x_i}v_j^e) + (\partial_{x_j}u_i^{(0)}+\partial_{x_j}v_i^e)\partial_{x_i}h_j^{(1)}\\&\qquad + \partial_{x_j}w_i^{(1)}\partial_{x_i}H_j^{(0)} + \partial_{x_j}H_i^{(0)}\partial_{x_i}w_j^{(1)}).\end{aligned} \tag{2.14}$$

2.3. *A priori estimate*

We now derive a $\mathbb{L}^2$-weighted estimate of the solution for the linear system (2.12)–(2.13).

Let $\phi(x_1)$ be a function defined in $(0, +\infty)$ such that

$$0 < \kappa \le \phi''(x_1) - (\phi'(x_1))^2 < +\infty, \tag{2.15}$$

and $e^{-\phi(x_1)}$ is bounded in $(0, +\infty)$. The condition (2.15) implies $\phi''(x_1) \ge \kappa$. In fact, there are many functions satisfying the above conditions. As a simple example, we consider the function

$$\phi(x_1) = -\ln|\cos(\sqrt{\kappa}x_1)|, \quad x_1 \ne 2i\pi + \frac{\pi}{2}, \quad \text{for } i \in \mathbb{Z}.$$

Lemma 2.1. *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds, and the initial approximation function* $(\boldsymbol{u}^{(0)}, \boldsymbol{H}^{(0)})$ *satisfies conditions* (2.4)–(2.8). *Then the solution* $(\boldsymbol{h}^{(1)}(\lambda x), \boldsymbol{w}^{(1)}(\lambda x))$ *of the linear system* (2.12)–(2.13) *satisfies*

$$\begin{aligned}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((\partial_{x_i}h_j^{(1)})^2 + (\partial_{x_i}w_j^{(1)})^2)dx + \sum_{j=1}^{3}\int_\Omega((h_j^{(1)})^2 + (w_j^{(1)})^2)dx \\ \le \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2)dx. \end{aligned}\tag{2.16}$$

Proof. Multiplying both sides of the six equations in (2.12)–(2.13) by $h_j^{(1)}e^{-\phi(x_1)}$ and $w_j^{(1)}e^{-\phi(x_1)}$, respectively, integrating over Ω, and using the boundary condition (2.11), for $j = 1, 2, 3$, we obtain

$$\begin{aligned}&\nu\lambda^2\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2e^{-\phi(x_1)}dx + \frac{\nu\lambda^2}{2}\int_\Omega(\phi''(x_1) - (\phi'(x_1))^2)(h_j^{(1)})^2e^{-\phi(x_1)}dx \\ &\quad + \lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)} + v_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx \\ &\quad + \Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(h_i^{(1)}(\partial_{x_i}u_j^{(0)} + \partial_{x_i}v_j^e))h_j^{(1)}e^{-\phi(x_1)}dx \\ &\quad + \lambda\Pi_{N_1}\int_\Omega\partial_{x_j}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx \\ &\quad - \lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(H_i^{(0)}\partial_{x_i}w_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx\end{aligned}$$

$$-\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega (w_i^{(1)}\partial_{x_i}H_j^{(0)})h_j^{(1)}e^{-\phi(x_1)}dx$$
$$=\Pi_{N_1}\int_\Omega E_j^{(0)}h_j^{(1)}e^{-\phi(x_1)}dx, \tag{2.17}$$

and

$$\nu\lambda^2\sum_{i=1}^{3}\int_\Omega (\partial_{x_i}w_j^{(1)})^2e^{-\phi(x_1)}dx+\frac{\nu\lambda^2}{2}\int_\Omega(\phi''(x_1)-(\phi'(x_1))^2)(w_j^{(1)})^2e^{-\phi(x_1)}dx$$
$$+\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}w_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx$$
$$+\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(h_i^{(1)}\partial_{x_i}H_j^{(0)})w_j^{(1)}e^{-\phi(x_1)}dx$$
$$-\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(H_i^{(0)}\partial_{x_i}h_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx$$
$$-\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(w_i^{(1)}(\partial_{x_i}u_j^{(0)}+\partial_{x_i}v_j^e))w_j^{(1)}e^{-\phi(x_1)}dx$$
$$=\Pi_{N_1}\int_\Omega\overline{E}_j^{(0)}w_j^{(1)}e^{-\phi(x_1)}dx. \tag{2.18}$$

We sum up (2.17)–(2.18) from $j=1$ to $j=3$. It follows that

$$\nu\lambda^2\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((\partial_{x_i}h_j^{(1)})^2+(\partial_{x_i}w_j^{(1)})^2)e^{-\phi(x_1)}dx$$
$$+\frac{\nu\lambda^2}{2}\sum_{j=1}^{3}\int_\Omega(\phi''(x_1)-(\phi'(x_1))^2)((h_j^{(1)})^2+(w_j^{(1)})^2)e^{-\phi(x_1)}dx$$
$$+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx$$
$$+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(h_i^{(1)}(\partial_{x_i}u_j^{(0)}+\partial_{x_i}v_j^e))h_j^{(1)}e^{-\phi(x_1)}dx$$
$$+\lambda\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega\partial_{x_j}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx$$
$$-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(H_i^{(0)}\partial_{x_i}w_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx$$

$$
\begin{aligned}
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}w_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx\\
&-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(H_i^{(0)}\partial_{x_i}h_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx\\
&-\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(w_i^{(1)}(\partial_{x_i}u_j^{(0)}+\partial_{x_i}v_j^e))w_j^{(1)}e^{-\phi(x_1)}dx\\
&=\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(E_j^{(0)}h_j^{(1)}+\overline{E}_j^{(0)}w_j^{(1)})e^{-\phi(x_1)}dx. \qquad (2.19)
\end{aligned}
$$

In what follows, we estimate each term in equality (2.19). Note that $\nabla\cdot\mathbf{u}^{(0)}=0$, and we have chosen the initial approximation function $(\mathbf{u}^{(0)},\mathbf{H}^{(0)})$ satisfying (2.4)–(2.8). We integrate by parts and we find

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad=-\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}u_i^{(0)}+\partial_{x_i}v_i^e)(h_j^{(1)})^2e^{-\phi(x_1)}dx\\
&\qquad+\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\phi'(x_1)(u_1^{(0)}+v_1^e)(h_j^{(1)})^2e^{-\phi(x_1)}dx\\
&\quad=\frac{1}{2}\sum_{j=1}^{3}\int_\Omega\phi'(x_1)(u_1^{(0)}+v_1^e)(h_j^{(1)})^2e^{-\phi(x_1)}dx. \qquad (2.20)
\end{aligned}
$$

By direct computation we obtain

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega h_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)h_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad=\sum_{j=1}^{3}\int_\Omega\partial_{x_j}(u_j^{(0)}+v_j^e)(h_j^{(1)})^2e^{-\phi(x_1)}dx\\
&\qquad+\sum_{j=1}^{3}\sum_{i\neq j}\int_\Omega h_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)h_j^{(1)}e^{-\phi(x_1)}dx. \qquad (2.21)
\end{aligned}
$$

Noticing the incompressible condition

$$\nabla\cdot\mathbf{h}^{(1)}=0,\quad \nabla\cdot\mathbf{w}^{(1)}=0,$$

it follows that

$$\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx$$
$$=-\sum_{j=1}^{3}\int_{\Omega}((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})\partial_{x_j}h_j^{(1)}e^{-\phi(x_1)}dx$$
$$+\int_{\Omega}\phi'(x_1)((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_1^{(1)}e^{-\phi(x_1)}dx$$
$$=\int_{\Omega}\phi'(x_1)((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_1^{(1)}e^{-\phi(x_1)}dx. \tag{2.22}$$

Furthermore, from (2.14), using the standard Calderón–Zygmund theory and Young's inequality, it follows that

$$\left|\int_{\Omega}\phi'(x_1)((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_1^{(1)}e^{-\phi(x_1)}dx\right|$$
$$\leq\left|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\phi'(x_1)\Delta^{-1}(\partial_{x_j}h_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)\right.$$
$$\left.+\partial_{x_j}(u_i^{(0)}+v_i^e)\partial_{x_i}h_j^{(1)})h_1^{(1)}e^{-\phi(x_1)}dx\right|$$
$$+\left|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\phi'(x_1)\Delta^{-1}(\partial_{x_j}w_i^{(1)}\partial_{x_i}H_j^{(0)}+\partial_{x_j}H_i^{(0)}\partial_{x_i}w_j^{(1)})h_1^{(1)}e^{-\phi(x_1)}dx\right|$$
$$\lesssim\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\phi'(x_1)(\partial_{x_i}(u_j^{(0)}+v_j^e)+\partial_{x_j}(u_i^{(0)}+v_i^e))|(h_1^{(1)})^2e^{-\phi(x_1)}dx$$
$$+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\phi'(x_1)|(|\partial_{x_i}u_j^{(0)}|(\partial_{x_j}h_i^{(1)})^2+|\partial_{x_j}u_i^{(0)}|(\partial_{x_i}h_j^{(1)})^2)e^{-\phi(x_1)}dx$$
$$+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\phi'(x_1)|(|\partial_{x_i}H_j^{(0)}|(\partial_{x_j}w_i^{(1)})^2$$
$$+|\partial_{x_j}H_i^{(0)}|(\partial_{x_i}w_j^{(1)})^2)e^{-\phi(x_1)}dx. \tag{2.23}$$

Using Young's inequality and integrating by parts, we obtain

$$\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)h_j^{(1)}e^{-\phi(x_1)}dx\leq\frac{3\varepsilon}{2}\sum_{j=1}^{3}\int_{\Omega}(h_j^{(1)})^2e^{-\phi(x_1)}dx, \tag{2.24}$$

and

$$\left|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(H_i^{(0)}\partial_{x_i}w_j^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx\right|$$
$$\leq\left|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(w_j^{(1)}\partial_{x_i}H_i^{(0)})h_j^{(1)}e^{-\phi(x_1)}dx\right|$$
$$+\left|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(w_j^{(1)}H_i^{(0)})\partial_{x_i}h_j^{(1)}e^{-\phi(x_1)}dx\right|$$
$$\lesssim\frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}((h_j^{(1)})^2+2(w_j^{(1)})^2+(\partial_{x_i}h_j^{(1)})^2)e^{-\phi(x_1)}dx, \tag{2.25}$$

and

$$\left|\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(H_i^{(0)}\partial_{x_i}h_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx\right|$$
$$\lesssim\frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}((w_j^{(1)})^2+(\partial_{x_i}h_j^{(1)})^2)e^{-\phi(x_1)}dx. \tag{2.26}$$

Meanwhile, similar to (2.20)–(2.21), we derive

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}((u_i^{(0)}+v_i^e)\partial_{x_i}w_j^{(1)})w_j^{(1)}e^{-\phi(x_1)}dx$$
$$=\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\phi'(x_1)(u_1^{(0)}+v_1^e)(w_j^{(1)})^2e^{-\phi(x_1)}dx, \tag{2.27}$$

and

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(w_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e))w_j^{(1)}e^{-\phi(x_1)}dx$$
$$=\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}(u_j^{(0)}+v_j^e)(w_j^{(1)})^2e^{-\phi(x_1)}dx$$
$$+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}w_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)w_j^{(1)}e^{-\phi(x_1)}dx$$
$$\lesssim\frac{\varepsilon}{2}\sum_{j=1}^{3}\int_{\Omega}(w_j^{(1)})^2e^{-\phi(x_1)}dx, \tag{2.28}$$

and

$$\sum_{j=1}^{3}\int_\Omega (E_j^{(0)}h_j^{(1)} + \overline{E}_j^{(0)}w_j^{(1)})e^{-\phi(x_1)}dx$$
$$\leq \frac{1}{2}\sum_{j=1}^{3}\int_\Omega [(E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2 + |\phi''(x_1)|((h_j^{(1)})^2 + (w_j^{(1)})^2)]e^{-\phi(x_1)}dx. \tag{2.29}$$

Next, we substitute (2.20)–(2.29) into (2.19) and we deduce that

$$\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\left(\nu\lambda - \frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}(u_j^{(0)} + v_j^e)| + |\partial_{x_j}(u_i^{(0)} + v_i^e)|)\right)$$
$$\times((\partial_{x_i}h_j^{(1)})^2 + (\partial_{x_i}w_j^{(1)})^2)e^{-\phi(x_1)}dx$$
$$+\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega A_j(t,x)((h_j^{(1)})^2 + (w_j^{(1)})^2)e^{-\phi(x_1)}dx$$
$$\lesssim \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega ((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dx, \tag{2.30}$$

with the coefficients

$$A_1(t,x) := \frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) + \frac{\lambda}{2}(u_1^{(0)} + v_1^e)\phi'(x_1) + \partial_{x_1}(u_1^{(0)} + v_1^e)$$
$$-\frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|\phi'(x_1)(\partial_{x_i}(u_j^{(0)} + v_j^e) + \partial_{x_j}(u_i^{(0)} + v_i^e))|$$
$$-\frac{1}{2}|\phi''(x_1)| - c_1\varepsilon,$$
$$A_2(t,x) := \frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) + \frac{\lambda}{2}(u_1^{(0)} + v_1^e)\phi'(x_1) + \partial_{x_2}(u_2^{(0)} + v_2^e)$$
$$-\frac{1}{2}|\phi''(x_1)| - c_1\varepsilon,$$
$$A_3(t,x) := \frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) + \frac{\lambda}{2}(u_1^{(0)} + v_1^e)\phi'(x_1) + \partial_{x_3}(u_3^{(0)} + v_3^e)$$
$$-\frac{1}{2}|\phi''(x_1)| - c_1\varepsilon.$$

Since the weighted function $\phi(x_1)$ satisfies (2.15), the main term of $A_i(t,x)$ $(i = 1, 2, 3)$ is

$$\frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) - \frac{1}{2}|\phi''(x_1)|.$$

Thus, noticing that $u_j^{(0)} + v_j^e$ decays faster in x_1 than the function $\phi(x_1)$, there is a suitable constant $\lambda > 1$ such that

$$\begin{aligned} A_1(t,x) \quad \geq \quad & \frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) - \frac{1}{2}|\phi''(x_1)| \\ & - \frac{\lambda}{2}|\phi'(x_1)|(\|u_1^{(0)}\|_{L^\infty(\Omega)} + \|v_1^e\|_{L^\infty(\Omega)}) \\ & - \|\partial_{x_1}(u_1^{(0)} + v_1^e)\|_{L^\infty(\Omega)} - \frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|\phi'(x_1)|(\|\partial_{x_i}(u_j^{(0)} + v_j^e)\|_{L^\infty} \\ & + \|\partial_{x_j}(u_i^{(0)} + u_i^e)\|_{L^\infty}) - c_1\varepsilon \\ \gtrsim \; & \tfrac{\nu\lambda^2}{2}\,(\phi''(x_1) - (\phi'(x_1))^2) - \frac{1}{2}|\phi''(x_1)| - \varepsilon. \end{aligned}$$

By this relation and (2.15), we deduce that there exists a positive constant $C_{\nu,\lambda,\varepsilon}$ depending on $\nu, \lambda, \varepsilon$ such that

$$A_1(t,x) \gtrsim \frac{\nu\lambda^2}{2}(\phi''(x_1) - (\phi'(x_1))^2) - \varepsilon \geq \frac{\nu\kappa\lambda^2}{2} - \varepsilon \geq C_{\nu,\lambda,\varepsilon} > 0,$$

where $\kappa \in (0, \frac{1}{4})$. With similar arguments we deduce that

$$A_2(t,x), \quad A_3(t,x) \gtrsim C_{\nu,\lambda,\varepsilon},$$

and

$$\nu\lambda - \frac{1}{2}|\phi'(x_1)|(|\partial_{x_i}(u_j^{(0)} + v_j^e)| + |\partial_{x_j}(u_i^{(0)} + v_j^e)|) \gtrsim C_{\nu,\lambda}.$$

Therefore, note that $e^{-\phi(x_1)}$ is a bounded smooth function in $(0, +\infty)$, we reduce the inequality (2.30) into

$$\begin{aligned} &\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((\partial_{x_i}h_j^{(1)})^2 + (\partial_{x_i}w_j^{(1)})^2)dx + \sum_{j=1}^{3}\int_\Omega((h_j^{(1)})^2 + (w_j^{(1)})^2)dx \\ &\quad \leq \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega((E_j^{(0)})^2 + (\overline{E}_j^{(0)})^2)dx. \end{aligned}$$

The proof is now complete $\square$

Furthermore, we derive higher order derivative estimates. For a fixed integer $s \geq 1$, we apply $D_i^s := \partial_{x_i}^s$ $(\forall k = 1, 2, 3)$ to both sides of (2.12). It follows that

$$\begin{aligned} &-\nu\lambda^2\Delta D_i^s h_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^{3}(u_i^{(0)} + v_i^e)\partial_{x_i}D_i^s h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3}D_i^s h_i^{(1)}\partial_{x_i}(u_j^{(0)} + v_j^e) \\ &\quad + \Pi_{N_1}\partial_{x_j}D_i^s((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)}) \end{aligned}$$

$$
\begin{aligned}
&- \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_j} D_i^s w_j^{(1)} \\
&- \Pi_{N_1} \sum_{i=1}^{3} D_i^s w_i^{(1)} \partial_{x_i} H_j^{(0)} = F_j, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.31}
$$

and

$$
\begin{aligned}
-\nu\lambda^2 \Delta D_i^s w_j^{(1)} &+ \lambda \Pi_{N_1} \sum_{i=1}^{3} (u_i^{(0)} + v_i^e) \partial_{x_i} D_i^s w_j^{(1)} + \Pi_{N_1} \sum_{i=1}^{3} D_i^s h_i^{(1)} \partial_{x_i} H_j^{(0)} \\
&- \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_i} D_i^s h_j^{(1)} \\
&- \Pi_{N_1} \sum_{i=1}^{3} D_i^s w_i^{(1)} \partial_{x_i} (u_j^{(0)} + v_j^e) = \overline{F}_j, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.32}
$$

with the boundary condition

$$
\begin{aligned}
D_i^l h_j^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0, \\
D_i^l w_j^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0,
\end{aligned} \tag{2.33}
$$

where the integer l satisfies $1 \leq l \leq s$, and

$$
\begin{aligned}
F_j := \Pi_{N_1} D_i^s E_j^{(0)} &- \lambda \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} D_i^{s_1} (u_i^{(0)} + v_i^e) \partial_{x_i} D_i^{s_2} h_j^{(1)} \\
&- \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(D_i^{s_1} \partial_{x_i} (u_j^{(0)} + v_j^e)) \\
&+ \lambda \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} (D_i^{s_1} H_i^{(0)})(\partial_{x_j} D_i^{s_2} w_j^{(1)}) \\
&+ \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} (D_i^{s_2} w_i^{(1)})(\partial_{x_i} D_i^{s_1} H_j^{(0)}), \\
\overline{F}_j := \Pi_{N_1} D_i^s \overline{E}_j^{(0)} &- \lambda \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} D_i^{s_1} (u_i^{(0)} + v_i^e)(\partial_{x_i} D_i^{s_2} w_j^{(1)}) \\
&- \Pi_{N_1} \sum_{s_1+s_2=s, 0 \leq s_2 \leq s-1} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(\partial_{x_i} D_i^{s_1} H_j^{(0)})
\end{aligned}
$$

$$+\lambda\Pi_{N_1}\sum_{s_1+s_2=s,0\le s_2\le s-1}\sum_{i=1}^{3}(D_i^{s_1}H_i^{(0)})(\partial_{x_i}D_i^{s_2}h_j^{(1)})$$

$$+\Pi_{N_1}\sum_{s_1+s_2=s,0\le s_2\le s-1}\sum_{i=1}^{3}(D_i^{s_2}w_i^{(1)})(\partial_{x_i}D_i^{s_1}(u_j^{(0)}+v_j^e)).$$

Next, we derive higher derivative estimates of solutions to problem (2.12)–(2.13).

Lemma 2.2. *Let* $0<\nu\ll 1$*. Assume that* (1.5) *holds and the initial approximation function* $(\boldsymbol{u}^{(0)},\boldsymbol{H}^{(0)})$ *satisfies conditions* (2.4)–(2.8)*. Then the solution* $(\boldsymbol{h}^{(1)}(\lambda x),\boldsymbol{w}^{(1)}(\lambda x))$ *of the linear system* (2.12)–(2.13) *satisfies*

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2)dx$$

$$\left.\lesssim\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{s}\int_\Omega((D_i^k E_j^{(0)})^2+(D_i^k\overline{E}_j^{(0)})^2)dxdt\right]. \tag{2.34}$$

Proof. This proof is based on an induction argument.

Let $s=1$. By (2.31)–(2.32), we have

$$-\nu\lambda^2\Delta D_i^1 h_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}(u_i^{(0)}+v_i^e)\partial_{x_i}D_i^1 h_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}D_i^1 h_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e)$$
$$+\Pi_{N_1}\partial_{x_j}D_i^1((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})$$
$$+\lambda\Pi_{N_1}\sum_{i=1}^{3}D_i^1(u_i^{(0)}+v_i^e)\partial_{x_i}h_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}h_i^{(1)}D_i^1\partial_{x_i}(u_j^{(0)}+v_j^e)$$
$$-\lambda\Pi_{N_1}\sum_{i=1}^{3}H_i^{(0)}\partial_{x_j}D_i^1 w_j^{(1)}-\Pi_{N_1}\sum_{i=1}^{3}D_i^1 w_i^{(1)}\partial_{x_i}H_j^{(0)}$$
$$-\lambda\Pi_{N_1}\sum_{i=1}^{3}D_i^1 H_i^{(0)}\partial_{x_j}w_j^{(1)}$$
$$-\Pi_{N_1}\sum_{i=1}^{3}w_i^{(1)}\partial_{x_i}D_i^1 H_j^{(0)}=\Pi_{N_1}D_i^1 E_j^{(0)},\quad\text{for } j=1,2,3, \tag{2.35}$$

and

$$-\nu\lambda^2\Delta D_i^1 w_j^{(1)}+\lambda\Pi_{N_1}\sum_{i=1}^{3}(u_i^{(0)}+v_i^e)\partial_{x_i}D_i^1 w_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}D_i^1 h_i^{(1)}\partial_{x_i}H_j^{(0)}$$
$$+\lambda\Pi_{N_1}\sum_{i=1}^{3}D_i^1(u_i^{(0)}+v_i^e)\partial_{x_i}w_j^{(1)}+\Pi_{N_1}\sum_{i=1}^{3}h_i^{(1)}\partial_{x_i}D_i^1 H_j^{(0)}$$

$$
\begin{aligned}
&- \lambda \Pi_{N_1} \sum_{i=1}^{3} H_i^{(0)} \partial_{x_i} D_i^1 h_j^{(1)} \\
&- \Pi_{N_1} \sum_{i=1}^{3} D_i^1 w_i^{(1)} \partial_{x_i} (u_j^{(0)} + v_j^e) - \lambda \Pi_{N_1} \sum_{i=1}^{3} D_i^1 H_i^{(0)} \partial_{x_i} h_j^{(1)} \\
&- \Pi_{N_1} \sum_{i=1}^{3} w_i^{(1)} \partial_{x_i} D_i^1 (u_j^{(0)} + v_j^e) = \Pi_{N_1} D_i^1 \overline{E}_j, \quad \text{for } j = 1, 2, 3,
\end{aligned} \tag{2.36}
$$

with the boundary condition

$$
D_i^1 h_j^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0, \quad D_i^1 w_j^{(1)}(\lambda x)|_{x \in \partial\Omega} = 0. \tag{2.37}
$$

Multiplying both sides of (2.35)–(2.33) by $D_i^1 h_j^{(1)} e^{-\phi(x_1)}$ and $D_i^1 w_j^{(1)} e^{-\phi(x_1)}$, respectively, then integrating over Ω by noticing (2.37), and summing up those equalities from $j = 1$ to $j = 3$, we deduce that

$$
\begin{aligned}
&\nu\lambda^2 \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega ((\partial_{x_k} D_i^1 h_j^{(1)})^2 + (\partial_{x_k} D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx \\
&\quad + \frac{\nu\lambda^2}{2} \sum_{j=1}^{3} \int_\Omega (\phi''(x_1) - (\phi'(x_1))^2)((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx \\
&\quad + \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (u_i^{(0)} + v_i^e)(\partial_{x_i} D_i^1 h_j^{(1)})(D_i^1 h_j^{(1)}) e^{-\phi(x_1)} dx \\
&\quad + \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (D_i^1 h_i^{(1)})(\partial_{x_i} (u_j^{(0)} + v_j^e))(D_i^1 h_j^{(1)}) e^{-\phi(x_1)} dx \\
&\quad + \lambda \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (u_i^{(0)} + v_i^e)(\partial_{x_i} D_i^1 w_j^{(1)})(D_i^1 w_j^{(1)}) e^{-\phi(x_1)} dx \\
&\quad - \Pi_{N_1} \sum_{j=1}^{3} \sum_{i=1}^{3} \int_\Omega (D_i^1 w_i^{(1)})(\partial_{x_i} (u_j^{(0)} + v_j^e))(D_i^1 w_j^{(1)}) e^{-\phi(x_1)} dx \\
&\quad + \Pi_{N_1} \sum_{j=1}^{3} \int_\Omega \partial_{x_j} D_i^1 ((\mathcal{D}_{\mathbf{u}^{(0)}} P)\mathbf{h}^{(1)} + (\mathcal{D}_{\mathbf{H}^{(0)}} P)\mathbf{w}^{(1)}) D_i^1 h_j^{(1)} e^{-\phi(x_1)} dx \\
&\quad + \Pi_{N_1} \sum_{k=1}^{11} I_k = 0,
\end{aligned} \tag{2.38}
$$

where

$$I_1 := \lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1(u_i^{(0)}+v_i^e))(\partial_{x_i}h_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_2 := \sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega h_i^{(1)}(D_i^1\partial_{x_i}(u_j^{(0)}+v_j^e))(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_3 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega H_i^{(0)}(\partial_{x_j}D_i^1w_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_4 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1H_i^{(0)}(\partial_{x_j}w_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_5 := -\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega w_i^{(1)}(\partial_{x_i}D_i^1H_j^{(0)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_6 := \lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1(u_i^{(0)}+v_i^e))(\partial_{x_i}w_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_7 := \sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega h_i^{(1)}(\partial_{x_i}D_i^1H_j^{(0)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_8 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega H_i^{(0)}(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_9 := -\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1H_i^{(0)})(\partial_{x_i}h_j^{(1)})(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_{10} := -\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega w_i^{(1)}(\partial_{x_i}D_i^1(u_j^{(0)}+v_j^e))(D_i^1w_j^{(1)})e^{-\phi(x_1)}dx,$$

$$I_{11} := \sum_{j=1}^{3}\int_\Omega ((D_i^1E_j^{(0)})(D_i^1h_j^{(1)})+(D_i^1\overline{E}_j^{(0)})(D_i^1w_j^{(1)}))dx.$$

We now estimate each term in (2.38). On the one hand, since we have chosen the initial approximation function $(\mathbf{u}^{(0)},\mathbf{H}^{(0)})^T$ satisfying (2.4)–(2.8), using the similar method of getting (2.20)–(2.22), we get

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (u_i^{(0)}+v_i^e)(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1h_j^{(1)})e^{-\phi(x_1)}dx$$
$$\lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (|\phi'(x_1)|+1)(D_i^1h_j^{(1)})^2e^{-\phi(x_1)}dx, \tag{2.39}$$

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_i^{(1)})(\partial_{x_i}(u_j^{(0)}+v_j^e))(D_i^1 h_j^{(1)})e^{-\phi(x_1)}dx\\
&\quad=\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_i}(u_i^{(0)}+v_i^e))(D_i^1 h_i^{(1)})^2 e^{-\phi(x_1)}dx\\
&\qquad+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}(\partial_{x_i}(u_j^{(0)}+v_j^e))(D_i^1 h_i^{(1)})(D_i^1 h_j^{(1)})e^{-\phi(x_1)}dx\\
&\quad\lesssim \varepsilon\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_j^{(1)})^2 e^{-\phi(x_1)}dx, \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(u_i^{(0)}+v_i^e)(\partial_{x_i}D_i^1 w_j^{(1)})(D_i^1 w_j^{(1)})e^{-\phi(x_1)}dx\\
&\quad\lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(|\phi'(x_1)|+1)(D_i^1 w_j^{(1)})^2 e^{-\phi(x_1)}dx, \qquad (2.41)
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 w_i^{(1)})(\partial_{x_i}(u_j^{(0)}+v_j^e))(D_i^1 w_j^{(1)})e^{-\phi(x_1)}dx\\
&\quad\lesssim \varepsilon\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 w_j^{(1)})^2 e^{-\phi(x_1)}dx. \qquad (2.42)
\end{aligned}$$

By the incompressible condition, we have

$$\begin{aligned}
&\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}D_i^1((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})D_i^1 h_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad\lesssim \varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\phi'(x_1)|((D_i h_j^{(1)})^2+(\partial_{x_j}D_i^1 h_j^{(1)})^2+(\partial_{x_j}D_i^1 w_j^{(1)})^2)e^{-\phi(x_1)}dx. \qquad (2.43)
\end{aligned}$$

Moreover, we use Young's inequality to derive

$$I_1\lesssim \varepsilon\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 h_j^{(1)})^2 e^{-\phi(x_1)}dx,$$

$$I_2\lesssim \frac{\varepsilon}{2}\int_{\Omega}\left(\sum_{i=1}^{3}(h_i^{(1)})^2+\sum_{j=1}^{3}(D_i^1 h_j^{(1)})^2\right)e^{-\phi(x_1)}dx,$$

$$I_3 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((\partial_{x_j} D_i^1 w_j^{(1)})^2 + (D_i^1 h_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_4 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((\partial_{x_j} w_j^{(1)})^2 + (D_i^1 h_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_5 \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((w_i^{(1)})^2 + (D_i^1 h_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_6 \lesssim \varepsilon\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_i} w_j^{(1)})^2 e^{-\phi(x_1)} dx,$$

$$I_7 \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((h_i^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_8 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((\partial_{x_i} D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_9 \lesssim \frac{\varepsilon\lambda}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((\partial_{x_i} h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$I_{10} \lesssim \frac{\varepsilon}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((w_i^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx,$$

$$\begin{aligned} I_{11} \lesssim \frac{1}{2}\sum_{j=1}^{3}\int_\Omega [(D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2 \\ + |\phi''(x_1)|((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2)] e^{-\phi(x_1)} dx, \end{aligned} \tag{2.44}$$

thus, by noticing the weighted function $\phi(x_1)$ satisfying (2.15), and using (2.39)–(2.44), we reduce the inequality (2.38) into

$$\begin{aligned} &C_{\nu,\lambda}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((\partial_{x_k} D_i^1 h_j^{(1)})^2 + (\partial_{x_k} D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx \\ &\quad + C_{\nu,\lambda,\varepsilon}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((D_i^1 h_j^{(1)})^2 + (D_i^1 w_j^{(1)})^2) e^{-\phi(x_1)} dx \\ &\lesssim \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega ((D_i^1 E_j^{(0)})^2 + (D_i^1 \overline{E}_j^{(0)})^2) e^{-\phi(x_1)} dx \\ &\quad + \varepsilon\sum_{j=1}^{3}\int_\Omega ((h_j^{(1)})^2 + (w_j^{(1)})^2) e^{-\phi(x_1)} dx. \end{aligned} \tag{2.45}$$

We observe that the last term in the right-hand side of (2.45) can be controlled by using (2.16). Hence, by noticing $e^{-\phi(x_1)}$ is a bounded smooth function in $(0,+\infty)$, it follows from (2.45) that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega((D_i^1h_j^{(1)})^2+(D_i^1w_j^{(1)})^2)e^{-\phi(x_1)}dx \lesssim \Pi_{N_1}\sum_{j=1}^{3}\Bigg[\int_\Omega((E_j^{(0)})^2+(\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dx + \sum_{i=1}^{3}\int_\Omega((D_i^1E_j^{(0)})^2+(D_i^1\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dx\Bigg]. \tag{2.46}$$

Assume that the $2\le l\le s-1$ derivative case holds, that is,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega((D_i^lh_j^{(1)})^2+(D_i^lw_j^{(1)})^2)e^{-\phi(x_1)}dx \lesssim \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{l}\int_\Omega((D_i^kE_j^{(0)})^2+(D_i^k\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dx\Bigg]. \tag{2.47}$$

We now prove that the sth derivative case holds. Multiplying both sides of equations (2.31)–(2.32) by $D_i^sh_j^{(1)}e^{-\phi(x_1)}$ and $D_i^sw_j^{(1)}e^{-\phi(x_1)}$, respectively, then integrating over Ω by using the boundary condition (2.33), and summing up these equalities from $j=1$ to $j=3$, we obtain

$$\begin{aligned}
&\nu\lambda^2\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((\partial_{x_k}D_i^sh_j^{(1)})^2+(\partial_{x_k}D_i^sw_j^{(1)})^2)e^{-\phi(x_1)}dx\\
&\quad+\frac{\nu\lambda^2}{2}\sum_{j=1}^{3}\int_\Omega(\phi''(x_1)-(\phi'(x_1))^2)((D_i^sh_j^{(1)})^2+(D_i^sw_j^{(1)})^2)e^{-\phi(x_1)}dx\\
&\quad+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}D_i^sh_j^{(1)})D_i^sh_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(D_i^sh_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e))D_i^sh_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\partial_{x_j}D_i^s((\mathcal{D}_{\mathbf{u}^{(0)}}P)\mathbf{h}^{(1)}+(\mathcal{D}_{\mathbf{H}^{(0)}}P)\mathbf{w}^{(1)})h_j^{(1)}e^{-\phi(x_1)}dx\\
&\quad+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((u_i^{(0)}+v_i^e)\partial_{x_i}D_i^sw_j^{(1)})D_i^sw_j^{(1)}e^{-\phi(x_1)}dx
\end{aligned}$$

$$-\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(H_i^{(0)}\partial_{x_i}D_i^s h_j^{(1)})D_i^s w_j^{(1)}e^{-\phi(x_1)}dx$$

$$-\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^s w_i^{(1)}\partial_{x_i}(u_j^{(0)}+v_j^e))D_i^s w_j^{(1)}e^{-\phi(x_1)}dx$$

$$=\sum_{j=1}^{3}\int_{\Omega}(F_jD_i^s h_j^{(1)}+\overline{F}_jD_i^s w_j^{(1)})e^{-\phi(x_1)}dx. \tag{2.48}$$

We notice that

$$\sum_{s_1+s_2=s,0\le s_2\le s-1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}(u_i^{(0)}+v_i^e)(\partial_{x_i}D_i^{s_2}h_j^{(1)})(D_i^s h_j^{(1)})e^{-\phi(x_1)}dx$$

$$=\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}(u_i^{(0)}+v_i^e))(D_i^s h_j^{(1)})^2e^{-\phi(x_1)}dx+\sum_{s_1+s_2=s,0\le s_2\le s-2}\sum_{j=1}^{3}\sum_{i=1}^{3}$$

$$\times\int_{\Omega}(D_i^{s_1}(u_i^{(0)}+v_i^e))(\partial_{x_i}D_i^{s_2}h_j^{(1)})(D_i^s h_j^{(1)})e^{-\phi(x_1)}dx. \tag{2.49}$$

Next, with the same arguments as for obtaining (2.45), we find

$$C_{\nu,\lambda}\sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}((\partial_{x_k}D_i^s h_j^{(1)})^2+(\partial_{x_k}D_i^s w_j^{(1)})^2)e^{-\phi(x_1)}dxdt$$

$$+C_{\nu,\lambda,\varepsilon}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2)e^{-\phi(x_1)}dxdt$$

$$\lesssim\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_0^t\int_{\Omega}((D_i^s E_j^{(0)})^2+(D_i^s\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dxdt$$

$$+\varepsilon\sum_{j=1}^{3}\sum_{l=0}^{s-1}\int_0^t\int_{\Omega}((D_i^l h_j^{(1)})^2+(D_i^l w_j^{(1)})^2)e^{-\phi(x_1)}dxdt. \tag{2.50}$$

Hence, by (2.4)–(2.8) and (2.47), the inequality (2.50) can be reduced into

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}((D_i^s h_j^{(1)})^2+(D_i^s w_j^{(1)})^2)e^{-\phi(x_1)}dx$$

$$\lesssim\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=0}^{s}\int_{\Omega}((D_i^k E_j^{(0)})^2+(D_i^k\overline{E}_j^{(0)})^2)e^{-\phi(x_1)}dx\Bigg],$$

which combining with $e^{-\phi(x_1)}$ being a bounded smooth function in $(0,+\infty)$ gives (2.34). The proof is now complete. □

2.4. *The existence of the first approximation step*

Based on above *a priori* estimates, we are ready to prove the existence of first approximation step by the classical theory of elliptic equations [14, 36].

Proposition 2.1. *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds and the initial approximation function* $(\boldsymbol{u}^{(0)}, \boldsymbol{H}^{(0)})$ *satisfies conditions* (2.4)–(2.8). *Then the linearized system* (2.10) *with the boundary condition* (2.11) *admits a Sobolev regular solution*

$$(\boldsymbol{h}^{(1)}(\lambda x), \boldsymbol{w}^{(1)}(\lambda x)) \in H^s(\Omega) \times H^s(\Omega),$$

and

$$\partial^l_{x_i} \boldsymbol{h}^{(1)}(x)|_{x\in\partial\Omega} = 0, \quad \partial^l_{x_i} \boldsymbol{w}^{(1)}(x)|_{x\in\partial\Omega} = 0.$$

Moreover, this solution satisfies

$$\|\boldsymbol{h}^{(1)}\|^2_{H^s} + \|\boldsymbol{w}^{(1)}\|^2_{H^s} \lesssim \|\Pi_{N_1} E^{(0)}\|^2_{H^s} + \|\Pi_{N_1} \overline{E}^{(0)}\|^2_{H^s}. \tag{2.51}$$

Proof. Let $\mathbb{P}$ the Leray projector onto the space of divergence free functions. We apply the Leray projector to equations (2.10), hence

$$\begin{aligned} &-\nu\lambda^2\mathbb{P}\Delta\mathbf{h} + \mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \mathbb{P}E_0, \\ &-\nu\lambda^2\mathbb{P}\Delta\mathbf{w} + \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) = \mathbb{P}\overline{E}_0, \end{aligned} \tag{2.52}$$

where

$$\begin{aligned} \mathbb{N}_1(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) &:= \mathbb{P}\Pi_{N_1}[\lambda((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{h}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e) \\ &\quad - \lambda(\mathbf{H}^{(0)}\cdot\nabla)\mathbf{w}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)\mathbf{H}^{(0)}], \\ \mathbb{N}_2(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) &:= \mathbb{P}\Pi_{N_1}[\lambda((\mathbf{u}^{(0)} + \mathbf{v}^e)\cdot\nabla)\mathbf{w}^{(1)} + (\mathbf{h}^{(1)}\cdot\nabla)\mathbf{H}^{(0)} \\ &\quad - \lambda(\mathbf{H}^{(0)}\cdot\nabla)\mathbf{h}^{(1)} - (\mathbf{w}^{(1)}\cdot\nabla)(\mathbf{u}^{(0)} + \mathbf{v}^e)]. \end{aligned}$$

By (2.16) in Lemma 2.1 and (2.34) in Lemma 2.2, we can get the uniform bound estimate

$$\|\mathbf{h}^{(1)}\|^2_{H^s} + \|\mathbf{w}^{(1)}\|^2_{H^s} \lesssim \|\Pi_{N_1} E^{(0)}\|^2_{H^s} + \|\Pi_{N_1} \overline{E}^{(0)}\|^2_{H^s}.$$

From the standard theory of elliptic equations of general order [14, 36], the linear elliptic equations (2.52) admit a unique weak solution $(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \in H^1 \times H^1$ if $(E^{(0)}, \overline{E}^{(0)}) \in H^1 \times H^1$. Since the error term $(E^{(0)}, \overline{E}^{(0)}) \in H^s \times H^s$ for $s \geq 1$, we obtain the solution $(\mathbf{h}^{(1)}, \mathbf{w}^{(1)}) \in H^s \times H^s$.

Noticing that

$$\partial^l_{x_i} E^{(0)}(x)|_{x\in\partial\Omega} = 0, \quad \partial^l_{x_i} \overline{E}^{(0)}(x)|_{x\in\partial\Omega} = 0,$$

it follows that

$$\partial^l_{x_i} \mathbf{h}^{(1)}(x)|_{x\in\partial\Omega} = 0, \quad \partial^l_{x_i} \mathbf{w}^{(1)}(x)|_{x\in\partial\Omega} = 0. \qquad \square$$

3. The mth Approximation Step

Let $\varepsilon \in (0,1)$ be a fixed constant. We define

$$\mathcal{B}_\varepsilon := \{(\mathbf{u}^{(k)}, \mathbf{w}^{(k)}) : \|\mathbf{u}^{(k)}\|_{H^s} + \|\mathbf{w}^{(k)}\|_{H^s} \lesssim \varepsilon < 1\} \tag{3.1}$$

with the integers $2 \le k \le m-1$ and $s \ge 1$.

Assume that the mth approximation solution of (2.2) is denoted by $(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x))^T$ with $m = 2, 3, \ldots$. Let

$$\mathbf{h}^{(m)}(\lambda x) := \mathbf{u}^{(m)}(\lambda x) - \mathbf{u}^{(m-1)}(\lambda x),$$
$$\mathbf{w}^{(m)}(\lambda x) := \mathbf{H}^{(m)}(\lambda x) - \mathbf{H}^{(m-1)}(\lambda x).$$

It follows that

$$\mathbf{u}^{(m)}(\lambda x) = \mathbf{u}^{(0)}(x) + \sum_{i=1}^{m} \mathbf{h}^{(i)}(\lambda x),$$
$$\mathbf{H}^{(m)}(\lambda x) = \mathbf{H}^{(0)}(x) + \sum_{i=1}^{m} \mathbf{w}^{(i)}(\lambda x).$$

We linearize the nonlinear system (2.2) around $(\mathbf{u}^{(m-1)}(\lambda x), \mathbf{H}^{(m-1)}(\lambda x))^T$ to get the following initial value problem:

$$\begin{aligned} &\mathcal{J}_1[\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = \Pi_{N_m} E^{(m-1)}, \\ &\mathcal{J}_2[\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = \Pi_{N_m} \overline{E}^{(m-1)}, \\ &\nabla \cdot \mathbf{h}^{(m)} = 0, \quad \nabla \cdot \mathbf{w}^{(m)} = 0, \end{aligned} \tag{3.2}$$

with the boundary conditions

$$\mathbf{h}^{(m)}(\lambda x)|_{x\in\partial\Omega} = 0, \quad \mathbf{w}^{(m)}(\lambda x)|_{x\in\partial\Omega} = 0. \tag{3.3}$$

The error terms are given by

$$\begin{aligned} E^{(m-1)} &:= \mathcal{L}_1[\mathbf{u}^{m-1}(\lambda x), \mathbf{H}^{m-1}(\lambda x)] = \mathcal{R}_1(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x)), \\ \overline{E}^{(m-1)} &:= \mathcal{L}_2[\mathbf{u}^{m-1}(\lambda x), \mathbf{H}^{m-1}(\lambda x)] = \mathcal{R}_2(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x)), \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\mathcal{R}_1(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x)) \\ &\quad := \mathcal{L}_1(\mathbf{u}^{(m-1)} + \mathbf{h}^{(m)}, \mathbf{H}^{(m-1)} + \mathbf{w}^{(m)}) - \mathcal{L}_1(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}) \\ &\qquad - \mathcal{L}_1[\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) \\ &\quad = \lambda(-(\mathbf{h}^{(m)} \cdot \nabla)\mathbf{h}^{(m)} + (\mathbf{w}^{(m)} \cdot \nabla)\mathbf{w}^{(m)} + \nabla P^{(m)}), \end{aligned}$$

$$\begin{aligned}&\mathcal{R}_2(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x))\\&\quad := \mathcal{L}_2(\mathbf{u}^{(m-1)} + \mathbf{h}^{(m)}, \mathbf{H}^{(m-1)} + \mathbf{w}^{(m)}) - \mathcal{L}_2(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)})\\&\qquad - \mathcal{L}_2[\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}](\mathbf{h}^{(m)}, \mathbf{w}^{(m)})\\&\quad = \lambda(-(\mathbf{h}^{(m)} \cdot \nabla)\mathbf{w}^{(m)} + (\mathbf{w}^{(m)} \cdot \nabla)\mathbf{h}^{(m)}),\end{aligned} \tag{3.5}$$

where the approximation pressure takes the form

$$P^{(m)}(x) = -\Delta^{-1}\mathrm{div}(\mathbf{u}^{(m)} \cdot \nabla \mathbf{u}^{(m)} - (\mathbf{H}^{(m)} \cdot \nabla)\mathbf{H}^{(m)}),$$

which is also the nonlinear term in approximation problem (2.2) at $(\mathbf{u}^{(m-1)}(\lambda x), \mathbf{H}^{(m-1)}(\lambda x))^T$.

The following result establishes how to construct the mth approximation solution.

Proposition 3.1. *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds*, *the initial approximation function* $(\boldsymbol{u}^{(0)}, \boldsymbol{H}^{(0)})$ *satisfies conditions* (2.4)–(2.8), *and* $(\boldsymbol{u}^{(m-1)}(\lambda x), \boldsymbol{H}^{(m-1)}(\lambda x))^T \in \mathcal{B}_\varepsilon$, *and*

$$\sum_{i=1}^{m-1} (\|\boldsymbol{h}^{(i)}\|_{H^s}^2 + \|\boldsymbol{w}^{(i)}\|_{H^s}^2) \lesssim \varepsilon^2,$$

Then the linearized problem (3.2) *with the boundary condition* (3.3) *admits a Sobolev regular solution*

$$(\boldsymbol{h}^{(m)}(\lambda x), \boldsymbol{w}^{(m)}(\lambda x)) \in H^s(\Omega) \times H^s(\Omega),$$

which satisfies

$$\|\boldsymbol{h}^{(m)}\|_{H^s}^2 + \|\boldsymbol{w}^{(m)}\|_{H^s}^2 \lesssim \|\Pi_{N_m} E^{(m-1)}\|_{H^s}^2 + \|\Pi_{N_m} \overline{E}^{(m-1)}\|_{H^s}^2, \tag{3.6}$$

where the error term satisfies

$$\|E^{(m)}\|_{H^s} + \|\overline{E}^{(m)}\|_{H^s} \lesssim \lambda^{s+2} N_m^2 (\|\boldsymbol{h}^{(m)}\|_{H^s}^2 + \|\boldsymbol{w}^{(m)}\|_{H^s}^2). \tag{3.7}$$

Proof. By the assumption $\sum_{i=1}^{m-1}(\|\mathbf{h}^{(i)}\|_{H^s}^2 + \|\mathbf{w}^{(i)}\|_{H^s}^2) \lesssim \varepsilon^2$ for any $s \geq 1$, we find

$$\begin{aligned}\sum_{k=0}^{s} \partial_{x_i}^k \mathbf{u}_j^{(m-1)} &= \sum_{k=0}^{s} \partial_{x_i}^k \mathbf{u}_j^{(0)}(x) + \sum_{i=1}^{m-1} \sum_{k=0}^{s} \partial_{x_i}^k \mathbf{h}^{(i)}(\lambda x)\\&:= \sum_{k=0}^{s} \partial_{x_i}^k \mathbf{u}_j^{(0)}(x) + \mathcal{O}(\varepsilon^2),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{s}\partial_{x_i}^k\mathbf{H}_j^{(m-1)} &= \sum_{k=0}^{s}\partial_{x_i}^k\mathbf{H}_j^{(0)}(x)+\sum_{i=1}^{m-1}\sum_{k=0}^{s}\partial_{x_i}^k\mathbf{w}^{(i)}(\lambda x)\\ &:= \sum_{k=0}^{s}\partial_{x_i}^k\mathbf{H}_j^{(0)}(x)+\mathcal{O}(\varepsilon^2),\end{aligned}$$

Thus, noticing that $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$ satisfies (2.4)–(2.8), it follows that

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k\mathbf{u}_j^{(m-1)}(x)\|_{L^\infty}\lesssim\varepsilon^2, \tag{3.8}$$

and

$$\sum_{k=0}^{s}\|\Pi_{N_m}\partial_{x_i}^k\mathbf{H}_j^{(m-1)}(x)\|_{L^\infty}\lesssim\varepsilon^2. \tag{3.9}$$

The $(m-1)$th approximation solution is

$$\mathbf{u}^{(m-1)}(\lambda x)=\mathbf{u}^{(0)}(x)+\sum_{i=1}^{m-1}\mathbf{h}^{(i)}(\lambda x),\quad \mathbf{H}^{(m-1)}(\lambda x)=\mathbf{H}^{(0)}(x)+\sum_{i=1}^{m-1}\mathbf{w}^{(i)}(\lambda x).$$

Moreover, we have

$$\begin{cases}\nabla\cdot\mathbf{u}^{(m-1)}(x)=0,\\ \|\mathbf{u}^{(m-1)}\|_{H^s}\lesssim\varepsilon^2,\\ \mathbf{u}^{(m-1)}(x)|_{x\in\partial\Omega}=0,\end{cases} \tag{3.10}$$

and

$$\begin{cases}\nabla\cdot\mathbf{H}^{(m-1)}(x)=0,\\ \|\mathbf{H}^{(m-1)}\|_{H^s}\lesssim\varepsilon^2,\\ \mathbf{H}^{(m-1)}(x)|_{x\in\partial\Omega}=0.\end{cases} \tag{3.11}$$

Next, we find the mth approximation solution $(\mathbf{u}^{(m)}(\lambda x), \mathbf{H}^{(m)}(\lambda x))^T$, which is equivalent to find $(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x))^T$ such that

$$\begin{aligned}\mathbf{u}^{(m)}(\lambda x)&=\mathbf{u}^{(m-1)}(\lambda x)+\mathbf{h}^{(m)}(\lambda x),\\ \mathbf{H}^{(m)}(\lambda x)&=\mathbf{H}^{(m-1)}(\lambda x)+\mathbf{w}^{(m)}(\lambda x).\end{aligned} \tag{3.12}$$

Substituting (3.12) into (2.2), we obtain

$$\begin{aligned}\mathcal{L}_1(\mathbf{u}^{(m)},\mathbf{H}^{(m)})&=\mathcal{L}_1(\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)})+\mathcal{L}_1[(\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})\\ &\quad+\mathcal{R}_1(\mathbf{h}^{(m)},\mathbf{w}^{(m)}),\\ \mathcal{L}_2(\mathbf{u}^{(m)},\mathbf{H}^{(m)})&=\mathcal{L}_2(\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)})+\mathcal{L}_2[(\mathbf{u}^{(m-1)},\mathbf{H}^{(m-1)})](\mathbf{h}^{(m)},\mathbf{w}^{(m)})\\ &\quad+\mathcal{R}_2(\mathbf{h}^{(m)},\mathbf{w}^{(m)}).\end{aligned}$$

Set

$$\mathcal{L}_1[(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)})](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = -\mathcal{L}_1(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}) = -E^{(m-1)},$$

$$\mathcal{L}_2[(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)})](\mathbf{h}^{(m)}, \mathbf{w}^{(m)}) = -\mathcal{L}_2(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}) = -\overline{E}^{(m-1)},$$

which we supplement with the boundary conditions (3.3).

Since we assume $(\mathbf{u}^{(m-1)}(\lambda x), \mathbf{H}^{(m-1)}(\lambda x))^T \in \mathcal{B}_\varepsilon$, then there is the same structure between the linear system (2.10) and the linear system of mth approximation solutions. Thus, by means of the same proof process as in Proposition 2.1, we can show that the above problem admits a solution $(\mathbf{h}^{(m)}(\lambda x), \mathbf{w}^{(m)}(\lambda x))^T \in H^s(\Omega) \times H^s(\Omega)$. Here, we should use (2.1). Furthermore, similar to (2.51), we can use (3.8)–(3.11) to derive

$$\|\mathbf{h}^{(m)}\|_{H^s}^2 + \|\mathbf{w}^{(m)}\|_{H^s}^2 \lesssim \|E^{(m-1)}\|_{H^s}^2 + \|\overline{E}^{(m-1)}\|_{H^s}^2, \quad \forall t > 0,$$

where one can see the $(m-1)$th error term $(E^{(m-1)}, \overline{E}^{(m-1)})^T$ such that

$$E^{(m-1)} := \mathcal{L}_1(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}) = \mathcal{R}_1(\mathbf{h}^{(m)}, \mathbf{w}^{(m)}),$$

$$\overline{E}^{(m-1)} := \mathcal{L}_2(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)}) = \mathcal{R}_2(\mathbf{h}^{(m)}, \mathbf{w}^{(m)}).$$

Furthermore, by (3.5) and the Calderón–Zygmund theory, it follows that

$$\begin{aligned}
&\|E^{(m)}\|_{H^s} + \|\overline{E}^{(m)}\|_{\mathbb{H}^s} \\
&\quad = \lambda\|\Pi_{N_m}(\mathbf{h}^{(m)} \cdot \nabla \mathbf{h}^{(m)})\|_{H^s} + \lambda\|\Pi_{N_m}(\mathbf{w}^{(m)} \cdot \nabla \mathbf{w}^{(m)})\|_{H^s} \\
&\qquad + \lambda\|\Pi_{N_m}(\mathbf{h}^{(m)} \cdot \nabla \mathbf{w}^{(m)})\|_{H^s} \\
&\qquad + \lambda\|\Pi_{N_m}(\mathbf{w}^{(m)} \cdot \nabla \mathbf{h}^{(m)})\|_{H^s} + \lambda\|\Pi_{N_m}\nabla P^{(m)}\|_{H^s} \\
&\quad \lesssim \lambda^{s+2} N_m^2 (\|\mathbf{h}^{(m)}\|_{H^s}^2 + \|\mathbf{w}^{(m)}\|_{H^s}^2).
\end{aligned}$$

The proof is now complete. □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 depends on the existence of Sobolev regular solution of the nonlinear perturbation equations (1.12). In order to complete it, we should prove that the series $\sum_{i=1}^m \mathbf{h}^{(i)}(\lambda x)$ and $\sum_{i=1}^m \mathbf{w}^{(i)}(\lambda x)$ are convergent, and it holds

$$\sum_{i=1}^m \mathbf{h}^{(i)}(\lambda x) \sim \mathcal{O}(\sqrt{\nu}), \quad \sum_{i=1}^m \mathbf{w}^{(i)}(\lambda x) \sim \mathcal{O}(\sqrt{\nu}).$$

For a fixed integer $s \geq 1$, let $1 \leq s = \bar{k} < k_0 \leq k$ and

$$k_m := \bar{k} + \frac{k - \bar{k}}{2^m}, \quad k_{+\infty} = \bar{k}, \quad \alpha_{m+1} := k_m - k_{m+1} = \frac{k - \bar{k}}{2^{m+1}}.$$

Therefore

$$k_0 > k_1 > \cdots > k_m > k_{m+1} > \cdots. \tag{4.1}$$

We now prove the existence of a Sobolev regular solution of the nonlinear perturbation equations (1.12). This is based on an induction argument. For convenience, we first deal with the case of zero initial data, that is, $\mathbf{u}_0(x) = \mathbf{H}_0(x) = (0, 0, 0)$. Note that $N_m = N_0^m$ with $N_0 > 1$. For all $m = 1, 2, \ldots$, we claim that there exists a small positive constant ε such that

$$\begin{gathered}
\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{w}^{(m)}\|_{H^{k_{m-1}}} < \varepsilon^{2^{m-1}}, \\
\|E^{(m)}\|_{H^{k_{m-1}}} + \|\overline{E}^{(m)}\|_{H^{k_{m-1}}} < \varepsilon^{2^m}, \\
(\mathbf{u}^{(m)}, \mathbf{H}^{(m)})^T \in \mathcal{B}_\varepsilon.
\end{gathered} \tag{4.2}$$

For the case of $m = 1$, we recall the assumptions (2.4)–(2.8) on the initial approximation function $(\mathbf{u}^{(0)}(x), \mathbf{H}^{(0)}(x))$. By (2.51), let $0 < \varepsilon_0 < \lambda^{-(s+2)} N_0^{-(8+k-\bar{k})} \varepsilon^2 < \frac{\varepsilon}{2} \ll 1$, hence

$$\|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \|E^{(0)}\|_{H^{k_0}} + \|\overline{E}^{(0)}\|_{H^{k_0}} < \varepsilon.$$

Moreover, by (3.7) and the above estimate, we deduce that

$$\begin{aligned}
\|E^{(1)}\|_{H^{k_0}} + \|\overline{E}^{(1)}\|_{H^{k_0}} &\lesssim \|\mathcal{R}_1(\mathbf{h}^{1)}, \mathbf{w}^{1)})\|_{H^{k_0}} + \|\mathcal{R}_2(\mathbf{h}^{1)}, \mathbf{w}^{1)})\|_{H^{k_0}} \\
&\lesssim \lambda^{s+2} N_1^2 (\|\mathbf{h}^{(1)}\|_{H^{k_0}}^2 + \|\mathbf{w}^{(1)}\|_{H^{k_0}}^2) \\
&< \varepsilon^2,
\end{aligned}$$

and

$$\|\mathbf{u}^{(1)}\|_{H^{k_0}} + \|\mathbf{H}^{(1)}\|_{H^{k_0}} \lesssim \|\mathbf{u}^{(0)}\|_{H^{k_0}} + |\mathbf{H}^{(0)}\|_{H^{k_0}} + \|\mathbf{h}^{(1)}\|_{H^{k_0}} + \|\mathbf{w}^{(1)}\|_{H^{k_0}} \lesssim \varepsilon,$$

which means that $(\mathbf{u}^{(1)}, \mathbf{H}^{(1)}) \in \mathcal{B}_\varepsilon$.

Assume that the case of $m - 1$ holds, that is,

$$\begin{gathered}
\|\mathbf{h}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{w}^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-2}}, \\
\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-1}}, \\
(\mathbf{u}^{(m-1)}, \mathbf{H}^{(m-1)})^T \in \mathcal{B}_\varepsilon.
\end{gathered} \tag{4.3}$$

We now prove that the case of m holds. Using (2.1), (3.6) and the second inequality of (4.3), we derive

$$\begin{aligned}
\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} + \|\mathbf{w}^{(m)}\|_{H^{k_{m-1}}} &\lesssim \|\Pi_{N_m} E^{(m-1)}\|_{H^{k_{m-1}}} + \|\Pi_{N_m} \overline{E}^{(m-1)}\|_{H^{k_{m-1}}} \\
&\lesssim N_m^{\alpha_m} (\|E^{(m-1)}\|_{H^{k_m}} + \|\overline{E}^{(m-1)}\|_{H^{k_m}}) \\
&< N_m^{\alpha_m} \varepsilon^{2^{m-1}} < \varepsilon^{2^{m-2}},
\end{aligned} \tag{4.4}$$

which combined with (2.1), (3.7) and (4.1) yields

$$\begin{aligned}\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} &\lesssim \lambda^{2(s+2)} N_m^2 (\|\mathbf{h}^{(m)}\|^2_{H^{k_m}} + \|\mathbf{w}^{(m)}\|^2_{H^{k_m}}) \\ &\lesssim \lambda^{s+2} N_m^{2+\alpha_{m+1}} (\|E^{(m-1)}\|_{H^{k_{m+1}}} + \|\overline{E}^{(m-1)}\|_{H^{k_{m+1}}})^2 \\ &\lesssim (\lambda^{s+2} N_0)^{(2+\alpha_{m+1})m + 2(2+\alpha_{m+2})(m-1)} (\|E^{(m-2)}\|_{H^{k_{m+2}}} \\ &\quad + \|\overline{E}^{(m-2)}\|_{H^{k_{m+2}}})^{2^2} \\ &\lesssim \dots, \\ &\lesssim [\lambda^{s+2} N_0^{8+k-\bar{k}} (\|E^{(0)}\|_{H^{k_{2m}}} + \|\overline{E}^{(0)}\|_{H^{k_{2m}}})]^{2^m}. \end{aligned} \tag{4.5}$$

We choose a sufficient small positive constant ε_0 such that

$$0 < \lambda^{s+2} N_0^{8+k-\bar{k}} (\|E^{(0)}\|_{H^{\bar{k}}} + \|\overline{E}^{(0)}\|_{H^{\bar{k}}}) < 2N_0^4 \varepsilon_0 < \varepsilon^2.$$

Thus, by (4.5), we have

$$\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}} < \varepsilon^{2^m},$$

and

$$\begin{aligned}0 &\le \lim_{m\to+\infty} (\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}) \\ &\lesssim [\lambda^{s+2} N_0^{8+k-\bar{k}} (\|E^{(0)}\|_{H^{k_{+\infty}}} + \|\overline{E}^{(0)}\|_{H^{k_{+\infty}}})]^{2^{+\infty}} \to 0.\end{aligned}$$

So, the error term goes to 0 as $m \to \infty$, that is,

$$\lim_{m\to\infty} (\|E^{(m)}\|_{H^{k_m}} + \|\overline{E}^{(m)}\|_{H^{k_m}}) = 0.$$

On the other hand, note that $N_m = N_0^m$, by (4.3)–(4.4). Therefore

$$\begin{aligned}&\|\mathbf{u}^{(m)}\|_{H^{k_m}} + \|\mathbf{H}^{(m)}\|_{H^{k_m}} \\ &\quad\lesssim \|\mathbf{u}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{H}^{(m-1)}\|_{H^{k_m}} + \|\mathbf{h}^{(m)}\|_{H^{k_m}} + \|\mathbf{w}^{(m)}\|_{H^{k_m}} \\ &\quad\lesssim \varepsilon + N_m^3 \varepsilon^{2^{m-1}} \lesssim \varepsilon.\end{aligned}$$

This means that $(\mathbf{u}^{(m)}, \mathbf{H}^{(m)})^T \in \mathcal{B}_\varepsilon$. Hence, we conclude that (4.2) holds.

Therefore, the nonlinear perturbation equations (1.12) admits a Sobolev regular solution

$$\mathbf{u}^{(\infty)}(x) = \mathbf{u}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(\lambda x) \in H^s(\Omega),$$

$$\mathbf{H}^{(\infty)}(x) = \mathbf{H}^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{w}^{(m)}(\lambda x) \in H^s(\Omega),$$

from which, one can see that the solution depends on the initial approximation function. Finally, we use (1.2) and apply the Calderón–Zygmund theory. Thus, for the Riesz operator $\mathcal{R}$, there exists $\|\mathcal{R}w\|_{\mathbb{L}^{s_0}} \leq \|w\|_{\mathbb{L}^{s_0}}$ with $1 < s_0 < \infty$ such that

$$\|P\|_{H^s} \lesssim \varepsilon.$$

Furthermore, since the initial approximation function $(\mathbf{u}^{(0)}, \mathbf{H}^{(0)})$ satisfies the conditions (2.4)–(2.8) and $\varepsilon \leq \sqrt{\nu}$, in Sobolev space H^s, it holds

$$\mathbf{u}^{(\infty)}(x) \sim \mathcal{O}(\sqrt{\nu}),$$
$$\mathbf{H}^{(\infty)}(x) \sim \mathcal{O}(\sqrt{\nu}),$$
$$P(x) \sim \mathcal{O}(\sqrt{\nu}),$$

thus, it holds

$$\lim_{\nu\to 0}(\|\mathbf{v}^\nu - \mathbf{v}^e\|_{H^s(\Omega)} + \|\mathbf{H}^\nu\|_{H^s(\Omega)}) = 0,$$

and

$$\lim_{\nu\to 0}\|P^\nu - P^e\|_{H^s(\Omega)} = 0.$$

This completes the proof.

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