Research Article

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Multiple solutions of double phase variational problems with variable exponent

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Abstract: This paper deals with the existence of multiple solutions for the quasilinear equation

$$-\operatorname{div}\mathbf{A}(x,\nabla u)+|u|^{\alpha(x)-2}u=f(x,u)\quad \text{in } \mathbb{R}^N,$$

which involves a general variable exponent elliptic operator $\mathbf{A}$ in divergence form. The problem corresponds to double phase anisotropic phenomena, in the sense that the differential operator has various types of behavior like $|\xi|^{q(x)-2}\xi$ for small $|\xi|$ and like $|\xi|^{p(x)-2}\xi$ for large $|\xi|$, where $1<\alpha(\cdot)\le p(\cdot)<q(\cdot)<N$. Our aim is to approach variationally the problem by using the tools of critical points theory in generalized Orlicz–Sobolev spaces with variable exponent. Our results extend the previous works [A. Azzollini, P. d'Avenia and A. Pomponio, Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz–Sobolev embeddings, *Calc. Var. Partial Differential Equations* **49** (2014), no. 1–2, 197–213] and [N. Chorfi and V. D. Rădulescu, Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electron. J. Qual. Theory Differ. Equ.* **2016** (2016), Paper No. 37] from cases where the exponents p and q are constant, to the case where $p(\cdot)$ and $q(\cdot)$ are functions. We also substantially weaken some of the hypotheses in these papers and we overcome the lack of compactness by using the weighting method.

Keywords: Variable exponent elliptic operator, integral functionals, variable exponent Orlicz–Sobolev spaces, critical point

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1 Introduction

In this paper, we deal with the following variable exponent elliptic equation:

$$-\operatorname{div}\mathbf{A}(x,\nabla u)+|u|^{\alpha(x)-2}u=f(x,u):=\lambda a(x)|u|^{\delta(x)-2}u+\mu w(x)g(x,u),\tag{$\mathcal{E}$}$$

where $\lambda>0$ and $\mu\ge 0$ are parameters, $\mathbf{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ admits a potential $\mathscr{A}$, with respect to its second variable ξ, satisfying the following assumption.

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($\mathcal{A}_1$) The potential $\mathscr{A} = \mathscr{A}(x, \xi)$ is a continuous function in $\mathbb{R}^N \times \mathbb{R}^N$, with continuous derivative with respect to ξ, $\mathbf{A} = \partial_\xi \mathscr{A}(x, \xi)$, and satisfies the following conditions:
(i) $\mathscr{A}(x, 0) = 0$ and $\mathscr{A}(x, \xi) = \mathscr{A}(x, -\xi)$ for all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$,
(ii) $\mathscr{A}(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$,
(iii) there exist constants $C_1, C_2 > 0$ and variable exponents p and q such that for all $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\left.\begin{matrix} C_1|\xi|^{p(x)} & \text{if } |\xi| \gg 1, \\ C_1|\xi|^{q(x)} & \text{if } |\xi| \ll 1, \end{matrix}\right\} \le \mathbf{A}(x, \xi) \cdot \xi \quad \text{and} \quad |\mathbf{A}(x, \xi)| \le \begin{cases} C_2|\xi|^{p(x)-1} & \text{if } |\xi| \gg 1, \\ C_2|\xi|^{q(x)-1} & \text{if } |\xi| \ll 1, \end{cases} \tag{1.1}$$

(iv) $1 \ll p(\cdot) \ll q(\cdot) \ll \min\{N, p^*(\cdot)\}$, and $p(\cdot)$, $q(\cdot)$ are Lipschitz continuous in $\mathbb{R}^N$,
(v) $\mathbf{A}(x, \xi) \cdot \xi \le s(x)\mathscr{A}(x, \xi)$ for any $(x, \xi) \in \mathbb{R}^{2N}$, where s is a Lipschitz continuous function and satisfies $q(\cdot) \le s(\cdot) \ll p^*(\cdot)$.

($\mathcal{A}_2$) The potential $\mathscr{A}$ is uniformly convex, that is, for any $\varepsilon \in (0, 1)$, there exists $\delta(\varepsilon) \in (0, 1)$ such that $|u - v| \le \varepsilon \max\{|u|, |v|\}$ or $\mathscr{A}(x, \frac{u+v}{2}) \le \frac{1}{2}(1 - \delta(\varepsilon))(\mathscr{A}(x, u) + \mathscr{A}(x, v))$ for any $x, u, v \in \mathbb{R}^N$.

In this paper, for any $v : \mathbb{R}^N \to \mathbb{R}$, we denote

$$v^+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} v(x), \qquad v^- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^N} v(x),$$

and we denote by $v_1 \ll v_2$ the fact that

$$\operatorname*{ess\,inf}_{x \in \mathbb{R}^N} (v_2(x) - v_1(x)) > 0.$$

Remark 1. A typical example of $\mathbf{A}$ is

$$\mathbf{A}(x, \nabla u) = \begin{cases} |\nabla u|^{p(x)-2}\nabla u & \text{if } |\nabla u| > 1, \\ |\nabla u|^{q(x)-2}\nabla u & \text{if } |\nabla u| \le 1. \end{cases}$$

Then

$$-\operatorname{div} \mathbf{A}(x, \nabla u) = \begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) & \text{if } |\nabla u| > 1, \\ -\operatorname{div}(|\nabla u|^{q(x)-2}\nabla u) & \text{if } |\nabla u| \le 1, \end{cases}$$

and

$$\mathscr{A}(x, \xi) = \begin{cases} \frac{1}{p(x)}|\xi|^{p(x)} + \frac{1}{q(x)} - \frac{1}{p(x)} & \text{if } |\xi| > 1, \\ \frac{1}{q(x)}|\xi|^{q(x)} & \text{if } |\xi| \le 1. \end{cases}$$

From [56, Lemma A.2 in Appendix A], it is clear that this typical potential $\mathscr{A}$ satisfies assumptions ($\mathcal{A}_1$)–($\mathcal{A}_2$), $1 < p^- \le p^+ < N$ and $1 < q^- \le q^+ < N$.

It is well known that the main difficulty in studying the elliptic equations in $\mathbb{R}^N$ is the lack of compactness. To overcome this difficulty, many methods can be used. One type of methods is that under some additional conditions the required compact embedding theorem holds, for example, the symmetry method, the coercive coefficient method and the weighting method. In [4], the authors consider equation ($\mathcal{E}$) with constant exponent by the symmetry method to rebuild the required compact embedding theorem. In [56], in order to rebuild the required compact embedding theorem, the authors consider equation ($\mathcal{E}$) with coercive coefficient $V(x)$ of $|u|^{\alpha(x)-2}u$, namely, $V(x) \to +\infty$ as $|x| \to \infty$. In this paper, we will apply the weighting method, namely if the coefficients w and a satisfy some integrable conditions, then we can rebuild the required compact embedding theorem.

We also make the following assumptions:

($\mathcal{H}_f^1$) The function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, $0 \le g(x, u)u = o(|u|^{\alpha(x)})$ as $u \to 0$, and $|g(x, u)| \le C(1 + |u|^{\gamma(x)-1})$, where $\gamma(\cdot)$ is Lipschitz continuous and $\alpha \le \gamma(\cdot) \ll p^*(\cdot)$.

($\mathcal{H}_f^2$) There exists a constant $\theta > s^+$ such that

$$0 < G(x, t) \le \frac{1}{\theta} t g(x, t), \quad t \in \mathbb{R} \setminus \{0\}, \ x \in \mathbb{R}^N,$$

where $G(x, t) = \int_0^t g(x, s)\, ds$, and $s(\cdot)$ is defined in ($\mathcal{A}_1$) (v).

($\mathcal{H}_f^3$) $g(x, -u) = -g(x, u)$.

$(\mathcal{H}_w)$ $w \in L^{r(\cdot)}(\mathbb{R}^N)$, $w > 0$ a.e. in $\mathbb{R}^N$, $1 \ll r(x) \ll \infty$, and

$$r'(x) \le \frac{p^*(x)}{\gamma(x)}, \quad x \in \mathbb{R}^N,$$

where $r'(x)$ is the conjugate function of $r(x)$, namely $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$, and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

$(\mathcal{H}_a)$ $1 \ll \delta(x) \le \delta^+ < \alpha^-$, $a \in L^{r_*(\cdot)}(\mathbb{R}^N)$, $a > 0$ a.e. in $\mathbb{R}^N$, $1 \ll r_*(x) \ll \infty$, and

$$\alpha(x) \le \frac{r_*(x)}{r_*(x) - 1}\delta(x) \le p^*(x), \quad x \in \mathbb{R}^N.$$

This paper generalizes some results contained in [4] and [10] to the case of partial differential equations with variable exponent. If $p(\cdot) \equiv p$, $q(\cdot) \equiv q$ and $\alpha(\cdot) \equiv \alpha$ are constants, then $(\mathcal{E})$ becomes the usual constant exponent differential equation in divergence form discussed in [10]. But if either $p(\cdot)$ or $q(\cdot)$ is a nonconstant function, then $(\mathcal{E})$ has a more complicated structure, due to its non-homogeneities and to the presence of several nonlinear terms.

This paper was motivated by double phase nonlinear problems with variational structure, which have been introduced by Marcellini [35] and developed by Baroni, Colombo and Mingione [5, 12] in the framework of non-homogeneous problems driven by a differential operator with variable growth described by nonconstant functions $p(x)$ and $q(x)$. In the case of two different materials that involve power hardening exponents $p(\cdot)$ and $q(\cdot)$, the differential operator $\operatorname{div} \mathbf{A}(x, \nabla u)$ describes the geometry of a composite of these two materials. Compare hypothesis (1.1), the $p(\cdot)$-material is present if $|\xi| \gg 1$. In the opposite case, the $q(\cdot)$-material is the only one describing the composite.

In recent years, the study of differential equations and variational problems with variable exponent growth conditions have been an interesting topic, which has the background in image processing, nonlinear electrorheological fluids and elastic mechanics etc. We refer the reader to [1, 9, 27, 29, 44, 46, 58] and the references therein for more background on applications. There are many reference papers related to the study of variational problems with variable exponent growth conditions. Far from being complete, we refer the readers to [2, 3, 6, 7, 14–31, 33–40, 43, 45, 47–56].

Our main results can be stated as follows.

Theorem 1.1. *Assume that* $1 \ll \alpha \le p \ll q \ll \min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, $\mu > 0$, λ *is small enough, and that hypotheses* $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{H}_f^1)$–$(\mathcal{H}_f^2)$, $(\mathcal{H}_w)$ *and* $(\mathcal{H}_a)$ *hold. Then problem* $(\mathcal{E})$ *has two pairs of nontrivial nonnegative and nonpositive solution.*

Theorem 1.2. *Assume that* $1 \ll \alpha \le p \ll q \ll \min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, $\mu > 0$, *and that hypotheses* $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{H}_f^1)$–$(\mathcal{H}_f^3)$, $(\mathcal{H}_w)$ *and* $(\mathcal{H}_a)$ *hold. Then problem* $(\mathcal{E})$ *has infinitely many nontrivial solutions with energy tending to* $+\infty$.

Theorem 1.3. *Assume that* $1 \ll \alpha \le p \ll q \ll \min\{N, p^*\}$, $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, $\mu = 0$, *and that hypotheses* $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{H}_w)$ *and* $(\mathcal{H}_a)$ *hold. Then problem* $(\mathcal{E})$ *has infinitely many nontrivial solutions with negative energy tending to* 0.

This paper is divided into five sections. Section 2 contains some properties of function spaces with variable exponent. Section 3 includes several basic properties of Orlicz–Sobolev spaces. In Section 4 we establish some qualitative properties of the operators involved in our analysis. In Section 5 we give the proofs of Theorems 1.1–1.3. We refer to [11] for the basic analytic tools used in this paper.

2 Variable exponent spaces theory

Nonlinear problems with non-homogeneous structure are motivated by numerous models in the applied sciences that are driven by partial differential equations with one or more variable exponents. In some circumstances, the standard analysis based on the theory of usual Lebesgue and Sobolev function spaces, L^p and $W^{1,p}$, is not appropriate in the framework of materials that involve non-homogeneities. For instance, both electrorheological "smart fluids" and phenomena arising in image processing are described in a correct way by nonlinear models in which the exponent p is not necessarily constant. The variable exponent describes the geometry of a material which is allowed to change its hardening exponent according to the point. This leads to the analysis of variable exponents Lebesgue and Sobolev function spaces (denoted by $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$), where p is a real-valued (nonconstant) function.

Throughout this paper, the letters c, c_i, C, C_i, $i = 1, 2, \dots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

In order to discuss problem $(\mathcal{E})$, we need some theory of variable exponent Lebesgue spaces and Sobolev spaces. In the sequel, we will give some properties of these variable exponent spaces. Let $\Omega \subset \mathbb{R}^N$ be an open domain. Let $S(\Omega)$ be the set of all measurable real valued functions defined on Ω. Let

$$C_+(\overline{\Omega}) = \{v : v \in C(\overline{\Omega}),\ v(x) > 1 \text{ for } x \in \overline{\Omega}\},$$

$$L^{p(\cdot)}(\Omega) = \left\{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

The function space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg norm

$$|u|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Then $(L^{p(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)})$ becomes a Banach space; we call it the variable exponent Lebesgue space. If $\Omega = \mathbb{R}^N$, we simply denote $(L^{p(\cdot)}(\mathbb{R}^N), |\cdot|_{L^{p(\cdot)}(\mathbb{R}^N)})$ as $(L^{p(\cdot)}, |\cdot|_{L^{p(\cdot)}})$.

Proposition 2.1 (see [22, Theorem 1.15]). *The space $(L^{p(\cdot)}(\Omega), |u|_{L^{p(\cdot)}(\Omega)})$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have the following Hölder inequality:*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{L^{p(\cdot)}(\Omega)}|v|_{L^{p'(\cdot)}(\Omega)}.$$

Proposition 2.2 (see [22, Theorem 1.16]). *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies*

$$|f(x,s)| \le d(x) + b|s|^{p_1(x)/p_2(x)} \quad \text{for any } x \in \Omega,\ s \in \mathbb{R},$$

where $p_1, p_2 \in C_+(\overline{\Omega})$, $d(x) \in L^{p_2(\cdot)}(\Omega)$, $d(x) \ge 0$, $b \ge 0$, then the Nemytsky operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3 (see [22, Theorem 1.3]). *If we denote*

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\, dx, \quad u \in L^{p(\cdot)}(\Omega),$$

then
(i) $|u|_{L^{p(\cdot)}(\Omega)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,
(ii) $|u|_{L^{p(\cdot)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le |u|^{p^+}_{L^{p(\cdot)}(\Omega)}$, $|u|_{L^{p(\cdot)}(\Omega)} < 1 \Rightarrow |u|^{p^-}_{L^{p(\cdot)}(\Omega)} \ge \rho_{p(\cdot)}(u) \ge |u|^{p^+}_{L^{p(\cdot)}(\Omega)}$,
(iii) $|u|_{L^{p(\cdot)}(\Omega)} \to \infty \Leftrightarrow \rho_{p(\cdot)}(u) \to \infty$.

Proposition 2.4 (see [22, Theorem 1.14]). *If $u, u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \dots$, then the following statements are equivalent:*
(1) $\lim_{k\to\infty} |u_k - u|_{L^{p(\cdot)}(\Omega)} = 0$,
(2) $\lim_{k\to\infty} \rho_{p(\cdot)}(u_k - u) = 0$,
(3) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho_{p(\cdot)}(u_k) = \rho_{p(\cdot)}(u)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \nabla u \in [L^{p(\cdot)}(\Omega)]^N\}$$

and it is equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = |u|_{L^{p(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)}, \quad u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces provided that p is constant. According to [44, pp. 8–9], these function spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ have some non-usual properties, such as:

(i) Assuming that $1 < p^- \le p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is a smooth function, then the co-area formula

$$\int_\Omega |u(x)|^p \, dx = p \int_0^\infty t^{p-1} \, |\{x \in \Omega : |u(x)| > t\}| \, dt$$

has no analogue in the framework of variable exponents.

(ii) The spaces $L^{p(\cdot)}$ do *not* satisfy the *mean continuity property*. More precisely, if p is nonconstant and continuous in an open ball B, then there is some $u \in L^{p(\cdot)}(B)$ such that $u(x+h) \notin L^{p(\cdot)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponent are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(\cdot)} \le C \, |f|_{p(\cdot)} \, \|g\|_{L^1}$$

remains valid if and only if p is constant.

Conditions $(\mathcal{A}_1)$ (i)–(ii) imply that

$$\mathscr{A}(x, \xi) \le \mathbf{A}(x, \xi) \cdot \xi \quad \text{for all } (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.1}$$

Furthermore, $(\mathcal{A}_1)$ (ii) is weaker than the assumption that $\mathscr{A}$ is uniformly convex, that is, for any $\varepsilon \in (0, 1)$, there exists a constant $\delta(\varepsilon) \in (0, 1)$ such that

$$\mathscr{A}\Big(x, \frac{\xi + \eta}{2}\Big) \le (1 - \delta(\varepsilon)) \frac{\mathscr{A}(x, \xi) + \mathscr{A}(x, \eta)}{2}$$

for all $x \in \mathbb{R}^N$ and $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfy $|u - v| \ge \varepsilon \max\{|u|, |v|\}$. By $(\mathcal{A}_1)$ (i)–(iii), we have

$$\mathscr{A}(x, \xi) = \int_0^1 \frac{d}{dt} \mathscr{A}(x, t\xi) \, dt = \int_0^1 \frac{1}{t} \mathbf{A}(x, t\xi) \cdot t\xi \, dt \ge \begin{cases} c_1 |\xi|^{p(x)}, & |\xi| > 1, \\ c_1 |\xi|^{q(x)}, & |\xi| \le 1. \end{cases}$$

This estimate in combination with (1.1) and (2.1) yields

$$\left.\begin{matrix} c_1 |\xi|^{p(x)}, & |\xi| > 1, \\ c_1 |\xi|^{q(x)}, & |\xi| \le 1, \end{matrix}\right\} \le \mathscr{A}(x, \xi) \le \mathbf{A}(x, \xi) \cdot \xi \le \begin{cases} c_2 |\xi|^{p(x)}, & |\xi| > 1, \\ c_2 |\xi|^{q(x)}, & |\xi| \le 1, \end{cases} \quad (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.2}$$

Denote

$$L_w^{\vartheta(\cdot)}(\Omega) = \Big\{u : \int_\Omega w(x) |u(x)|^{\vartheta(x)} \, dx < \infty\Big\},$$

with the norm

$$|u|_{L_w^{\vartheta(\cdot)}(\Omega)} = \inf \Big\{\lambda > 0 : \int_\Omega w(x) \Big|\frac{u(x)}{\lambda}\Big|^{\vartheta(x)} \, dx \le 1\Big\}.$$

If $\Omega = \mathbb{R}^N$, we simply denote $(L_w^{\vartheta(\cdot)}(\mathbb{R}^N), |\cdot|_{L_w^{\vartheta(\cdot)}(\mathbb{R}^N)})$ as $(L_w^{\vartheta(\cdot)}, |\cdot|_{L_w^{\vartheta(\cdot)}})$.

From now on, we denote by B_R the ball in $\mathbb{R}^N$ centered at the origin and of radius $R > 0$.

Lemma 2.5 (see [42, Lemma 2.2]). *Assume that $\vartheta^- > 1$ and $\vartheta^+ < \infty$. Then $L_w^{\vartheta(\cdot)}$ is a separable uniformly convex Banach space.*

Theorem 2.6 (see [56, Interpolation Theorem]). *If $p(\cdot) < \alpha(\cdot) < q(\cdot)$, then for any $u \in L^{\alpha(\cdot)}(\Omega)$, there exists a constant $\lambda = \lambda(\Omega, \alpha, p, q, u) \in [\vartheta^-, \vartheta^+]$, where $\vartheta(\cdot) = \frac{p(q-\alpha)}{\alpha(q-p)}$, such that*

$$|u|_{L^{\alpha(\cdot)}(\Omega)} \leq 2|u|^{\lambda}_{L^{p(\cdot)}(\Omega)} \cdot |u|^{1-\lambda}_{L^{q(\cdot)}(\Omega)}.$$

Moreover, if $\vartheta^- < \vartheta^+$, then $\lambda \in (\vartheta^-, \vartheta^+)$.

Proposition 2.7 (see [16, 23]). *If Ω is a bounded domain, we have:*
(i) *$W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable reflexive Banach spaces,*
(ii) *if $\vartheta \in C_+(\overline{\Omega})$ and $\vartheta(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(\cdot)}(\Omega)$ to $L^{\vartheta(\cdot)}(\Omega)$ is compact and continuous,*
(iii) *there is a constant $C > 0$ such that*

$$|u|_{L^{p(\cdot)}(\Omega)} \leq C|\nabla u|_{L^{p(\cdot)}(\Omega)}, \quad u \in W_0^{1,p(\cdot)}(\Omega).$$

3 Variable exponent Orlicz–Sobolev spaces theory

Let $\Omega \subset \mathbb{R}^N$ be an open domain.

Definition 3.1. We define the following real-valued linear space:

$$L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega) = \{u : u = v + w, v \in L^{p(\cdot)}(\Omega),\ w \in L^{q(\cdot)}(\Omega)\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} = \inf\{|v|_{L^{p(\cdot)}(\Omega)} + |w|_{L^{q(\cdot)}(\Omega)} : v \in L^{p(\cdot)}(\Omega),\ w \in L^{q(\cdot)}(\Omega),\ v + w = u\}. \tag{3.1}$$

If $\Omega = \mathbb{R}^N$, we simply denote $(L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ as $(L^{p(\cdot)} + L^{q(\cdot)}, |\cdot|_{L^{p(\cdot)}+L^{q(\cdot)}})$.

We also define the linear space

$$L^{p(\cdot)}(\Omega) \cap L^{q(\cdot)}(\Omega) = \{u : u \in L^{p(\cdot)}(\Omega) \text{ and } u \in L^{q(\cdot)}(\Omega)\},$$

which is endowed with the norm

$$|u|_{L^{p(\cdot)}(\Omega)\cap L^{q(\cdot)}(\Omega)} = \max\{|u|_{L^{p(\cdot)}(\Omega)}, |u|_{L^{q(\cdot)}(\Omega)}\}.$$

Throughout this paper, we denote

$$\Lambda_u = \{x \in \Omega : |u(x)| > 1\} \quad \text{and} \quad \Lambda_u^c = \{x \in \Omega : |u(x)| \leq 1\}.$$

Proposition 3.2 (see [56, Proposition 3.2]). *Assume $(\mathcal{A}_1)$(iv). Let $\Omega \subset \mathbb{R}^N$ and $u \in L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega)$. Then the following properties hold:*
(i) *If $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^{p(\cdot)}(\Omega')$.*
(ii) *If $\Omega' \subset \Omega$ is such that $u \in L^{\infty}(\Omega')$, then $u \in L^{q(\cdot)}(\Omega')$.*
(iii) *$|\Lambda_u| < +\infty$.*
(iv) *$u \in L^{p(\cdot)}(\Lambda_u) \cap L^{q(\cdot)}(\Lambda_u^c)$.*
(v) *The infimum in (3.1) is attained.*
(vi) *If $B \subset \Omega$, then $|u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \leq |u|_{L^{p(\cdot)}(B)+L^{q(\cdot)}(B)} + |u|_{L^{p(\cdot)}(\Omega\backslash B)+L^{q(\cdot)}(\Omega\backslash B)}$.*
(vii) *We have*

$$\begin{aligned}\max\left\{\frac{1}{1+2|\Lambda_u|^{\frac{1}{p(\xi)}-\frac{1}{q(\xi)}}}|u|_{L^{p(\cdot)}(\Lambda_u)}, c\min\left\{|u|_{L^{q(\cdot)}(\Lambda_u^c)}, |u|^{\frac{q(\xi)}{p(\xi)}}_{L^{q(\cdot)}(\Lambda_u^c)}\right\}\right\} &\leq |u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)} \\ &\leq |u|_{L^{p(\cdot)}(\Lambda_u)} + |u|_{L^{q(\cdot)}(\Lambda_u^c)},\end{aligned}$$

where $\xi \in \mathbb{R}^N$ and c is a small positive constant.

Proposition 3.3 (see [32, Theorem 2]). *If $(X, \|\cdot\|)$ is a Banach space, then the following two statements are equivalent:*
(i) *$(X, \|\cdot\|)$ is reflexive.*
(ii) *Any bounded sequence of $(X, \|\cdot\|)$ has a weak convergent subsequence.*

Proposition 3.4 (see [56, Proposition 3.8]). *Assume that hypothesis $(\mathcal{A}_1)$ (iv) is fulfilled. Then*

$$(L^{p'(\cdot)}(\Omega) \cap L^{q'(\cdot)}(\Omega))' = L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega).$$

Proposition 3.5 (see [56, Proposition 3.9]). *Assume $(\mathcal{A}_1)$ (iv). Then $(L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega), |\cdot|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)})$ is a reflexive Banach space.*

Define

$$X(\Omega) = \{u \in L^{\alpha(\cdot)}(\Omega) : \nabla u \in (L^{p(\cdot)}(\Omega) + L^{q(\cdot)}(\Omega))^N\}$$

with the norm

$$\|u\|_\Omega = |u|_{L^{\alpha(\cdot)}(\Omega)} + |\nabla u|_{L^{p(\cdot)}(\Omega)+L^{q(\cdot)}(\Omega)}.$$

If $\Omega = \mathbb{R}^N$, we simply denote $(X(\Omega), \|u\|_\Omega)$ as $(X, \|u\|)$.

Proposition 3.6 (see [56, Proposition 3.10]). *Assume $(\mathcal{A}_1)$ (iv). Then $(X(\Omega), \|u\|_\Omega)$ is a Banach space.*

Proposition 3.7 (see [56, Proposition 3.11]). *Assume $(\mathcal{A}_1)$ (iv). Then $(X(\Omega), \|u\|_\Omega)$ is reflexive.*

Theorem 3.8 (see [56, Theorem 3.12]). *Assume $(\mathcal{A}_1)$ (iv), $1 \ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$, α satisfies $1 \ll \alpha(\cdot) \ll p^*(\cdot)\frac{N-1}{N}$ and $1 \ll \alpha(\cdot) \leq p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$. Then the space $X(\Omega)$ is continuously embedded into $L^{p^*(\cdot)}(\Omega)$.*

Corollary 3.9 (see [56, Corollary 3.13]). *Assume the conditions of Theorem 3.8. We have the following properties:*
(i) *For any $u \in X(\Omega)$, $\psi_n u \to u$ in $X(\Omega)$.*
(ii) *For any $u \in X$, we have $u_\varepsilon = u * \mathbf{j}_\varepsilon \to u$ in X (where $\mathbf{j}_\varepsilon(x) = \varepsilon^{-N}\mathbf{j}(\frac{x}{\varepsilon})$ and $\mathbf{j} : \mathbb{R}^N \to \mathbb{R}^+$ is in $C_c^\infty(\mathbb{R}^N)$, a function inducing a probability measure).*
(iii) *For any $u \in X$, there exists a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in X.*

Theorem 3.10. *Assume conditions of Theorem 3.8.*
(i) *For any $\alpha \leq s \leq p^*$, the space $X(\Omega)$ is continuously embedded into $L^{s(\cdot)}(\Omega)$.*
(ii) *For any bounded subset $\Omega \subset \mathbb{R}^N$, there is a compact embedding $X(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ for any $1 \leq s \leq p^*$.*
(iii) *We also assume that $\vartheta(\cdot) \in C(\mathbb{R}^N)$ is Lipschitz continuous, $w \in L^{r(\cdot)}$ and*

$$\alpha(\cdot) \leq r'(\cdot)\vartheta(\cdot) \leq p^*(\cdot) \quad \text{in } \mathbb{R}^N.$$

Then there is a compact embedding $X \hookrightarrow L_w^{\vartheta(\cdot)}$.

Proof. The proofs of (i) and (ii) are trivial from Proposition 2.7. We only need to prove (iii). Since X is embedded into $L^{r'(\cdot)\vartheta(\cdot)}(\mathbb{R}^N)$ for $\alpha(\cdot) \leq r'(\cdot)\vartheta(\cdot) \ll p^*(\cdot)$, we may assume that $u_n \rightharpoonup u$ in X. Then $\{\|u_n\|\}$ is bounded and the continuous embedding $X \hookrightarrow L^{r'(\cdot)\vartheta(\cdot)}(\mathbb{R}^N)$ guarantees the boundedness of $\{|u_n|_{L^{r'(\cdot)\vartheta(\cdot)}}\}$. So, there is a positive constant M such that

$$\sup\{||u_n|^{\vartheta(x)}|_{L^{r'(\cdot)}}, ||u|^{\vartheta(x)}|_{L^{r'(\cdot)}}\} \leq M.$$

Set $B_k = \{x \in \mathbb{R}^N : |x| < k\}$. If $w \in L^{r(\cdot)}(\mathbb{R}^N)$, then

$$|w|_{L^{r(\cdot)}(\mathbb{R}^N \setminus B_k)} \to 0 \quad \text{as } k \to \infty.$$

For any $\varepsilon > 0$, we can find large enough $k_1 > 0$ such that

$$|w|_{L^{r(\cdot)}(\mathbb{R}^N \setminus B_k)} \leq \frac{\varepsilon}{8M} \quad \text{for all } k \geq k_1.$$

From (ii) of this theorem, there is a compact embedding $X(B_{k_1}) \hookrightarrow L^{r'(\cdot)\vartheta(\cdot)}(B_{k_1})$, so $u_n \rightharpoonup u$ implies

$$\int_{B_{k_1}} |w(x)||u_n - u|^{\vartheta(x)}\,dx \leq |w(x)|_{L^{r(\cdot)}(B_{k_1})}||u_n - u|^{\vartheta(x)}|_{L^{r'(\cdot)}(B_{k_1})} \to 0 \quad \text{as } n \to +\infty.$$

Thus, there exists $n_1 > 0$ such that for all $n \geq n_1$ we have

$$\int_{B_{k_1}} |w(x)||u_n - u|^{\vartheta(x)}\,dx < \frac{\varepsilon}{2}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^N} |w(x)||u_n - u|^{\vartheta(x)}\,dx &\leq \int_{B_{k_1}} |w(x)||u_n - u|^{\vartheta(x)}\,dx + \int_{\mathbb{R}^N\setminus B_{k_1}} |w(x)||u_n - u|^{\vartheta(x)}\,dx \\ &\leq \frac{\varepsilon}{2} + 2|w|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_k)}\big||u_n - u|^{\vartheta(x)}\big|_{L^{r'(\cdot)}(\mathbb{R}^N\setminus B_k)} \\ &\leq \frac{\varepsilon}{2} + 2|w|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_k)}\big\{\big||u_n|^{\vartheta(x)}\big|_{L^{r'(\cdot)}(\mathbb{R}^N\setminus B_k)} + \big||u|^{\vartheta(x)}\big|_{L^{r'(\cdot)}(\mathbb{R}^N\setminus B_k)}\big\} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We conclude that $u_n \to u$ in $L_w^{\vartheta(\cdot)}$. This completes the proof. □

4 Properties of functionals and operators

By (vii) of Proposition 3.2, we deduce that $\mathscr{A}(x, \nabla u)$ is integrable on $\mathbb{R}^N$ for all $u \in X$. Thus, $\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx$ is well defined. For $u \in X$, it follows by (2.2) that

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u)\cdot\nabla u\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)}\,dx \geq c_1\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}^c} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)}\,dx\Big)$$

and

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u)\cdot\nabla u\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)}\,dx \leq c_2\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}^c} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)}\,dx\Big),$$

where c_1 and c_2 are positive constants. Similarly, using (2.2), we get for all $u \in X$,

$$\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx \geq c_1\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}^c} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx\Big)$$

and

$$\int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx \leq c_2\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}^c} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx\Big).$$

From $(\mathcal{H}_f^1)$, we have $|g(x, u)| \leq |u|^{\alpha(x)-1} + C|u|^{\gamma(x)-1}$. Notice that $\alpha(\cdot) \leq \gamma(\cdot) \leq \frac{p^*(\cdot)}{r'(\cdot)}$. Combining $(\mathcal{H}_w)$ and $(\mathcal{H}_a)$, it is easy to check that $f(x, u)v$ and $F(x, u)$ are integrable on $\mathbb{R}^N$ for all $u, v \in X$.

We say that $u \in X$ is a solution of problem $(\mathcal{E})$ if

$$\int_{\mathbb{R}^N} \mathbf{A}(x, \nabla u)\cdot\nabla v\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)-2}uv\,dx = \int_{\mathbb{R}^N} f(x, u)v\,dx, \qquad v \in X.$$

It follows that solutions of $(\mathcal{E})$ correspond to the critical points of the Euler–Lagrange energy functional $\Phi : X \to \mathbb{R}$, defined by

$$\Phi = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} F(x, u)\,dx,$$

where $F(x, u) = \int_0^u f(x, s)\,ds$.

Define the functionals $\Phi_{\mathscr{A}}, \Phi_\alpha, \Phi_f : X \to \mathbb{R}$ by

$$\Phi_{\mathscr{A}}(u) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\, dx,$$

$$\Phi_\alpha(u) = \int_{\mathbb{R}^N} \frac{1}{\alpha(x)} |u|^{\alpha(x)}\, dx,$$

$$\Phi_f(u) = \int_{\mathbb{R}^N} F(x, u)\, dx.$$

Lemma 4.1 (see [56, Lemma 4.1]). *Assume the structure conditions $(\mathcal{A}_1)$. Then the functional $\Phi_{\mathscr{A}}$ is convex, of class C^1, and sequentially weakly lower semicontinuous in X. Moreover, $\Phi'_{\mathscr{A}} : X \to X^*$ is bounded.*

Lemma 4.2 (see [56, Lemma 4.2]). *Assume the structure conditions $(\mathcal{A}_1)$ (iv). Then the functional Φ_α is convex, of class C^1 and sequentially weakly lower semicontinuous. Moreover, if $u_n, u \in X$ and $u_n \rightharpoonup u$ in X, then $\Phi'_\alpha(u_n) \overset{*}{\rightharpoonup} \Phi'_\alpha(u)$ in X^*.*

Lemma 4.3 (see [56, Lemma 4.3]). *Assume $(\mathcal{A}_1)$, $(\mathcal{H}_f^1)$, $(\mathcal{H}_w)$ and $(\mathcal{H}_a)$. Then Φ_f is of class C^1 and sequentially weakly-strongly continuous, that is, if $u_n \rightharpoonup u$ in X, then $\Phi_f(u_n) \to \Phi_f(u)$ and $\Phi'_f(u_n) \to \Phi'_f(u)$ in X^*.*

Proof. Since X is embedded into $L^{\gamma(\cdot)}(\mathbb{R}^N)$ for $\alpha(\cdot) \le \gamma(\cdot) \le \frac{p^*(\cdot)}{r'(\cdot)}$, we deduce that $F(x, u)$ is integrable on $\mathbb{R}^N$, hence $\Phi_f(u)$ is well defined.

Now, let us prove that is Φ_f weakly-strongly continuous. Assume that $u_n \rightharpoonup u$ in X; then $\{u_n\}$ is bounded in X, hence $\{|u_n|_{L^{r'(\cdot)\alpha(\cdot)}}\}$ and $\{|u_n|_{L^{r'(\cdot)\gamma(\cdot)}}\}$ are bounded. Since

$$|G(x, t)| \le \frac{1}{\alpha(x)} |t|^{\alpha(x)} + \frac{c}{\gamma(x)} |t|^{\gamma(x)},$$

we have

$$w(x)|G(x, u_n) - G(x, u)| \le w(x)\{|u|^{\alpha(x)} + |u_n|^{\alpha(x)} + c|u_n|^{\gamma(x)} + c|u|^{\gamma(x)}\}.$$

Therefore $\{w(x)|G(x, u_n) - G(x, u)|\}$ is uniformly integrable in $\mathbb{R}^N$. By Theorem 3.10, $u_n \to u$ a.e. in $\mathbb{R}^N$. Thus, by Vitali's theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} w(x)|G(x, u_n) - G(x, u)|\, dx = \int_{\mathbb{R}^N} \lim_{n\to\infty} w(x)|G(x, u_n) - G(x, u)|\, dx = 0.$$

Similarly, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{a(x)}{\delta(x)} \big||u_n|^{\delta(x)} - |u|^{\delta(x)}\big|\, dx = \int_{\mathbb{R}^N} \lim_{n\to\infty} \frac{a(x)}{\delta(x)} \big||u_n|^{\delta(x)} - |u|^{\delta(x)}\big|\, dx = 0.$$

We conclude that $\Phi_f(u_n) \to \Phi_f(u)$. In a similar way, we can obtain the weakly-strongly continuity of Φ'_f. □

Lemma 4.4. *Assume $(\mathcal{A}_1)$, $(\mathcal{H}_f^1)$, $(\mathcal{H}_w)$ and $(\mathcal{H}_a)$. Then the functional Φ is of class C^1 and sequentially weakly lower semicontinuous in X, that is, if $u_n \rightharpoonup u_0$ in X, then*

$$\Phi(u_0) \le \liminf_{n\to\infty} \Phi(u_n).$$

Proof. Invoking Lemmas 4.1–4.3, we obtain the C^1-continuity of Φ. Next, we will prove that Φ is the sequentially weakly lower semicontinuous in X. By Lemma 4.3, $\Phi_f(u)$ is weakly continuous. Obviously,

$$\begin{aligned}
\liminf_{n\to\infty} \Phi(u_n) &\ge \liminf_{n\to\infty} (\Phi_{\mathscr{A}}(u_n) + \Phi_\alpha(u_n)) - \limsup_{n\to\infty} \Phi_f(u_n) \\
&\ge \Phi_{\mathscr{A}}(u_0) + \Phi_\alpha(u_0) - \Phi_f(u_0) \\
&= \Phi(u_0).
\end{aligned}$$

Thus, Φ is sequentially weakly lower semicontinuous in X. □

Lemma 4.5 (see [56, Lemma 4.6]). *Suppose that $\mathscr{A}$ satisfies $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ (namely $\mathscr{A}(x,\cdot):\mathbb{R}^N\to\mathbb{R}$ is a uniformly convex function), that is, for any $\varepsilon\in(0,1)$ there exists $\delta(\varepsilon)\in(0,1)$ such that*

$$\mathscr{A}\left(x,\frac{u+v}{2}\right)\le\frac{(1-\delta(\varepsilon))}{2}(\mathscr{A}(x,u)+\mathscr{A}(x,v))$$

for all $x\in\mathbb{R}^N$ and all $(u,v)\in\mathbb{R}^N$ with $|u-v|\le\varepsilon\max\{|u|,|v|\}$. Then we have:

(i) *$\Phi_{\mathscr{A}}(\cdot):X\to\mathbb{R}$ is uniformly convex, that is, for any $\varepsilon\in(0,1)$ there exists $\delta(\varepsilon)\in(0,1)$ such that for all $u,v\in X$,*

$$\Phi_{\mathscr{A}}\left(\frac{u-v}{2}\right)\le\varepsilon\frac{\Phi_{\mathscr{A}}(u)+\Phi_{\mathscr{A}}(v)}{2}\quad\text{or}\quad\Phi_{\mathscr{A}}\left(\frac{u+v}{2}\right)\le(1-\delta(\varepsilon))\frac{\Phi_{\mathscr{A}}(u)+\Phi_{\mathscr{A}}(v)}{2}.$$

(ii) *If $u_n\rightharpoonup u$ in X and $\overline{\lim}_{n\to\infty}(\Phi'_{\mathscr{A}}(u_n)-\Phi'_{\mathscr{A}}(u),u_n-u)\le 0$, then*

$$\Phi_{\mathscr{A}}(u_n-u)\to 0\quad\text{and}\quad|\nabla u_n-\nabla u|_{L^{p(\cdot)}+L^{q(\cdot)}}\to 0.$$

Define $\rho(\cdot):X\to\mathbb{R}$ as

$$\rho(u)=\int_{\mathbb{R}^N}\mathscr{A}(x,\nabla u)\,dx+\int_{\mathbb{R}^N}\frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx,$$

and we denote the derivative operator by L, that is, $L=\rho':X\to X^*$ with

$$(L(u),v)=\int_{\mathbb{R}^N}\mathbf{A}(x,\nabla u)\nabla v\,dx+\int_{\mathbb{R}^N}|u|^{\alpha(x)-2}uv\,dx,\quad u,v\in X.$$

Lemma 4.6 (see [56, Lemma 4.7]). *Under the structure conditions $(\mathcal{A}_1)$, we have the following property:*

(i) *$L:X\to X^*$ is a continuous, bounded and strictly monotone operator.*

If $(\mathcal{A}_2)$ is also satisfied, then we have:

(ii) *L is a mapping of type (S_+), that is, if $u_n\rightharpoonup u$ in X and $\overline{\lim}_{n\to\infty}(L(u_n)-L(u),u_n-u)\le 0$, then $u_n\to u$ in X.*

(iii) *$L:X\to X^*$ is a homeomorphism.*

Lemma 4.7. *We assume the structure conditions $(\mathcal{A}_1)$–$(\mathcal{A}_2)$, $(\mathcal{H}_f^1)$–$(\mathcal{H}_f^2)$, $(\mathcal{H}_w)$, $(\mathcal{H}_a)$, $1\ll\alpha(\cdot)\ll p^*(\cdot)\frac{q'(\cdot)}{p'(\cdot)}$ and $\alpha\le p$. Then Φ satisfies the* (PS) *condition, that is, if $\{u_n\}\subset X$ satisfies $\Phi(u_n)\to c$ and $\|\Phi'(u_n)\|_{X^*}\to 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. Assume that $\{u_n\}$ is bounded. Then, up to a subsequence, $u_n\rightharpoonup u_0$. By Lemma 4.3, again up to a subsequence, we have $\Phi'_f(u_n)\to\Phi'_f(u_0)$ in X^*. By Lemma 4.6, L^{-1} is continuous from X^* to X, hence $u_n\to L^{-1}\circ\Phi'_f(u_0)$ in X. We only need to prove that $\{u_n\}$ is bounded in X. We argue by contradiction. Suppose not; then there exist $c\in\mathbb{R}$ and $\{u_n\}\subset X$ satisfying

$$\Phi(u_n)\to c,\quad\|\Phi'(u_n)\|_{X^*}\to 0,\quad\|u_n\|\to+\infty.$$

Since $(\Phi'(u_n),\frac{1}{\theta}u_n)\to 0$, we may assume that

$$\begin{aligned}c+\|u_n\|&\ge\Phi(u_n)-(\Phi'(u_n),\tfrac{1}{\theta}u_n)\\&=\int_{\mathbb{R}^N}\mathscr{A}(x,\nabla u_n)\,dx+\int_{\mathbb{R}^N}\frac{1}{\alpha(x)}|u_n|^{\alpha(x)}\,dx-\int_{\mathbb{R}^N}F(x,u_n)\,dx\\&\qquad-\left\{\int_{\mathbb{R}^N}\frac{1}{\theta}\mathbf{A}(x,\nabla u_n)\nabla u_n\,dx+\int_{\mathbb{R}^N}\frac{1}{\theta}|u_n|^{\alpha(x)}\,dx-\int_{\mathbb{R}^N}\frac{1}{\theta}f(x,u_n)u_n\,dx\right\}\\&\ge\int_{\mathbb{R}^N}\left(1-\frac{s(x)}{\theta}\right)\mathscr{A}(x,\nabla u_n)\,dx+\int_{\mathbb{R}^N}\left(\frac{1}{\alpha(x)}-\frac{1}{\theta}\right)|u_n|^{\alpha(x)}\,dx+\int_{\mathbb{R}^N}\left\{\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\right\}dx.\end{aligned}$$

It follows that

$$\begin{aligned}c+\|u_n\|\ge\Phi(u_n)-(\Phi'(u_n),\tfrac{1}{\theta}u_n)&\ge c_1\int_{\mathbb{R}^N}\mathscr{A}(x,\nabla u_n)+|u_n|^{\alpha(x)}\,dx-\int_{\mathbb{R}^N}\left(\frac{1}{\delta(x)}-\frac{1}{\theta}\right)\lambda a(x)|u_n|^{\delta(x)}\,dx\\&\ge c_1\|u_n\|^{\alpha^-}-C\|u_n\|^{\delta^+}-C.\end{aligned}$$

Notice that $\alpha^->\delta^+>1$. Thus, we obtain a contradiction. The proof is complete. □

5 Proof of the theorems

In this section, we will give the proofs of Theorems 1.1–1.3.

5.1 Proof of Theorem 1.1

Let us consider the following auxiliary problem:

$$-\operatorname{div}\mathbf{A}(x,\nabla u)+|u|^{\alpha(x)-2}u=f^{+}(x,u), \tag{$\mathcal{E}^+$}$$

where

$$f^{+}(x,u)=\begin{cases} f(x,u) & \text{if } f(x,u)\geq 0,\\ 0 & \text{if } f(x,u)<0.\end{cases}$$

The corresponding Euler–Lagrange functional is

$$\Phi^{+}(u)=\int\limits_{\mathbb{R}^N}\mathscr{A}(x,\nabla u)\,dx+\int\limits_{\mathbb{R}^N}\frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx-\int\limits_{\mathbb{R}^N}F^{+}(x,u)\,dx,$$

where $F^{+}(x,u)=\int_0^u f^{+}(x,t)\,dt$. Similar to the proof of Lemma 4.7, we deduce that Φ^{+} satisfies the (PS) condition.

Next, we prove that $\Phi^{+}(u)$ satisfies the conditions of the mountain pass lemma. By assumption $(\mathcal{H}_f^2)$, we have

$$g(x,tu)tu\geq\theta G(x,tu)>0,\quad u\neq 0,\ t\neq 0,\ x\in\mathbb{R}^N,$$

and

$$\frac{g(x,tu)u}{G(x,tu)}\geq\frac{\theta}{t}>0,\quad u\neq 0,\ t>0,\ x\in\mathbb{R}^N.$$

Integrating with respect to t from 1 to t, we have

$$G(x,tu)\geq|t|^{\theta}G(x,u)\geq 0,\quad u\in\mathbb{R},\ t\geq 1,\ x\in\mathbb{R}^N, \tag{5.1}$$

and

$$0\leq G(x,tu)\leq|t|^{\theta}G(x,u),\quad u\in\mathbb{R},\ t\in(0,1],\ x\in\mathbb{R}^N.$$

Hence, if $0<\|u\|\leq 1$, then

$$\begin{aligned}\Phi^{+}(u)&\geq\frac{1}{s^{+}}\|u\|^{s^{+}}-\|u\|^{\theta}\int\limits_{\mathbb{R}^N}\mu w(x)G^{+}\Big(x,\frac{u}{\|u\|}\Big)\,dx-\int\limits_{\mathbb{R}^N}\lambda\frac{a(x)}{\delta(x)}|u|^{\delta(x)}\,dx\\&\geq\frac{1}{s^{+}}\|u\|^{s^{+}}-c\|u\|^{\theta}-\lambda\|u\|^{\delta(\xi)}\quad\text{for some }\xi\in\mathbb{R}^N.\end{aligned}$$

Let $\epsilon>0$ and $\lambda>0$ be small enough. Then

$$\Phi(u)\geq c>0\quad\text{for all }\|u\|=\epsilon. \tag{5.2}$$

For $0\leq u\in X\setminus\{0\}$ and $t>1$, we have

$$\begin{aligned}\Phi^{+}(tu)&=\int\limits_{\mathbb{R}^N}\mathscr{A}(x,\nabla tu)\,dx+\int\limits_{\mathbb{R}^N}\frac{1}{\alpha(x)}|tu|^{\alpha(x)}\,dx-\int\limits_{\mathbb{R}^N}F(x,tu)\,dx\\&\leq t^{s^{+}}\int\limits_{\mathbb{R}^N}\mathscr{A}(x,\nabla tu)\,dx+t^{\alpha^{+}}\int\limits_{\mathbb{R}^N}\frac{1}{\alpha(x)}|tu|^{\alpha(x)}\,dx-t^{\theta}\int\limits_{\mathbb{R}^N}\mu w(x)G(x,u)\,dx.\end{aligned}$$

Since $\alpha^{+}\leq s^{+}<\theta$, it follows that $\Phi(tw)\to-\infty$ $(t\to+\infty)$. Obviously, $\Phi^{+}(0)=0$, hence Φ^{+} satisfies the conditions of the mountain pass lemma.

So, Φ^+ admits at least one nontrivial critical point u_1 that satisfies $\Phi^+(u_1) > 0$. Thus $(\mathcal{E}^+)$ has a solution u_1, and it is easy to see that $u_1 \geq 0$, so u_1 is a nonnegative solution u_1 of $(\mathcal{E})$ with $\Phi(u_1) > 0$. Similarly, we can establish the existence of a nonpositive solution u_2 of $(\mathcal{E})$ with $\Phi(u_2) > 0$. Define $h_0 \in C_0(\overline{B(x_0, \varepsilon_0)})$ as

$$h_0(x) = \begin{cases} 0, & |x - x_0| \geq \varepsilon_0, \\ \varepsilon_0 - |x - x_0|, & |x - x_0| < \varepsilon_0. \end{cases}$$

Let $\varepsilon_0 > 0$ be small enough. By $(\mathcal{H}_a)$, we have $\Phi^+(th_0) < 0$ for small enough $t > 0$. Combining (5.2) and Lemma 4.4, we deduce that Φ^+ attains its infimum on $\{u \in X : \|u\| < \epsilon\}$. Therefore, Φ^+ admits at least one nontrivial critical point u_3 satisfying $\Phi^+(u_3) < 0$. It is easy to see that $u_3 \geq 0$, so u_3 is a nonnegative solution of $(\mathcal{E})$ with $\Phi(u_3) < 0$. Similarly, we can establish the existence of a nonpositive solution u_4 of $(\mathcal{E})$ with $\Phi(u_4) < 0$. The proof is complete.

5.2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need to recall some preliminary results. Since X is a reflexive and separable Banach space (see [57, Section 17, Theorems 2–3], there exist sequences $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}}\{e_j : j = 1, 2, \ldots\}, \quad X^* = \overline{\operatorname{span}}^{w^*}\{e_j^* : j = 1, 2, \ldots\},$$

and

$$\langle e_j^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For convenience, we write

$$X_j = \operatorname{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{5.3}$$

Let

$$\Theta(u) = |u|_{L_a^{\delta(\cdot)}} + |u|_{L_w^{\alpha(\cdot)}} + |u|_{L_w^{\gamma(\cdot)}}.$$

By Theorem 3.10, similar to the proof of Lemma 4.3, we deduce that $\Theta : X \to \mathbb{R}$ is weakly-strongly continuous and $\Theta(0) = 0$.

Lemma 5.1 (see [42, Lemma 5.1]). *Assume that $\Theta : X \to \mathbb{R}$ is weakly-strongly continuous and $\Theta(0) = 0$, $\gamma_0 > 0$ is a given number. Let*

$$\beta_k = \beta_k(\gamma_0) = \sup\{\Theta(u) : \|u\| \leq \gamma_0,\ u \in Z_k\}.$$

Then $\beta_k \to 0$ as $k \to \infty$.

To complete the proof of Theorem 1.2, we recall the following critical point lemma (see, e.g., [59, Theorem 4.7]).

Lemma 5.2. *Suppose that $\Phi \in C^1(X, R)$ is even and satisfies the* (PS) *condition. Let $V^+, V^- \subset X$ be closed subspaces of X with* $\operatorname{codim} V^+ + 1 = \dim V^-$, *and suppose that the following conditions are fulfilled:*
(1^0) $\Phi(0) = 0$.
(2^0) *There exist $\tau > 0$ and $\gamma_0 > 0$ such that for all $u \in V^+$, $\|u\| = \gamma_0 \Rightarrow \Phi(u) \geq \tau$.*
(3^0) *There exists $\rho > 0$ such that for all $u \in V^-$, $\|u\| \geq \rho \Rightarrow \Phi(u) \leq 0$.*
Consider the following set:

$$\Gamma = \{h \in C^0(X, X) : h \text{ is odd, } h(u) = u \text{ if } u \in V^- \text{ and } \|u\| \geq \rho\}.$$

Then:
(a) *For all $\delta_0 > 0$ and $h \in \Gamma$ one has $S_{\delta_0}^+ \cap h(V^-) \neq \varnothing$, where $S_{\delta_0}^+ = \{u \in V^+ : \|u\| = \delta_0\}$.*
(b) *The number $\varpi := \inf_{h \in \Gamma} \sup_{u \in V^-} \Phi(h(u)) \geq \tau > 0$ is a critical value for Φ.*

Proof of Theorem 1.2. According to our assumptions, Φ is an even functional and it satisfies the (PS) compactness condition. Let $V_k^+ = Z_k$, which is a closed linear subspace of X and $V_k^+ \oplus Y_{k-1} = X$.

Set $V_k^- = X_k$. We will prove that there are infinitely many pairs of V_k^+ and V_k^- such that φ satisfies the conditions of Lemma 5.2. We also show that the corresponding critical value $\varpi_k := \inf_{h\in\Gamma}\sup_{u\in V_k^-}\Phi(h(u))$ tends to $+\infty$ as $k\to\infty$, which implies that there are infinitely many pairs of solutions to problem $(\mathcal{E})$.

For any $k = 1, 2, \ldots$, we prove that there exist $\rho_k > \tau_k > 0$ and large enough k such that

(A1) $b_k := \inf\{\Phi(u) : u \in V_k^+,\ \|u\| = \tau_k\} \to +\infty$ as $k \to +\infty$,

(A2) $a_k := \max\{\Phi(u) | u \in V_k^-,\ \|u\| = \rho_k\} \le 0$.

We first show that (A1) holds. Let $\sigma \in (0, 1)$ be small enough. By $(\mathcal{H}_f^1)$, there exists $C(\sigma) > 0$ such that

$$G(x, u) \le \sigma|u|^{\alpha(x)} + C(\sigma)|u|^{\gamma(x)}, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}.$$

By computation, for any $u \in Z_k$ with $\|u\| = \tau_k = (2C_2\frac{1}{c_1}\beta_k^{\delta^+})^{\frac{1}{\alpha^- -\gamma^+}}$, we have

$$
\begin{aligned}
\Phi(u) &= \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla u)\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|u|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} F(x, u)\,dx \\
&\ge 2c_1\Big(\int_{\mathbb{R}^N\cap\Lambda_{\nabla u}} |\nabla u|^{p(x)}\,dx + \int_{\mathbb{R}^N\cap\Lambda_{\nabla u}^c} |\nabla u|^{q(x)}\,dx + \int_{\mathbb{R}^N} |u|^{\alpha(x)}\,dx\Big) \\
&\quad - \int_{\mathbb{R}^N} \lambda\frac{a(x)}{\delta(x)}|u|^{\delta(x)}\,dx - \sigma\int_{\mathbb{R}^N} \mu w(x)|u|^{\alpha(x)}\,dx - C(\sigma)\int_{\mathbb{R}^N} \mu w(x)|u|^{\gamma(x)}\,dx \\
&\ge 2c_1\|u\|^{\alpha^-} - |u|^{\delta(\xi_1)}_{L_a^{\delta(\cdot)}} - \sigma|u|^{\alpha(\xi_2)}_{L_w^{\alpha(\cdot)}} - C(\sigma)|u|^{\gamma(\xi_3)}_{L_w^{\gamma(\cdot)}} \quad (\text{where } \xi_1, \xi_2, \xi_3 \in \mathbb{R}^N) \\
&\ge 2c_1\|u\|^{\alpha^-} - |u|^{\delta^+}_{L_a^{\delta(\cdot)}} - \sigma|u|^{\alpha^+}_{L_w^{\alpha(\cdot)}} - C(\sigma)|u|^{\gamma^+}_{L_w^{\gamma(\cdot)}} - C_1 \\
&\ge 2c_1\|u\|^{\alpha^-} - \beta_k^{\delta^+}\|u\|^{\delta^+} - \sigma\beta_k^{\alpha^+}\|u\|^{\alpha^+} - C(\sigma)\beta_k^{\gamma^+}\|u\|^{\gamma^+} - C_1 \\
&\ge c_1\|u\|^{\alpha^-} - C_2\beta_k^{\delta^+}\|u\|^{\gamma^+} - C_3 \\
&= c_1\Big(2C_2\frac{1}{c_1}\beta_k^{\delta^+}\Big)^{\frac{\alpha^-}{\alpha^- -\gamma^+}} - C_2\beta_k^{\delta^+}\Big(2C_2\frac{1}{c_1}\beta_k^{\delta^+}\Big)^{\frac{\gamma^+}{\alpha^- -\gamma^+}} - C_3 \\
&= \frac{c_1}{2}\Big(2C_2\frac{1}{c_1}\beta_k^{\gamma^+}\Big)^{\frac{\alpha^-}{\alpha^- -\gamma^+}} - C_3 \to +\infty \quad (\text{as } k \to \infty),
\end{aligned}
$$

because $1 < \delta^+ < \alpha^- < \gamma^+$ and $\beta_k \to 0^+$ as $k \to \infty$. Therefore, $b_k \to +\infty$ as $k \to \infty$.

Now, we show that (A2) holds. By $(\mathcal{H}_f^2)$ and (5.1), we deduce that

$$\Phi(tu) \to -\infty \text{ as } t \to +\infty \quad \text{for all } h \in V_k^- \text{ with } \|u\| = 1,$$

which implies that (A2) holds.

We conclude that the proof of Theorem 1.2 is complete. □

5.3 Proof of Theorem 1.3

We first prove that Φ is coercive on X. Note that $\mu = 0$. Since $1 < \delta^+ < \alpha^-$, we have

$$\Phi(u) \ge \frac{1}{q^+}\|u\|^{\alpha^-} - c\|u\|^{\delta^+} \to +\infty \quad \text{as } \|u\| \to \infty.$$

By Lemma 4.4, Φ is weakly lower semicontinuous. Then Φ attains its minimum on X, which provides a solution of problem $(\mathcal{E})$.

Since Φ is coercive, it follows that Φ satisfies the (PS) condition on X. By assumption $(\mathcal{A}_1)$ (i), Φ is an even functional. Denote by $\gamma(A)$ the genus of A (see [8, p. 215]). Set

$$
\begin{aligned}
\mathbb{R} &= \{A \subset X \setminus \{0\} : A \text{ is compact and } A = -A\}, \\
\mathbb{R}_k &= \{A \subset \mathbb{R} : \gamma(A) \ge k\}, \\
c_k &= \inf_{A\in\mathbb{R}_k}\sup_{u\in A}\Phi(u), \quad k = 1, 2, \ldots .
\end{aligned}
$$

We have

$$-\infty < c_1 \le c_2 \le \cdots \le c_k \le c_{k+1} \le \cdots .$$

In what follows we prove that $c_k < 0$ for every k.

For fixed k, we can choose a k-dimensional linear subspace E_k of X such that $E_k \subset C_0^\infty(B_R)$. Since the norms on E_k are equivalent, for any given $\delta_0 > 0$, there exists $\rho_k \in (0,1)$ such that $u \in E_k$ with $\|u\| \le \rho_k$ implies $|u|_{L^\infty} \le \delta_0$. Set

$$S_{\rho_k}^{(k)} = \{u \in E_k : \|u\| = \rho_k\}.$$

From the compactness of $S_{\rho_k}^{(k)}$, there exists $\theta_k > 0$ such that

$$\int_{\mathbb{R}^N} F(x,u)\,dx = \int_{\mathbb{R}^N} \frac{\lambda a(x)|u|^{\delta(x)}}{\delta(x)}\,dx \ge \theta_k, \quad u \in S_{\rho_k}^{(k)}.$$

For $u \in S_{\rho_k}^{(k)}$ and $t \in (0,1)$, we have

$$\Phi(tu) = \int_{\mathbb{R}^N} \mathscr{A}(x, \nabla tu)\,dx + \int_{\mathbb{R}^N} \frac{1}{\alpha(x)}|tu|^{\alpha(x)}\,dx - \int_{\mathbb{R}^N} \frac{\lambda a(x)|tu|^{\delta(x)}}{\delta(x)}\,dx \le C_1 \frac{t^{\alpha^-}}{\alpha^-}\rho_k^{\alpha^-} - t^{\delta^+}\theta_k.$$

Since $1 < \delta^+ < \alpha^-$, we can find $t_k \in (0,1)$ and $\varepsilon_k > 0$ such that

$$\Phi(t_k u) \le -\varepsilon_k < 0 \quad \text{for all } u \in S_{\rho_k}^{(k)},$$

that is,

$$\Phi(u) \le -\varepsilon_k < 0 \quad \text{for all } u \in S_{t_k\rho_k}^{(k)}.$$

Obviously, $\gamma(S_{t_k\rho_k}^{(k)}) = k$, so $c_k \le -\varepsilon_k < 0$.

By the genus theory (see [8, p. 219, Theorem 3.3]), each c_k is a critical value of Φ, hence there is a sequence of solutions $\{\pm u_k : k = 1,2,\dots\}$ such that $\Phi(\pm u_k) < 0$. It only remains to prove that $c_k \to 0^-$ as $k \to \infty$. Since Φ is coercive, there exists a constant $R > 0$ such that $\Phi(u) > 0$ when $\|u\| \ge R$. Taking $A \in \mathbb{R}_k$ arbitrarily, we have $\gamma(A) \ge k$. Let Y_k and Z_k be the subspaces of X as mentioned in (5.3). According to the properties of genus, we know that $A \cap Z_k \ne \varnothing$. Let $\beta_k = \sup\{|\Phi_f(u)| : u \in Z_k,\ \|u\| \le R\}$. By Lemma 5.1, we have $\beta_k \to 0$ as $k \to \infty$. For all $u \in Z_k$ with $\|u\| \le R$, we have

$$\Phi(u) = \Phi_{\mathscr{A}}(u) + \Phi_\alpha(u) - \Phi_f(u) \ge -\Phi_f(u) \ge -\beta_k.$$

Hence $\sup_{u\in A}\Phi(u) \ge -\beta_k$, and thus $c_k \ge -\beta_k$. We conclude that $c_k \to 0^-$ as $k \to \infty$.

6 Perspectives and open problems

We now address to the readers several comments, perspectives, and open problems.

(i) Hypothesis $(\mathcal{A}_1)$ (iv) establishes that problem $(\mathcal{E})$ is described in the *subcritical* setting. To the best of our knowledge, there is no result in the literature corresponding to the following *almost critical* framework described in what follows. Assume that condition $q(\cdot) \ll \min\{N, p^*(\cdot)\}$ in $(\mathcal{A}_1)$ (iv) is replaced with the following hypothesis: there exists a finite set $A \subset \mathbb{R}^N$ such that $q(a) = \min\{N, p^*(a)\}$ for all $a \in A$ and $q(x) < \min\{N, p^*(x)\}$ for all $x \in \mathbb{R}^N \setminus A$.

Open problem. Study if Theorems 1.1–1.3 established in this paper still remain true in the above almost critical abstract setting.

(ii) Another very interesting research direction is to extend the approach developed in this paper to the case of *double phase* problems studied in [5, 12, 13]. This corresponds to the non-homogeneous potential

$$\mathscr{A}(x,\xi) = \frac{a(x)}{p(x)}|\xi|^{p(x)} + \frac{b(x)}{q(x)}|\xi|^{q(x)},$$

where the coefficients $a(x)$ and $b(x)$ are nonnegative and at least one is strictly positive for all $x \in \mathbb{R}^N$. At present, we do not know any multiplicity results for double phase problems of this type.

We also refer to the pioneering papers by Marcellini [35, 36] on (p, q)-growth conditions, which involve integral functionals of the type

$$W^{1,1} \ni u \mapsto \int_{\Omega} f(x, \nabla u)\, dx,$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set. The integrand $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies unbalanced polynomial growth conditions of the type

$$|\xi|^p \lesssim f(x, \xi) \lesssim |\xi|^q + 1 \quad \text{with } 1 < p < q$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

(iii) The differential operator $\mathscr{A}(x, \xi)$ considered in problem $(\mathcal{E})$ falls in the realm of those related to the so-called Musielak–Orlicz spaces (see [40, 41]), more generally, of the operators having non-standard growth conditions (which are widely considered in the calculus of variations). These function spaces are Orlicz spaces whose defining Young function exhibits an additional dependence on the x variable. Indeed, classical Orlicz spaces L^Φ are defined requiring that a member function f satisfies

$$\int_{\Omega} \Phi(|f|)\, dx < \infty,$$

where $\Phi(t)$ is a Young function (convex, non-decreasing, $\Phi(0) = 0$). In the new case of Musielak–Orlicz spaces, the above condition becomes

$$\int_{\Omega} \Phi(x, |f|)\, dx < \infty.$$

The problems considered in this paper are indeed driven by the function

$$\Phi(x, |\xi|) := \begin{cases} |\xi|^{p(x)} & \text{if } |\xi| \le 1, \\ |\xi|^{q(x)} & \text{if } |\xi| \ge 1. \end{cases} \tag{6.1}$$

When $p(x) = q(x)$, we find the so-called variable exponent spaces, which are defined by

$$\Phi(x, |\xi|) := |\xi|^{p(x)}.$$

We conclude these comments by pointing out that the present paper is concerned with a double phase variant of the operators stemming from the energy generated by the function defined in (6.1).

(iv) An interesting double phase-type operator considered in the papers of Baroni, Colombo and Mingione [5, 12, 13] addresses functionals of the type

$$w \mapsto \int_{\Omega} (|\nabla w|^p + a(x)|\nabla w|^q)\, dx, \tag{6.2}$$

where $a(x) \ge 0$. The meaning of this functional is also to give a sharper version of the energy

$$w \mapsto \int_{\Omega} |\nabla w|^{p(x)}\, dx,$$

thereby describing sharper phase transitions. Composite materials with locally different hardening exponents p and q can be described using the energy defined in (6.2). Problems of this type were also motivated by applications to elasticity, homogenization, modelling of strongly anisotropic materials, Lavrentiev phenomenon, etc.

Accordingly, a new double phase model can be given by

$$\Phi_d(x, |\xi|) := \begin{cases} |\xi|^p + a(x)|\xi|^q & \text{if } |\xi| \le 1, \\ |\xi|^{p_1} + a(x)|\xi|^{q_1} & \text{if } |\xi| \ge 1, \end{cases}$$

with $a(x) \ge 0$.

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