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The Impact of the Mountain Pass Theory in Nonlinear Analysis: a Mathematical Survey

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Dedicated to Antonio Ambrosetti
on the occasion of his 65th birthday

Abstract. – *We provide a survey on the mountain pass theory, viewed as a central tool in the modern nonlinear analysis. The abstract results are illustrated with relevant applications to nonlinear partial differential equations.*

1. – Introduction.

The mountain pass theorem of A. Ambrosetti and P. Rabinowitz [2] is a result of great intuitive appeal which is very useful to find the critical points of functionals, particularly those that occur in the theory of ordinary and partial differential equations.

The original version of A. Ambrosetti and P. Rabinowitz corresponds to the case of mountains of *positive altitude*. Their proof relies on some deep deformation techniques developed by R. Palais and S. Smale [25, 26], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. H. Brezis and L. Nirenberg provided in [5] a simpler proof which combines two major tools: Ekeland's variational principle and the pseudogradient lemma. Ekeland's variational principle is the nonlinear version of the Bishop-Phelps theorem and it may be also viewed as a generalization of Fermat's theorem. The case of mountains of *zero altitude* is due to P. Pucci and J. Serrin [28, 29, 31].

In many nonlinear problems we are interested in finding solutions as stationary points of some associated "energy" functionals. Often such a mapping is unbounded from above and below, so that it has no maximum or minimum. This forces us to look for saddle points, which are obtained by minimax arguments. In such a case one maximizes a functional f over a closed set A belonging to some family Γ of sets (a rigorous definition of this set of "paths" Γ will be provided in relation (3)) and then one minimizes with respect to the set A in the family. Thus, it is natural to define

$$c = \inf_{A \in \Gamma} \sup_{u \in A} f(u)\,. \tag{1}$$

Under various hypotheses, one tries to prove that this number c is a critical value of f, hence there is a point u such that $f(u) = c$ and $f'(u) = 0$. Indeed, it seems intuitively obvious that c defined in (1) is a critical value of f. However, this is not true in general, as showed by the following example in the plane: let $f(x, y) = x^2 - (x-1)^3 y^2$. Then $(0, 0)$ is the only critical point of f but c is not a critical value. Indeed, looking for sets A lying in a small neighborhood of the origin, then $c > 0$. This example shows that the heart of the matter is to find appropriate conditions on f and on the family Γ.

One of the most important minimax results is the so-called *mountain pass theorem*. In this result one considers a function $f : X \to \mathbb{R}$ of class C^1, where X is a real Banach space. It is assumed that f satisfies the following geometric conditions:

(H1) *there exist two numbers* $R > 0$ *and* $c_0 \in \mathbb{R}$ *such that* $f(u) \geq c_0$ *for every* $u \in S_R := \{v \in X;\ \|v\| = R\}$;
(H2) $f(0) < c_0$ *and* $f(e) < c_0$ *for some* $e \in X$ *with* $\|e\| > R$.

With an additional compactness condition of Palais-Smale type it then follows that the function f has a critical point $u_0 \in X \setminus \{0,\ e\}$. With critical value $c \geq c_0$. In essence, this critical value occurs because 0 and e are low points on either side of the *mountain* S_R, so that between 0 and e there must be a lowest critical point, or *mountain pass*. Condition (H2) signifies that the mountain should have *positive altitude*. P. Pucci and J. Serrin [28, 29] proved that the mountain pass theorem continues to hold for a mountain of *zero altitude*, provided it also has nonzero thickness. In addition, if $c = c_0$, then the “pass” itself occurs precisely on the mountain. Roughly speaking, P. Pucci and J. Serrin showed that the mountain pass theorem still remains true if (H1) is strengthened a little, to the form

(H1)$'$ *there exist real numbers* $c_0,\ R,\ r$ *such that* $0 < r < R$ *and* $f(u) \geq c_0$ *for every* $u \in A := \{v \in X;\ r < \|v\| < R\}$,

while hypothesis (H2) is weakened and replaced with

(H2)$'$ $f(0) \leq c_0$ *and* $f(e) \leq c_0$ *for some* $e \in X$ *with* $\|e\| > R$.

The geometrical interpretation will be roughly described in the sequel. Denote by f the function which measures the altitude of a mountain terrain and assume that there are two points in the horizontal plane L_1 and L_2, representing the coordinates of two locations such that $f(L_1)$ and $f(L_2)$ are the deepest points of two separated valleys. Roughly speaking, our aim is to walk along an optimal path on the mountain from the point $(L_1, f(L_1))$ to $(L_2, f(L_2))$ spending the least amount of energy by passing the mountain ridge between the two valleys. Walking on a path $(\gamma, f(\gamma))$ from $(L_1, f(L_1))$ to $(L_2, f(L_2))$ such that the maximal altitude along γ is the smallest among all such continuous paths connecting

$(L_1, f(L_1))$ and $(L_2, f(L_2))$, we reach a point L on γ passing the ridge of the mountain which is called a *mountain pass point*. As pointed out by H. Brezis and F. Browder [3], the mountain pass theorem "*extends ideas already present in Poincaré and Birkhoff*".

We refer to the books by A. Ambrosetti and A. Malchiodi [1], M. Ghergu and V. Rădulescu [16], Y. Jabri [18], A. Kristály, V. Rădulescu, and Cs. Varga [21], J. Mawhin and M. Willem [24], P. Rabinowitz [35], M. Schechter [39], M. Struwe [41], M. Willem [42], and W. Zou [43] for relevant applications of the mountain pass theory.

This survey paper is organized as follows. In the next section we prove the mountain pass theorem with arguments relying on the deformation lemma. Section 3 contains a proof of Ekeland's variational principle, a central tool in a more recent proof of the mountain pass theorem, which is due H. Brezis and L. Nirenberg [5]. Section 4 is devoted to the mountain pass theorem in a non-smooth setting. We start this section with some basic properties of locally Lipschitz functionals, such as directional derivative and Clarke generalized gradient. Next, we are concerned with the proofs of the mountain pass theorem both for positive altitude and for mountains of zero altitude. In this respect we deduce by means of variational arguments the Ambrozetti-Rabinowitz, the Pucci-Serrin, and the Ghoussoub-Preiss theorems. Section 5 includes three relevant applications to PDEs of the mountain pass theorem: the subcritical Lane-Emden equation, a perturbed Lane-Emden equation with sign-changing solution, and a bifurcation problem.

2. – A deformation approach of the mountain pass theorem.

We start with a simplified approach of the original setting of the mountain pass theorem, as introduced by A. Ambrosetti and P. Rabinowitz [2]. It relies on a version of the deformation lemma.

Let X be a real Banach space and assume that $f : X \to \mathbb{R}$ is a function of class C^1 satisfying the following assumption: there exists an open neighborhood $\mathcal{N}$ of some $e_0 \in X$ and there are $e_1 \notin \bar{\mathcal{N}}$ and $c_0 \in \mathbb{R}$ such that

$$\max\{f(e_0), f(e_1)\} < c_0 \leq f(u) \qquad \text{for all } u \in \partial\mathcal{N}. \tag{2}$$

Next, we consider the family $\mathcal{P}$ of all continuous paths $p : [0,1] \to X$ joining e_0 and e_1, that is, $p(0) = e_0$ and $p(1) = e_1$. Denote

$$c := \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} f(p(t)). \tag{3}$$

Since each path $p \in \mathcal{P}$ crosses the boundary of $\mathcal{N}$, we have $\max_{t \in [0,1]} f(p(t)) \geq c_0$, hence $c \geq c_0$. In fact, in the original version of the mountain pass theorem it is assumed that the mountain has *positive altitude*, that is,

$$c > \max\{f(e_0), f(e_1)\}\,. \tag{4}$$

The number c defined by relation (3) is an "approximate critical value" of the functional f. The sense of this notion appears in the following version of the mountain pass theorem.

THEOREM 1. – *Assume that $f \in C^1(X, \mathbb{R})$ satisfies condition* (4). *There there exists a sequence (u_n) in X such that*

$$f(u_n) \to c \quad \textit{and} \quad \|f'(u_n)\|_{X^*} \to 0 \qquad \textit{as } n \to \infty. \tag{5}$$

We refer to Figure 1 for a geometric illustration of Theorem 1.

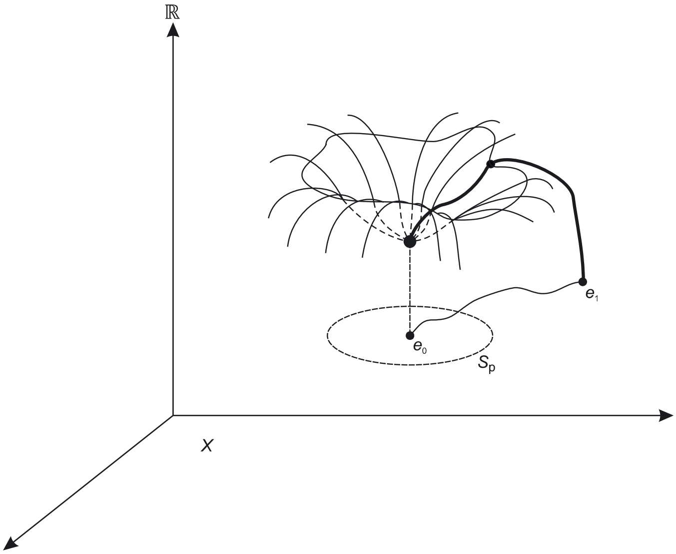

Fig. 1. – Mountain pass landscape between "villages" e_0 and e_1.

In order to assert that c is really a critical value of f it has become very standard to assume the following *compactness condition*, originally introduced by R. Palais and S. Smale [26]: the function f is said to satisfy the Palais-Smale condition $(\mathrm{PS})_a$ at level $a \in \mathbb{R}$ provided that

$$\begin{cases} \text{any sequence } (u_n) \text{ in } X \text{ such that } f(u_n) \to a \text{ and } \|f'(u_n)\|_{X^*} \to 0 \\ \text{has a convergent subsequence.} \end{cases}$$

If we now assume that Palais-Smale condition $(\mathrm{PS})_c$ is fulfilled with c defined in relation (3), we obtain the following version of the mountain pass theorem with compactness assumption.

THEOREM 2. – *Assume that $f \in C^1(X, \mathbb{R})$ satisfies conditions* (4) *and (PS)$_c$. There the number c defined in* (3) *is a critical value of f.*

The proof of Theorem 1 relies on a version of the deformation lemma, which is obtained by applying the following *pseudo-gradient* lemma.

LEMMA 1. – *Let M be a metric space and assume that $\Phi : M \to X^* \setminus \{0\}$ is a continuous function. Then, for any $\varepsilon > 0$, there exists a continuous function $v : M \to X$ such that for all $x \in M$,*

$$\|v(x)\| \leq (1+\varepsilon)\,\|\Phi(x)\| \tag{6}$$

and

$$\langle \Phi(x), v(x) \rangle \geq \|\Phi(x)\|^2 . \tag{7}$$

PROOF. – Fix $x \in M$. Then there exists $z \in M$ such that

$$\|z\| < (1+\varepsilon)\,\|\Phi(x)\|$$

and

$$\langle \Phi(x), z \rangle > \|\Phi(x)\|^2 .$$

Now, for fixed z and using the continuity of Φ, we deduce that these relations hold true for a whole open neighborhood $\mathcal{N}_x$ of x. But $M = \bigcup_{x \in M} \mathcal{N}_x$. Thus, by Theorem 5.3 in Dugundji [11], there exists a locally finite subcovering U_j of M with associated z_j. Set $\rho_j := \operatorname{dist}(x, M \setminus U_j)$. Then the mapping ρ_j is continuous, $\rho_j \equiv 0$ on $M \setminus U_j$, and (φ_j) is a partition of unity associated to the covering (U_j), where

$$\varphi_j(x) := \frac{\rho_j(x)}{\sum_k \rho_k(x)} .$$

Then the mapping $v : M \to X$ defined by

$$v(x) = \sum_j \varphi_j(x) z_j$$

satisfies relations (6) and (7). □

If $\Phi : X \to \mathbb{R}$ is an arbitrary function and $a \in \mathbb{R}$, we set

$$\Phi_a := \{u \in X;\ \Phi(u) \leq a\} .$$

The key point in the proof of Theorem 1 is the following deformation lemma.

LEMMA 2. – *Let X be a real Banach space and assume that $f : X \to \mathbb{R}$ is a function of class C^1. Assume that there exist $c \in \mathbb{R}$, $\varepsilon > 0$, and $\delta > 0$ such that*

$$\|f'(u)\| \geq \delta \qquad \text{for all } u \in X \text{ with } f(u) \in [c-\varepsilon, c+\varepsilon]. \tag{8}$$

Then there exists a continuous deformation $\eta : [0,1] \times X \to X$ such that

$$\eta(0,u) = 0 \qquad \text{for every } u \in X; \tag{9}$$

$$\eta(t,u) = u \text{ for every } (t,u) \in [0,1] \times X \text{ with } f(u) \notin [c-\varepsilon, c+\varepsilon]; \tag{10}$$

$$\eta(1, f_{c+\varepsilon/2}) \subset f_{c-\varepsilon/2}\,. \tag{11}$$

PROOF. – Set

$$M := \{u \in X;\ c - \varepsilon < f(u) < c + \varepsilon\}$$

and

$$M_0 := \{u \in X;\ c - \varepsilon/2 \leq f(u) \leq c + \varepsilon/2\}\,.$$

The continuous function $h : X \to [0,1]$ defined by

$$h(u) := \frac{\operatorname{dist}(u, X \setminus M)}{\operatorname{dist}(u, X \setminus M) + \operatorname{dist}(u, M_0)}$$

satisfies $h \equiv 1$ on M_0 and $h \equiv 0$ on $X \setminus M$.

According to Lemma 1, there is a pseudo-gradient $v : \{u \in X; f'(u) \neq 0\} \to X$. Define the vector field $V : X \to X$ by

$$V(u) := \begin{cases} -h(u)\,\dfrac{v(u)}{\|v(u)\|^2} & \text{if } u \in M \\ 0 & \text{if } u \notin M. \end{cases}$$

Then V locally Lipschitz on X and, for any $u \in X$, $\|V(u)\| \leq 1/\delta$. Thus, for any fixed $u \in X$, the problem

$$\begin{cases} \eta'(t) = V(\eta(t)) & \text{if } t > 0 \\ \eta(0) = u \end{cases}$$

has a unique solution $\eta(t) = \eta(t,u)$, defined for all $0 \leq t < \infty$. Moreover, $\eta(t,u) = u$ for all $t \geq 0$ and every $u \in X \setminus M$.

We can assume without loss of generality that $\varepsilon \in (0, 1/4)$. Then, for all $(t,u) \in [0,\infty) \times X$,

$$f'(\eta(t)) = \langle f'(\eta(t)), V(\eta(t)) \rangle \leq -\frac{1}{4}\, h(\eta(t))\,.$$

The mapping η has all the required properties. □

We now have all ingredients to prove Theorem 1. Arguing by contradiction, we assume that there is no sequence (u_n) in X satisfying (5). Thus, there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\|f'(u)\| \geq \delta \quad \text{for all } u \in X \text{ with } c - \varepsilon < f(u) < c + \varepsilon.$$

We can assume without loss of generality that $\varepsilon \leq 1/4$, $f(e_0) < c - \varepsilon$, and $f(e_1) < c - \varepsilon$. Using the definition of c given in relation (2), we deduce that there is a path $p \in \mathcal{P}$ such that

$$f(p(t)) \leq c + \varepsilon/2 \qquad \text{for all } t \in [0,1].$$

Thus, by Lemma 2, there is a continuous deformation η satisfying relations (9)-(11). Define the path $q(t) = \eta(1, p(t))$ for all $t \in [0,1]$. Since $q(0) = \eta(1, e_0) = e_0$, $q(1) = \eta(1, e_1) = e_1$, we deduce that $q \in \mathcal{P}$. On the other hand, by Lemma 2, $q(t) \in f_{c-\varepsilon/2}$ for all $t \in [0,1]$, which contradicts our basic assumption (4). This concludes the proof of Theorem 1. □

A straightforward argument shows that the conclusions of Lemma 2 still remain true provided that assumption (8) is replaced with the weaker hypothesis: there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$(1 + \|u\|)\, \|f'(u)\| \geq \delta \qquad \text{for all } u \in X \text{ with } f(u) \in [c - \varepsilon, c + \varepsilon]. \tag{12}$$

This enables us to show that the conclusion (5) of Theorem 1 can be strengthened to the following *Cerami compactness condition* (see [6])

$$f(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\, \|f'(u_n)\|_{X^*} \to 0 \qquad \text{as } n \to \infty. \tag{13}$$

More generally, the same conclusions remain true if $(1 + \|u\|)$ is replaced in relations (12) and (13) with $\psi(\|u\|)$, where $\psi : [0, \infty) \to [1, \infty)$ is a continuous function satisfying

$$\int_0^\infty \frac{\mathrm{d}t}{\psi(t)} = \infty\,.$$

3. – Ekeland's variational principle.

The following basic theorem is due to I. Ekeland [12, 13].

THEOREM 3. – *Let (M, d) be a complete metric space and $\psi : M \to (-\infty, \infty]$, $\psi \not\equiv \infty$, be a lower semicontinuous function which is bounded from below.*

Then the following properties hold true: for every $\varepsilon > 0$ and for any $z_0 \in M$ there exists $z \in M$ such that

(i) $\psi(z) \leq \psi(z_0) - \varepsilon\, d(z, z_0)$;
(ii) $\psi(x) \geq \psi(z) - \varepsilon\, d(x, z)$, *for any $x \in M$.*

PROOF. – We may assume without loss of generality that $\varepsilon = 1$. Define the following binary relation on M:

$$y \leq x \qquad \text{if and only if} \qquad \psi(y) - \psi(x) + d(x, y) \leq 0\,.$$

We verify that "$\leq$" is an order relation, that is,

(a) $x \leq x$, for any $x \in M$;
(b) if $x \leq y$ and $y \leq x$ then $x = y$;
(c) if $x \leq y$ and $y \leq z$ then $x \leq z$.

For arbitrary $x \in M$, set

$$S(x) = \{y \in M;\ y \leq x\}\,.$$

Let (ε_n) be a sequence of positive numbers such that $\varepsilon_n \to 0$ and fix $z_0 \in M$. For any $n \geq 0$, let $z_{n+1} \in S(z_n)$ be such that

$$\psi(z_{n+1}) \leq \inf_{S(z_n)} \psi + \varepsilon_{n+1}\,.$$

The existence of z_{n+1} follows by the definition of the set $S(x)$. We prove that the sequence (z_n) converges to some element z which satisfies (i) and (ii).

Let us first remark that $S(y) \subset S(x)$, provided that $y \leq x$. Hence, $S(z_{n+1}) \subset S(z_n)$. It follows that, for any $n \geq 0$,

$$\psi(z_{n+1}) - \psi(z_n) + d(z_n, z_{n+1}) \leq 0\,,$$

which implies $\psi(z_{n+1}) \leq \psi(z_n)$. Since ψ is bounded from below, it follows that the sequence $\{\psi(z_n)\}$ converges.

We prove in what follows that (z_n) is a Cauchy sequence. Indeed, for any n and p we have

$$\psi(z_{n+p}) - \psi(z_n) + d(z_{n+p}, z_n) \leq 0\,. \tag{14}$$

Therefore

$$d(z_{n+p}, z_n) \leq \psi(z_n) - \psi(z_{n+p}) \to 0\,, \quad \text{as } n \to \infty\,,$$

which shows that (z_n) is a Cauchy sequence, so it converges to some $z \in M$. Now, taking $n = 0$ in (14), we find

$$\psi(z_p) - \psi(z_0) + d(z_p, z_0) \leq 0\,.$$

So, as $p \to \infty$, we get (i).

In order to prove (ii), let us choose an arbitrary $x \in M$. We distinguish the following situations.

CASE 1. – $x \in S(z_n)$, for any $n \geq 0$. It follows that $\psi(z_{n+1}) \leq \psi(x) + \varepsilon_{n+1}$ which implies that $\psi(z) \leq \psi(x)$.

CASE 2. – There exists an integer $N \geq 1$ such that $x \notin S(z_n)$, for any $n \geq N$ or, equivalently,

$$\psi(x) - \psi(z_n) + d(x, z_n) > 0\,, \qquad \text{for every } \; n \geq N\,.$$

Passing at the limit in this inequality as $n \to \infty$ we find (ii). □

COROLLARY 1. – *Assume the same hypotheses on M and ψ. Then, for any $\varepsilon > 0$, there exists $z \in M$ such that*

$$\psi(z) < \inf_M \psi + \varepsilon$$

and

$$\psi(x) \geq \psi(z) - \varepsilon\, d(x, z)\,, \qquad \textit{for any } \; x \in M\,.$$

The conclusion follows directly from Theorem 3.

The following consequence of Ekeland's variational principle is of particular interest in our next arguments. Roughly speaking, this property establishes the existence of *almost critical points* for bounded from below C^1–functionals defined on Banach spaces. In order words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

COROLLARY 2. – *Let E be a Banach space and let $\psi : E \to \mathbb{R}$ be a C^1 function which is bounded from below. Then, for any $\varepsilon > 0$, there exists $z \in E$ such that*

$$\psi(z) \leq \inf_E \psi + \varepsilon \qquad \textit{and} \qquad \|\psi'(z)\|_{E^\star} \leq \varepsilon\,.$$

PROOF. – The first part of the conclusion follows directly from Theorem 3. For the second part we have

$$\|\psi'(z)\|_{E^\star} = \sup_{\|u\|=1} \langle \psi'(z), u \rangle\,.$$

But

$$\langle \psi'(z), u \rangle = \lim_{\delta \to 0} \frac{\psi(z + \delta u) - \psi(z)}{\delta \|u\|}\,.$$

So, by Theorem 3,

$$\langle \psi'(z), u \rangle \geq -\varepsilon\,.$$

Replacing now u by $-u$ we find

$$\langle \psi'(z), u \rangle \leq \varepsilon\,,$$

which concludes our proof. □

We point out that in the setting of Corollary 2, the Ekeland variational principle can also be proved by using the deformation lemma, as in the Willem's book [42].

We give in what follows a variant of Ekeland's variational principle in the case of *finite dimensional* Banach spaces. We also state an alternative proof, which relies on elementary arguments.

THEOREM 4. – *Let* $\psi : \mathbb{R}^N \to (-\infty, \infty]$ *be a lower semicontinuous function,* $\psi \not\equiv \infty$, *bounded from below. Let* $x_\varepsilon \in \mathbb{R}^N$ *be such that*

$$\inf \psi \leq \psi(\chi_\varepsilon) \leq \inf \psi + \varepsilon\,. \tag{15}$$

Then, for every $h > 0$, *there exists* $z_\varepsilon \in \mathbb{R}^N$ *such that*

(i) $\psi(z_\varepsilon) \leq \psi(x_\varepsilon)$;
(ii) $\|z_\varepsilon - x_\varepsilon\| \leq h$;
(iii) $\psi(z_\varepsilon) \leq \psi(x) + \dfrac{\varepsilon}{h}\,\|z_\varepsilon - x\|$, *for every* $x \in \mathbb{R}^N$.

PROOF. – Given x_ε satisfying (15), let us consider $\varphi : \mathbb{R}^N \to (-\infty, \infty]$ defined by

$$\varphi(x) = \psi(x) + \frac{\varepsilon}{h}\,\|x - x_\varepsilon\|\,.$$

By our hypotheses on ψ it follows that φ is lower semicontinuous and bounded from below. Moreover, $\varphi(x) \to \infty$ as $\|x\| \to \infty$. Therefore there exists $z_\varepsilon \in \mathbb{R}^N$ which minimizes φ on $\mathbb{R}^N$, that is, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) + \frac{\varepsilon}{h}\,\|z_\varepsilon - x_\varepsilon\| \leq \psi(x) + \frac{\varepsilon}{h}\,\|x - x_\varepsilon\|\,. \tag{16}$$

By letting $x = x_\varepsilon$ we find

$$\psi(z_\varepsilon) + \frac{\varepsilon}{h}\,\|z_\varepsilon - x_\varepsilon\| \leq \psi(x_\varepsilon)\,,$$

and (i) follows. Next, since $\psi(x_\varepsilon) \leq \inf \psi + \varepsilon$, we deduce from the above inequality that $\|z_\varepsilon - x_\varepsilon\| \leq h$, that is (ii) holds.

We infer from (16) that, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) \leq \psi(x) + \frac{\varepsilon}{h}\,(\|x - x_\varepsilon\| - \|z_\varepsilon - x_\varepsilon\|) \leq \psi(x) + \frac{\varepsilon}{h}\,\|x - z_\varepsilon\|,$$

which is exactly the desired inequality (iii). □

The above result shows that, the closer to x_ε we desire z_ε to be, the larger the perturbation of ψ that must be accepted. In practise, a good compromise is to take $h = \sqrt{\varepsilon}$.

It is striking to remark that the Ekeland variational principle characterizes the completeness of a metric space in a certain sense. More precisely, we have the following property.

THEOREM 5. – *Let* (M, d) *be a metric space. Then* M *is complete if and only if the following holds: for every lower semicontinuous function* $\psi : M \to (-\infty, \infty]$, $\psi \not\equiv \infty$, *which is bounded from below and for every* $\varepsilon > 0$, *there exists* $z_\varepsilon \in M$ *such that*

(i) $\psi(z_\varepsilon) \leq \inf_M \psi + \varepsilon$,
(ii) $\psi(z) > \psi(z_\varepsilon) - \varepsilon\, d(z, z_\varepsilon)$, *for any* $z \in M \setminus \{z_\varepsilon\}$.

PROOF. – The "only if" part follows directly from Corollary 1.

For the converse, let us assume that M is an arbitrary metric space satisfying the hypotheses. Let $(v_n) \subset M$ be an arbitrary Cauchy sequence and consider the function $f : M \to \mathbb{R}$ defined by

$$f(u) = \lim_{n\to\infty} d(u, v_n)\,.$$

The function f is continuous and $\inf f = 0$, since (v_n) is a Cauchy sequence. In order to justify the completeness of M it is enough to find $v \in M$ such that $f(v) = 0$. For this aim, choose arbitrarily $\varepsilon \in (0, 1)$. Now, from our hypotheses (i) and (ii), there exists $v \in M$ such that $f(v) \leq \varepsilon$ and

$$f(w) + \varepsilon\, d(w, v) > f(v)\,, \qquad \text{for any } \; w \in M \setminus \{v\}\,.$$

From the definition of f and the fact that (v_n) is a Cauchy sequence we can take $w = v_k$ for k large enough such that $f(w)$ is arbitrarily small and $d(w, v) \leq \varepsilon + \eta$, for any $\eta > 0$, because $f(v) \leq \varepsilon$. Using (ii) we obtain that, in fact, $f(v) \leq \varepsilon^k$. Repeating the argument we may conclude that $f(v) \leq \varepsilon^n$, for all $n \geq 1$ and so $f(v) = 0$, as required. □

4. – Nonsmooth extensions of the mountain pass theorem.

We give in this section several generalizations of the mountain pass theorem in the framework of locally Lipschitz functionals. We are concerned with many nonsmooth extensions of this celebrated result, including in the cases of finite and infinite dimensional linking, presence of symmetries, etc. Theorems 6, 7, and 8 were proved in Rădulescu [36].

We start by recalling some basic properties of locally Lipschitz functionals defined on real Banach spaces.

4.1 – *Basic properties of locally Lipschitz functionals.*

Throughout this section, X denotes a real Banach space. Let X^* be its dual and, for every $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle$ be the duality pairing between X^* and X.

DEFINITION 1. – *A functional $f : X \to \mathbb{R}$ is said to be locally Lipschitz provided that, for every $x \in X$, there exists a neighbourhood V of x and a positive constant $k = k(V)$ depending on V such that*

$$|f(y) - f(z)| \leq k \,\|y - z\|,$$

for each $y, z \in V$.

The set of all locally Lipschitz mappings defined on X with real values is denoted by $\mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$.

DEFINITION 2. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x, v \in X$, $v \neq 0$. We call the generalized directional derivative of f in x with respect to the direction v the number*

$$f^0(x, v) = \limsup_{\substack{y \to x \\ h \searrow 0}} \frac{f(y + h\,v) - f(y)}{h}\,.$$

We first observe that if f is a locally Lipschitz functional, then $f^0(x, v)$ is a finite number and

$$|f^0(x, v)| \leq k \,\|v\|\,. \tag{17}$$

Moreover, if $x \in X$ is fixed, then the mapping $v \mapsto f^0(x, v)$ is positive homogeneous and subadditive, so it is convex continuous. By the Hahn-Banach theorem, there exists a linear map $x^* : X \to \mathbb{R}$ such that for every $v \in X$,

$$x^*(v) \leq f^0(x, v)\,.$$

The continuity of x^* is an immediate consequence of the above inequality and of (17).

DEFINITION 3. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x \in X$. The generalized gradient (Clarke subdifferential) of f at the point x is the nonempty subset $\partial f(x)$ of X^* which is defined by*

$$\partial f(x) = \{x^* \in X^*;\ f^0(x, v) \geq \langle x^*, v\rangle, \ \ \textit{for all} \ \ v \in X\}\,.$$

We point out that if f is convex then $\partial f(x)$ coincides with the subdifferential of f in x in the sense of the convex analysis, that is

$$\partial f(x) = \{x^* \in X^*;\ f(y) - f(x) \geq \langle x^*, y - x\rangle, \ \ \text{for all} \ \ y \in X\}\,.$$

We list in what follows the main properties of the Clarke gradient of a locally Lipschitz functional. We refer to [7, 8, 10] for further details and proofs.

a) For every $x \in X$, $\partial f(x)$ is a convex and $\sigma(X^*, X)$-compact set.

b) For every $x, v \in X$, $v \neq 0$, the following holds

$$f^0(x, v) = \max\{\langle x^*, v\rangle;\ x^* \in \partial f(x)\}\,.$$

c) The multivalued mapping $x \mapsto \partial f(x)$ is upper semicontinuous, in the sense that for every $x_0 \in X,\ \varepsilon > 0$ and $v \in X$, there exists $\delta > 0$ such that, for any $x^* \in \partial f(x)$ satisfying $\|x - x_0\| < \delta$, there is some $x_0^* \in \partial f(x_0)$ satisfying $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

d) The functional $f^0(\cdot,\cdot)$ is upper semicontinuous.

e) If x is an extremum point of f, then $0 \in \partial f(x)$.

f) The mapping

$$\lambda(x) = \min_{x^* \in \partial f(x)} \|x^*\|$$

is well defined and lower semicontinuous.

g) $\partial(-f)(x) = -\partial f(x)$.

h) Lebourg's mean value theorem (see [22]): if x and y are two distinct points in X then there exists a point z situated on the open segment joining x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x\rangle\,.$$

i) If f has a Gâteaux derivative f' which is continuous in a neighbourhood of x, then $\partial f(x) = \{f'(x)\}$. If X is finite dimensional, then $\partial f(x)$ reduces at one point if and only if f is Fréchet-differentiable to x.

DEFINITION 4. – *A point $x \in X$ is said to be a critical point of the locally Lipschitz functional $f : X \to \mathbb{R}$ if $0 \in \partial f(x)$, that is $f^0(x, v) \geq 0$, for every $v \in X \setminus \{0\}$. A number c is a critical value of f provided that there exists a critical point $x \in X$ such that $f(x) = c$.*

Note that any minimum point is a critical point. Indeed, if x is a local minimum point, then for every $v \in X$,

$$0 \leq \limsup_{h \searrow 0} \frac{f(x + hv) - f(x)}{h} \leq f^0(x, v)\,.$$

We now introduce a compactness condition for locally Lipschitz functionals. This condition was used for the first time, in the case of C^1-functionals, by H. Brezis, J.M. Coron and L. Nirenberg [4].

DEFINITION 5. – *If $f : X \to \mathbb{R}$ is a locally Lipschitz functional and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short, $(\mathrm{PS})_c$) if any sequence (x_n) in X, satisfying $f(x_n) \longrightarrow c$ and $\lambda(x_n) \longrightarrow 0$, contains a convergent subsequence. The mapping f satisfies the Palais-Smale condition (in short, (PS)) if every sequence (x_n), which satisfies $(f(x_n))$ is bounded and $\lambda(x_n) \longrightarrow 0$, has a convergent subsequence.*

4.2 – *Mountains of positive altitude.*

Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional. Consider K a compact metric space and K^* a closed nonempty subset of K. If $p^* : K^* \to X$ is a continuous mapping, set

$$\mathcal{P} = \{p \in C(K,X);\ p = p^* \text{ on } K^*\} .$$

By a celebrated theorem of Dugundji [11], the set $\mathcal{P}$ is nonempty.

Define

$$c = \inf_{p\in\mathcal{P}} \max_{t\in K} f(p(t)) . \tag{18}$$

Obviously, $c \geq \max_{t\in K^*} f(p^*(t))$.

THEOREM 6. – *Assume that*

$$c > \max_{t\in K^*} f(p^*(t)) . \tag{19}$$

Then there exists a sequence (x_n) in X such that

i) $\lim_{n\to\infty} f(x_n) = c$;

ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

For the proof of this theorem we need the following auxiliary result.

LEMMA 3. – *Let M be a compact metric space and let $\varphi : M \to 2^{X^*}$ be an upper semicontinuous mapping such that, for every $t \in M$, the set $\varphi(t)$ is convex and $\sigma(X^*,X)$-compact. For $t \in M$, denote*

$$\gamma(t) = \inf\{ \|x^*\|;\ \ x^* \in \varphi(t)\} \quad \text{and} \quad \gamma = \inf_{t\in M} \gamma(t) .$$

Then, for every fixed $\varepsilon > 0$, there exists a continuous mapping $v : M \to X$ such that for every $t \in M$ and $x^* \in \varphi(t)$,

$$\|v(t)\| \leq 1 \quad \textit{and} \quad \langle x^*, v(t)\rangle \geq \gamma - \varepsilon .$$

PROOF. – Assume, without loss of generality, that $\gamma > 0$ and $0<\varepsilon<\gamma$. Denoting by B_r the open ball in X^* centered at the origin and with radius r then, for every $t \in M$ we have

$$B_{\gamma-\varepsilon/2} \cap \varphi(t) = \emptyset .$$

Since $\varphi(t)$ and $B_{\gamma-\varepsilon/2}$ are convex, disjoint and $\sigma(X^*,X)$-compact sets, it follows from the separation theorem in locally convex spaces (Theorem 3.4 in W. Rudin [38]), applied to the space $(X^*, \sigma(X^*,X))$, and from the fact that the dual of this space is X, that: for every $t \in M$, there exists $v_t \in X$ such that

$$\|v_t\| = 1 \quad \text{and} \quad \langle \xi, v_t \rangle \leq \langle x^*, v_t \rangle \,,$$

for any $\xi \in B_{\gamma-\varepsilon/2}$ and for every $x^* \in \varphi(t)$.

Hence, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t \rangle \geq \sup_{\xi \in B_{\gamma-\varepsilon/2}} \langle \xi, v_t \rangle = \gamma - \varepsilon/2 \,.$$

Since φ is upper semicontinuous, there exists an open neighbourhood $V(t)$ of t such that for every $t' \in V(t)$ and all $x^* \in \varphi(t')$,

$$\langle x^*, v_t \rangle > \gamma - \varepsilon \,.$$

Therefore, since M is compact and $M = \bigcup_{t \in M} V(t)$, there exists an open covering $\{V_1, \ldots, V_n\}$ of M. Let $v_1, \ldots, v_n$ be on the unit sphere of X such that

$$\langle x^*, v_i \rangle > \gamma - \varepsilon,$$

for every $1 \leq i \leq n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \mathrm{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum\limits_{j=1}^{n} \rho_j(t)} \quad \text{and} \quad v(t) = \sum_{i=1}^{n} \zeta_i(t) v_i .$$

A straightforward computation shows that v satisfies our conclusion. This completes the proof of Lemma 3. □

Proof of Theorem 6. – We apply Ekeland's variational principle to the functional

$$\psi(p) = \max_{t \in K} f(p(t)) \,,$$

defined on $\mathcal{P}$, which is a complete metric space if it is endowed with the metric

$$d(p,q) = \max_{t \in K} \|p(t) - q(t)\| \,, \qquad \text{for any } p,\, q \in \mathcal{P}.$$

The mapping ψ is continuous and bounded from below because, for every $p \in \mathcal{P}$,

$$\psi(p) \geq \max_{t \in K^*} f(p^*(t))$$

Since

$$c = \inf_{p \in \mathcal{P}} \psi(p),$$

it follows that for every $\varepsilon > 0$, there is some $p \in \mathcal{P}$ such that

$$\psi(q) - \psi(p) + \varepsilon d(p,q) \geq 0, \quad \text{for all } \; q \in \mathcal{P}\,; \tag{20}$$

$$c \leq \psi(p) \leq c + \varepsilon \,.$$

Set

$$B(p) = \{t \in K;\ f(p(t)) = \psi(p)\}\,.$$

For concluding the proof it is sufficient to show that there exists $t' \in B(p)$ such that

$$\lambda(p(t')) \le 2\varepsilon\,.$$

Indeed the conclusion of the theorem follows then easily by choosing $\varepsilon = 1/n$ and $x_n = p(t')$.

Applying Lemma 3 for $M = B(p)$ and $\varphi(t) = \partial f(p(t))$, we obtain a continuous map $v : B(p) \to X$ such that, for every $t \in B(p)$ and $x^* \in \partial f(p(t))$, we have

$$\|v(t)\| \le 1 \quad \text{and} \quad \langle x^*, v(t)\rangle \ge \gamma - \varepsilon\,,$$

where

$$\gamma = \inf_{t \in B(p)} \lambda(p(t))\,.$$

It follows that for every $t \in B(p)$,

$$\begin{aligned} f^0(p(t), -v(t)) &= \max\{\langle x^*, -v(t)\rangle;\ x^* \in \partial f(p(t))\} \\ &= -\min\{\langle x^*, v(t);\ x^* \in \partial f(p(t))\} \le -\gamma + \varepsilon\,. \end{aligned}$$

By (19) we have $B(p) \cap K^* = \emptyset$. So, there exists a continuous extension $w : K \to X$ of v such that $w = 0$ on K^* and, for every $t \in K$,

$$\|w(t)\| \le 1\,.$$

Choose in the place of q in (20) small perturbations q_h of the path p:

$$q_h(t) = p(t) - hw(t),$$

where $h > 0$ is small enough.

We deduce from (20) that, for every $h > 0$,

$$-\varepsilon \le -\varepsilon\|w\|_\infty \le \frac{\psi(q_h) - \psi(p)}{h}\,. \tag{21}$$

In what follows, $\varepsilon > 0$ will be fixed, while $h \to 0$. Let $t_h \in K$ be such that $f(q_h(t_h)) = \psi(q_h)$. We may also assume that the sequence (t_{h_n}) converges to some t_0, which, obviously, is in $B(p)$. Observe that

$$\frac{\psi(q_h) - \psi(p)}{h} = \frac{\psi(p - hw) - \psi(p)}{h} \le \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}\,.$$

It follows from this relation and from (21) that

$$\begin{aligned} -\varepsilon &\le \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h} \\ &\le \frac{f(p(t_h) - hw(t_0)) - f(p(t_h))}{h} + \frac{f(p(t_h) - hw(t_h)) - f(p(t_h) - hw(t_0))}{h}\,. \end{aligned}$$

Using the fact that f is a locally Lipschitz map and $t_{h_n} \to t_0$, we find that

$$\lim_{n\to\infty} \frac{f(p(t_{h_n}) - h_n w(t_{h_n})) - f(p(t_{h_n}) - h_n w(t_0))}{h_n} = 0\,.$$

Therefore

$$-\varepsilon \le \limsup_{n\to\infty} \frac{f(p(t_0) + z_n - h_n w(t_0)) - f(p(t_0) + z_n)}{h_n}\,,$$

where $z_n = p(t_{h_n}) - p(t_0)$. Consequently,

$$-\varepsilon \le f^0(p(t_0), -w(t_0)) = f^0(p(t_0), -v(t_0)) \le -\gamma + \varepsilon\,.$$

It follows that

$$\gamma = \inf\{\|x^*\|;\ x^* \in \partial f(p(t)), t \in B(p)\} \le 2\varepsilon\,.$$

Taking into account the lower semicontinuity of λ, we deduce the existence of some $t' \in B(p)$ such that

$$\lambda(p(t')) = \inf\{\|x^*\|;\ x^* \in \partial f(p(t'))\} \le 2\varepsilon\,.$$

This concludes the proof. □

COROLLARY 3. – *If f satisfies the condition* $(\mathrm{PS})_c$ *and the hypotheses of Theorem* 6, *then c is a critical value of f corresponding to a critical point which is not in $p^*(K^*)$.*

The proof of this result follows easily by applying Theorem 6 and the fact that λ is lower semicontinuous. □

The following result generalizes the classical mountain pass theorem of A. Ambrosetti and P. Rabinowitz.

COROLLARY 4. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional such that $f(0) = 0$ and there exists $v \in X \setminus \{0\}$ so that $f(v) \le 0$. Set*

$$\mathcal{P} = \{p \in C([0,1], X);\ p(0) = 0 \ \ \textit{and} \ \ p(1) = v\}$$

and

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t))\,.$$

If $c > 0$ and f satisfies the condition $(\mathrm{PS})_c$, *then c is a critical value of f.*

For the proof of this result it is sufficient to apply Corollary 3 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = 0$ and $p^*(1) = v$. □

COROLLARY 5. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz mapping. Assume that there exists a subset S of X such that, for every $p \in \mathcal{P}$,*

$$p(K) \cap S \neq \emptyset .$$

If

$$\inf_{x\in S} f(x) > \max_{t\in K^*} f(p^*(t)),$$

then the conclusion of Theorem 6 *holds.*

PROOF. – It suffices to observe that

$$\inf_{p\in\mathcal{P}} \max_{t\in K} f(p(t)) \geq \inf_{x\in S} f(x) > \max_{t\in K^*} f(p^*(t)) .$$

Then our conclusion follows directly. □

Using now Theorem 6 we may prove the following result, which is originally due to H. Brezis, J.-M. Coron, and L. Nirenberg (see Theorem 2 in [4]):

COROLLARY 6. – *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable functional such that $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. If f satisfies* (19), *then there exists a sequence (x_n) in X such that*

i) $\lim_{n\to\infty} f(x_n) = c$;

ii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.

Moreover, if f satisfies the condition $(\mathrm{PS})_c$, *then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

PROOF. – Observe first that f' is locally bounded. Indeed, if (x_n) is a sequence converging to x_0, then

$$\sup_n |\langle f'(x_n), v\rangle| < \infty,$$

for every $v \in X$. Thus, by the Banach-Steinhaus theorem,

$$\limsup_{n\to\infty} \|f'(x_n)\| < \infty .$$

For $h > 0$ small enough and $w \in X$ sufficiently small we have

$$|f(x_0 + w + h\,v) - f(x_0 + w)| = |h\langle f'(x_0 + w + h\,\theta\,v), v\rangle| \leq C\,h\,\|v\| , \tag{22}$$

where $\theta \in (0, 1)$. Therefore $f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ and $f^0(x_0, v) = \langle f'(x_0), v\rangle$, by the continuity assumption on f'. In relation (22) the existence of C follows from the local boundedness property of f'.

Since f^0 is linear in v, we obtain

$$\partial f(x) = \{f'(x)\}\,.$$

To conclude the proof, it remains to apply Theorem 6 and Corollary 3. □

A very useful result in applications is the following variant of the *saddle point* theorem of P. Rabinowitz.

COROLLARY 7. – *Assume that X admits a decomposition of the form $X = X_1 \oplus X_2$, where X_2 is a finite dimensional subspace of X. For some fixed $z \in X_2$, suppose that there exists $R > \|z\|$ such that*

$$\inf_{x\in X_1} f(x+z) > \max_{x\in K^*} f(x)\,,$$

where

$$K^* = \{x \in X_2;\ \|x\| = R\}\,.$$

Set

$$K = \{x \in X_2;\ \|x\| \le R\},$$
$$\mathcal{P} = \{p \in C(K,X);\ p(x) = x \quad \textit{if} \quad \|x\| = R\}.$$

If c is chosen as in (18) *and f satisfies the condition* $(\mathrm{PS})_c$, *then c is a critical value of f.*

PROOF. – Applying Corollary 5 for $S = z + X_1$, we observe that it is sufficient to prove that, for every $p \in \mathcal{P}$,

$$p(K) \cap (z + X_1) \neq \emptyset\,.$$

If $P : X \to X_2$ is the canonical projection, then the above condition is equivalent to the fact that for every $p \in \mathcal{P}$, there exists $x \in K$ such that

$$P(p(x) - z) = P(p(x)) - z = 0\,.$$

To prove this claim, we use an argument based on the topological degree theory. Indeed, for every fixed $p \in \mathcal{P}$ we have

$$P \circ p = \mathrm{Id} \quad \text{on } K^* = \partial K\,.$$

Hence

$$d(P\circ p - z, \mathrm{Int}\, K, 0) = d(P \circ p, \mathrm{Int}\, K, z) = d(\mathrm{Id}, \mathrm{Int}\, K, z) = 1\,.$$

Now, by the existence property of the Brouwer degree, we may find $x \in \mathrm{Int}\, K$ such that

$$(P \circ p)(x) - z = 0,$$

which concludes our proof. □

4.3 – *Mountains of zero altitude: the Pucci-Serrin theorem.*

It is natural to ask us what happens if the condition (19) fails to be valid, more precisely, if

$$c = \max_{t\in K^*} f(p^*(t))\,.$$

The following example shows that in this case the conclusion of Theorem 6 does not hold.

EXAMPLE 1. – Let $X=\mathbb{R}^2,\ K=[0,1]\times\{0\},\ K^*=\{(0,0),(1,0)\}$ and let p^* be the identic map of K^*. As locally Lipschitz functional we choose

$$f : X \to \mathbb{R}, \quad f(x,y) = |y|\,.$$

Then f satisfies the Palais-Smale condition and

$$c = \max_{t\in K^*} f(p^*(t)) = 0\,.$$

However, f has no critical point.

In the smooth framework, the mountain pass theorem in the *zero altitude* case was proved by P. Pucci and J. Serrin [29] for a functional $J : X \to \mathbb{R}$ of class C^1 satisfying the following geometric conditions:

(a) *there exist real numbers a, r, R such that* $0<r<R$ *and* $J(u)\geq a$ *for every* $u\in X$ *with* $r<\|u\|<R$;

(b) $J(0)\leq a$ *and* $J(v)\leq a$ *for some* $v\in X$ *with* $\|v\|>R$.

Under these hypotheses, combined with the standard Palais-Smale compactness condition, P. Pucci and J. Serrin established the existence of a critical point $u_0\in X\setminus\{0,v\}$ of J with corresponding critical value $c\geq a$. Moreover, if $c=a$ then the critical point can be chosen with $r<\|u_0\|<R$. Roughly speaking, *the mountain pass theorem continues to hold for a mountain of zero altitude, provided it also has non-zero thickness*; in addition, *if* $c=a$, *then the pass itself occurs precisely on the mountain*, in the sense that it satisfies $r<\|u_0\|<R$.

The following result gives a sufficient condition so that Theorem 6 holds even if (19) fails.

THEOREM 7. – *Assume that for every* $p\in\mathcal{P}$ *there exists* $t\in K\setminus K^*$ *such that* $f(p(t))\geq c$.

Then there exists a sequence (x_n) *in* X *such that*

i) $\displaystyle\lim_{n\to\infty} f(x_n) = c\,;$

ii) $\displaystyle\lim_{n\to\infty} \lambda(x_n) = 0\,.$

Moreover, if f satisfies the condition $(\mathrm{PS})_c$*, then c is a critical value of f. Furthermore, if (p_n) is an arbitrary sequence in $\mathcal{P}$ satisfying*

$$\lim_{n\to\infty} \max_{t\in K} f(p_n(t)) = c,$$

then there exists a sequence (t_n) in K such that

$$\lim_{n\to\infty} f(p_n(t_n)) = c \quad \textit{and} \quad \lim_{n\to\infty} \lambda(p_n(t_n)) = 0\,.$$

PROOF. – For every $\varepsilon > 0$ we apply Ekeland's variational principle to the perturbed functional

$$\psi_\varepsilon : \mathcal{P} \to \mathbb{R}, \quad \psi_\varepsilon(p) = \max_{t\in K} (f(p(t)) + \varepsilon d(t))\,,$$

where

$$d(t) = \min\{\mathrm{dist}\,(t, K^*), 1\}\,.$$

If

$$c_\varepsilon = \inf_{p\in\mathcal{P}} \psi_\varepsilon(p),$$

then

$$c \le c_\varepsilon \le c + \varepsilon\,.$$

Thus, by Ekeland's variational principle, there exists a path $p \in \mathcal{P}$ such that for every $q \in \mathcal{P}$,

$$\psi_\varepsilon(q) - \psi_\varepsilon(p) + \varepsilon d(p, q) \ge 0, \tag{23}$$

$$c \le c_\varepsilon \le \psi_\varepsilon(p) \le c_\varepsilon + \varepsilon \le c + 2\varepsilon\,.$$

Denoting

$$B_\varepsilon(p) = \{t \in K;\ f(p(t)) + \varepsilon d(t) = \psi_\varepsilon(p)\},$$

it remains to show that there is some $t' \in B_\varepsilon(p)$ such that $\lambda(p(t')) \le 2\varepsilon$. Indeed, the conclusion of the first part of the theorem follows easily, by choosing $\varepsilon = 1/n$ and $x_n = p(t')$.

Now, by Lemma 3 applied for $M = B_\varepsilon(p)$ and $\varphi(t) = \partial f(p(t))$, we find a continuous mapping $v : B_\varepsilon(p) \to X$ such that, for every $t \in B_\varepsilon(p)$ and all $x^* \in \partial f(p(t))$,

$$\|v(t)\| \le 1 \quad \text{and} \quad \langle x^*, v(t)\rangle \ge \gamma_\varepsilon - \varepsilon,$$

where

$$\gamma_\varepsilon = \inf_{t\in B_\varepsilon(p)} \lambda(p(t))\,.$$

On the other hand, it follows by our hypothesis that

$$\psi_\varepsilon(p) > \max_{t\in K^*} f(p(t))\,.$$

Hence

$$B_\varepsilon(p) \cap K^* = \emptyset\,.$$

So, there exists a continuous extension w of v, defined on K and such that

$$w = 0 \quad \text{on} \quad K^* \quad \text{and} \quad \|w(t)\| \le 1, \quad \text{for any} \quad t \in K\,.$$

Choose as paths q in relation (23) small variations of the path p:

$$q_h(t) = p(t) - h\,w(t),$$

for $h > 0$ sufficiently small.

In what follows $\varepsilon > 0$ will be fixed, while $h \to 0$.

Let $t_h \in B_\varepsilon(p)$ be such that

$$f(q(t_h)) + \varepsilon d(t_h) = \psi_\varepsilon(q_h)\,.$$

There exists a sequence (h_n) converging to 0 and such that the corresponding sequence (t_{h_n}) converges to some t_0, which, obviously, lies in $B_\varepsilon(p)$. It follows that

$$-\varepsilon \le -\varepsilon\,\|w\|_\infty \le \frac{\psi_\varepsilon(q_h) - \psi_\varepsilon(p)}{h} = \frac{f(q_h(t_h)) + \varepsilon d(t_h) - \psi_\varepsilon(p)}{h}$$

$$\le \frac{f(q_h(t_h)) - f(p(t_h))}{h} = \frac{f(p(t_h) - h\,w(t_h)) - f(p(t_h))}{h}\,.$$

With the same arguments as in the proof of Theorem 6 we obtain the existence of some $t' \in B_\varepsilon(p)$ such that

$$\lambda(p(t')) \le 2\varepsilon\,.$$

Furthermore, if f satisfies $(\mathrm{PS})_c$ then c is a critical value of f, since λ is lower semicontinuous.

For the second part of the proof, applying again Ekeland's variational principle, we deduce the existence of a sequence of paths (q_n) in $\mathcal{P}$ such that, for every $q \in \mathcal{P}$,

$$\psi_{\varepsilon_n^2}(q) - \psi_{\varepsilon_n^2}(q_n) + \varepsilon_n d(q, q_n) \ge 0$$

and

$$\psi_{\varepsilon_n^2}(q_n) \le \psi_{\varepsilon_n^2}(p_n) - \varepsilon_n d(p_n, q_n)\,,$$

where (ε_n) is a sequence of positive numbers converging to 0 and (p_n) are paths in $\mathcal{P}$ such that

$$\psi_{\varepsilon_n^2}(p_n) \le c + 2\varepsilon_n^2\,.$$

Applying the same argument for q_n, instead of p, we find $t_n \in K$ such that

$$c - \varepsilon_n^2 \le f(q_n(t_n)) \le c + 2\varepsilon_n^2$$

and

$$\lambda(q_n(t_n)) \le 2\varepsilon_n .$$

We argue that this is the desired sequence (t_n). Indeed, by the Palais-Smale condition $(\mathrm{PS})_c$, there exists a subsequence of $(q_n(t_n))$ which converges to a critical point. The corresponding subsequence of $(p_n(t_n))$ converges to the same limit. A standard argument, based on the continuity of f and the lower semicontinuity of λ shows that for all the sequence we have

$$\lim_{n\to\infty} f(p_n(t_n)) = c \quad \text{and} \quad \lim_{n\to\infty} \lambda(p_n(t_n)) = 0 .$$

This concludes our proof. $\square$

COROLLARY 8. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional which satisfies the Palais-Smale condition.*

If f has two different minimum points, then f possesses a third critical point.

PROOF. – Let x_0 and x_1 be two different minimum points of f.

CASE 1. – $f(x_0) = f(x_1) = a$. Choose $0 < R < \frac{1}{2}\|x_1 - x_0\|$ such that $f(x) \ge a$, for all $x \in B(x_0, R) \cup B(x_1, R)$. Set $A = \overline{B}(x_0, R/2) \cup \overline{B}(x_1, R/2)$.

CASE 2. – $f(x_0) > f(x_1)$. Choose $0 < R < \|x_1 - x_0\|$ such that $f(x) \ge f(x_0)$, for every $x \in B(x_0, R)$. Put $A = \overline{B}(x_0, R/2) \cup \{x_1\}$.

In both cases, fix $p^* \in C([0,1], X)$ such that $p^*(0) = x_0$ and $p^*(1) = x_1$. If $K^* = (p^*)^{-1}(A)$ then, by Theorem 7, we obtain the existence of a critical point of f, which is different from x_0 and x_1, as we can easily deduce by examining the proof of Theorem 7. $\square$

With the same proof as in Corollary 6 one can deduce the following mountain pass property which extends the Pucci-Serrin theorem [29, Theorem 1].

COROLLARY 9. – *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a Gâteaux-differentiable functional such that the operator $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Assume that for every $p \in \mathcal{P}$ there exists $t \in K \setminus K^*$ such that $f(p(t)) \ge c$.*

Then there exists a sequence (x_n) in X so that

i) $\lim_{n\to\infty} f(x_n) = c\,;$

ii) $\lim_{n\to\infty} \|f'(x_n)\| = 0\,.$

If, furthermore, f satisfies $(\mathrm{PS})_c$, *then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

4.4 – *An extension due to Ghoussoub and Preiss.*

We recall that if $f : X \to \mathbb{R}$ is a locally Lipschitz functional, K is a compact metric space, K^* is a closed nonempty subset of K, $p^* : K^* \to X$ is a continuous mapping, then

$$c = \inf_{p \in \mathcal{P}} \max_{t \in K} f(p(t)) .$$

We have defined

$$\mathcal{P} = \{p \in C(K, X);\ p = p^* \quad \text{on} \quad K^*\} .$$

The following result is a strengthened variant of Theorems 6 and 7. The smooth case corresponding to C^1-functionals is due to N. Ghoussoub and D. Preiss [17].

THEOREM 8. – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and let F be a closed subset of X, with no common point with $p^*(K^*)$. Assume that*

$$f(x) \geq c, \quad \textit{for every} \quad x \in F, \tag{24}$$

and

$$p(K) \cap F \neq \emptyset \ , \quad \textit{for all} \quad p \in \mathcal{P}. \tag{25}$$

Then there exists a sequence (x_n) in X such that

i) $\lim_{n\to\infty} \operatorname{dist}\,(x_n, F) = 0$;

ii) $\lim_{n\to\infty} f(x_n) = c$;

iii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

PROOF. – Fix $\varepsilon > 0$ such that

$$\varepsilon < \min\{1;\ \operatorname{dist}(p^*(K^*), F)\} .$$

Choose $p \in \mathcal{P}$ so that

$$\max_{t \in K} f(p(t)) \leq c + \varepsilon^2/4.$$

The set

$$K_0 = \{t \in K;\ \operatorname{dist}(p(t), F) \geq \varepsilon\}$$

is bounded and contains K^*. Define

$$\mathcal{P}_0 = \{q \in C(K, X);\ q = p \ \text{ on } \ K_0\}\,.$$

Set

$$\eta : X \to \mathbb{R}, \quad \eta(x) = \max\{0;\ \varepsilon^2 - \varepsilon\, \mathrm{dist}\,(x, F)\}\,.$$

Define $\psi : \mathcal{P}_0 \to \mathbb{R}$ by

$$\psi(q) = \max_{t \in K}\ (f + \eta)(q(t))\,.$$

The functional ψ is continuous and bounded from below. By Ekeland's variational principle, there exists $p_0 \in \mathcal{P}_0$ such that for every $q \in \mathcal{P}_0$,

$$\psi(p_0) \leq \psi(q)\,,$$

$$d(p_0, q) \leq \varepsilon/2, \tag{26}$$

$$\psi(p_0) \leq \psi(q) + \varepsilon\, d(q, p_0)/2. \tag{27}$$

The set

$$B(p_0) = \{t \in K;\ \ (f + \eta)(p_0(t)) = \psi(p_0)\}$$

is closed. To conclude the proof, it is sufficient to show that there exists $t \in B(p_0)$ such that

$$\mathrm{dist}\,(p_0(t), F) \leq 3\varepsilon/2, \tag{28}$$

$$c \leq f(p_0(t)) \leq c + 5\varepsilon^2/4, \tag{29}$$

$$\lambda(p_0(t)) \leq 5\varepsilon/2. \tag{30}$$

Indeed, it is enough to choose then $\varepsilon = 1/n$ and $x_n = p_0(t)$.

PROOF OF (28). – It follows by the definition of $\mathcal{P}_0$ and (25) that, for every $q \in \mathcal{P}_0$, we have

$$q(K \setminus K_0) \cap F \neq \emptyset\,.$$

Therefore, for any $q \in \mathcal{P}_0$,

$$\psi(q) \geq c + \varepsilon^2\,.$$

On the other hand,

$$\psi(p) \leq c + \varepsilon^2/4 + \varepsilon^2 = c + 5\varepsilon^2/4.$$

Hence

$$c + \varepsilon^2 \leq \psi(p_0) \leq \psi(p) \leq c + 5\varepsilon^2/4. \tag{31}$$

So, for each $t \in B(p_0)$,

$$c + \varepsilon^2 \le \psi(p_0) = (f + \eta)(p_0(t)).$$

Moreover, if $t \in K_0$, then

$$(f + \eta)(p_0(t)) = (f + \eta)(p(t)) = f(p(t)) \le c + \varepsilon^2/4.$$

This implies that

$$B(p_0) \subset K \setminus K_0.$$

By the definition of K_0 it follows that for every $t \in B(p_0)$ we have

$$\operatorname{dist}(p(t), F) \le \varepsilon.$$

Now, the relation (26) yields

$$\operatorname{dist}(p_0(t), F) \le \frac{3\varepsilon}{2}\,.$$

PROOF OF (29). – For every $t \in B(p_0)$ we have

$$\psi(p_0) = (f + \eta)(p_0(t)).$$

Using (31) and taking into account that

$$0 \le \eta \le \varepsilon^2,$$

it follows that

$$c \le f(p_0(t)) \le c + 5\varepsilon^2/4.$$

PROOF OF (30). – Applying Lemma 3 for $\varphi(t) = \partial f(p_0(t))$, we find a continuous mapping $v : B(p_0) \to X$ such that for every $t \in B(p_0)$,

$$\|v(t)\| \le 1.$$

Moreover, for any $t \in B(p_0)$ and $x^* \in \partial f(p_0(t))$,

$$\langle x^*, v(t)\rangle \ge \gamma - \varepsilon, \quad \text{where} \quad \gamma = \inf_{t \in B(p_0)} \lambda(p_0(t)).$$

Hence for every $t \in B(p_0)$,

$$\begin{aligned} f^0(p_0(t), -v(t)) &= \max\{\langle x^*, -v(t)\rangle; x^* \in \partial f(p_0(t))\} \\ &= -\min\{\langle x^*, v(t)\rangle; x^* \in \partial f(p_0(t))\} \le -\gamma + \varepsilon. \end{aligned}$$

Since $B(p_0) \cap K_0 = \emptyset$, there exists a continuous extension w of v to the set K such that $w = 0$ on K_0 and $\|w(t)\| \le 1$, for all $t \in K$.

Now, by relation (27), it follows that for every $h > 0$,

$$-\frac{\varepsilon}{2} \le -\frac{\varepsilon}{2}\,\|w\|_\infty \le \frac{\psi(p_0 - h\,w) - \psi(p_0)}{h}\,. \tag{32}$$

For every n, there exists $t_n \in K$ such that

$$\psi(p_0 - w/n) = (f+\eta)(p_0(t_n) - w(t_n)/n).$$

Passing eventually to a subsequence, we may suppose that (t_n) converges to t_0, which, clearly, lies in $B(p_0)$. On the other hand, for every $t \in K$ and $h > 0$ we have

$$f(p_0(t) - h\,w(t)) \le f(p_0(t)) + h\,\varepsilon.$$

Hence

$$n[\psi(p_0 - h\,w) - \psi(p_0)] \le n\left[f(p_0(t_n) - w(t_n)/n) + \frac{\varepsilon}{n} - f(p_0(t_n))\right].$$

Therefore, by (32), it follows that

$$\begin{aligned} -\frac{3\varepsilon}{2} &\le n[\psi(p_0(t_n) - w(t_n)/n) - f(p_0(t_n))] \\ &\le n[\psi(p_0(t_n) - w(t_0)/n) - f(p_0(t_n))] \\ &\qquad + n[f(p_0(t_n) - w(t_n)/n) - f(p_0(t_n) - w(t_0)/n]. \end{aligned}$$

Since f is locally Lipschitz and $t_n \to t_0$ we deduce that

$$\limsup_{n\to\infty} n[f(p_0(t_n) - w(t_n)/n) - f(p_0(t_n) - w(t_0)/n)] = 0.$$

Therefore

$$-\frac{3\varepsilon}{2} \le \limsup_{n\to\infty} n\,[f(p_0(t_0) + z_n - w(t_0)/n) - f(p_0(t_0) + z_n)],$$

where $z_n = p_0(t_n) - p_0(t_0)$. Hence

$$-\frac{3\varepsilon}{2} \le f^0(p_0(t_0), -w(t_0)) \le -\gamma + \varepsilon.$$

So

$$\gamma = \inf\{\|x^*\|;\ x^* \in \partial f(p_0(t)), t \in B(p_0)\} \le 5\varepsilon/2.$$

Now, by the lower semicontinuity of λ, we find $t \in B(p_0)$ such that

$$\lambda(p_0(t)) = \inf_{x^* \in \partial f(p_0(t))} \|x^*\| \le 5\varepsilon/2,$$

which concludes our proof. □

COROLLARY 10. – *Assume that the hypotheses of Theorem* 8 *are fulfilled and, moreover,* f *satisfies the Palais-Smale condition* $(\mathrm{PS})_c$. *Then* c *is a critical value of* f.

REMARK 1. – If

$$\inf_{x\in X_1} f(x+z) = \max_{x\in K^*} f(x),$$

then the conclusion of Corollary 7 remains valid, with an argument based on Theorem 9.

COROLLARY 11 (Ghoussoub-Preiss Theorem). – *Let $f : X \to \mathbb{R}$ be a locally Lipschitz Gâteaux-differentiable functional such that $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Let e_0 and e_1 be in X and define*

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t)),$$

where $\mathcal{P}$ is the set of continuous paths $p : [0,1] \to X$ such that $p(0) = e_0$ and $p(1) = e_1$. Let F be a closed subset of X which does not contain e_0 and e_1 and $f(x) \geq c$, for all $x \in F$. Suppose, in addition that, for every $p \in \mathcal{P}$,

$$p([0,1]) \cap F \neq \emptyset.$$

Then there exists a sequence (x_n) in X, such that

i) $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0$;

ii) $\lim_{n\to\infty} f(x_n) = c$;

iii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.

Moreover, if f satisfies (PS)$_c$*, then there exists $x \in F$ such that $f(x) = c$ and $f'(x) = 0$.*

PROOF. – Using the arguments in the proof of Corollary 6 we deduce that the functional f is locally Lipschitz and

$$\partial f(x) = \{f'(x)\}.$$

Applying Theorem 8 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = e_0$, $p^*(1) = e_1$, our conclusion follows.

The last part of the theorem follows from Corollary 10. □

5. – Applications of the mountain pass theorem.

We are concerned in what follows with some relevant applications of the mountain pass theorem in the framework of nonlinear partial differential equations. We mainly refer to the paper by Brezis and Nirenberg [5], which develops pioneering directions for the qualitative analysis of nonlinear elliptic equations by means of the mountain pass theorem. We also refer to the books by

Ambrosetti and Malchiodi [1], Ghergu and Rădulescu [16], Schechter [39], Struwe [41], and Willem [42] for related results and further extensions.

In this section p denotes a real exponent, with $1<p<(N+2)/(N-2)$, if $N \geq 3$, and $1<p<\infty$, if $N=1,2$.

5.1 – *The subcritical Lane-Emden equation.*

Consider the nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta u = u^p\,, & \text{in } \Omega \\ u > 0\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{33}$$

This equation was introduced by Emden [14] and Fowler [15]. Existence results for problem (33) are related not only to the values of p, but also to the geometry of Ω. For instance, problem (33) has no solution if $p \geq (N+2)/(N-2)$ (if $N \geq 3$) and if Ω is star-shaped with respect to some point (say, with respect to the origin). This property was observed by Rellich [37] in 1940 and refound by Pohozaev [27] in 1965, and its proof relies on the Rellich–Pohozaev identity. More precisely, after multiplication by $x \cdot \nabla u$ in equation (33) and integration by parts we find

$$\int_\Omega \left(\frac{N}{p+1}\, u^{p+1} - \frac{N-2}{2}\, u^{p+1} \right) dx = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 (x \cdot n) d\sigma(x). \tag{34}$$

Because Ω is starshaped with respect to the origin, then the right-hand side of (34) is positive; hence, problem (33) has no solution if $N/(p+1) - (N-2)/2 \leq 0$, which is equivalent to $p \geq (N+2)/(N-2)$. We point out that a very general variational identity was discovered in 1986 by P. Pucci and J. Serrin [30].

The situation is different if we are looking for *entire solutions* (that is, solutions on the whole space) of the Emden-Fowler equation either in the critical or in the supercritical case. For instance, the equation

$$-\Delta u = u^{(N+2)/(N-2)} \quad \text{in } \mathbb{R}^N,\ (N \geq 3),$$

admits the family of solutions

$$u_C(x) = \left(\frac{C\sqrt{N(N-2)}}{C^2 + |x|^2} \right)^{(N-2)/2},$$

for any $C > 0$.

If Ω is not star-shaped, Kazdan and Warner [20] showed in 1975 that problem (33) has a solution for any $p > 1$, provided Ω is an annulus in $\mathbb{R}^N$. We also point out that if $\Omega \subset \mathbb{R}^N$ is an *arbitrary* bounded domain with smooth boundary, then the perturbed Emden–Fowler problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{in } \Omega, \\ \quad u \not\equiv 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, if λ is an *arbitrary* real number and $1<p<(N+2)/(N-2)$ if $N \geq 3$ or $1<p<\infty$ if $N \in \{1,2\}$. The proof combines the mountain pass theorem (if $\lambda<\lambda_1$) and the *dual variational method* (if $\lambda \geq \lambda_1$). We refer to the pioneering paper by Clarke [9] for the dual action principle and its applications to the existence of periodic solutions to Hamilton's equations. As usually, λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Our aim here is to prove the following result.

THEOREM 9. – *There exists a solution of the problem* (33), *which is not necessarily unique. Furthermore, this solution is unstable and of class $C^2(\overline{\Omega})$.*

PROOF. – We first establish the instability of an arbitrary solution u. So, in order to argue that the first eigenvalue μ_1 of $(-\Delta - pu^{p-1})$ is *negative*, let $\varphi > 0$ be an eigenfunction corresponding to μ_1. We have

$$-\Delta\varphi - pu^{p-1}\varphi = \mu_1\varphi\,, \qquad \text{in } \Omega\,.$$

Integrating by parts this equality we find

$$(1-p)\int_\Omega u^p\varphi = \mu_1\int_\Omega \varphi u\,,$$

which implies $\mu_1<0$, since $u > 0$ in Ω and $p > 1$.

Next, we prove the existence of a solution by using two different methods. The first proof uses variational techniques, which are then replaced by tools relying on the mountain pass theorem.

1. A VARIATIONAL PROOF. – Let

$$m = \inf\left\{ \int_\Omega |\nabla v|^2;\ v \in H_0^1(\Omega) \text{ and } \|v\|_{L^{p+1}} = 1 \right\}.$$

First step: $m > 0$ is achieved. Let $(u_n) \subset H_0^1(\Omega)$ be a minimizing sequence. Since $p<(N+2)/(N-2)$ then $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$. It follows that

$$\int_\Omega |\nabla u_n|^2 \to m \qquad \text{as } n \to \infty$$

and, for all positive integer n,

$$\|u_n\|_{L^{p+1}} = 1\,.$$

So, up to a subsequence,

$$u_n \rightharpoonup v\,, \qquad \text{weakly in } H_0^1(\Omega)$$

and

$$u_n \to v\,, \qquad \text{strongly in } L^{p+1}(\Omega)\,.$$

By the lower semicontinuity of the functional $\|\cdot\|_{L^2}$ we find that

$$\int_\Omega |\nabla v|^2 \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 = m$$

which implies $\int_\Omega |\nabla v|^2 = m$. Since $\|v\|_{L^{p+1}} = 1$, it follows that $m > 0$ is achieved by v.

We remark that we have even $u_n \to v$, strongly in $H_0^1(\Omega)$. This follows by the weak convergence of (u_n) in $H_0^1(\Omega)$ and by the fact that $\|u_n\|_{H_0^1} \to \|v\|_{H_0^1}$.

Second step: $v \ge 0$ a.e. in Ω. We may assume that $v \ge 0$ a.e. in Ω. Indeed, if not, we may replace v by $|v|$. This is possible since $|v| \in H_0^1(\Omega)$ and so, by Stampacchia's theorem,

$$\nabla |v| = (\operatorname{sign} v)\, \nabla v\,, \qquad \text{if } v \ne 0\,.$$

Moreover, on the level set $[v = 0]$ we have $\nabla v = 0$, so

$$|\nabla |v|| = |\nabla v|\,, \qquad \text{a.e. in } \Omega\,.$$

Third step: v verifies $-\Delta v = m\, v^p$ in the weak sense. We have to prove that for every $w \in H_0^1(\Omega)$,

$$\int_\Omega \nabla v \nabla w = m \int_\Omega v^p w\,.$$

Put $z = v + \varepsilon w$ in the definition of m. It follows that

$$\int_\Omega |\nabla z|^2 = \int_\Omega |\nabla v|^2 + 2\varepsilon \int_\Omega \nabla v \nabla w + \varepsilon^2 \int_\Omega |\nabla w|^2 = m + 2\varepsilon \int_\Omega \nabla v \nabla w + o(\varepsilon)$$

and

$$\begin{aligned}\int_\Omega |v + \varepsilon w|^{p+1} &= \int_\Omega |v|^{p+1} + \varepsilon(p+1) \int_\Omega v^p w + o(\varepsilon)\\ &= 1 + \varepsilon(p+1) \int_\Omega v^p w + o(\varepsilon)\,.\end{aligned}$$

Therefore

$$\|z\|^2_{L^{p+1}} = \left(1+\varepsilon(p+1)\int_\Omega v^p w + o(\varepsilon)\right)^{2/(p+1)} = 1+2\varepsilon\int_\Omega v^p w + o(\varepsilon).$$

Hence

$$m = \int_\Omega |\nabla v|^2 \le \frac{m+2\varepsilon\displaystyle\int_\Omega \nabla v\nabla w + o(\varepsilon)}{1+2\varepsilon\displaystyle\int_\Omega v^p w + o(\varepsilon)} = m+2\varepsilon\left(\int_\Omega \nabla v\nabla w - m\int_\Omega v^p w\right) + o(\varepsilon)\,,$$

which implies

$$\int_\Omega \nabla v\nabla w = m\int_\Omega v^p w\,, \qquad \text{for every } \ w\in H^1_0(\Omega)\,.$$

Consequently, *the function* $u = m^a v$, *where* $a = -1/(p-1)$, *is a weak solution of the problem* (33).

Fourth step: regularity of u. Until now we know only that $u\in H^1_0(\Omega)\subset L^{2^*}(\Omega)$. In a general framework, assuming that $u\in L^q(\Omega)$, it follows that $u^p\in L^{q/p}(\Omega)$, that is, by the Schauder regularity and the Sobolev embeddings, $u\in W^{2,q/p}(\Omega)\subset L^s(\Omega)$, where $1/s = p/q - 2/N$. So, assuming that $q_1 > (p-1)N/2$, we have $u\in L^{q_2}(\Omega)$, where $1/q_2 = p/q_1 - 2/N$. In particular, $q_2 > q_1$. Let (q_n) be the increasing sequence we construct in this manner and set $q_\infty = \lim_{n\to\infty} q_n$. Assuming, by contradiction, that $q_n < Np/2$ we obtain, passing at the limit as $n\to\infty$, that $q_\infty = N(p-1)2 < q_1$, a contradiction. This shows that there exists $r > N/2$ such that $u\in L^r(\Omega)$ which implies $u\in W^{2,r}(\Omega)\subset L^\infty(\Omega)$. Therefore $u\in W^{2,r}(\Omega)\subset C^k(\overline{\Omega})$, where k denotes the integer part of $2-N/r$. Now, by the Hölder continuity, $u\in C^2(\overline{\Omega})$.

2. Second proof. – The below arguments rely upon the mountain pass theorem. Set

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega (u^+)^{p+1}\,, \qquad u\in H^1_0(\Omega)\,.$$

Standard arguments show that $\mathcal{F}$ is a C^1 functional and u is a critical point of $\mathcal{F}$ if and only if u is a solution to the problem (33). We observe that $\mathcal{F}'(u) = -\Delta u - (u^+)^p \in H^{-1}(\Omega)$. So, if u is a critical point of $\mathcal{F}$ then $-\Delta u = (u^+)^p \ge 0$ in Ω and hence, by the maximum principle, $u\ge 0$ in Ω.

We verify the hypotheses of the mountain pass theorem. Clearly, $\mathcal{F}(0) = 0$. On the other hand,

$$\int_\Omega (u^+)^{p+1} \le \int_\Omega |u|^{p+1} = \|u\|^{p+1}_{L^{p+1}} \le C\|u\|^{p+1}_{H^1_0} .$$

Therefore

$$\mathcal{F}(u) \ge \frac{1}{2}\,\|u\|^2_{H^1_0} - \frac{C}{p+1}\,\|u\|^{p+1}_{H^1_0} \ge \rho > 0\,,$$

provided that $\|u\|_{H^1_0} = R$ is small enough.

Let us now prove the existence of some v_0 such that $\|v_0\| > R$ and $\mathcal{F}(v_0) \le 0$. For this aim, choose an arbitrary $w_0 \ge 0$, $w_0 \not\equiv 0$. We have

$$\mathcal{F}(tw_0) = \frac{t^2}{2}\int_\Omega |\nabla w_0|^2 - \frac{t^{p+1}}{p+1}\int_\Omega (w_0^+)^{p+1} \le 0\,,$$

for $t > 0$ large enough.

To complete the existence of a weak solution to problem (33), it remains to argue that the associated energy functional $\mathcal{F}$ satisfies the Palais-Smale condition. For this purpose, let (u_k) be a sequence in $H^1_0(\Omega)$ such that

$$\sup_k |\mathcal{F}(u_k)| < +\infty \tag{35}$$

and

$$\lim_{k\to\infty} \|\mathcal{F}'(u_k)\|_{H^{-1}(\Omega)} = 0. \tag{36}$$

Relation (36) implies that for all $\varphi \in H^1_0(\Omega)$,

$$\int_\Omega \nabla u_k \cdot \nabla\varphi\,dx = \int_\Omega (u_k^+)^p\varphi\,dx + o(1)\,\|\varphi\|\,. \tag{37}$$

Next, we apply this estimate to obtain a bound for $\|u_k\|$. Indeed, taking $\varphi = u_k$ in relation (37) we find

$$\int_\Omega |\nabla u_k|^2\,dx = \int_\Omega (u_k^+)^p u_k\,dx + o(1)\,\|u_k\|\,. \tag{38}$$

Combining now relations (35) and (38) we deduce that

$$\frac{p-1}{2}\int_\Omega (u_k^+)^p u_k\,dx = O(1) + o(1)\,\|u_k\|\,.$$

Using now again relation (35) and taking into account the expression of the energy functional $\mathcal{F}$, we conclude that (u_k) is bounded in $H^1_0(\Omega)$. This guarantees that, up to a subsequence, (u_k) is weakly convergent in $H^1_0(\Omega)$.

Next, we claim that any *bounded* sequence (u_k) in $H_0^1(\Omega)$ that satisfies relation (37) has the remarkable property that contains a *strongly* convergent sequence in $H_0^1(\Omega)$. By relation (37) and using the fact that $(-\Delta)^{-1}$ is a linear and continuous operator from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$, it suffices to prove that a subsequence of $(u_k^+)^p$ converges in $H^{-1}(\Omega)$. By Sobolev embeddings, this is achieved by showing that a subsequence of $(u_k^+)^p$ converges in $L^{2N/(N+2)}(\Omega)$, which is the dual space of $L^{2N/(N-2)}(\Omega)$. We first observe that, up to a subsequence, u_k converges a.e. to $u \in L^{2N/(N-2)}(\Omega)$. Applying Egorov's theorem we find that for any $\delta > 0$, there exists a subset A of Ω with $|A| < \delta$ and such that u_k converges uniformly to u in $\Omega \setminus A$. Thus, it is enough to prove that the quantity

$$\int_A |(u_k^+)^p - (u^+)^p|^{2N/(N+2)}\, dx$$

can be made arbitrarily small. On the one hand, by Young's inequality,

$$\int_A |(u^+)^p|^{2N/(N+2)}\, dx \le C \int_A (|(u^+)^p|^{2N/(N-2)} + 1)\, dx\,.$$

Hence, by choosing $\delta > 0$ arbitrarily small, the right hand-side of the above relation can be made as small as we desire. On the other hand, we have

$$\int_A |(u_k^+)^p|^{2N/(N+2)}\, dx \le \varepsilon \int_A |u_k|^{2N/(N-2)}\, dx + C_\varepsilon\, |A|\,.$$

Applying now the boundedness of (u_k) in $H_0^1(\Omega)$ combined with Sobolev embeddings, we deduce that the right hand-side above is bounded by $\varepsilon\, C + C_\varepsilon\, |A|$, which can be made arbitrarily small. This completes the proof of the Palais-Smale property. □

Theorem 9 can be extended to the following more general class of Lane-Emden equations. Consider the nonlinear problem

$$\text{(39)} \qquad \begin{cases} -\Delta u + a(x)u = g(x,u)\,, & \text{in } \Omega \\ u > 0\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

where $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying the following hypotheses:

$$\text{(40)} \qquad |g(x,u)| \le C\,(1 + |u|^p) \qquad \text{for all } (x,u) \in \Omega \times [0,\infty),$$

where as usual $p < (N+2)/(N-2)$ if $N \ge 3$ and $p < \infty$ if $N \in \{1, 2\}$;

$$\text{(41)} \qquad g(x,0) = g_u(x,0) = 0;$$

$$\text{(42)} \qquad 0 < \mu G(x,u) \le u g(x,u) \quad \text{for } u > 0 \text{ large enough and some } \mu > 2,$$

where $G(x,u) := \int_0^u g(x,t)\,dt$. We point out that hypothesis (42) is frequently referred as the *Ambrosetti-Rabinowitz growth condition.*

We also assume that $a : \Omega \to \mathbb{R}$ is a smooth function and there exists $C > 0$ such that

$$\int_\Omega \left(|\nabla v|^2 + a(x)v^2 \right) \geq C\, \|v\|^2_{H^1_0(\Omega)} \qquad \text{for all } v \in H^1_0(\Omega). \tag{43}$$

We observe that condition (42) implies that the nonlinear term g has a superlinear growth at infinity, in the sense that there are positive constants C_1 and C_2 such that for all $(x,u) \in \Omega \times [0,\infty)$,

$$g(x,u) \geq C_1\, u^{\mu-1} - C_2\,.$$

We also remark that condition (43) means that the linear operator $-\Delta + a(x)I$ is coercive in $H^1_0(\Omega)$.

The counterpart of Theorem 9 in this general setting is the following.

THEOREM 10. – *Assume conditions* (40)-(43) *are fulfilled. Then problem* (39) *has at least one solution.*

The proof relies on the same ideas as for Theorem 9.

5.2 – *The dual variational method versus mountain pass.*

We are now concerned with a linear perturbation of the Lane-Emden equation. Consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \lambda u\,, & \text{in } \Omega \\ u \not\equiv 0\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{44}$$

where $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$.

By Theorem 10, problem (44) has a positive solution, provided that $\lambda < \lambda_1$, where $\lambda_1 > 0$ denotes the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$. The basic assumption $\lambda < \lambda_1$ is used to check the existence of the *valley condition* in the mountain pass theorem. Next, we observe that problem (44) does not have any positive solution if $\lambda \geq \lambda_1$. This conclusion follows easily after multiplication with the first eigenfunction φ_1 in (33) and integration over Ω. Moreover, the assumption $\lambda \geq \lambda_1$ implies that problem (44) *does not have a mountain pass geometry*. This means that the associated energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) - \frac{1}{p+1}\int_\Omega |u|^{p+1}$$

does not fulfill the valley condition of the mountain pass theorem near the origin. The next result shows that problem (44) has a nontrivial solution, which, in view of these remarks, changes sign.

THEOREM 11. – *Assume that $\lambda \geq \lambda_1$. Then problem* (44) *has at least one solution.*

PROOF. – The case where λ is an eigenvalue is more difficult to handle. That is why, for simplicity, we assume that λ *is not an eigenvalue of the Laplace operator.*

The proof combines the dual variational method with the mountain pass theorem. This is not applied to the original energy $\mathcal{E}$ (we have already observed that this does not satisfy the mountain pass theorem). For this purpose we first introduce a new unknown v, defined by $v = |u|^{p-1}u$. Thus, $u = |v|^{q-1}v$ with $q = 1/p < 1$. Problem (44) becomes

$$u = Kv, \tag{45}$$

where K denotes the linear operator $K = (-\Delta - \lambda I)^{-1}$. We observe that K is well-defined, since λ is not an eigenvalue of $-\Delta$. Problem (45) can be rewritten as

$$|v|^{q-1}v = Kv\,. \tag{46}$$

Thus, any solution of problem (44) is also a solution of problem (46) and conversely.

All solutions of problem (46) are critical points of the functional

$$\mathcal{F}(v) = \frac{1}{q+1}\int_\Omega |v|^{q+1} - \frac{1}{2}\int_\Omega vKv\,.$$

We prove the following basic properties of $\mathcal{F}$:

(i) $\mathcal{F}$ is well-defined and of class C^1 on $L^{q+1}(\Omega)$. Indeed, the operator K maps $L^{q+1}(\Omega)$ into $W^{2,q+1}(\Omega) \subset L^{p+1}(\Omega)$, since $p < (N+2)/(N-2)$.

(ii) $\mathcal{F}$ satisfies the assumptions of the mountain pass theorem. Indeed, since $q+1 < 2$ then the dominant term of $\mathcal{F}$ near the origin is $(q+1)^{-1}\int_\Omega |v|^{q+1}$. Next, assuming that v_0 is an eigenfunction of $-\Delta$ associated with an eigenvalue larger than λ, then $\int_\Omega v_0Kv_0 > 0$, which shows that $\mathcal{F}(tv_0) < 0$, provided that $t > 0$ is large enough.

(iii) $\mathcal{F}$ satisfies the Palais-Smale condition. Indeed, let us assume that the sequence $(v_n) \subset L^{q+1}(\Omega)$ verifies

$$\sup_n |\mathcal{F}(v_n)| < \infty \tag{47}$$

and

$$\mathcal{F}'(v_n) \to 0 \qquad \text{in } \left(L^{q+1}(\Omega)\right)^* = L^{p+1}(\Omega). \tag{48}$$

Thus, by relation (48),

$$|v_n|^{q-1}v_n - Kv_n \to 0 \qquad \text{in } L^{p+1}(\Omega). \tag{49}$$

Now, multiplying relation (49) with v_n, integrating, and comparing with (48), we deduce that the sequence (v_n) is bounded in $L^{q+1}(\Omega)$. Thus, up to a subsequence,

$$v_n \to v \qquad \text{weakly in } L^{q+1}(\Omega).$$

Since $K : L^{q+1}(\Omega) \to L^{p+1}(\Omega)$ is a compact operator and the embedding $W^{2,q+1}(\Omega) \subset L^{p+1}(\Omega)$ is compact, we conclude that $Kv_n \to Kv$ in $L^{p+1}(\Omega)$, hence $v_n \to v$ in $L^{q+1}(\Omega)$.

Using (i), (ii), (iii), and applying the mountain pass theorem, we deduce that $\mathcal{F}$ has a critical point v_0 and, moreover, $\mathcal{F}(v_0) > 0$. This concludes the proof of Theorem 11. □

5.3 – *A bifurcation problem.*

Let us consider a C^1 convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and $f'(0) > 0$. We assume that f has a subcritical growth, that is, there exists $1 < p < (N+2)/(N-2)$ such that for all $u \in \mathbb{R}$,

$$|f(u)| \le C(1 + |u|^p).$$

We also suppose that there exist $\mu > 2$ and $A > 0$ such that

$$\mu \int_0^u f(t)dt \le uf(u), \qquad \text{for every } u \ge A.$$

A standard example of function satisfying these conditions is $f(u) = (1+u)^p$.

Consider the bifurcation problem

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{50}$$

The implicit function theorem implies that there exists $\lambda^\star > 0$ such that for every $\lambda < \lambda^\star$, there exists a minimal and stable solution $\underline{u}$ to the problem (50).

THEOREM 12. – *Under the above hypotheses on f, for every $\lambda \in (0, \lambda^\star)$, there exists a second solution $u \ge \underline{u}$ and, furthermore, u is unstable.*

SKETCH OF THE PROOF. – We find a solution u of the form $u = \underline{u} + v$ with $v \geq 0$. It follows that v satisfies

$$\text{(51)} \qquad \begin{cases} -\Delta v = \lambda[f(\underline{u}+v) - f(\underline{u})], & \text{in } \Omega \\ v > 0\,, & \text{in } \Omega \\ v = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Hence v fulfills an equation of the form

$$-\Delta v + a(x)v = g(x,v)\,, \qquad \text{in } \Omega\,,$$

where $a(x) = -\lambda f'(\underline{u})$ and

$$g(x,v) = \lambda[f(\underline{u}(x)+v) - f(\underline{u}(x))] - \lambda f'(\underline{u}(x))v\,.$$

We verify easily the following properties:

(i) $g(x,0) = g_v(x,0) = 0$;

(ii) $|g(x,v)| \leq C(1+|v|^p)$;

(iii) $\mu \int_0^v g(x,t)dt \leq vg(x,v)$, for every $v \geq A$ large enough;

(iv) the operator $-\Delta - \lambda f'(\underline{u})$ is coercive, since $\lambda_1(-\Delta - \lambda f'(\underline{u})) > 0$, for every $\lambda < \lambda^\star$.

So, by the mountain pass theorem, the problem (50) has a solution which is, *a fortiori*, unstable. □

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