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An infinity of nodal solutions for superlinear Robin problems with an indefinite and unbounded potential

Nikolaos S. Papageorgiou [a,b], Vicenţiu D. Rădulescu [c,d,*]

[a] *National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece*
[b] *King Saud University, Department of Mathematics, P.O. Box 2454, Riyadh 11451, Saudi Arabia*
[c] *Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[d] *Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania*

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We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential using a suitable version of the symmetric mountain pass theorem, we show that the problem has an infinity of nodal solutions whose energy level diverges to $+\infty$.

* Corresponding author.
E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@imar.ro (V.D. Rădulescu).

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{1}$$

In this problem, the potential function $\xi \in L^s(\Omega)$ with $s > N$ and is indefinite (that is, $\xi(\cdot)$ is sign changing). The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for almost all $z \in \Omega$ $x \to f(z, x)$ is continuous), which is superlinear in the $x \in \mathbb{R}$ variable, but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). In addition, for almost all $z \in \Omega$ $f(z, \cdot)$ satisfies a one-sided Lipschitz condition and it is odd. In the boundary condition $\frac{\partial u}{\partial n}$ denotes the usual normal derivative defined by extension of the map

$$C^1(\overline{\Omega}) \ni u \to \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, then we recover the Neumann problem.

We are looking for the existence of multiple nodal (that is, sign changing) solutions for problem (1). Using a version of the symmetric mountain pass theorem due to Qian and Li [13, Theorem 4.2], we show the existence of a sequence of distinct nodal solutions with energies diverging to $+\infty$.

In the past, an infinity of nodal solutions for superlinear Dirichlet problems with $\xi \equiv 0$, we proved by Qian and Li [13, Theorem 5.4] using AR-condition and with more restrictive conditions on the reaction term f. Subsequently, Qian [12, Theorem 1.1] produced an infinity of nodal solutions for a superlinear Neumann problem with $\xi \equiv a \in (0, +\infty)$. So, in Qian [12] the differential operator (right-hand side of the equation), is coercive and this simplifies the arguments considerably. Qian [12] did not use the AR-condition and instead employed a condition which was first introduced by Jeanjean [4]. This condition is global in nature and for this reason not entirely satisfactory. For Robin problems, there is the work of Qian and Li [14], who assume that $\xi \equiv 0$ and $f \in C(\overline{\Omega} \times \mathbb{R})$ satisfies the Jeanjean condition. They produce an infinity of distinct solutions, but they do not show that these solutions are nodal (see [14, Theorem 1.3]).

Problems with indefinite linear part (that is, the potential function $\xi(\cdot)$ is indefinite), were investigated by Zhang and Liu [18], Qin, Tang and Zhang [15], Zhang, Tang and Zhang [19]. All the aforementioned works deal with Dirichlet problems and use a nonquadraticity condition analogous to the one employed by Costa and Magalhaes [2]. They produce infinitely many nontrivial solutions, but the not show that they are nodal. Multiple nodal solutions for problems with indefinite linear part, were produced by Papageorgiou and Papalini [7] (Dirichlet problems), Papageorgiou and Rădulescu [8]

(Neumann problems) and Papageorgiou and Rădulescu [10] (Robin problems). None of the above works produces infinitely many nodal solutions. Finally we mention the very recent paper of Papageorgiou and Rădulescu [11], who produce a sequence of nodal solutions for nonlinear Robin problems but under different conditions and using different tools.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on φ, more general than the usual Palais–Smale condition. Nevertheless, it leads to the same deformation theorem from which one can derive the minimax theory of the critical values of φ.

The following spaces will be important in our analysis:

- The Sobolev space $H^1(\Omega)$;
- The Banach space $C^1(\overline{\Omega})$;
- The "boundary" Lebesgue spaces $L^q(\partial\Omega),\ 1 \leqslant q \leqslant \infty$.

The Sobolev space $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, h)_{H^1(\Omega)} = \int_\Omega uh dz + \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega)$$

and corresponding norm

$$||u|| = [||u||_2^2 + ||Du||_2^2]^{1/2} \text{ for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an order Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega} : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega})\}$$

This cone has a nonempty interior containing

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the Lebesgue spaces $L^q(\partial\Omega)$ $1 \leqslant q \leqslant \infty$. According to the theory of Sobolev spaces, there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ known as the "trace map", which satisfies

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns "boundary values" to all Sobolev functions. This map is compact into $L^q(\partial\Omega)$ for all $q \in \left[1, \frac{2N-2}{N-2}\right)$ if $N \geqslant 3$ and into $L^q(\partial\Omega)$ for all $q \geqslant 1$ if $N = 1, 2$. In addition we have

$$ker\gamma_0 = H_0^1(\Omega) \text{ and } im\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega).$$

In the sequel, for the sake of notational economy, we drop the use of the map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Next we consider the following linear eigenvalue problem:

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{array} \right\} \tag{2}$$

We impose the following conditions on the data of this eigenvalue problem

- $\xi \in L^{\frac{N}{2}}(\Omega)$ if $N \geqslant 3, \xi \in L^q(\Omega)$ with $q > 1$ if $N = 2$ and $\xi \in L^1(\Omega)$ if $N = 1$.
- $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Let $\vartheta : H^1(\Omega) \to \mathbb{R}$ be the C^1-functions defined by

$$\vartheta(u) = ||Du||_2^2 + \int_\Omega \xi(z)u^2dz + \int_{\partial\Omega} \beta(z)u^2d\sigma \text{ for all } u \in H^1(\Omega).$$

We know (see [10]) that there exists $\mu > 0$ such that

$$\vartheta(u) + \mu||u||_2^2 \geqslant c_0||u||^2 \text{ for all } u \in H^1(\Omega), \text{ some } c_0 > 0. \tag{3}$$

Using (3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we define the spectrum of (2) consisting of a sequence $\{\hat{\lambda}_k\}_{k\geqslant 1} \subseteq \mathbb{R}$ such that $\hat{\lambda}_k \to +\infty$. By $E(\hat{\lambda}_k)$ $k \in \mathbb{N}$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. We know that each $E(\hat{\lambda}_k)$ is finite dimensional and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{\underset{k\geqslant 1}{\oplus} E(\hat{\lambda}_k)}.$$

We know that

- $\hat{\lambda}_1$ is simple (that is, $dimE(\hat{\lambda}_1) = 1$).
- $\hat{\lambda}_1 = inf\left[\frac{\vartheta(u)}{||u||_2^2} : u \in H^1(\Omega), u \neq 0\right]$. (4)
- $\hat{\lambda}_m = inf\left[\frac{\vartheta(u)}{||u||_2^2} : u \in \overline{\underset{k\geqslant m}{\oplus} E(\hat{\lambda}_k)}, u \neq 0\right]$

$$= sup\left[\frac{\vartheta(u)}{||u||_2^2} : u \in \overset{m}{\underset{k=1}{\oplus}} E(\hat{\lambda}_k), u \neq 0\right] \quad m \geqslant 2. \tag{5}$$

The infimum in (4) is realized on $E(\hat{\lambda}_1)$. Both the infimum and supremum in (5) are realized on $E(\hat{\lambda}_m)$. Evidently the elements of $E(\hat{\lambda}_1)$ do not change sign, while the elements of $E(\hat{\lambda}_m)$ $m \geqslant 2$ are nodal (that is sign changing).

In what follows $A : H^1(\Omega) \to H^1(\Omega)^*$ is the bounded linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega).$$

Also, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and $2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geqslant 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$

3. A sequence of nodal solutions

Our hypotheses on the data of (1) are the following:

$H(\xi)$: $\xi \in L^s(\Omega), s > N$ and $\xi^+ \in L^\infty(\Omega)$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$ $f(z, \cdot)$ is odd and

(i) $|f(z, x)| \leqslant a(z)(1 + |x|^{r-1})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega), r \in (2, 2^*)$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{x^2} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) if $e(z, x) = f(z, x)x - 2F(z, x)$, then there exists $d \in L^1(\Omega)$ such that

$$e(z, x) \leqslant e(z, y) + d(z) \text{ for almost all } z \in \Omega, \text{ all } 0 \leqslant x \leqslant y \text{ or } y \leqslant x \leqslant 0;$$

(iv) there exists $\hat{\eta} > 0$ such that for almost all $z \in \Omega$ the function

$$x \to f(z, x) + \hat{\eta}x$$

is increasing on $\mathbb{R}$;

(v) there exist $\hat{c}_0, \hat{c}_1 > 0$ such that

$$-\hat{c}_0 \leqslant \liminf_{x\to 0} \frac{f(z,x)}{x} \leqslant \limsup_{x\to 0} \frac{f(z,x)}{x} \leqslant \hat{c}_1 \text{ uniformly for almost all } z \in \Omega.$$

Remark 1. Hypothesis $H(f)(ii)$ implies that the primitive $F(z,\cdot)$ is superquadratic near $+\infty$. Hypotheses $H(f)(ii), (iii)$ imply that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{x} = +\infty \text{ uniformly for almost all } z \in \Omega$$

So, the reaction term $f(z,\cdot)$ is superlinear. However, this superlinearity is not expressed via the classical AR-condition, which says that there exist $q > 2$ and $M > 0$ such that

$$0 < qF(z,x) \leqslant f(z,x)x \text{ for almost all } z \in \Omega, \text{ all } |x| \geqslant M \tag{6a}$$

$$0 < ess \inf_{\Omega} F(\cdot, \pm M) \tag{6b}$$

(see Ambrosetti and Rabinowitz [1] and Mugnai [6]). Integrating (6a) and using (6b), we obtain the weaker condition

$$c_1|x|^q \leqslant F(z,x) \text{ for almost all } z \in \Omega, \text{ all } |x| \geqslant M, \text{ some } c_1 > 0$$

This means that under the AR-condition $f(z,\cdot)$ has at least $(q-1)$-polynomial growth near $\pm\infty$. The Jeanjean condition used in some works mentioned in the Introduction, says that there exist $\eta \geqslant 1$ and $s \in [0,1]$ such that

$$e(z,sx) \leqslant \eta e(z,x) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}$$

We mention the global nature of this condition. This is a feature which we would like to avoid. Here instead of the AR-condition and the Jeanjean condition we employ a quasimonotonicity condition on $e(z,\cdot)$ (see hypothesis $H(f)(iii)$). This condition is a slightly more general version of a condition used by Li and Yang [5]. It is satisfied if there exists $M \geqslant 0$ such that

$$e(z,\cdot) \text{ is nondecreasing on } x \geqslant M \text{ and nonincreasing on } x \leqslant -M.$$

In turn, this is implied by the following condition

$$x \to \frac{f(z,x)}{x} \text{ is nondecreasing on } x \geqslant M,$$
$$x \to \frac{f(z,x)}{x} \text{ is nonincreasing on } x \leqslant -M$$

We stress the local character of the last two conditions. Hypothesis $H(f)(iv)$ is a one-sided Lipschitz condition. Finally hypothesis $H(f)(v)$ implies that for almost all $z \in \Omega$, $f(z, \cdot)$ is linear near zero.

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\vartheta(u) - \int_\Omega F(z, u)dz \text{ for all } u \in H^1(\Omega).$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

Proposition 1. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then the functional φ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$|\varphi(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{7}$$

$$(1 + ||u||)\varphi'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty \tag{8}$$

From (8) we have

$$\left| \langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial\Omega} \beta(z)u_n h d\sigma - \int_\Omega f(z, u_n)h dz \right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \tag{9}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$

In (9) we choose $h = u_n \in H^1(\Omega)$. Then

$$\vartheta(u_n) + \int_\Omega f(z, u_n)u_n dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N} \tag{10}$$

From (7), we have

$$-\vartheta(u_n) - \int_\Omega 2F(z, u_n)dz \leqslant 2M_1 \text{ for all } n \in \mathbb{N} \tag{11}$$

Adding (10)and (11), we obtain

$$\int_\Omega e(z, u_n)dz \leqslant M_2 \text{ for some } M_2 > 0, \text{ all } n \in \mathbb{N}. \tag{12}$$

Claim 1. *$\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded.*

We argue by contradiction. So, suppose that the Claim is not true. By passing to a suitable subsequence if necessary, we may assume that

$$||u_n|| \to \infty. \tag{13}$$

Let $y_n = \frac{u_n}{||u_n||}$ $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{x} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and } L^2(\partial\Omega) \tag{14}$$

(note that we can always assume $r \geqslant \frac{2s}{s-1}$, see hypothesis $H(f)(i)$).

First assume that $y \neq 0$ and let $\Omega_0 = \{z \in \Omega : y(z) \neq 0\}$. We have $|\Omega_0|_N > 0$ and

$$|u_n(z)| \to +\infty \text{ for almost all } z \in \Omega_0.$$

Then hypothesis $H(f)(ii)$ implies that

$$\frac{F(z, u_n(z))}{||u_n||^2} = \frac{F(z, u_n(z))}{u_n(z)^2} y_n(z)^2 \to +\infty \text{ for almost all } z \in \Omega_0 \text{ as } n \to \infty. \tag{15}$$

Using (15) and Fatou's lemma (it can be used on account of hypothesis $H(f)(iii)$), we have

$$\frac{1}{||u_n||^2} \int_{\Omega_0} F(z, u_n)dz \to +\infty \text{ as } n \to \infty. \tag{16}$$

Hypothesis $h(f)(ii)$ implies that we can find $M_3 > 0$ such that

$$F(z, x) \geqslant 0 \text{ for almost all } z \in \Omega, \text{ all } |x| \geqslant M_3. \tag{17}$$

We have

$$\begin{aligned}
\frac{1}{||u_n||^2} \int_{\Omega} F(z, u_n)dz &= \frac{1}{||u_n||^2} \int_{\Omega_0} F(z, u_n)dz + \frac{1}{||u_n||^2} \int_{\Omega_0^c \cap \{|u_n| \geqslant M_3\}} F(z, u_n)dz + \\
&\quad \frac{1}{||u_n||^2} \int_{\Omega_0^c \cap \{|u_n| < M_3\}} F(z, u_n)dz \\
&\geqslant \frac{1}{||u_n||^2} \int_{\Omega_0} F(z, u_n)dz - \frac{c_2}{||u_n||^2} \text{ for some } c_2 > 0, \text{ all } n \in \mathbb{N} \\
&\quad \text{(see (17 and use hypothesis) } H(f)(i)\text{)}, \\
\Rightarrow \lim_{n \to \infty} & \frac{1}{||u_n||^2} \int_{\Omega} F(z, u_n)dz = +\infty \text{ (see (16))}.
\end{aligned} \tag{18}$$

On the other hand from (7) we have

$$\frac{1}{||u_n||^2}\int_\Omega F(z,u_n)dz \leqslant \frac{M_1}{||u_n||^2}+\frac{1}{2}\vartheta(y_n) \text{ for all } n\in\mathbb{N}$$
$$\Rightarrow \frac{1}{||u_n||^2}\int_\Omega F(z,u_n)dz \leqslant M_4 \text{ for some } M_4>0,\ \text{all } n\in\mathbb{N} \tag{19}$$
(see hypotheses $H(\xi), H(\beta)$ and recall that $||y_n||=1,\ n\in\mathbb{N}$)

Comparing (16) and (19), we reach a contradiction.
Next suppose that $y=0$. Given $\tau>0$, let

$$v_n=(2\tau)^{1/2}y_n \text{ for all } n\in\mathbb{N}.$$

We have

$$v_n\to 0 \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega) \text{ (see (14) and recall that y=0).}$$

It follows that

$$\int_\Omega F(z,v_n)dz\to 0 \text{ as } n\to\infty. \tag{20}$$

From (13) we see that we can find $n_0\in\mathbb{N}$ such that

$$0<(2\tau)^{1/2}\frac{1}{||u_n||}\leqslant 1 \text{ for all } n\geqslant n_0 \tag{21}$$

Choose $t_n\in[0,1]$ such that

$$\varphi(t_nu_n)=max[\varphi(tu):0\leqslant t\leqslant 1] \text{ for all } n\in\mathbb{N}. \tag{22}$$

Taking into account (21), we have

$$\begin{aligned}\varphi(t_nu_n)&\geqslant\varphi(v_n)\\ &=\tau\vartheta(y_n)-\int_\Omega F(z,u_n)dz\\ &\geqslant\tau[c_0-\mu||y_n||_2^2]-\int_\Omega F(z,v_n)dz \text{ for all } n\geqslant n_0 \text{ (see (3)).}\end{aligned} \tag{23}$$

Passing to the limit as $n\to\infty$ in (23) and using (14) and (20) and recalling that $y=0$, we obtain

$$\liminf_{n\to\infty}\varphi(t_nu_n)\geqslant\tau c_0.$$

But $\tau > 0$ is arbitrary. So, it follows that

$$\varphi(t_n u_n) \to +\infty \text{ as } n \to \infty. \tag{24}$$

We have

$$\varphi(0) = 0 \text{ and } \varphi(u_n) \leqslant M_1 \text{ for all } n \in \mathbb{N} \text{ (see (7))}.$$

The from (24) we infer that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0,1) \text{ for all } n \geqslant n_2 \tag{25}$$

From (22) and (25) it follows that

$$\begin{aligned}
&\frac{d}{dt}\varphi(tu_n)\Big|_{t=t_n} = 0 \text{ for all } n \geqslant n_2,\\
\Rightarrow\ &\langle \varphi'(t_n u_n), t_n u_n\rangle = 0 \text{ for all } n \geqslant n_2 \text{ (bu the chain rule)},\\
\Rightarrow\ &\vartheta(t_n u_n) = \int_\Omega f(z, t_n u_n)(t_n u_n)dz \text{ for all } n \geqslant n_2.
\end{aligned} \tag{26}$$

Hypothesis $H(f)(iii)$ and (25) imply that

$$\begin{aligned}
&\int_\Omega e(z, tu_n)dz\\
&= \int_\Omega [e(z, tu_n^+) + e(z, -t_n u_n^-)]dz \text{ (note that } e(z,0) = 0 \text{ for almost all } z \in \Omega)\\
&\leqslant \int_\Omega [e(z, u_n^+) + e(z, -u_n^-)]dz + ||d||_1 \text{ (see hypothesis } H(f)(iii))\\
&= \int_\Omega e(z, u_n)dz + ||d||_1 \text{ for all } n \geqslant n_2.
\end{aligned}$$

Using this inequality in (26), we obtain

$$\begin{aligned}
2\varphi(t_n u_n) &\leqslant \int_\Omega e(z, u_n)dz + ||d||_1\\
&\leqslant M_2 + ||d||_1 = M_5 \text{ for all } n \geqslant n_2 \text{ (see (12))}.
\end{aligned} \tag{27}$$

Comparing (24) and (27), we have a contradiction.
This proves the Claim.

On account of the Claim, we may assume that

$$u_n \xrightarrow{w} u \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega). \tag{28}$$

In (9) we choose $h = -u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (28) (recall $r \geqslant \frac{2s}{s-1}$). We obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ &\Rightarrow ||Du||_2 \to ||Du||_2, \\ &\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (by the Kadec–Klee property for Hilbert spaces, see (28))}, \\ &\Rightarrow \varphi \text{ satisfies the C-condition.} \quad \square \end{aligned}$$

For every $m \in \mathbb{N}$, we define

$$Y_m = \overset{m}{\underset{\mathrm{k}=1}{\oplus}} E(\hat{\lambda}_k) \text{ and } V_m = \overline{\underset{k\geqslant m}{\oplus} E(\hat{\lambda}_k)}.$$

Let

$$\beta_m = sup[||u||_r : u \in V_m, ||u|| = 1] \tag{29}$$

As in the proof of Lemma 3.8 of Willem [17, p. 60], we show that

$$\beta_m \to 0^+ \text{ as } m \to +\infty. \tag{30}$$

Proposition 2. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then we can find $\{l_m\}_{m\in\mathbb{N}} \subseteq (0, +\infty)$ such that*

$$\gamma_m = inf[\varphi(u) : u \in V_m, ||u|| = l_m] \to +\infty \text{ as } m \to \infty.$$

Proof. Let $u \in V_m$. We have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\vartheta(u) - \int_\Omega F(z,u)dz \\ &= \frac{1}{2}\vartheta(u) + \frac{\mu}{2}||u||_2^2 - \frac{\mu}{2}||u||_2^2 - \int_\Omega F(z,u)dz \text{ (with } \mu > 0 \text{ as in (3))} \\ &\geqslant \frac{c_0}{2}||u||^2 - \frac{\mu}{2}||u||_2^2 - c_3||u||_r^r - c_3 \text{ for some } c_3 > 0 \\ &\quad \text{(see (3 and hypothesis H(f)(i))} \end{aligned} \tag{31}$$

Recall that $r > 2$ (see hypothesis $H(f)(i)$). So, we can find $c_4 > 0$ such that

$$||u||_2 \leqslant c_4||u||_r \text{ for all } u \in H^1(\Omega) \tag{32}$$

Using (32) and (31) we obtain

$$\varphi(u) \geqslant \frac{c_0}{2}||u||^2 - c_5(||u||_r^r + ||u||_r^2) - c_3 \text{ for some } c_5 > 0, \text{ all } u \in V_m$$

Suppose that $||u|| \geqslant 1$. Then using once more the fact that $r > 2$, we obtain

$$\varphi(u) \geqslant \frac{c_0}{2}||u||^2 - c_6||u||_r^r - c_3 \text{ with } c_6 = 2c_5 > 0, \text{ all } u \in V_m, ||u|| \geqslant 1 \tag{33}$$

From (29) we have

$$\beta_m||u|| \geqslant ||u||_r \text{ for all } u \in V_m$$

Using this inequality in (33), we obtain

$$\varphi(u) \geqslant \frac{c_0}{2}||u||^2 - c_6\beta_m^r||u||^r - c_3 \text{ for all } u \in V_m, ||u|| \geqslant 1. \tag{34}$$

Let $l_m = \left(\frac{c_6}{c_0} r\beta_m^r\right)^{\frac{1}{2-r}}$, we have

$$l_m \to +\infty \text{ as } m \to +\infty \text{ (see (30) and recall that } r > 2).$$

Hence we may assume that $l_m \geqslant 1$ for all $m \in \mathbb{N}$. Then from (34) we see that for all $u \in V_m$ with $||u|| = l_m$, we have

$$\begin{aligned}
\varphi(u) &\geqslant \frac{c_0}{2}\left(\frac{c_6}{c_0} r\beta_m^r\right)^{\frac{2}{2-r}} - c_6\beta_m^r\left(\frac{c_6}{c_0} r\beta_m^r\right)^{\frac{r}{2-r}} \\
&= \left[\frac{c_0}{2} - c_6\beta_m^r\frac{c_0}{c_6 r\beta_m^r}\right]\left(\frac{c_6}{c_0} r\beta_m^r\right)^{\frac{2}{2-r}} \\
&= c_0\left[\frac{1}{2} - \frac{1}{r}\right]\left(\frac{c_6}{c_0} r\beta_m^r\right)^{\frac{2}{2-r}}, \\
&\Rightarrow l_m \to +\infty \text{ as } m \to \infty \text{ (see (30) and recall that } r > 2). \quad \square
\end{aligned}$$

Proposition 3. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then we can find $\{\rho_m\}_{m\in\mathbb{N}} \subseteq (0,\infty)$, $\rho_0 > l_m > 0$ for all $m \in \mathbb{N}$ such that*

$$\Im_m = sup[\varphi(u) : u \in Y_m, ||u|| = \rho_m] \leqslant 0 \text{ for all } m \in \mathbb{N}.$$

Proof. Let $u \in Y_m$. We have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\vartheta(u) - \int_\Omega F(z,u)dz \\
&\leqslant \frac{1}{2}||Du||_2^2 + \frac{1}{2}\int_\Omega \xi^+(z)u^2dz + \frac{1}{2}\int_{\partial\Omega}\beta(z)u^2d\sigma - \int_\Omega F(z,u)dz.
\end{aligned} \tag{35}$$

Hypotheses $H(f)(i),(ii)$ imply that given any $\eta > 0$, we can find $c_7 = c_7(\eta) > 0$ such that

$$F(z,x) \geqslant \eta x^2 - c_7 \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Using this unilateral growth estimate and hypothesis $H(\xi)$ in (35), we obtain

$$\varphi(u) \leqslant c_8 ||u||^2 - \eta ||u||_2^2 + c_7 |\Omega|_N \text{ for some } c_8 > 0, \text{ all } u \in Y_m.$$

But Y_m is finite dimensional. So, all norms are equivalent. Hence we can find $c_9 > 0$ such that

$$\begin{aligned}\varphi(u) &\leqslant c_8 ||u||^2 - \eta c_9 ||u||^2 + c_7 |\Omega|_N \\ &= [c_8 - \eta c_9] ||u||^2 + c_7 |\Omega|_N \text{ for all } u \in Y_m.\end{aligned} \tag{36}$$

Recall that $\eta > 0$ is arbitrary. So, we choose $\eta > \frac{c_8}{c_9}$. Then from (36) it is clear that we can find $\rho_m > l_m$ $m \in \mathbb{N}$ such that

$$\begin{aligned}&\varphi(u) \leqslant 0 \text{ for all } u \in Y_m, ||u|| = \rho_m, \\ \Rightarrow\ &\Im_m \leqslant 0 \text{ for all } m \in \mathbb{N}. \quad \square\end{aligned}$$

Proposition 4. *If hypotheses $H(\xi), H(\beta), H(f)$ hold and $u \in H^1(\Omega)$ is a solution of (1), then $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$ (see hypothesis $H(\xi)$).*

Proof. Hypotheses $H(f)(i),(v)$ imply that

$$|f(z,x)| \leqslant c_{10}(|x| + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10} > 0. \tag{37}$$

By hypothesis we have

$$\begin{aligned}&-\Delta u(z) + \xi(z) u(z) = f(z, u(z)) \text{ for almost all } z \in \Omega \\ &\text{(see also Papageorgiou and Rădulescu [9])}, \\ \Rightarrow\ &-\Delta u(z) = \left[\frac{f(z,u(z))}{u(z)} - \xi(z)\right] u(z) \text{ for almost all } z \in \Omega.\end{aligned}$$

Note that $f(z,0) = 0$ for almost all $z \in \Omega$ (see hypothesis $H(f)(v)$) and let

$$\hat{a}(z) = \begin{cases} \frac{f(z,u(z))}{u(z)} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0 \end{cases}$$

Then

$$
\begin{aligned}
|\hat{a}(z)| &\leqslant \frac{|f(z,u(z))|}{|u(z)|} + |\xi(z)| \\
&\leqslant c_{10}(1+|u(z)|^{r-2}) + |\xi(z)| \text{ for almost all } z \in \Omega \text{ (see (37))}
\end{aligned}
$$

Note that $|u(\cdot)|^{r-2} \in L^{\frac{2^*}{r-2}}(\Omega)$ (recall that $u \in H^1(\Omega)$ and use the Sobolev embedding theorem) and observe that $\frac{2^*}{r-2} > \frac{N}{2}$ (recall that $r < 2^*$). Therefore

$$
\hat{a} \in L^q(\Omega) \text{ with } q > \frac{N}{2} \text{ (see hypothesis } H(\xi)).
$$

Then Lemma 5.1 of Wang [16] implies that

$$
u \in L^\infty(\Omega).
$$

Using this fact and hypotheses $H(f)(i)$ and $H(\xi)$, we have

$$
f(\cdot,u(\cdot)) - \xi(\cdot)u(\cdot) \in L^s(\Omega) \text{ with } s > N.
$$

So, the Calderon–Zygmund estimates (see Wang [16, Lemma 5.2]), we have

$$
u \in W^{2,s}(\Omega).
$$

The Sobolev embedding theorem implies that

$$
u \in C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0. \quad \square
$$

Proposition 5. *If hypotheses $H(\xi), H(\beta), H(f)$ hold and $u, v \in H^1(\Omega)$ are distinct solutions of (1) such that $v \leqslant u$, then $u - v \in D_+$.*

Proof. From Proposition 4, we know that

$$
u, v \in C^1(\overline{\Omega}).
$$

Let $\hat{\eta} > 0$ be as in hypothesis $H(f)(iv)$. Then

$$
\begin{aligned}
& -\Delta v(z) + (\xi(z)+\hat{\eta})v(z) \\
=\ & f(z,v(z)) + \hat{\eta}v(z) \\
\leqslant\ & f(z,u(z)) + \hat{\eta}u(z) \text{ (see hypothesis } H(f)(iv) \text{ and recall that } v \leqslant u) \\
=\ & -\Delta u(z) + (\xi(z)+\hat{\eta})u(z) \text{ for almost all } z \in \Omega,
\end{aligned}
$$

$$\Rightarrow \Delta(u-v)(z) \leqslant [||\xi^+||_\infty + \hat{\eta}](u-v)(z) \text{ for almost all } z \in \Omega$$
$$\text{(see hypothesis } H(\xi)),$$
$$\Rightarrow u - v \in D_+$$

(by the strong maximum principle, see Gasinski and Papageorgiou [3, p. 738]). □

Corollary 6. *If hypotheses $H(\xi), H(\beta), H(f)$ hold and $u \in H^1(\Omega), u \neq 0, u \geqslant 0$ is a solution of (1), then $u \in D_+$.*

From Proposition 5, Corollary 6 and Proposition 5.4 of Qian and Li [13] (see also the proof of Theorem 2 in Papageorgiou and Papalini [7]), we obtain the following result.

Proposition 7. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then C_+ is an admissible invariant set of φ.*

We make a final observation before formulating our multiplicity theorem.

Proposition 8. *If hypotheses $H(\xi), H(\beta), H(f)$ and $l_m > 0$ $m \in \mathbb{N}$ is as in Proposition 2, then $V_m \cap \partial B_{l_m} \cap C_+ = \emptyset$ for all $m \geqslant 2$ (hence $\partial B_{l_m} = \{u \in H^1(\Omega) : ||u|| = l_m\}$).*

Proof. Let $\hat{u}_1$ be the positive, L^2-normalized (that is, $||\hat{u}_1||_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The regularity theory (see [16]) and the strong maximum principle (see [3]), imply that $\hat{u}_1 \in D_+$.

For $u \in C_+\backslash\{0\}$ we have

$$\int_\Omega u\hat{u}_1 dz > 0$$

On the other hand for every $u \in V_m$ with $m \geqslant 2$, we have

$$\int_\Omega u\hat{u}_1 dz = 0 \text{ (since } V_m^1 \supseteq E(\hat{\lambda}_1)).$$

Therefore $V_m \cap \partial B_{l_m} \cap C_+ = \emptyset$. □

All these auxiliary results permit the use of Theorem 4.2 of Qian and Li [13] (the symmetric mountain pass theorem). So, we have the following multiplicity theorem.

Theorem 9. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, the problem (1) admits a sequence $\{u_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$ of distinct nodal solutions such that $\varphi(u_n) \to +\infty$.*

Conflict of interest statement

There is no conflict of interest.

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